ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 9, Number 2, Spring 1979

DISCONTINUITY OF THE ALTERNATING CHEBYSHEV OPERATOR

ECKARD SCHMIDT* AND C. B. DUNHAM

1. **Introduction.** Let $[\alpha, \beta]$ be a closed interval and $\| \quad \|$ the Chebyshev norm on $C[\alpha, \beta]$. Consider Chebyshev approximation of $f \in C[\alpha, \beta]$ by an approximating function F such that each approximation $F(A, \cdot)$ has a *degree* $\rho(A)$ such that $F(A, \cdot)$ is best to f if and only if f-$F(A, \cdot)$ alternates $\rho(A)$ times on $[\alpha, \beta]$. Such approximating functions were first considered in full generality by J. Rice [**6**, p. 17ff]. The best known examples where ρ is variable are ordinary rational approximation and exponential approximation,

$$F(A, x) = \sum_{k=1}^{n} a_k \exp(a_{n+k}x). \tag{0}$$

It is known that a best approximation is unique (if it exists). Denote the best approximation to f by Tf, defining the alternating Chebyshev operator. Even when Tf always exists, T may be discontinuous, as discovered by Maehly and Witzgall [**5**], who studied approximation by ordinary rational functions. The behavior of T for this family has been characterized by H. Werner [**7**], who showed that T is continuous at f if and only if Tf was of maximum degree or f is an approximant. The first *general* continuity results were those of Dunham [**1**], [**3**, p. 106], who proved that T is continuous at f if Tf is "non-degenerate", which happens if Tf is of maximum degree, or f is an approximant. An example is given in [**3**, p. 106] to show that discontinuity need not occur if Tf is degenerate and f is not an approximant. Thus it appears that a solution of the problem of continuity of T will require further hypotheses. Dunham also obtained the first general discontinuity result [**3**, p. 107]. In the present paper, Schmidt obtains another general discontinuity result, using a generalization of the property of *irregularity*, first given by Dunham in [**2**; **4**]. By Theorem 3 of [**3**], a non-degenerate approximant cannot be (monotone) irregular. It should be noted that E. Schmidt has studied continuity of T in approximation by exponential sums (limits of families of the form (0)), for which neither alternation nor uniqueness hold, in [**8**].

Received by the editors on February 7, 1977.

*This paper is being published posthumously. Correspondence concerning the paper should be addressed to Charles B. Dunham at the University of Western Ontario.

Following the text of Cheney [9, 80, 165] we define $F(A, \cdot)$ to be a *strongly unique* best approximation to f if there exists a constant $\gamma > 0$ such that

$$\|f - F(B, \cdot)\| \geqq \|f - F(A, \cdot)\| + \gamma \|F(B, \cdot) - F(A, \cdot)\|.$$

The above inequality implies Lipschitz continuity of T at f [9, 82], which in turn implies continuity of T at f. Hence if T has a discontinuity at f, Tf cannot be strongly unique to f. Thus the theorem following gives sufficient conditions for strong uniqueness to fail.

1. **Definitions and Result.**

DEFINITION. F is irregular resp. monotone-irregular at A if for any triple (x, y, ϵ), where $\alpha < x \leqq \beta, y \in \mathbf{R}, \epsilon > 0$, there is a parameter B satisfying the following conditions (i) to (iii) resp. (i) to (iv).

(i) $\rho(B) \leqq \rho(A) + 1$,

(ii) $|F(B, \alpha) - y| < \epsilon$,

(iii) $|F(B, t) - F(A, t)| < \epsilon$ for all $t \geqq x$,

$$\text{(iv)} \quad \begin{cases} \max\limits_{t \in [\alpha, x]} F(B, t) - \max \{F(B, \alpha), F(B, x)\} \leqq \epsilon \\ \\ \min\limits_{t \in [\alpha, x]} F(B, t) - \min \{F(B, \alpha), F(B, x)\} \geqq -\epsilon. \end{cases}$$

Condition (iv) means that $F(B, t)$ is almost monotone in $[\alpha, x]$ in the sense that putting $\epsilon = 0$ would specify monotonicity. The ordinary rationals and exponential sums of the form (0) are monotone-irregular where they are degenerate.

THEOREM. *Let A be best for $f \in C[\alpha, \beta]$, $f \neq F(A, \cdot)$. If F is monotone-irregular at A then T is not continuous at f.*

PROOF. For simplicity and without loss of generality we carry out the proof for $\alpha = 0$, $\beta = 1$. Assuming $f(0) - F(A, 0) \neq \|f - F(A, \cdot)\| =: \eta$ we will construct a sequence of functions $f_m \in C[\alpha, \beta]$, such that $\|f_m - f\| \to 0$ and $\|Tf_m - Tf\| \not\to 0$ by having $f_m - Tf_m$ take the value $\|f_m - Tf_m\|$ at $x = 0$. An obvious change can be done if $f(0) - F(A, 0) = \eta$.

Let $\epsilon_m > 0$ and $\{\epsilon_m\}$ be a null-sequence. By uniform continuity of f and $F(A, \cdot)$ on $[0, 1]$ there is a $d_m > 0$ such that for arbitrary $x, t \in \epsilon$ $[0, 1]$ we have

$$|f(x) - f(t)| \leqq \epsilon_m \tag{1}$$

and

$$|F(A, x) - F(A, t)| \leqq \epsilon_m \tag{2}$$

whenever $|x - t| \leqq d_m$.

Let B_m be a parameter such that conditions (i) to (iv) are satisfied for the triple $(d_m, f(0) - \eta, \epsilon_m)$.

We define

$$\tilde{f}_m(x) := \begin{cases} f(0) & \text{for } 0 \leqq x \leqq d_m \\ f\left(\dfrac{x - d_m}{1 - d_m}\right) & \text{for } d_m \leqq x \leqq 1. \end{cases} \tag{3}$$

Let $N := \rho(A)$, $\{x_0, \cdots, x_N\}$ be an alternant of $f - F(A, \cdot)$ and

$$x_i^m := \frac{x_i - d_m}{1 - d_m}. \tag{4}$$

We now change $\tilde{f}_m$ into a function f_m such that $f_m - F(B_m, \cdot)$ has an alternant $\{0, x_0^m, \cdots, x_N^m\}$ and norm $l_m := \tilde{f}_m(0) - F(B_m, 0)$. Using (1), (2) and $|\eta - l_m| < \epsilon_m$, which is implied by (ii), we have for those indices i where $f(x_i) - F(A, x_i) = +\eta$:

$$\begin{aligned} |\tilde{f}_m(x_i^m) - F(B_m, x_i^m) - l_m| \leqq & |\tilde{f}_m(x_i^m) - f(x_i)| \\ & + |f(x_i) - F(A, x_i) - l_m| \\ & + |F(A, x_i) - F(A, x_i^m)| \\ & + |F(A, x_i^m) - F(B_m, x_i^m)|, \end{aligned}$$

hence

$$|\tilde{f}_m(x_i^m) - F(B_m, x_i^m) - l_m| \leqq 4\,\epsilon_m. \tag{5}$$

In the same manner we get for those indices i such that $f(x_i) - F(A, x_i) = -\eta$,

$$|\tilde{f}_m(x_i^m) - F(B_m, x_i^m) + l_m| \leqq 4\,\epsilon_m. \tag{6}$$

Let $\tilde{E}_m(x) := \tilde{f}_m(x) - F(B_m, x)$ and put

$$I^+ := \{i \mid 0 \leqq i \leqq N, 0 < \tilde{E}_m(x_i^m) < l_m\} \tag{7}$$

and

$$I^- := \{i \mid 0 \leqq i \leqq N, 0 > \tilde{E}_m(x_i^m) > -l_m\}. \tag{8}$$

For every $i \in I^+$ with $x_i^m \in (0, 1)$ there exists a pair u_i, v_i with $u_i < x_i^m < v_i$ such that

$$\tilde{E}_m(u_i) = \tilde{E}_m(v_i) = l_m - 5\,\epsilon_m \tag{9}$$

and

$$\tilde{E}_m(x) \geqq \tilde{E}_m(u_i) \text{ for } u_i \leqq x \leqq v_i. \tag{10}$$

We replace now $E_m(x)$, $u_i \leqq x \leqq v_i$, by a quadratic parabola which interpolates the three points $(u_i, \tilde{E}_m(u_i))$, (t_i, l_m), $(v_i, \tilde{E}_m(v_i))$ with t_i arbitrary in (u_i, v_i). A similar construction is done for I^-. Further consideration deserves only x_N^m since $x_0^m > 0$. If $x_N^m = 1$ there exists a u_N either as point of intersection of $\tilde{E}_m$ with the line $l_m - 5\,\epsilon_m$, such that $\tilde{E}_m(x) \geqq \tilde{E}_m(u_N)$ for $u_N \leqq x \leqq 1$, or as point of intersection of $\tilde{E}_m$ with the line $-l_m + 5\,\epsilon_m$ such that $\tilde{E}_m(x) \leqq \tilde{E}_m(u_N)$ for $u_N \leqq x \leqq 1$. If $x_N^m = 1$ we replace $\tilde{E}_m$ by the straight line connecting $(u_N, \tilde{E}_m(u_N))$ with $(1, l_m)$ resp. $(1, -l_m)$. Furthermore we replace $\tilde{E}_m(x)$ by l_m if $\tilde{E}_m(x) > l_m$ and by $-l_m$ if $\tilde{E}_m(x) < -l_m$.

Considering all these changes in $\tilde{E}_m$ as applied to $\tilde{f}_m$ we have defined a function f_m which is continuous on $[0, 1]$. Furthermore $\{0, x_0^m, \cdots, x_N^m\}$ is an alternant of $f_m - F(B_m, \cdot)$. Since by (i) B_m has a degree not greater than $N + 1$ it is best for f_m.

We now show that for all x, $0 \leqq x \leqq 1$,

$$\tilde{E}_m(x) - l_m \leqq 4\,\epsilon_m \tag{11}$$

and

$$\tilde{E}_m(x) + l_m \geqq -4\,\epsilon_m \tag{12}$$

which then implies $\|f_m - \tilde{f}_m\| \leqq 5\,\epsilon_m$, hence

$$\|f_m - f\| \leqq \|f_m - \tilde{f}_m\| + \|f_m - f\| \leqq 6\,\epsilon_m,$$

that is,

$$\|f_m - f\| \to 0 \text{ as } m \to \infty. \tag{13}$$

Since

$$\max_{[0,d_m]} \left| \frac{x - d_m}{1 - d_m} - x \right| = d_m, \tag{14}$$

we have for $d_m \leqq x \leqq 1$

$$E_m(x) - l_m = \tilde{f}_m(x) - F(B_m, x) - l_m$$

$$= \tilde{f}\left(\frac{x - d_m}{1 - d_m} \right) - F(B_m, x) - l_m$$

$$= f\left(\frac{s - d_m}{1 - d_m} \right) - f(x)$$

$$+ f(x) - F(A, x)$$

$$+ F(A, x) - F(B_m, x) - l_m$$

$$\leqq 2\,\epsilon_m + \eta - l_m \leqq 3\,\epsilon_m.$$

For $0 \leqq x \leqq d_m$ we have by (iv)

$$\begin{aligned}\tilde{E}_m(x) - l_m &= f(0) - F(B_m, x) - l_m \\ &\leqq f(0) - \min_{[0,d_m]} \{F(B_m, x)\} - l_m \\ &\leqq f(0) - \min\,\{F(B_m, 0), F(B_m, d_m)\} + \epsilon_m - l_m.\end{aligned}$$

If the minimum is taken at $x = 0$ we get

$$\tilde{E}_m(x) - l_m \leqq f(0) - F(B_m, 0) + \epsilon_m - l_m = \epsilon_m.$$

If the minimum is taken at $x = d_m$ we have

$$\begin{aligned}\tilde{E}_m(x) - l_m \leqq\ & f(0) - f(d_m) \\ &+ f(d_m) - F(A, d_m) \\ &+ F(A, d_m) - F(B, d_m) \\ &+ \epsilon_m - l_m \\ \leqq\ & 3\,\epsilon_m + \eta - l_m \leqq 4\,\epsilon_m.\end{aligned}$$

This shows the validity of (11); (12) is obtained in a similar way. From (ii) we have

$$f(0) - F(B_m, 0) \longrightarrow \eta \text{ as } m \longrightarrow \infty.$$

With the assumption $f(0) - F(A, 0) \neq \eta$ it follows that

$$\lim_{m\to\infty} F(B_m, 0) \neq F(A, 0) \tag{15}$$

which finishes the proof.

References

1. C. Dunham, *Continuity of the varisolvent Chebyshev operator*, Bull. Amer. Math. Soc. 74 (1968), 606–608.

2. ———, *Approximation by alternating families on subsets*, Computing **9** (1972), 261–265.

3. ———, *Alternating Chebyshev approximation*, Trans. Amer. Math. Soc. **178** (1973), 95–109.

4. ———, *Nonexistence of best alternating approximations on subsets*, Proc. Amer. Math. Soc. **60** (1976), 203–206.

5. H. Maehly and C. Witzgall, *Tschebyscheff-Approximation in kleinen Intervallen* II, Numer. Math. **2** (1960), 293–307.

6. J. Rice, *The approximation of functions*, **2**, Addison-Wesley, Reading, Mass. 1969.

7. H. Werner, *On the rational Tschebyscheff operator*, Math. Z. **86** (1964), 317–326.

8. E. Schmidt, *Stetigkeitsaussagen bei der Tschebyscheff-Approximation mit Exponentialsummen*, Math. Z. **113** (1970), 159–170.

9. E. W. Cheney, *Introduction to Approximation Theory*, McGraw-Hill, New York, 1966.

UNIVERSITY OF CALGARY, CALGARY, ALBERTA
UNIVERSITY OF WESTERN ONTARIO, LONDON, ONTARIO