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A HOMOMORPHISM OF A PSEUDO PLANE ONTO A PROJECTIVE PLANE

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1. The purpose of this paper is to give an example of a homomorphism of a proper pseudo plane onto a projective plane. The pseudo plane used is coordinatized by Zemmer's nonplanar nearfield [5], and the image plane is a field plane. With the exception of the concept of place of a homomorphism the notation and terminology will follow that found in [4]. In §2 we give a characterization of place found in [1], and in §3 we give the example referred to above.

2. We let π and π' be pseudo planes and $\alpha : \pi \to \pi'$ be a homomorphism. We may choose a coordinatizing quadrangle for π such that its image is a coordinatizing quadrangle for π'. Call the quadrangle for π, (∞), (0), $(0, 0)$, $(1, 1)$. Let T and T' be the pseudo ternaries associated with these quadrangles for π and π' respectively. Pseudo ternaries are discussed in [3]. Then there is a mapping $\bar{\alpha} : T \to T' \cup \{\infty\}$ defined by

$$\bar{\alpha}b = \begin{cases} b' & \text{if } \alpha(0, b) = (0', b'), \\ \infty & \text{if } \alpha(0, b) = \alpha(\infty). \end{cases}$$

$\bar{\alpha}$ is called a place of α. Generally no confusion results from denoting $\bar{\alpha}$ by α.

If we assume that $(T, +, \cdot)$ is a nearfield, then the proof of Theorem 4.3 found in [1] suffices to show that α is a place of a homomorphism if and only if the following hold:

S1. $\alpha 0 = 0$, and $\alpha 1 = 1$.

S2. αa and $\alpha b \neq \infty$ implies $\alpha(a + b) = \alpha a + \alpha b$, and $\alpha(ab) = \alpha a \alpha b$.

S3. $\alpha a \neq \infty$ and $\alpha b = \infty$ implies $\alpha(a + b) = \alpha(b + a) = \infty$.

S4. $\alpha a \neq 0$ and $\alpha b = \infty$ implies $\alpha(ab) = \alpha(ba) = \infty$.

S5. $\alpha(-ax + a^*x) \neq \infty$ and $\alpha x = \infty$ implies $\alpha a = \alpha a^*$.

S6. $\alpha(ax - ax^*) \neq \infty$ and $\alpha a = \infty$ implies $\alpha x = \alpha x^*$.

S7. $a^*x + ax^* = ax$ and $\alpha a = \alpha x = \alpha(a^*x) = \alpha(ax^*) = \infty$ implies $\alpha a^* = \infty$ or $\alpha x^* = \infty$.

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We will use S1—S7 in the next section to show that a mapping is a place.

3. The pseudo plane under consideration in this section is the pseudo plane coordinatized by Zemmer's nonplanar nearfield [5].

DEFINITION. Let $N = F(\lambda)$ be the set of rational functions over F, a field of characteristic 0. For $a(\lambda) \in F(\lambda)$ let $n(a) =$ degree of numerator of a, and $d(a) =$ degree of denominator of a. Let $\delta(a) = n(a) - d(a)$. Define $a(\lambda) + b(\lambda)$ to be the sum in $F(\lambda)$. For $a(\lambda) \neq 0$, let $a(\lambda) \circ b(\lambda) = a(\lambda)b(\lambda + \delta(a))$, and also let $0 \circ b(\lambda) = 0$.

It is known, [5], that $(N, +, \circ)$ is a near field in which the equation $x = \lambda \circ x + \lambda$ has no solution. Hence $(N, +, \circ)$ cannot coordinatize a projective plane. We will construct a place from $(N, +, \circ)$ to $(F, +, \cdot)$, and the latter does coordinatize a projective plane as it is a field. $(N, +, \circ)$ coordinatizes a pseudo plane [2].

The following facts will be used, often without mention.

$$\delta(a+b) = \begin{cases} \delta(a) & \text{if } n(a) + d(b) > n(b) + d(a), \\ \delta(b) & \text{if } n(a) + d(b) < n(b) + d(a), \\ k \leqq \delta(a) & \text{if } n(a) + d(b) = n(b) + d(a). \end{cases} \tag{1}$$

$$\delta(a \circ b) = \delta(a) + \delta(b). \tag{2}$$

Let $a = (a_n\lambda^n + \cdots + a_0)/(c_m\lambda^m + \cdots + c_0)$, and define

$$\alpha(a) = \begin{cases} a_n/c_m & \text{if } \delta(a) = 0, \\ 0 & \text{if } \delta(a) < 0, \\ \infty & \text{if } \delta(a) > 0. \end{cases}$$

We will prove that α is a place of a homomorphism from the pseudo plane coordinatized by $(N, +, \circ)$ onto the plane coordinatized by $(F, +, \cdot)$ by showing that S1—S7 are satisfied.

S1 is immediate.

To verify S2, let a be given as above, and let

$$b = (b_u\lambda^u + \cdots + b_0)/(e_v\lambda^v + \cdots + e_0).$$

If we write only the terms of largest degree in each product, we obtain

$$a + b = \frac{a_n e_v \lambda^{n+v} + b_u c_m \lambda^{m+u} + \cdots}{c_m e_v \lambda^{m+v} + \cdots}.$$

Take $\alpha a \neq \infty \neq \alpha b$. Then $n \leqq m$ and $u \leqq v$. So $n + v \leqq m + v$

and $m + u \leqq m + v$. If $n + v > m + u$, then $v > u$, and thus $\alpha b = 0$. But $\alpha(a + b) = (a_n e_v)/(c_m e_v) = a_n/c_m = \alpha a = \alpha a + \alpha b$. Similarly, the desired result holds if $n + v < m + u$.

Let $n + v = m + u$. If $n < m$, then $v > u$, and $\alpha a + \alpha b = 0$. But then $\delta(a + b) = k \leqq \delta(a) < 0$. So $\alpha(a + b) = 0$ as required. If $n = m$, then $u = v$ and $\alpha a = a_n/c_m$ and $\alpha b = b_u/e_v$. Thus $\alpha a + \alpha b = (a_n e_v + b_u c_m)/(c_m e_v)$. If $a_n e_v + b_u c_m \neq 0$, then

$$\alpha(a + b) = (a_n e_v + b_u c_m)/(c_m e_v),$$

as required. Suppose that $a_n e_v + b_u c_m = 0$, then $k < 0$, and in this case $\alpha(a + b) = 0$, and $\alpha a + \alpha b = 0/(c_m e_v) = 0$.

That $\alpha(a \circ b) = \alpha a \alpha b$ is immediate.

Now consider S3. Use a and b as above with $\alpha a = \infty$, and $\alpha b \neq \infty$. Then $n > m$, and $u \leqq v$. So as before,

$$a + b = \frac{a_n e_v \lambda^{n+v} + b_u c_m \lambda^{m+u} + \cdots}{c_m e_v \lambda^{m+v} + \cdots}.$$

We have $n + v > m + u$, and $n + v > m + v$, so $n(a + b) = n + v$ and $\alpha(a + b) = \infty$.

For S4 we take a and b as before with $\alpha a \neq 0$, and $\alpha b = \infty$. Then $m \leqq n$, and $u > v$. Considering only leading terms we have

$$a \circ b = \frac{a_n b_u \lambda^{n+u} + \cdots}{c_m e_v \lambda^{m+v} + \cdots}.$$

Thus, $\alpha(a \circ b) = \infty$ as $n + u > m + v$. Similarly, $\alpha(b \circ a) = \infty$.

For S5 let a and b be as above, and let

$$x = \frac{x_r \lambda^r + \cdots + x_0}{w_s \lambda^s + \cdots + w_0}, \qquad a^* = \frac{y_t \lambda^t + \cdots + y_0}{z_p \lambda^p + \cdots + z_0},$$

$$x^* = \frac{f_q \lambda^q + \cdots + f_0}{g_h \lambda^h + \cdots + g_0}.$$

We take $\alpha(-a \circ x + a^* \circ x) \neq \infty$, and $\alpha x = \infty$. Then

$$\begin{aligned} a^* \circ x - a \circ x = & \left(\frac{y_t \lambda^t + \cdots + y_0}{z_p \lambda^p + \cdots + z_0}\right) \circ \left(\frac{x_r \lambda^r + \cdots + x_0}{w_s \lambda^s + \cdots + w_0}\right) \\ & - \left(\frac{a_n \lambda^n + \cdots + a_0}{c_m \lambda^m + \cdots + c_0}\right) \circ \left(\frac{x_r \lambda^r + \cdots + x_0}{w_s \lambda^s + \cdots + w_0}\right) \end{aligned}$$

$$= \frac{(y_t\lambda^t + \cdots + y_0)(x_r(\lambda + t - p)^r + \cdots + x_0)}{(z_p\lambda^p + \cdots + z_0)(w_s(\lambda + t - p)^s + \cdots + w_0)}$$

$$- \frac{(a_n\lambda^n + \cdots + a_0)(x_r(\lambda + n - m)^r + \cdots + x_0)}{(c_m\lambda^m + \cdots + c_0)(w_s(\lambda + n - m)^s + \cdots + w_0)}$$

$$= \frac{y_t x_r \lambda^{t+r} + \cdots}{z_p w_s \lambda^{p+s} + \cdots} - \frac{a_n x_r \lambda^{n+r} + \cdots}{c_m w_s \lambda^{m+s} + \cdots}$$

$$= \frac{y_t x_r c_m w_s \lambda^{t+r+m+s} - a_n x_r z_p w_s \lambda^{n+r+p+s} + \cdots}{z_p w_s c_m w_s \lambda^{p+m+s+a} + \cdots},$$

considering only leading terms. Since $\alpha x = \infty$ we have $r > s$.

Case 1. $t + r + m + s > n + r + p + s$. Then $t + r + m + s \leqq p + m + 2s$, since $\alpha(-a \circ x + a^* \circ x) \neq \infty$. Thus, $t + m > n + p$, and $t + r \leqq p + s$. Thus, $t < p$, so $m > n$. Hence $\alpha a = \alpha a^* = 0$.

Case 2. $t + r + m + s < n + r + p + s$ yields as in Case 1, $\alpha a = \alpha a = 0$.

Case 3. $t + r + m + s = n + r + p + s$. If $y_t x_r c_m w_s - a_n x_r z_p w_s \neq 0$, then $t + r + m + s \leqq p + m + 2s$. Thus $t + r < p + s$. Hence $t < p$. But $t + m = n + p$, so $m > n$. Again $\alpha a^* = \alpha a^* = 0$. Finally, take $y_t x_r c_m w_s - a_n x_r z_p w_s = 0$. Then $y_t c_m = a_n z_p$, and thus $a_n/c_m = y_t/z_p$. Hence $\alpha a = \alpha a^*$, as $m - n \geqq 0$ if and only if $p - t \geqq 0$.

Next, we consider S6. By the definition of α, $\alpha(-g) = -\alpha g$. Let $\alpha(a \circ x - a \circ x^*) \neq \infty$ and $\alpha a = \infty$, where x^* is given above. Now, $(N, +, \circ)$ is a nearfield, so we have

$$\alpha(a \circ x - a \circ x^*) = \alpha(a \circ (x - x^*)),$$

So by S4, $\alpha(x - x^*) = 0$. We show that $\alpha x = \alpha x^*$. If at least one of αx or αx^* is ∞, then by S3, they both must be ∞. So suppose that neither is ∞. Then by S2, $\alpha x = \alpha x^*$.

Lastly we consider S7. We suppose that the hypotheses for S7 are satisfied, and as for S5 we consider only leading terms in $a^* \circ x + a \circ x^* = a \circ x$. The result is

$$\frac{y_t x_r c_m g_h \lambda^{t+r+m+h} + a_n f_q z_p w_s \lambda^{n+q+p+s} + \cdots}{z_p w_s c_m g_h \lambda^{p+s+m+h} + \cdots} = \frac{a_n x_r \lambda^{n+r} + \cdots}{c_m w_s \lambda^{m+s} + \cdots},$$

which yields,

$$y_t x_r c_m g_h c_m w_s \lambda^{t+r+m+h+m+s} + a_n f_q z_p w_s c_m w_s \lambda^{n+q+p+s+m+s}$$
$$= a_n x_r z_p w_s c_m g_h \lambda^{n+r+p+s+m+h} + \cdots.$$

The assumptions yield $n > m$, $r > s$, $t + r > p + s$, and $n + q > m + h$.

Case 1. $t + r + m + h + m + s > n + q + p + s + m + s$. Then we have that $t + r + m + h + m + s = n + r + p + s + m + h$. Hence, $t + m = n + p$, so $t > p$. Hence, $\alpha a^* = \infty$.

Case 2. $t + r + m + h + m + s < n + q + p + s + m + s$. Then we have that $n + q + p + s + m + s = n + r + p + s + m + h$, so $q + s = r + h$. Hence $q > h$, which implies that $\alpha x^* = \infty$.

Case 3. $t + r + m + h + m + s = n + q + p + s + m + s$. Now, if $n + q + p + s + m + s = n + r + p + s + m + h$, then as in Case 2, $q > h$, and $\alpha x^* = \infty$. If $n + q + p + s + m + s < n + r + p + s + m + h$, we may apply Case 1. Suppose that $n + q + p + s + m + s > n + r + p + s + m + h$, then again $q + s > r + h$, so $q > h$. Hence $\alpha x^* = \infty$.

Thus by the discussion in §2, there is a homomorphism from the pseudo plane coordinatized by the nearfield to the projective plane coordinatized by $(F, +, \cdot)$. The fact that this homomorphism is onto is immediate.

References

1. J. André, *Über Homomorphismen projektiver Ebenen,* Abh. Math. Sem. Univ. Hamburg **34** (1969/70), 98–114. MR **42** #949.

2. E. Davis, *Incidence systems associated with non-planar nearfields,* Canad. J. Math. **22** (1970), 939–952.

3. R. Sandler, *Pseudo planes and pseudo ternaries,* J. Algebra **4** (1966), 300–316. MR **33** #7929.

4. R. Sandler, *On homomorphisms and images of projective planes and pseudo planes,* Acta Math. (to appear).

5. J. Zemmer, *Nearfields, planar, and non-planar,* Math. Student **31** (1964), 145–150. MR **31** #5888.

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