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RECIPROCITY ISOMORPHISMS FOR SEPARABLE FIELD EXTENSIONS

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0. **Introduction.** Throughout k is a field, k_s its separable algebraic closure, and $\mathcal{G} = \mathrm{gal}(k_s/k)$. We make frequent use, without further comment, of the equivalence between étale sheaves and discrete $\mathcal{G}$ modules given by $F \rightsquigarrow F(k_s)$. We remark particularly that étale sheaves are additive. An exposition may be found in [5]. We also use freely the standard facts about Amitsur cohomology summarized in the introduction to [8].

When $S \to T$ is a map of commutative R algebras we denote by inf the induced map $H^n(S/R, F) \to H^n(T/R, F)$ on cohomology. It is called inflation.

Amitsur cohomology coincides with Galois cohomology for (finite) Galois extensions and inf coincides with the usual inflation [8, § 4]. It is thus reasonable to ask whether Galois theoretic results which have cohomological statements can be extended to more general field extensions, replacing Galois with Amitsur cohomology. For example, Hilbert's Theorem 90 can be so extended, as can the classical isomorphism of Brauer groups with the second cohomology group [4, p. 26ff]. In this paper we extend the cohomological version [10, IX, § 8] of some of the reciprocity isomorphisms of class field theory to arbitrary finite separable field extensions.

Our technique is to sheafify the splitting module of Tate [11], and introduce a functor (not quite a sheaf) which plays the role of **Z** with trivial Galois action.

1. **Formation Sheaves.** Let F be an étale sheaf [5, I. 5]. By analogy with the terminology of class formations [10, XI] we will say F is a *field sheaf* if $H^1(M/L, F) = 0$ whenever $k \subseteq L \subseteq M \subseteq k_s$ with $[M : k]$ finite and M/L Galois.

PROPOSITION 1.1. *If F is a field sheaf then $H^1(M/L, F) = 0$ for any fields $k \subseteq L \subseteq M \subseteq k_s$ with $[M : k]$ finite.*

PROOF. Let M' be a finite Galois extension of L containing M. $\mathrm{Inf} : H^1(M/L, F) \to H^1(M'/L, F) = 0$ is a monomorphism [8, Thm. 3.2]

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and the assertion is proved.

Now let F be any abelian group valued functor on the category of finite separable k-algebras, and let $E \subseteq K \subseteq L \subseteq M$ be finite separable field extensions of k. Since inf is induced by the inclusion maps the following diagram is commutative:

$$\begin{tikzcd} H^2(K/E, F) \arrow[rr, "\inf"] \arrow[dr, "\inf"'] & & H^2(L/E, F) \arrow[dl, "\inf"] \\ & H^2(M/E, F) & \end{tikzcd}$$

We may therefore define $H^2(E, F)$, the *Brauer group over E of the functor F* by $\mathrm{H}^2(E, F) = \varinjlim H^2(K/E, F)$, the limit taken over the finite separable field extensions of E [10, XI]. Note that the Galois extensions are cofinal in the set of separable field extensions of E so that $H^2(E, F)$ can be defined by taking the limit over Galois extensions only.

PROPOSITION 1.2. *Let F be a field sheaf. Then* $\inf : H^2(K/E, F) \to H^2(L/E, F)$ *is a monomorphism and so $H^2(E, F)$ may be regarded as a union.*

PROOF. In deducing the classical group cohomology exactness of the inflation-restriction sequence from our exactness result [8, Thm. 3.2], we observed that the vanishing of $H^1(T/S, F)$ when T and S are merely fields sufficed to apply Theorem 3.2 of [8] which implies inf is monic.

It is known that if F is the functor which assigns to an algebra its group of units, then $H^2(E, F)$ is the classical Brauer group of central simple E-algebras. This is Theorem 5.4 of [1, p. 96] or Theorem 3 of [9] together with the fact ([2, 8.10.4.3]) that every central simple E-algebra has a splitting field which is finite Galois over E.

Now if M is a finite separable field extension of L, L a finite separable field extension of k, there is a restriction map [8, § 2] $\mathrm{res} : H^2(M/K, F) \to H^2(M/L, F)$ natural in M. That is, if $M' \supseteq M$ is another finite separable field extension of L then the following diagram commutes:

$$\begin{tikzcd} H^2(M/K, F) \arrow[r, "\mathrm{res}"] \arrow[d, "\inf"] & H^2(M/L, F) \arrow[d, "\inf"] \\ H^2(M'/K, F) \arrow[r, "\mathrm{res}"] & H^2(M'/L, F). \end{tikzcd}$$

This is immediate from the fact that

$$\begin{array}{ccc} M \otimes_K M \otimes_K M & \longrightarrow & M \otimes_L M \otimes_L M \\ \Big\downarrow & & \Big\downarrow \\ M' \otimes_K M' \otimes_K M' & \longrightarrow & M' \otimes_L M' \otimes_L M' \end{array}$$

commutes. Since the finite separable extensions of L are cofinal among those of K, these restrictions induce a map $\mathrm{res}_{L/K} : H^2(K, F) \to H^2(L, F)$ on direct limits. We will also call this map *restriction*.

We say a field sheaf is a *formation sheaf* if for each finite separable extension K of k there is a monomorphism $\mathrm{inv}_K : H^2(K, F) \to \mathbf{Q}/\mathbf{Z}$ satisfying

(a) If L/K is finite Galois of degree n then inv_K maps $H^2(L/K, F)$ onto the subgroup generated by $(1/n)\mathbf{Z}$.

(b) For any finite separable extension L/K, $\mathrm{inv}_L\ \mathrm{res}_{L/K} = n\ \mathrm{inv}_K$, i.e., the following diagram commutes

$$\begin{array}{ccc} H^2(K, F) & \xrightarrow{\mathrm{res}_{L/K}} & H^2(L, F) \\ \Big\downarrow \mathrm{inv}_K & & \Big\downarrow \mathrm{inv}_L \\ \mathbf{Q}/\mathbf{Z} & \xrightarrow{\ n\ } & \mathbf{Q}/\mathbf{Z}. \end{array}$$

The maps inv are called *invariants*.

If L/K is Galois the element of $H^2(L/K, F)$ whose invariant is $(1/n)\mathbf{Z}$ is the *fundamental class*. A cocycle representing it is a *fundamental cocycle*. Condition (a) says that $H^2(L/K, F)$ is cyclic of order n, generated by the fundamental class.

The following results allow us to extend these notions to the case where L need not be Galois.

PROPOSITION 1.3. *Let $k \subseteq K \subseteq L \subseteq M \subseteq k_s$ where M is finite over k and Galois over K. Let F be a formation sheaf. If α is the fundamental class in $H^2(M/K,F)$ then* $\mathrm{res}_{L/K}\alpha$ *is the fundamental class in $H^2(M/L, F)$.*

PROOF. Let $n = [M : K]$ and $m = [M : L]$. Then $[L : K] = (n/m)$ and the definition states

$$\mathrm{inv}_L \mathrm{res}_{L/K}\alpha = (n/m)\,\mathrm{inv}_K\alpha \equiv (n/m)(1/n) \equiv (1/m)(\mathrm{mod}\ \mathbf{Z}).$$

Thus since M/L is Galois with $[M : L] = m$, $\mathrm{res}_{L/K}\alpha$ is the fundamental class in $H^2(M/L, F)$.

COROLLARY 1.4. *For any finite separable field extension L of K,* inv_K *maps $H^2(L/K, F)$ onto the subgroup generated by* $(1/[L : K])\mathbf{Z}$.

PROOF. Let M be a finite Galois extension of K containing L. Consider the diagram

$$\begin{array}{ccccc} 0 \to H^2(L/K, F) & \xrightarrow{\text{inf}} & H^2(M/K, F) & \xrightarrow{\text{res}} & H^2(M/L, F) \\ \Big\downarrow \mathrm{inv}_K & & \Big\downarrow \mathrm{inv}_K & & \\ \mathbf{Q}/\mathbf{Z} & \xrightarrow{\text{id}} & \mathbf{Q}/\mathbf{Z} & & \end{array}$$

As we remarked in Prop. 1.2, the hypotheses of Theorem 3.2 of [8] apply, since F is a field sheaf, so that the upper row is exact. By Prop. 1.3, res is a surjection of $H^2(M/K, F)$ onto $H^2(M/L, F)$. M/K and M/L are Galois, so $H^2(M/K, F)$ and $H^2(M/L, F)$ are cyclic of order $[M:K]$ and $[M:L]$ respectively and a counting argument shows that $H^2(L/K, F)$ is cyclic of order $[L:K]$. Hence, since the square commutes by the definition of inv_K and since inv_K maps $H^2(M/K, F)$ onto $(1/[M:K])\mathbf{Z}/\mathbf{Z}$, it follows that inv_K maps $H^2(L/K, F)$ onto the subgroup generated by $(1/[L:K])\mathbf{Z}$, completing the proof.

In view of the corollary the definition of *fundamental class* and *cocycle* may be extended by replacing "Galois" with "separable".

COROLLARY 1.5. *Let K, L, M and F be as in Prop.* 1.3 *but assume only that M/L is separable. If α is the fundamental class in $H^2(M/K, F)$, then $\mathrm{res}_{L/K}\alpha$ is the fundamental class in $H^2(M/L, F)$.*

PROOF. Cor. 1.4 is condition (a) of the definition of inv_L, but with the restriction that M/L be Galois removed. The proof of Prop. 1.3 carries over with the application of condition (a) replaced by Cor. 1.4.

2. **Splitting Sheaves.** In this section we extend a construction of Tate [11, p. 294] to étale sheaves.

Let G be a finite group, H a normal subgroup and A a G-module. If g is in G, we denote by $\bar{g}$ the left coset gH.

Let $f: G/H \times G/H \to A^H$ be a normalized two cocycle, i.e., $f(\bar{1}, \bar{g}) = f(\bar{g}, \bar{1}) = 0$ for all g in G. It is known that every two cocycle is cohomologous to a normalized one [7, § 15.7]. Then $\lambda f = \mathrm{inf}(f): G \times G \to A$ given by $\lambda f(a, b) = f(\bar{a}, \bar{b})$ is still a normalized two cocycle (in fact, $\lambda f(h, a) = \lambda f(a, h) = 0$ for all a in G, h in H).

Let $\tilde{A}_{\lambda f}$ be the *splitting module for* λf [11]. That is, as abelian groups

$$\tilde{A}_{\lambda f} = A \oplus \sum_{1 \neq g \in G} \mathbf{Z} x_g.$$

$\tilde{A}_{\lambda f}$ is a G-module with G acting in the given way on A and with

$$a x_b = x_{ab} - x_a + \lambda f(a, b) = x_{ab} - x_a + f(\bar{a}, \bar{b}).$$

That this gives a module action is a consequence of the two cocycle identity

$$a\lambda f(b, c) - \lambda f(ab, c) + \lambda f(a, bc) - \lambda f(a, b) = 0.$$

Similarly let $\tilde{A}_f^H$ be the splitting module for f. Thus

$$\tilde{A}_f^H = A^H \oplus \sum_{\bar{1} \neq \bar{g} \in G/H} \mathbf{Z} y_{\bar{g}}$$

with the natural G/H action on A^H extended by

$$\bar{a} \cdot y_{\bar{b}} = y_{\overline{ab}} - y_{\bar{a}} + f(\bar{a}, \bar{b}).$$

Now define a map $\varphi : \tilde{A}_f^H \to \tilde{A}_{\lambda f}$ as follows: Let

$$y = \alpha + \sum_{\bar{1} \neq \bar{g}} m_{\bar{g}} y_{\bar{g}}$$

with $m_{\bar{g}}$ in $\mathbf{Z}$, α in A^H. Let $\{g_i\}$ be a set of left coset representatives for H with $g_1 = 1$. Set $m_i = m_{\bar{g}_i}$ with $m_1 = 0$ for convenience. Then define

$$\varphi(y) = \alpha + \sum_i \sum_{h \in H} m_i(x_{g_i h} - x_h).$$

Clearly, this definition is independent of choice of coset representatives: any other representative of $\bar{g}_i$ is of the form $g_i h_i$ for some h_i in H. Since $\sum_{h \in H} x_{g_i h} = \sum_{h \in H} x_{g_i h_i h}$, the independence is immediate.

It is straightforward to check that φ in fact takes values in $\tilde{A}_{\lambda f}^H$ and is a G/H-module map.

The map φ is introduced for the following reason:

Suppose F is a sheaf, L/k a finite Galois field extension and f a two cocycle from $H^2(\mathrm{Gal}(L/k), F(L))$. If M/k is any finite Galois field extension containing L, let $G = \mathrm{Gal}(M/k)$, and $H = \mathrm{Gal}(M/L)$. We will see (Lemma 2.3) that, because F is a sheaf, $F(L) = F(M)^H$ and φ will provide a map from the splitting module for $F(L)$ to that for $F(M)$. Passing to the direct limit over all such M we will construct a discrete module for $\mathscr{G} = \mathrm{Gal}(k_s/k)$. Using the relation between such modules and sheaves, we still have a "splitting sheaf" for F with properties analogous to those in Theorem 1 of [11] from which we will deduce our reciprocity results.

We must first relate splitting modules to group rings and augmentation ideals.

For any group S, denote the group ring by $\mathbf{Z}S$ and its augmentation ideal by I_S, that is, the set $\{\sum n_s s \mid \sum n_s = 0\}$.

Define an abelian group map $\psi : \mathbf{Z}(G/H) \to \mathbf{Z}G$ by $\psi(\sum_i n_i \bar{g}_i) = \sum_i \sum_{h \in H} n_i(g_i h)$ (where again $\{g_i\}$ is a set of left coset representatives of H). This is easily seen to be independent of representative, since $\sum_{h\in H} g_i h_i h = \sum_{h \in H} g_i h$ for any collection $\{h_i\}$ of elements of H. For this reason also ψ has image contained in $(\mathbf{Z}H)^H$, since for any h_0 in H there are h_i in H such that $h_0 g_i h = g_i h_i h$ (because H is normal). A trivial computation shows that ψ is a G/H-module map.

Now ψ takes $I_{G/H}$ into $L_G{}^H$. To prove this it suffices to show $\psi(I_{G/H})$ is contained in I_G, since the image of ψ is fixed by H. Note that for any group S, I_S is generated by $\{s - 1 \mid s \in S\}$. The assertion then follows from the observation that $\psi(\sum_i n_i(\bar{g}_i - \bar{1}) = \sum_i \sum_{h\in H} n_i(g_i h - h)$ which lies in I_G.

The following diagram of abelian groups, which has exact rows, is then commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_G & \longrightarrow & \mathbf{Z}_G & \longrightarrow & \mathbf{Z} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle [H:1]} & & \\
0 & \longrightarrow & I_{G/H} & \longrightarrow & \mathbf{Z}(G/H) & \longrightarrow & \mathbf{Z} & \longrightarrow & 0\,.
\end{array}$$

The right square is commutative since $\psi(\bar{g}) = \sum_n gh$ has augmentation $[H:1]$.

In fact, we have

PROPOSITION 2.1.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_G{}^H & \longrightarrow & (\mathbf{Z}G)^H & \longrightarrow & [H:1]\mathbf{Z} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle [H:1]} & & \\
0 & \longrightarrow & I_{G/H} & \longrightarrow & \mathbf{Z}(G/H) & \longrightarrow & \mathbf{Z} & \longrightarrow & 0
\end{array}$$

is commutative with exact rows.

PROOF. The functor $(\ )^H$ is a left exact functor from the category of H-modules to abelian groups, so the upper row is exact at $I_G{}^H$ and $(\mathbf{Z}G)^H$.

Now an element $x = \sum_g n_g g$ of $\mathbf{Z}G$ is fixed by H if and only if $n_g = n_{gh}$ for each h in H. It then follows that the augmentation of $\mathbf{Z}G$, when restricted to $(\mathbf{Z}G)^H$, has image equal to $[H:1]\mathbf{Z}$. Consequently, the first row is also exact at $[H:1]\mathbf{Z}$.

Since the image of ψ is fixed by H, commutativity follows from the commutativity of the diagram preceding the proposition.

Now if, as above, $f: G/H \times G/H \to A^H$ is a (normalized) cocycle, then there is a map of abelian groups $\tilde{A}_{\lambda f} \to I_G$ given by $x_g \to g - 1$ whose kernel is clearly A. With the corresponding map for $\tilde{A}_f{}^H$, we have a commutative diagram with exact rows of abelian groups:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & \tilde{A}_{\lambda f} \arrow[r] & I_G \arrow[r] & 0 \\ 0 \arrow[r] & A^H \arrow[r] \arrow[u] & \tilde{A}_f^H \arrow[r] \arrow[u, "\varphi"'] & I_{G/H} \arrow[r] \arrow[u, "\psi"'] & 0. \end{tikzcd}$$

To see the commutativity of the right square note that $\varphi(x_{\bar{g}}) = \sum_h x_{gh} - h$ has image $\sum_{\bar{h}} (gh - 1) - \sum_h (h - 1) = \sum_h gh - \sum_h gh - \sum_h h = \psi(\bar{g} - \bar{1})$. Since $\bar{g} - \bar{1}$ is the image in $I_{G/H}$ of $x_{\bar{g}}$, the square commutes.

We have already mentioned that φ and ψ have images in the submodules fixed by H, and are G/H-module maps so, in fact, the above may be replaced by a commutative exact diagram of G/H-modules:

$$\begin{tikzcd} 0 \arrow[r] & A^H \arrow[r] & \tilde{A}^H_{\lambda f} \arrow[r] & I_G{}^H \arrow[r] & H^1(H,A) \\ 0 \arrow[r] & A^H \arrow[r] \arrow[u, "\mathrm{id}"'] & \tilde{A}_f{}^H \arrow[r] \arrow[u, "\varphi"'] & I_{G/H} \arrow[r] \arrow[u, "\psi"'] & 0\,. \end{tikzcd}$$

The upper row is simply the first part of the usual long exact sequence of group cohomology.

In the sequel we will principally be interested in modules for which $H^1(H, A) = 0$.

Finally, suppose G is itself a quotient, say $G = G'/H'$. Let $H = K'/H'$, so that $G/H = G'/H'/K'/H'$. Let A be a G'-module and λ'' and λ' denote the inflation of cocycles from G/H to G' and from G to G', respectively. Then $\lambda'\lambda f = \lambda'' f$ and one easily checks that

(1)
$$\begin{tikzcd} \tilde{A}_f^{K'} = (\tilde{A}^{H'})_f^H \arrow[rr] \arrow[dr] & & \tilde{A}_f^{H'} \arrow[dl] \\ & \tilde{A}_{\lambda'' f} = \tilde{A}_{\lambda'\lambda f} & \end{tikzcd}$$

commutes.

Also

(2)
$$\begin{tikzcd} I_{G'/K'} = I_{G/H} \arrow[rr, "\psi"] \arrow[dr, "\psi"'] & & I_G = I_{G'/H'} \arrow[dl, "\psi"] \\ & I_{G'} & \end{tikzcd}$$

commutes as does the corresponding diagram for group rings. Since $[H:1]=[K':H']$ we also have a commutative diagram of (trivial) G/H-modules:

$$\begin{tikzcd} \mathbf{Z} \arrow[rr, "{[H:1]}"] \arrow[dr, "{[K':1]}"'] & & \mathbf{Z} \arrow[dl, "{[H':1]}"] \\ & \mathbf{Z} & \end{tikzcd} \tag{3}$$

Now assume that F is a formation sheaf (§ 1). Recall that $\mathscr{G}=\mathrm{Gal}\,(k_s/k)$. Let L/k be a finite separable field extension and suppose α is a fundamental two cocycle representing the fundamental class of $H(L/k, F)$.

Let $\mathfrak{A}$ be the discrete $\mathscr{G}$-module corresponding to F [5, I, § 5]. Recall $\mathfrak{A}^{\mathfrak{B}}=F(M)$ if M is finite Galois with group $\mathscr{G}/\mathfrak{B}$. Since the set $\mathscr{S}$ of finite Galois field extensions within k_s of k which contain L is cofinal in the set of all finite Galois extensions of k, we have $\mathfrak{A}=\varinjlim_{M\in\mathscr{S}}F(M)$ with the natural $\mathscr{G}$ action arising from considering $\mathscr{G}=\varinjlim_{\mathfrak{B}_M}\mathscr{G}/\mathfrak{B}_M$, the limit taken over $\{\mathfrak{B}_M=\mathrm{Gal}\,(k_s/M)\mid M\in\mathscr{S}\}$.

Now let N be the smallest Galois extension of k which contains L. Let $M\supseteq N$ be any other finite Galois extension of k. Write $G=\mathscr{G}/\mathfrak{B}_M$ $=\mathrm{Gal}\,(M/k)$, $H=\mathfrak{B}_N/\mathfrak{B}_M=\mathrm{Gal}\,(M/N)$ so that $G/H\simeq\mathscr{G}/\mathfrak{B}_N=$ $\mathrm{Gal}\,(N/k)$.

In what follows we will regard natural isomorphisms such as $G/H\simeq\mathscr{G}/\mathfrak{B}_N$ as identifications.

We have that $F(M)=\mathfrak{A}^{\mathfrak{B}_M}$ is a G-module with $F(N)=\mathfrak{A}^{\mathfrak{B}_N}=(\mathfrak{A}^{\mathfrak{B}_M})^H$ $=F(M)^H$ as a G/H-module.

If ℓ denotes the inflation to N of the cocycle α, let f be the corresponding cocycle arising from the isomorphism of complexes $C(G/_H, F(N))\simeq C(N/k, F)$ [4, Thm. 5.4] and let λf be the inflation of this to G. We may form the splitting modules $A_N=F(M)^H_f=F(N)_i$ and $A_M=F(M)_{\lambda f}$ as above.

The commutative diagram (1) above, since it comprises maps of G/H-modules, gives $\{A_M\mid M\in\mathscr{S}\}$ the structure of a directed system compatible with the inverse structure on $\{\mathscr{G}/\mathfrak{B}_M\mid M\in\mathscr{S}\}$ so $\varinjlim A_M$ is a discrete module over $\varinjlim_{M\in\mathscr{S}}\mathscr{G}/\mathfrak{B}_M=\mathscr{G}$.

Similarly, let $R_M=\mathbf{Z}G$ be the group ring for G, $I_M=I_G$ be the augmentation ideal for G and $\mathbf{Z}_M=\mathbf{Z}$. Direct the R_M, I_M and $\mathbf{Z}_M$ according to the commutative diagrams above, for M in $\mathscr{S}$. (In particular, the map $\mathbf{Z}_M\to\mathbf{Z}_{M'}$ is multiplication by $[M':M]$.) As for A_M we have, using M are discrete $\mathscr{G}$-modules.

Let $\overline{F}, \mathscr{I}$ and $\mathscr{R}$ be the sheaves corresponding to $\varinjlim A_M$, $\varinjlim I_M$ and $\varinjlim R_M$ respectively. $\overline{F}$ is called the *splitting sheaf for F*.

PROPOSITION 2.2. *There is an exact sequence of functors* $0 \to F \to \overline{F} \to \mathscr{I} \to 0$.

PROOF. The remarks on splitting modules show that $0 \to \mathfrak{A}^{\mathfrak{B}_M} \to A_M \to I_M \to 0$. $M \in \mathscr{S}$ is an exact sequence of abelian groups natural in M.

Since $\varinjlim$ preserves exactness we have that

$$0 \to \mathfrak{A} \to \varinjlim_{M \in \mathscr{S}} A_M \to \varinjlim_{M \in \mathscr{S}} I_M \to 0$$

is an exact sequence of discrete $\mathscr{G}$-modules.

Suppose N/k is a finite separable field extension with $\mathfrak{B}_N = \mathrm{Gal}\,(k_s/N)$. We then have an exact sequence of abelian groups

$$0 \to \mathfrak{A}^{\mathfrak{B}_N} \to (\varinjlim A_M)^{\mathfrak{B}_N} \to (\varinjlim I_M)^{\mathfrak{B}_N} \to H^1(\mathfrak{B}_N, \mathfrak{A})$$

where the last term is a profinite group cohomology group [5, I, § 1]. Now $\mathfrak{A}$ is also a discrete $\mathfrak{B}_N$-module and $\mathfrak{B}_N = \varinjlim_M \mathfrak{B}_N/\mathfrak{B}_M$, the limit taken over all fields M which are finite Galois over N. Hence by Prop. $\varinjlim_M H^1(M/N, F(M))$ (recalling that $\mathfrak{A}$ is the module corresponding to F) [4, Thm. 5.4]. Since F is a field sheaf, each term in this direct limit is zero. Again using the correspondence between modules and sheaves the above exact sequence then becomes an exact sequence of abelian groups

$$0 \to F(N) \to \overline{F}(N) \to \mathscr{I}(N) \to 0,$$

for any finite separable field extension N of k.

Any finite separable k-algebra A is a direct sum of finite separable field extensions. Since F, $\overline{F}$, and $\mathscr{I}$ are additive, being étale sheaves, a sequence

$$0 \to F(A) \to \overline{F}(A) \to \mathscr{I}(A) \to 0$$

is the direct sum of sequences of the sort shown above to be exact, and is therefore itself exact, completing the proof.

Now let $\mathscr{Z}$ be the additive functor defined by $\mathscr{Z}(M) = \mathbf{Z}$ and if $f: M \to M'$ is a map of fields $\mathscr{Z}(M) \to \mathscr{Z}M')$ is given by multiplication by $[M' : f(M)]$. Additivity defines $\mathscr{Z}$ on the category of finite separable k-algebras.

REMARK. $\mathscr{Z}$ is not an étale sheaf. However, Dobbs [6] has shown that the inclusion functor Additive Functors $\rightsquigarrow$ Functors has a left adjoint *( ). $\mathscr{Z}$ is in fact the value of this "addification" of the constant functor **Z**.

Next we need a technical lemma. We will use it only with T the étale topology and B/A a finite galois extension of fields with group G. The idempotent decomposition mentioned in the proof is just the well known one arising from the isomorphism $B \otimes_A B \simeq \Pi B$, the product of $|G|$ copies of B.

LEMMA 2.3. *Let T be any R based topology* [5]. *Let A, B be in* Cat T *with B a Galois A-algebra with group G* [4]. *Let F be an additive sheaf on T. If the set containing only the structure map $i : A \to B$ is a cover then $F(i)$ is an isomorphism of $F(A)$ onto $F(B)^G$.*

PROOF. Since B is Galois, Theorem 3.1, of [4] gives orthogonal idempotents e_g, g in G in $B \otimes_A B$ with $\sum_{g \in G} e_g = 1$ and $s \otimes 1 = \sum_{g \in G} (1 \otimes g(s))e_g$ for any s in B. Hence

$$(*) \qquad \epsilon_1(s) = \sum \epsilon_0(g(s))e_g$$

for any s in B. Let $\pi_g : B^2 = \Pi B^2 e_g \to B^2 e_g$ denote the natural projection of R algebras. Now $(*)$ means $\pi_g \epsilon_1 = \pi_g \epsilon_0 g$ as maps $B^2 \to B^2 e_g$, so that if $y \in F(B)^G$ we have $F\pi_g F\epsilon_1(y) = F\pi_g F\epsilon_0(y)$ for all g. Since F is additive, this gives $F\epsilon_1(y) = F\epsilon_0(y)$. Since F is a sheaf it follows that y lies in $\mathrm{Im}\,(F(A) \to F(B))$. The opposite set inclusion is trivial.

PROPOSITION 2.4. *There is a short exact sequence of functors* $0 \to \mathscr{I} \to \mathscr{R} \to \mathscr{Z} \to 0$.

PROOF. As in Prop. 2.2, since $\mathscr{I}$, $\mathscr{R}$ and $\mathscr{Z}$ are additive it suffices to show that if N is any finite separable field extension of k then

$$0 \to \mathscr{I}(N) \to \mathscr{R}(N) \to \mathscr{Z}(N) \to 0$$

is exact.

Let M be any finite Galois extension of k which contains N and let $j : N \to M$ be the inclusion. Since $\varinjlim_{\mathrm{M}} I_M$ is discrete we have $\mathscr{I}(M) = (\lim_{\mathrm{M}'} I_{M'})^{\mathfrak{B}_M} = I_M$ (here $\varinjlim$ *is taken over all finite Galois extensions M' of* k containing N). Similarly $\mathscr{R}(M) = R_M$. Since $0 \to I_M \to R_M \to \mathbf{Z} \to 0$ is exact, the top row in the following commutative diagram is exact:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{I}(M) \arrow[r] & \mathscr{R}(M) \arrow[r] & \mathscr{Z}(M) \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{I}(N) \arrow[r] \arrow[u, "\mathscr{I}(j)"] & \mathscr{R}(N) \arrow[r] \arrow[u, "\mathscr{R}(j)"] & \mathscr{Z}(N) \arrow[r] \arrow[u, "\mathscr{Z}(j) = [M:N]"] & 0.
\end{tikzcd}$$

Let $G = \mathrm{Gal}\,(M/k)$, $H = \mathrm{Gal}\,(M/N)$. According to Lemma 2.3, $\mathscr{I}(j)$ and $\mathscr{R}(j)$ are isomorphisms onto $\mathscr{I}(M)^H$ and $\mathscr{R}(M)^H$ respectively. As we observed in Proposition 2.1, the map $\mathscr{R}(M)^H = R_M{}^H = (\mathbf{Z}G)^H \to \mathbf{Z} =$

$Z(M)$ has image $[H:1]\mathbf{Z} = [M:N]\mathbf{Z}$. Accordingly, the above diagram may be replaced by a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{I}(M)^H \arrow[r] & \mathscr{R}(M)^H \arrow[r] & {[M:N]\,\mathcal{Z}(M)} \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{I}(N) \arrow[r] \arrow[u, "\mathscr{I}(j)"'] & \mathscr{R}(N) \arrow[r] \arrow[u, "\mathscr{R}(j)"'] & \mathcal{Z}(N) \arrow[r] \arrow[u, "{\mathcal{Z}(j) = [M:N]}"'] & 0.
\end{tikzcd}$$

In this diagram the top row is exact (Proposition 2.1) and the vertical maps are isomorphisms.

Hence the bottom row is exact as was to be proved.

3. Reciprocity Isomorphisms.

THEOREM 3.1. *Let F be any étale sheaf. If $H^i(N/L, F) = 0$ for all $i > 0$ whenever $k \subseteq L \subseteq N \subseteq k_s$ with N/L Galois and N/k finite, then $H^i(M/L, F) = 0$ for all $i > 0$ and for any $k \subseteq L \subseteq M \subseteq k_s$ with M/k finite.*

PROOF. For $i = 1$ this is Proposition 1.1. Choose a finite galois extension N of L which contains M. Now to apply [8, Thm. 3.2] in higher dimensions, it suffices by the additivity of F to show that

$$H^j_{T_{k_s/k}}(E, F) = 0$$

for $j < i$ and for every field E with $L \subseteq E \subseteq N$. Here the cohomology is the Grothendieck cohomology, the derived functors of "evaluation at E". The algebras which appear in the hypothesis of [8, Thm. 3.2] are direct products of such fields and

$$H^j_{T_{k_s/k}}(A \times B, F) \simeq H^j_{T_{k_s/k}}(A, F) \oplus H^j_{T_{k_s/k}}(B, F).$$

Now

$$H^j_{T_{k_s/k}}(E, F) \simeq H^j(\mathfrak{B}_E, \mathfrak{A})$$

where $\mathfrak{A}$ is the module corresponding to F. But $H^j(\mathfrak{B}_E, \mathfrak{A}) \simeq \varinjlim H^j(\mathfrak{B}_E/\mathscr{R}_P, \mathfrak{A}^{\mathfrak{B}_P})$ the limit taken over all fields $P \subseteq k_s$ which are finite Galois over E. Since $\mathfrak{A}^{\mathfrak{B}_P} = F(P)$ and $\mathfrak{B}_E/\mathfrak{B}_P \simeq \mathrm{Gal}\,(P/E)$, $H^j(\mathfrak{B}_E, \mathfrak{A})$ is then isomorphic to $\varinjlim H^j(P/E, F)$ [4, Thm. 5.4]. Since P/E is Galois, $H^j(P/E, F) = 0$ by assumption, so that

$$H^j_{T_{k_s/k}}(E, F) = 0$$

for all j. Then [4] $0 \to H^i(M/L, F) \to H^i(N/L, F)$ is exact for any $i > 1$ [8, Thm. 3.2]. Since N/L is Galois, $H^i(N/L, F)$ is assumed zero. Hence $H^i(M/L, F) = 0$ as was to be shown.

COROLLARY 3.2. *Let $k \subseteq L \subseteq M \subseteq k_s$ be fields with $[M:k]$ finite. If F is a formation sheaf and $\overline{F}$ its splitting sheaf, then $H^i(M/L, \overline{F}) \neq 0$ for $i > 0$.*

PROOF. Let $N \subseteq k_s$ be any finite Galois extension of L with group G. Then $H^i(N/L, \overline{F}) \simeq H^i(G, \overline{F}(N))$. In the notation of § 2, $\overline{F}(N) = A_N$, the splitting module for the G-module $F(N)$ relative to a fundamental cocycle (that is, a class of a cocycle from $H^2(G, \overline{F}(N))$ corresponding to a fundamental cocycle class from $H^2(N/L, \overline{F})$. It follows from Theorem 1 of [11, p. 294] that $H^i(G, \overline{F}(N)) = 0$ for $i > 0$. Hence, Theorem 3.1 implies $H^i(M/L, \overline{F}) = 0$ for $i > 0$.

COROLLARY 3.3. *Let $\mathscr{R}$ be the functor described in § 2 which assigns to a Galois field extension N of k its group ring $\mathbf{Z}G$ where $G = \mathrm{Gal}\,(N/k)$. Let $k \subseteq L \subseteq M \subseteq k_s$ with $[M:k] < \infty$. Then $H^i(M/L, \mathscr{R}) = 0$ for $i > 0$.*

PROOF. By Theorem 3.1 it suffices to prove that $H^i(M/L, \mathscr{R}) = 0$ whenever M/k is Galois (for then M/L will be Galois and Theorem 3.1 applies directly). Suppose M/L has group G. Then by construction $\mathscr{R}(M) \simeq \mathbf{Z}G$ has the usual G-module structure so that $H^i(M/L, \mathscr{R}) \simeq H^i(H, \mathbf{Z}G)$ where $H = \mathrm{Gal}\,(M/L)$. $\mathbf{Z}G$ is a free H-module and so is induced [3, p. 98]. Since H is finite $\mathbf{Z}G$ is also then coinduced [3, p. 101] and so $H^i(H, \mathbf{Z}G) = 0$ completing the proof.

THEOREM 3.4. (RECIPROCITY). *Let F be a formation sheaf. Let $\mathscr{Z}$ be the additive functor which assigns $\mathbf{Z}$ to each finite separable field extension K and for which, if $K \xrightarrow{j} M$ is a monomorphism of fields with M/K finite separable, $\mathscr{Z}(j)$ is multiplication by $[M : j(K)]$. Then for any finite separable field extension M of K, $H^{i+2}\,(M/K, F) \simeq H^i(M/K, \mathscr{Z})$ for all $i > 0$.*

PROOF. By Proposition 2.2 the sequence

$$\cdots H^{i+1}(M/K, \overline{F}) \to H^{i+1}(M/K, \mathscr{I})$$
$$\to H^{i+2}(M/K, F) \to H^{i+2}(M/K, \overline{F}) \to \cdots$$

is exact for $i \geqq 0$. Hence by Corollary 3.2, $H^{i+1}(M/K, \mathscr{I}) \simeq H^{i+2}(M/K, F)$ for $i > 0$. Similarly, Proposition 2.4 and Corollary 3.3 imply $H^i(M/K, \mathscr{Z}) \simeq H^{i+1}(M/K, \mathscr{I})$ for $i > 0$, completing the proof.

REMARKS. If k is a local field, the functor U which assigns to an algebra its group of units is a formation sheaf. If k is a global field there is a sheaf $\mathscr{I}$ which assigns to each finite separable field extension its idèle class group and this is a formation sheaf. If F is U (in the local case) or $\mathcal{J}$ (in the global case), let $\mathfrak{A}$ be the discrete $\mathcal{G}$-module corresponding to F, and $k \subseteq L \subseteq N \subseteq k_s$ with N/k finite Galois. Then $H^i(N/L, F) \simeq H^i(\mathfrak{B}_L/\mathfrak{B}_N, \mathfrak{A}^{\mathfrak{B}_N})$ and this isomorphism preserves inflation and restriction (Proposition 3.1). The verification that F is a formation sheaf is then precisely the verification that $(\mathcal{G}, \{\mathfrak{B}_M \mid M/k$ is finite Galois$\}, \mathfrak{A})$ is a class formation in the sense of [10, XI]. This verification forms the deep part of local and global class field theory (see, for example, the articles of Tate and Serre in [3]).

If M/K is Galois with group G then $H^i(M/K, \mathcal{F})$ is isomorphic to $H^i(G, \mathcal{F}(M))$, for $\mathcal{F}$ any of the functors F, $\overline{F}$, $\mathcal{I}$, $\mathcal{R}$, or $\mathcal{Z}$ (note that $\mathbf{Z} = \mathcal{Z}(M)$ is a trivial G-module, since $[M:M] = 1$). The two long exact sequences used in the proof of the reciprocity theorem thus coincide with similar sequences for group cohomology which yield the usual reciprocity laws of class field theory in positive dimensions [10, IX, § 8].

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