ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 11, Number 1, winter 1981

QUOTIENTS OF $C(X)$ BY UNIFORM ALGEBRAS

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ABSTRACT. If A is a uniform algebra on X and M is a closed A-submodule of $C(X)$, the following are shown: that $C(X)/A$ is not separable provided A is antisymmetric and X is totally disconnected and non-metrizable (in particular giving yet another proof that L^∞/H^∞ is not separable); that $C(X)/A$ is not reflexive unless $A = C(X)$; and that, at least under a suitable additional hypothesis on A, $C(X)/M$ is not reflexive if M has infinitecodimension in $C(X)$.

1. **Introduction**. Several years ago, E. Berkson and L.A. Rubel published an article entitled "Seven different proofs that L^∞/H^∞ is not separable" [1]. The present paper consists of an eighth proof (which appears to be the simplest available) from the point of view of function algebras, and of a number of results arising from investigations related to or suggested by it. Ultimately, the objective is to study the Banach space structure (there is no obvious algebra structure) of $C(X)/A$, where A is a uniform algebra on X. We shall show that under certain conditions $C(X)/A$ cannot be separable, and that (if $A \neq C(X)$) it cannot be reflexive. We shall also briefly consider $C(X)/M$ for M a closed A-submodule of $C(X)$.

2. **Non-separability**. By a uniform algebra A on the compact Hausdorff space X we mean a (uniformly) closed subalgebra of $C(X)$ such that A separates the points of X and contains the constant functions (the latter is essentially irrelevent for most of what follows). As is well known, L^∞ can be viewed as $C(X)$ for a certain totally disconnected non-metrizable compact Hausdorff space X, and H^∞ as a uniform algebra A on this X; the last chapter of K. Hoffman's book [4] serves as a good reference on this representation. It is also well known that this A is antisymmetric, that is, the only real-valued functions in A are the constants. Thus the non-separability of L^∞/H^∞ is a particular instance of the following theorem.

THEOREM. *Let A be an antisymmetric uniform algebra on a compact Hausdorff space X which contains uncountably many distinct open-and-closed sets (which happens, for example, if X is totally disconnected and non-metrizable, and in any case implies non-metrizability of X). Then $C(X)/A$ is not separable.*

AMS 1970 (MOS) subject classification: Primary 46J10.
Key words and phrases: uniform algebra, antisymmetric, separable, reflexive.
Received by the editors on March 26, 1979.

PROOF. It is enough to show that if K and L are distinct open-and-closed subsets of X, then their indicator functions χ_K and χ_L are at distance $\geqq 1/2$ in $C(X)/A$, that is,

$$\|[\chi_K - \chi_L]\|_{C(X)/A} \geqq 1/2$$

where [.] denotes coset. But if this is false there is $g \in A$ such that

$$\|\chi_K - \chi_L - g\|_X = \delta < 1/2.$$

Thus if $B(z; r) = \{z' \in \mathbf{C}: |z' - z| \leqq r\}$ for $z \in \mathbf{C}$ and $r \geqq 0$, we have $g((K \cap L) \cup (X\backslash(K \cup L))) \subset B(0; \delta)$, $g(K|L) \subset B(1; \delta)$, and $g(L\backslash K) \subset B(-1; \delta)$. The $B(j; \delta)$ are disjoint closed discs, hence there is a sequence of polynomials (P_k) such that $P_k \to j$ uniformly on $B(j; \delta)$, $j = -1, 0, 1$. Thus $P_k \circ g \in A$ and $P_k \circ g \to \chi_K - \chi_L$ uniformly on X, hence $\chi_K - \chi_L \in A$, a contradiction since $\chi_K - \chi_L$ is realvalued and non-constant. The proof is complete.

REMARKS. (1) The idea of this proof goes back (at least) to H.S. Bear's proof of A.M. Gleason's result that if X is totally disconnected and $\{\mathrm{Re}(f): f \in A\}$ is uniformly dense in $C_R(X)$, then $A = C(X)$ [**4**, p. 182].

(2) Trivial examples, for instance $A = C(X)$, show the necessity of some hypothesis such as antisymmetry.

(3) Our theorem contains not only the non-separability of L^∞/H^∞ on the circle, but also that of L^∞/H^∞ on the disc; the latter was obtained by a separate argument in [**1**].

(4) The metrizability of X, the separability of $C(X)$, and the separability of A are mutually equivalent.

(5) We have been unable to ascertain whether the theorem remains true if the only topological assumption on the compact Hausdorff space X is non-metrizability (but see remark (1) in §3 for a special case). This problem really does belong in the context of subalgebras, and not merely subspaces, of $C(X)$, as we now indicate by an example. The direct sum $\ell^\infty \oplus c$ is $C(X)$ for an appropriate (totally disconnected non-metrizable) compact Hausdorff space X. Let $\mathbf{x} = (x_n)_1^\infty$ and $\mathbf{y} = (y_n)_1^\infty$ denote typical elements of ℓ^∞ and c. Let $B = \{(\mathbf{x}, \mathbf{y}) \in \ell^\infty \oplus c: y_n = ix_n/n\}$. Then B is a closed point-separating subspace of $C(X)$ which contains no non-zero real-valued functions, yet has a separable complement (namely c).

3. **Non-reflexivity**. In the remainder of this paper we shall be concerned for the most part with the existence of copies of c_0 in $C(X)/A$. Obviously this existence implies the non-reflexivity of $C(X)/A$. On the other hand, a result of A. Pelczynski [**5**] shows the reverse implication: every non-reflexive quotient of $C(X)$ must contain a copy of c_0.

We begin with a preliminary result about uniform algebras. Recall that a non-empty closed subset E of X is an interpolation set for A provided $A|E = C(E)$. We remark that trivial examples show the necessity of some extra hypothesis such as (a) or (b) in the following lemma.

LEMMA. *Let A be an antisymmetric uniform algebra on X, where X contains at least two points. Suppose that X is either* (a) $B(A)$ *the Silov boundary of A, or* (b) $M(A)$ *the maximal ideal space of A. Then any interpolation set for A has empty interior in X.*

PROOF. We first treat case (a), so $X = B(A)$. Suppose the interpolating set E contains a non-empty open set V. Choose a constant $M \geqq 1$ such that each $f \in C(E)$ has an extension $F \in A$ with $\|F\|_X \leq M\|f\|_E$. Since $X = B(A)$, we can find $f \in A$ such that $\|f\|_X = 1$ but $\|f\|_{X \setminus V} \leqq (3M)^{-1}$. Let $W = \{x \in X: |f(X)| > 1/2\}$, so W is non-empty and open, and $\bar{W} \subset V$. There is $g \in A$ such that $g = 1/f$ on $\bar{W}$, $\|g\|_E \leqq 2$, and $\|g\|_X \leqq 2M$. Thus $h = fg$ satisfies $h \in A$, $h = 1$ on $\bar{W}$, and $\|h\|_{X \setminus V} \leqq 2/3$. Suppose now that $u \in C(X)$ satisfies $u = 0$ on $X \setminus W$. Choose $U \in A$ such that $U = u$ on E. Then $h^n U \in A$ and $h^n U \to u$ uniformly, hence $u \in A$. In particular, u can be chosen non-constant and real-valued, giving a contradiction to the antisymmetry of A. This disposes of case (a).

For case (b), suppose that V is a non-empty open subset of $X = M(A)$. If $V \setminus B(A) \neq \emptyset$, then H. Rossi's local maximum modulus principle [**6**; cf. also **7**, §9] shows that V cannot be contained in an interpolation set for A, while if $V \subset B(A)$ case (a) gives the same conclusion.

THEOREM. *Let A be a uniform algebra on X, $A \neq C(X)$. Then $C(X)/A$ contains isomorphic copies of c_0, so is not reflexive. Furthermore, the isomorphism can be taken as close as desired to an isometry: given $\varepsilon > 0$ there is an isomorphism T of c_0 into $C(X)/A$ such that $\|T\| \leqq 1$ and $\|T^{-1}\| \leqq 1 + \varepsilon$.*

PROOF. By E. Bishop's generalized Stone-Weierstrass theorem [**2**; cf. also **3** and **7**, §12] there is a non-empty closed subset K of X such that $A|K$ is an antisymmetric uniform algebra on K and $A|K \neq C(K)$. Choose sequences $(V_n)_1^\infty$ and $(W_n)_1^\infty$ of open subsets of X such that $W_n \cap B(A|K) \neq \emptyset$, $\bar{W}_n \subset V_n$, and the V_n are pairwise disjoint. Now $A|\bar{W}_n \neq C(\bar{W}_n)$ by the lemma (applied to the restriction of A to $B(A|K)$), so by the standard geometric series approximation argument, given constants $0 < k < 1$ and $M > 0$ there is $u_n \in C(X)$ with $\|u_n\|_X = 1$ and $u_n = 0$ on $X \setminus V_n$ such that the conditions $f \in A$, $\|f\|_X \leqq M$ and $\|f - u_n\|_{\bar{W}_n} \leqq k$ are mutually incompatible. Apply this fact with $k = (1 + \varepsilon)^{-1}$ and $M = 2$ to get a sequence $(u_n)_1^\infty$ in $C(X)$. The u_n have pairwise disjoints supports, so we can define T: $c_0 \to C(X)/A$ by $T(\Lambda) = [\sum_1^\infty \lambda_n u_n]$ where $\Lambda = (\lambda_n) \in c_0$. Clearly T is linear and norm-decreasing. Suppose $\Lambda = (\lambda_n)$ is an element of c_0 of norm 1, and consider $f \in A$. Now $g = \sum \lambda_n u_n$ satisfies $\|g\|_X \leqq 1$, so if $\|f\|_X \geqq 2$, we have surely $\|g - f\|_X \geqq 1 > (1 + \varepsilon)^{-1}$. On the other hand, if $\|f\|_X \leqq$

2, then $\|u_n - f\|_{\bar{W}_n} > (1 + \varepsilon)^{-1}$ for all n and, choosing n so that $|\lambda_n| = 1$, we obtain

$$\|g - f\|_X \geqq \|g - f\|_{\bar{W}_n} = \|\lambda_n u_n - f\|_{\bar{W}_n} = \|u_n - \bar{\lambda}_n f\|_{\bar{W}_n} > (1 + \varepsilon)^{-1}.$$

Thus $\|g - f\|_X > (1 + \varepsilon)^{-1}$ for all $f \in A$, hence

$$\|T(\Lambda)\|_{C(X)/A} = \|[g]\|_{C(X)/A} \geqq (1 + \varepsilon)^{-1}.$$

The proof is complete.

REMARKS. (1) Obviously the same proof can be applied to show that $c_0(S)$ is isomorphically (and almost isometrically) contained in $C(X)/A$ for an uncountable index set S (and hence $C(X)/A$ is not separable) provided some maximal set of antisymmetry K for A satisfies either (a) $K = M(A|K)$ (automatic if $X = M(A)$) and K contains an uncountable family of pairwise disjoint (relatively) open non-empty subsets, or (b) $B(A|K)$ contains such a family.

(2) There appears to be no simple "formula" for obtaining $C(X)/A$ from the spaces $C(K)/(A|K)$ where K runs through the family $\mathscr{K}$ of maximal sets of antisymmetry for A. There is always an isometry

$$C(X)/A \to (\sum_{\mathscr{K}} \oplus C(K)/(A|K))_{\ell^\infty}$$

given by "restriction" in each coordinate. That this is an isometry (and not just norm-decreasing) follows from the fact [3; cf. also 7, §12] that each extreme point of the closed unit ball of $A^\perp$ is supported by a single member of $\mathscr{K}$. It is not hard to check that the projection of the image of this isometry onto any finite set of coordinates is onto, but beyond that the size of the image can vary considerably from instance to instance. To illustrate, we mention two simple examples. If S is any infinite index set, let Δ_s denote a closed disc ($s \in S$), let X be the one-point compactification of the disjoint union of the Δ_s (so if S is countable, X can be realized as $\bigcup\{B(1/n;\ 1/3n^2)\colon 1 \leqq n \leqq \infty\}$), and let A consist of those functions in $C(X)$ which are analytic on the interior of each Δ_s. Then the members of $\mathscr{K}$ are the discs Δ_s and the point at infinity, and the image of the isometry is exactly

$$(\sum_{\mathscr{K}} \oplus C(K)/(A|K))_{c_0} = (\sum_{S} \oplus C(\Delta_s)/(A|\Delta_s))_{c_0}.$$

If $X = I \times \Delta$ where $I = [0, 1]$ and $\Delta = B(0; 1)$ and if A consists of those functions in $C(X)$ which are analytic as a function of z, $|z| < 1$, on each slice $\{t\} \times \Delta$, then the members of $\mathscr{K}$ are the slices $\{t\} \times \Delta$, and the image of the isometry neither contains nor is contained in $(\sum_{\mathscr{K}} \oplus C(K)/(A|K))_{c_0}$.

4. **Quotients by modules.** Let A be a uniform algebra on X, M a closed

A-submodule of $C(X)$. Under suitable hypotheses, it is possible to extend some of the preceding results on $C(X)/A$ to $C(X)/M$. Recall that $M^\perp$ denotes the (weak $*$ closed subspace of) complex regular Borel measures on X which annihilate M. $M^\perp$ is itself an A-module, because M is. We begin with an easy equivalence.

PROPOSITION. *The following three assertions are equivalent:*

(a) *M has finite codimension in $C(X)$.*

(b) *There is a finite subset K of X such that $M = \{f \in C(X): f \equiv 0$ on $K\}$.*

(c) *$\bigcup\{\mathrm{supp}(\mu): \mu \in M^\perp\}$ is finite, where $\mathrm{supp}(\mu)$ denotes the closed support of the measure μ.*

PROOF. We may suppose $M \neq C(X)$. First assume (a), so by duality $M^\perp$ is finite-dimensional. Let $\mu \in M^\perp$ be non-zero with closed support E. Then $f|E \to f\mu$ gives a linear injection of $A|E$ into $M^\perp$, so $A|E$ is finite-dimensional, hence E is finite. Varying μ over a basis for $M^\perp$ yields the implication (a) $\Rightarrow$ (c).

Next assume (c), and let K denote the (finite) union therein so $K \neq \varnothing$. Then $M^\perp \subset \ell^1(K)$, so $M \supset \{f \in C(X): f \equiv 0$ on $K\}$. But the former inclusion (and hence the latter) is actually equality, since if $x \in K$, we can choose $\mu \in M^\perp$ for which $\mu(\{x\}) \neq 0$ and then (because $A|K = C(K)$) $f \in A$ such that $f\mu$ is point-mass at x. So we obtain the implication (c) $\Rightarrow$ (b).

Finally, the implication (b) $\Rightarrow$ (a) is obvious.

THEOREM. *Suppose that M has infinite codimension in $C(X)$ and that A satisfies the following separation condition: whenever $\eta > 0$ and E and F are disjoint non-empty closed subsets of X, it follows that there is $f \in A$ such that $|f| > 1$ on E but $|f| < \eta$ on F. Then $C(X)/M$ contains isomorphic copies of c_0. and $M^\perp$ contains isomorphic copies of ℓ^1. Furthermore, the isomorphisms can be taken as close as desired to isometries.*

PROOF. By the proposition, the closure K of $\bigcup\{\mathrm{supp}(\mu): \mu \in M^\perp\}$ is infinite, so we can choose a sequence $\{(W_n, V_n, \mu_n)\}_1^\infty$ where W_n and V_n, are open subsets of X, $\bar{W}_n \subset V_n$, the V_n are pairwise disjoint, $\mu_n \in M^\perp$, and $W_n \cap \mathrm{supp}(\mu_n) \neq \varnothing$.

Given a positive number $\sigma < 1$, choose a positive number $\delta < 1 - \sigma$ and then choose positive numbers

$$\eta_n < \frac{1 - \sigma - \delta}{1 + \sigma} \cdot \frac{|\mu_n|(W_n)}{|\mu_n|(X \setminus V_n)}.$$

Finally choose $f_n \in A$ so that $|f_n| > 1$ on $\bar{W}_n$ but $|f_n| < \eta_n$ on $X \setminus V_n$. Let $\nu_n = f_n\mu_n/\|f_n\mu_n\|$, so $\nu_n \in M^\perp$ and $\|\nu_n\| = 1$, where the norms on measures are the total variation norms. Choose $u_n \in C(X)$ such that $u_n \equiv 0$ on $X \backslash V_n$, $\|u_n\|_X = 1$, and

$$\left|\int u_n d\nu_n\right| > (1-\delta)|\nu_n|(V_n).$$

Define $T\colon c_0 \to C(X)$ and $U\colon \ell^1 \to M^\perp$ by $T(\Lambda) = \sum_1^\infty \lambda_n u_n$ and $U(\Lambda) = \sum_1^\infty \lambda_n \nu_n$. Clearly $\|T\| = \|U\| = 1$. It remains only to verify

(1) $\quad \operatorname{dist}_{C(X)}(T(\Lambda), M) \geqq \sigma \|\Lambda\|_{c_0}$ for all $\Lambda \in c_0$; and

(2) $\quad \|U(\Lambda)\| \geqq \sigma \|\Lambda\|_{\ell^1}$ for all $\Lambda \in \ell^1$.

To do this, we first calculate that for each n,

(3) $$|\nu_n|(V_n) - \sum_{k \neq n} |\nu_n|(V_k) \geqq (1-\delta)|\nu_n|(V_n) - \sum_{k\neq n} |\nu_n|(V_k) > \sigma.$$

Indeed,

$$\begin{aligned}
&(1-\delta)|\nu_n|(V_n) - \sum_{k\neq n} |\nu_n|(V_k) \geqq (1-\delta)|\nu_n|(V_n) - |\nu_n|(X\backslash V_n) \\
&= \frac{(1-\delta)|f_n\mu_n|(V_n) - |f_n\mu_n|(X\backslash V_n)}{\|f_n\mu_n\|} \geqq \frac{(1-\delta)|f_n\mu_n|(V_n) - \eta_n|\mu_n|(X\backslash V_n)}{|f_n\mu_n|(V_n) + \eta_n|\mu_n|(X\backslash V_n)} \\
&\geqq \frac{(1-\delta)|\mu_n|(W_n) - \eta_n|\mu_n|(X\backslash V_n)}{|\mu_n|(W_n) + \eta_n|\mu_n|(X\backslash V_n)} > \sigma.
\end{aligned}$$

the last inequality following from the choice of η_n.

Now suppose $\Lambda = (\lambda_n)_1^\infty \in c_0$ and $\|\Lambda\|_{c_0} = 1$. If n is chosen so that $|\lambda_n| = 1$, then

$$\begin{aligned}
\operatorname{dist}_{C(X)}(T(\Lambda), M) &\geqq \left|\int T(\Lambda) d\nu_n\right| \geqq |\lambda_n|\left|\int u_n d\nu_n\right| - \sum_{k\neq n} |\lambda_k|\left|\int u_k d\nu_n\right| \\
&\geqq 1\cdot(1-\delta)|\nu_n|(V_n) - \sum_{k\neq n} 1\cdot|\nu_n|(V_k) > \sigma
\end{aligned}$$

by (3), hence (1) holds.

Finally, let $\Lambda = (\lambda_n)_1^\infty \in \ell^1$. For each n,

$$|U(\Lambda)|(V_n) \geqq |\lambda_n||\nu_n|(V_n) - \sum_{j\neq n} |\lambda_j|\,|\nu_j|(V_n),$$

hence

$$\begin{aligned}
\|U(\Lambda)\| &\geqq \sum_{n=1}^\infty |U(\Lambda)|(V_n) \\
&\geqq \sum_{n=1}^\infty \Big(|\lambda_n|\,|\nu_n|(V_n) - \sum_{j\neq n} |\lambda_j|\,|\nu_j|(V_n)\Big) \\
&= \sum_{n=1}^\infty |\lambda_n| \Big(|\nu_n|(V_n) - \sum_{k\neq n} |\nu_n|(V_k)\Big) \\
&\geqq \sum_{n=1}^\infty |\lambda_n|\sigma = \sigma\|\Lambda\|_{\ell^1},
\end{aligned}$$

proving (2).

REMARKS. (1) The separation condition implies $X = B(A)$, and is clearly implied by (for instance) the uniform density of $\{|f|:f\in A\}$ in the set of all non-negative real-valued continuous functions on X.

(2) c_0 and ℓ^1 can be replaced by $c_0(S)$ and $\ell^1(S)$ if S is any set which indexes a family $\{V_s: s\in S\}$ of pairwise disjoint open subsets of X which have non-empty intersection with the set K of the proof. The proof given is for the case of countable S, but with the extension to uncountable S in mind, we chose not to simplify the notation of the proof slightly by selecting a single $\mu \in M^{\perp}$ which could act as μ_n for all n.

(3) Of course, the result of Pełczynski mentioned in §3 shows that $\ell^1 \subset M^{\perp}$ implies $c_0 \subset C(X)/M$.

(4) We have been unable to ascertain whether (as seems likely) the separation condition can be dispensed with in the hypothesis. Certainly it is not enough for M to be merely a closed subspace of $C(X)$, since it is known that $L^\infty = C(X)$ (and more generally $C(X)$ for any compact Hausdorff space which contains a perfect set) has ℓ^2 as a quotient, and ℓ^2 contains no isomorph of either c_0 or ℓ^1.

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