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THE MEHLER-FOCK TRANSFORM OF DISTRIBUTIONS

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1. **Introduction.** The classical Mehler-Fock transformaton of a function $f(x)$ is defined by

$$f^*(\tau) = \int_0^\infty f(x)P_{-(1/2)+i\tau}(\mathrm{ch}x)\mathrm{sh}x\,dx$$

where $P_{-(1/2)+i\tau}(\mathrm{ch}x)$ is the Legendre function of first kind. Although the corresponding inversion formula

$$f(x) = \int_0^\infty \tau \mathrm{th}\pi\tau P_{-(1/2)+i\tau}(\mathrm{ch}x)f^*(\tau)\,d\tau$$

was given by Mehler [7] in the year 1881 in a purely formal way, the research on this transformation has been rather slow due to the complex nature of this transformation. Fock [3] in the early forties established these formulas for a class of functions. Since the early sixties considerable interest has been shown in the use of this transformation in solving the boundary value problems in the mathematical theory of elasticity. The objective here is to extend this transformation to a class of distributions.

The notation and terminology used here is that of Zemanian [15]. In the following I denotes the open interval $(0, \infty)$. Spaces $\mathscr{D}(I)$ and $\mathscr{D}'(I)$ have their usual meaning.

The Legendre function $P_{-(1/2)+i\tau}(\mathrm{ch}x)$ possesses the following well known properties [3, p. 254].

$$\left| P_{-(1/2)+i\tau}(\mathrm{ch}x) \right| < \frac{x}{2\mathrm{Sh}(x/2)} \leqq 1, \qquad x \geqq 0, \tag{1}$$

$$P_{-(1/2)+i\tau}(\mathrm{ch}\theta) = \left(\frac{\theta}{\mathrm{Sh}\theta} \right)^{1/2} \left\{ J_0(\tau\theta) + \frac{1}{8\tau}\left(\coth\theta - \frac{1}{\theta}\right) J_1(\tau\theta) + \cdots \right\}, \tag{2}$$

where $J_n(\tau\theta)$ is the Bessel function of first kind [3, p. 253].

Using the asymptotic expansions of Bessel functions for large $\tau > 0$ and fixed $\theta > 0$ in (2) it follows that

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$$P_{-(1/2)+i\tau}(\mathrm{ch}\theta) = \left(\frac{2}{\pi\tau\mathrm{sh}\theta}\right)^{1/2}\left[\sin(\tau\theta + \pi/4) + \frac{\coth\theta}{8\tau}\sin(\tau\theta - \pi/4) + 0(1/\tau^2)\right]. \tag{3}$$

This asymptotic expansion is uniform with respect to θ on a compact subset of I.

2. The testing function space $M_\alpha(I)$ and its dual. For $\alpha > 0$, let $\zeta(t)$ be the function defined on I by

$$\begin{aligned} \zeta(t) &= t^\alpha, && 0 < t < 1 \\ &= 1, && 1 \leqq 1 \leqq \infty. \end{aligned}$$

Let $M_\alpha(I)$ be the collection of all infinitely differentiable, complex valued functions $\phi(t)$ defined on I such that for every non-negative integer k

$$\gamma_k(\phi) \triangleq \sup_{0<t<\infty} \left| \zeta(t)\Delta_t^k \phi(t) \right| < \infty; \ \Delta_t^k = [D^2 + \coth tD]^k; \ D \equiv \frac{d}{dt}.$$

It can be shown that $\{\gamma_k\}$ is a separating collection of seminorms and $M_\alpha(I)$ is a sequentially complete locally convex topological vector space. Let $M_\alpha'(I)$ denote the dual of $M_\alpha(I)$. The restriction of $f \in M'(I)$ to $\mathscr{D}(I)$ is in $\mathscr{D}'(I)$.

For $\tau \geqq 0$, $P_{-(1/2)+i\tau}(\mathrm{ch}t) \in M_\alpha(I)$ and the operator Δ_t satisfies the property

$$\Delta_t[P_{-(1/2)+i\tau}(\mathrm{ch}t)] = -((1/4) + \tau^2)P_{-(1/2)+i\tau}(\mathrm{ch}t). \tag{4}$$

3. The distributional Mehler-Fock transform. For $f \in M_\alpha'(I)$ and $\tau > 0$, define the distributional Mehler-Fock transformation $f^*(\tau)$ of f by

$$f^*(\tau) = \langle f(t), P_{-(1/2)+i\tau}(\mathrm{ch}t)\rangle. \tag{5}$$

Using convergence in $M_\alpha(I)$, equation (4) and commutativity of $\Delta_t^k \cdot \partial/\partial t$ we can establish the

THEOREM 1. *For $f \in M_\alpha'(I)$ and $\tau \geqq 0$ let $f^*(\tau)$ be defined by* (5). *Then*

$$\frac{d}{d\tau} f^*(\tau) = \left\langle f(t), \frac{\partial}{\partial\tau} P_{-(1/2)+i\tau}(\mathrm{ch}t) \right\rangle \qquad [9, \text{p. } 60].$$

LEMMA 2. *Let $f \in M_\alpha'(I)$. Then for a positive integer N,*

$$\int_0^N \tau \operatorname{th}\pi\tau P_{-(1/2)+i\tau}(\operatorname{ch}t) \langle f(x), P_{-(1/2)+i\tau}(\operatorname{ch}x)\rangle \, d\tau$$

$$= \left\langle f(x), \int_0^N \tau \operatorname{th}\pi\tau P_{-(1/2)+i\tau}(\operatorname{ch}x) P_{-(1/2)+i\tau}(\operatorname{ch}t)\, d\tau \right\rangle.$$

The proof follows from Riemman sum technique.

For x and $t \in I$ and $N > 0$ let

$$G_N(t, x) \triangleq \int_0^N \tau \operatorname{th}\pi\tau P_{-(1/2)+i\tau}(\operatorname{ch}x) P_{-(1/2)+i\tau}(\operatorname{ch}t) \operatorname{sh}t \, d\tau.$$

Then by Fock's inversion theorem [11], p. 390], we obtain the

LEMMA 3. *For real numbers a and b satisfying* $0 < a < b$

$$\lim_{N\to\infty} \int_a^b G_N(t, x)\, dt = \begin{matrix} 1 & \text{for } x \in (a, b) \\ 0 & \text{for } x \notin [a, b]. \\ \frac{1}{2} & \text{for } x = a, b \end{matrix}$$

LEMMA 4. *Let $\phi(t) \in \mathscr{D}(I)$ with its support contained in the interval $[a, b]$. Then*

$$\int_a^b G_N(t, x)\phi(t)\, dt \to \phi(x)$$

in $M_\alpha(I)$ as $N \to \infty$.

PROOF. Using integration by parts and relation (4) one can see that

$$\Delta_x^k \int_a^b G_N(t, x)\phi(t)\, dt = \int_a^b G_N(t, x)\phi_k(t)\, dt$$

where $\phi_k(t) \equiv \Delta_t^k \phi(t)$. Thus in view of Lemma 3

$$\lim_{N\to\infty} \zeta(x)\Delta_x^k \left[\int_a^b G_N(t, x)\phi(t)\, dt - \phi(x) \right]$$

$$= \lim_{N\to\infty} \zeta(x) \int_a^b G_N(t, x)[\phi_k(t) - \phi_k(x)]\, dt.$$

It is, therefore, reduced to proving that

$$\zeta(x) \int^b G_N(t, x)[\Psi(t) - \Psi(x)]\, dt \to 0$$

uniformly for all x as $N \to \infty$ where $\Psi(t) \in \mathscr{D}(I)$ with its support contained in $[a, b]$.

For a fixed $x \geqq 2\delta$, where $0 < \delta < \min(a/2, 1/2)$, we can write

$$\zeta(t) \int_a^b G_N(t, x)[\Psi(t) - \Psi(x)]\, dt$$

$$= \zeta(x) \left(\int_a^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{b} \right) G_N(t, x)[\Psi(t) - \Psi(x)]\, dt$$

$$= I_1 + I_2 + I_3$$

where I_i, $i = 1, 2, 3$ denotes corresponding integrals.

For $x \in (-\infty, a - \delta] \cup [b + \delta, \infty)$, I_2 is zero. Therefore

$$|I_2| \leqq \sup_{a \leqq \eta \leqq b} |\psi'(\eta)| \int_{x-\delta}^{x+\delta} |G_N(t, x)(t - x)|\, dt$$

$$\leqq \sup_{a \leqq \eta \leqq b} |\psi'(\eta)| \cdot \sup_{\substack{a-(1/2)\Delta x \Delta b+(1/2) \\ a-(1/2)\Delta t \Delta b+(1/2)}} |G_N(t, x)(t - x)|.$$

$G_N(t, x)(t - x)$ is uniformly bounded for $N > 0$ and for all x and all t in the closed interval $[a - (1/2), a + (1/2)]$. Hence we can find a constant $D > 0$ and independent of δ such that

$$|I_2| \leqq D\delta.$$

For a given $\epsilon > 0$ we can choose $\delta < \min(a/2, 1/2, \epsilon/D)$ to obtain

$$|I_2| \leqq \frac{\epsilon}{2}. \tag{6}$$

Let L be an arbitrary large number greater than b. Then, using the uniform boundedness of $(t - x)G_N(t, x)$ for all $N > 0$ and for all x and all t in $[2\delta, L]$ together with the convergence of the integral.

$$\int_{x+\delta}^{b} (t - x)G_N(t, x)\, dt \to 0$$

uniformly for all $x \in [\delta, b - \delta]$ as $N \to \infty$, we get that

$$I_3 \to 0 \tag{7}$$

uniformly for all $x \geqq 2\,\delta$ as $N \to \infty$ [9, p. 82]. Similarly

$$I_1 \to 0$$

uniformly for all $x \in (a, L]$ as $N \to \infty$. I_1 vanishes identically for all $x \leqq a$. Moreover, $\psi(t)$ is of compact support in $[a, b]$. Therefore for $x > L > b$ we write

$$I_1 = \zeta(x) \int_a^b \psi(t) \operatorname{sh} t\, dt \int_0^N \tau \operatorname{th} \pi\tau P_{-(1/2)+i\tau}(\operatorname{ch} x) P_{-(1/2)+i\tau}(\operatorname{ch} t)\, d\tau.$$

Let N_0 be a sufficiently large but fixed number greater than one. Then, using (1), we have

$$|I_1| \leqq \zeta(x) \int_a^b |\psi(t)|\, dt \int_0^{N_0} \tau \mathrm{th}\pi\tau \frac{x/2}{\mathrm{sh}x/2} \frac{t/2}{\mathrm{sh}t/2} \mathrm{sh}t\, d\tau$$

$$+ \zeta(x) \left| \int_a^b \psi(t)\, dt \int_{N_0}^N \tau \mathrm{th}\pi\tau P_{-(1/2)+i\tau}(\mathrm{ch}x) P_{-(1/2)+i\tau}(\mathrm{ch}t) \mathrm{sh}t\, d\tau \right|$$

$$= 0(xe^{-x/2}) + \frac{1}{\pi(\mathrm{sh}x)^{1/2}} \int_a^b \psi(t)(\mathrm{sh}t)^{1/2}\, dt \int_{N_0}^N \mathrm{th}\pi\tau \left[1 + 0\left(\frac{1}{\tau} \right)\right]$$

$$\times\ 2\sin \left(x\tau + \frac{\pi}{4}\right) \sin \left(t\tau + \frac{\pi}{4}\right) d\tau + \frac{1}{8} \int_{N_0}^N \mathrm{th}\pi\tau \left[1 + 0\left(\frac{1}{\tau} \right)\right]$$

$$\times\ \coth t\ 2\sin \left(x\tau + \frac{\pi}{4}\right) \sin \left(t\tau + \frac{\pi}{4}\right) d\tau + 2 \int_{N_0}^N \mathrm{th}\pi\tau \left[1 + 0\left(\frac{1}{\tau} \right)\right]$$

$$\times \sin \left(x\tau + \frac{\pi}{4}\right) 0\left(\frac{1}{\tau^2} \right) d\tau \qquad \text{[By equation (3)]}$$

$$= 0(xe^{-x/2}) + Q_1 + Q_2 + Q_3 \qquad \text{(say).}$$

Now, using integration by parts we have that

$$|Q_1| \leqq |0(e^{-x/2})|,\ x \to \infty.$$

Similarly,it can be shown that Q_2 and Q_3 are $0(x^{\alpha}e^{-x/2})$, where $\alpha \geqq 0$, as $x \to \infty$. Therefore we can make

$$|I_1| < \epsilon/2 \tag{8}$$

for all $N > 0$ by choosing $x > L$ for sufficiently large L.

Combining (6), (7) and (8) together, we have

$$\limsup |I| < \epsilon \text{ uniformly for all } x \geqq 2\delta. \tag{9}$$

For $0 < x \leqq 2\delta$, write

$$I = x^{\alpha} \left[\int_a^{x+\delta} + \int_{x+\delta}^b \right] G_N(t, x)[\psi(t) - \psi(x)]\, dt$$

$$= J_1 + J_2 \text{ (say).}$$

First, we consider J_1. Clearly $J_1 \equiv 0$ for $0 \leqq x \leqq \delta$, and for $\delta < x < 2\delta$

$$|J_1| \leqq x^{\alpha} \int_a^{x+\delta} |G_N(t, x)| \, |\psi(t) - \psi(x)| \, dt$$

$$\leqq x^{\alpha} \int_a^{x+\delta} |G_N(t, x)| \, dt \, \left| \int_x^{\tau} \psi'(\eta) \, d\eta \right|$$

$$\leqq \delta \sup_{a<\eta<b} |\psi'(\eta)| \int_a^{x+\delta} |G_N(t, x)| \, dt$$

$$\leqq \delta \sup_{a<\eta<b} |\psi'(\eta)| \int_a^{3/2} |G_N(t, x)| \, dt.$$

Since the last integral is bounded uniformly for all $x \in (\delta, 2\delta)$, we can find a positive constant C independent of ϵ satisfying

$$|J_1| \leqq \delta C = \delta D \frac{C}{D} < \frac{\epsilon}{2} \frac{C}{D}. \tag{10}$$

Next, consider J_2. Since $0 < x < 2\delta$ and $2\delta < \min(a, 1)$,

$$J_2 = x^{\alpha} \int_a^b G_N(t, x) \psi(t) \, dt.$$

Therefore,

$$|J_2| \leqq x^{\alpha} \int_a^b |\psi(t)| \operatorname{sh} t \, dt \int_0^{N_0} \tau \operatorname{th} \pi\tau \, \frac{x/2}{\operatorname{sh} x/2} \, \frac{t/2}{\operatorname{sh} t/2} \, d\tau$$

$$+ \left(\frac{2}{\pi} \right)^{1/2} x^{\alpha} \int_{N_0}^{N} (\tau)^{1/2} \operatorname{th} \pi\tau \, \Big| \, d\tau$$

$$\times \int_a^b \psi(t) (\operatorname{sh} t)^{1/2} \sin\left(\tau t - \frac{\pi}{4} \right) dt \, \Big|$$

$$+ \frac{1}{8\pi} x^{\alpha} \int_a^b \psi(t) (\operatorname{sh} t)^{1/2} \coth t \sin\left(t\tau - \frac{\pi}{4} \right) d\tau$$

$$+ \frac{A x^{\alpha}}{\tau^2} \int_a^b \psi(t) (\operatorname{sh} t)^{1/2} \, dt \mid d\tau \qquad (A \text{ is a constant})$$

$$\leqq 0(x^{\alpha}) + Q_1 + Q_2 + Q_3 (\text{say}).$$

Here, we have used the result (1) and the uniform asymptotic expansion (3) for $P^{-(1/2)+i\tau}(x)$ where τ is large and x is small. Now using integration by parts twice it can be shown that $|Q_1| = 0(x^{\alpha})$ as $x \to 0+$. Similarly, Q_2 and Q_3 are $0(x^{\alpha})$ as $x \to 0+$. Therefore,

$$|J_2| = 0(x^\alpha) \qquad \text{as } x \to 0+. \tag{11}$$

$$\leqq Mx^\alpha \leqq M2^\alpha\delta^\alpha \leqq \frac{M2^\alpha\epsilon^\alpha}{D^\alpha}.$$

Combining (10) and (11) together, we have

$$\limsup_{N\to\infty} |I| \leqq \frac{\epsilon}{2}\frac{C}{D} + \frac{M2^\alpha\epsilon^\alpha}{D^\alpha} \qquad \text{for } 0 < x \leqq 2\delta$$

Combining (9) and (12) together, we have

$$\lim_{N\to\infty} I = 0 \qquad \text{uniformly for all } x > 0.$$

This completes the proof of the lemma.

The following inversion theorem is a consequence of Theorem 1, Lemma 2 and Lemma 4.

THEOREM 5. *Let $f \in M_\alpha'(I)$ and let $f^*(\tau)$ be the distributional Mehler-Fock transformation of f defined by* (5). *Then, for each $\phi \in D(I)$*

$$\left\langle \int_0^N \tau \operatorname{th}\pi\tau f^*(\tau) P_{-(1/2)+i\tau}(\operatorname{ch}t)\operatorname{sh}t\, d\tau,\ \Phi(t) \right\rangle \to \langle f, \Phi\rangle$$

as $N \to \infty$.

The generalized classical Mehler-Fock transformation defined by

$$f_m^*(\tau) = \int_0^\infty f(x) P^{-m}_{-(1/2)+i\tau}(\operatorname{ch}x)\operatorname{sh}x\, dx$$

can similarly be extended to distributions. But the extension to distributions of the corresponding inversion formula defined by

$$f(x) = (-1)^m \int_0^\infty \tau \operatorname{th}\pi\tau P^m_{-(1/2)+i\tau}(\operatorname{ch}x) f_m^*(\tau)\, d\tau$$

where $P^{\pm m}_{-(1/2)+i\tau}(\operatorname{ch}x)$ are associated Legendre functions of first kind, offers considerable difficulty. For $m = 0$ this transform reduces to the case considered by us. It seems that with some restrictions on m the inversion theorem can be proved for distribution in a similar way. However, restrictions on m may destroy the usefulness of this transformation.

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