ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 2, Number 1, Winter 1972

SOME COUNTEREXAMPLES INVOLVING SELFADJOINT OPERATORS

JEROME A. GOLDSTEIN [1]

1. **Introduction.** We present several counterexamples related to the convergence, (generalized) addition, and (generalized) commutation of (unbounded) skew-adjoint operators.

2. **Convergence of skew-adjoint operators.** Let A_n $(n = 0, 1, 2, \cdots)$ be a skew-adjoint operator on a Hilbert space $\mathcal{H}$. We say that A_n *converges to* A_0 and we write $\lim_{n\to\infty} A_n = A_0$ iff

$$\lim_{n\to\infty} (\lambda I - A_n)^{-1} f = (\lambda I - A_0)^{-1} f \tag{1}$$

for all $f \in \mathcal{H}$ and all $\lambda \in \mathbf{R}\backslash\{0\}$ ($\mathbf{R}$ is the real line and I is the identity on $\mathcal{H}$). This is equivalent to

$$\lim_{n\to\infty} U_n(t) f = U_0(t) f \tag{2}$$

for all $t \in \mathbf{R}$ and all $f \in \mathcal{H}$ where $U_n = \{U_n(t); t \in \mathbf{R}\}$ is the (C_0) unitary group generated by A_n, $n = 0, 1, 2, \cdots$. The above result is an immediate consequence of Stone's theorem and the Trotter-Neveu-Kato approximation theorem for (C_0) semigroups of operators (cf. for instance Goldstein [**5**], Kato [**6**], Yosida [**9**]).

A useful sufficient condition for (1) to hold is given by the following well-known simple result.

LEMMA 1. *Let A_n be skew-adjoint operators on $\mathcal{H}$, $n = 0, 1, 2, \cdots$. Then* (1) *holds for all $f \in \mathcal{H}$ and all $\lambda \in \mathbf{R}\backslash\{0\}$ if there is a subspace $\mathfrak{D} \subset \mathfrak{D}(A_0)$ (= the domain of A_0) such that*

(i) *A_0 is the closure of $A_0 \mid \mathfrak{D}$,*

(ii) *for all $f \in \mathfrak{D}$, $f \in \mathfrak{D}(A_n)$ for n sufficiently large and $\lim_{n\to\infty} A_n f = A_0 f$.*

Our first example shows that the sufficient condition given in Lemma 1 is far from being necessary.

EXAMPLE 1. *There is a sequence U_n $(n = 0, 1, 2, \cdots)$ of (C_0)*

Received by the editors October 26, 1970.

AMS 1970 *subject classifications.* Primary 47C99, 47D10, 47B25, 47B47; Secondary 22A99.

[1]Supported by National Science Foundation grant GP-19824.

unitary groups on a Hilbert space $\mathcal{H}$ *with skew-adjoint generators* A_n *such that* $\lim_{n\to\infty} A_n = A_0$ *but*

$$\mathfrak{D}(A_0) \cap \bigcup_{n=1}^{\infty} \mathfrak{D}(A_n) = \{0\}.$$

CONSTRUCTION. Let $\mathcal{H}$ be the complex Hilbert space $L_2(\mathbf{R})$. ($\mathbf{R}$ is given Lebesgue measure.) Define

$$U(\sigma,\tau;t)f(x) = f(x+\sigma t)\exp\left\{i\tau\int_0^t q(x+\sigma s)\,ds\right\}$$

for $f \in \mathcal{H}$, σ, τ, t, $x \in \mathbf{R}$, where $q \in L^1_{\mathrm{loc}}(\mathbf{R})$. Then $U(\sigma,\tau) = \{U(\sigma,\tau;t);\ t \in \mathbf{R}\}$ is a (C_0) unitary group on $\mathcal{H}$ for each $\sigma, \tau \in \mathbf{R}$. Note that $U_0 \equiv U(1,0)$ is the translation group whose generator A_0 is given by $A_0 f = f'$ for $f \in \mathfrak{D}(A_0) = \{g \in \mathcal{H} : g$ absolutely continuous, $g' \in \mathcal{H}\}$.

Let $\tau \neq 0$. For $f \in \mathcal{H}$, $h \in \mathbf{R}\backslash\{0\}$,

$$h^{-1}(U(\sigma,\tau;h)f - f) = J_1 + J_2$$

where

$$J_1(x) = \exp\left\{i\tau\int_0^h q(x+\sigma s)\,ds\right\}h^{-1}(f(x+\sigma h) - f(x)),$$

$$J_2(x) = h^{-1}\left(\exp\left\{i\tau\int_0^h q(x+\sigma s)\,ds\right\} - 1\right)f(x).$$

If $f \in \mathfrak{D}(A_0)$, then $\lim_{h\to 0} J_1$ exists (in the norm topology of $\mathcal{H}$) and equals $\sigma f' = \sigma A_0 f$. If also f is the domain of the generator $A(\sigma,\tau)$ of $U(\sigma,\tau)$, then also $\lim_{h\to 0}(J_1 + J_2)$ exists, hence $\lim_{h\to 0} J_2$ necessarily exists in $\mathcal{H}$. But for almost all $x \in \mathbf{R}$, $\lim_{h\to 0} J_2(x) = q(x)f(x)$. Hence $qf \in \mathcal{H}$. But q can be chosen so that

$$\{g \in \mathfrak{D}(A_0) : qg \in \mathcal{H}\} = \{0\}, \tag{3}$$

whence $f = 0$ (cf. Chernoff [1], Goldstein [5]). The construction is well known, but we indicate it for completeness. Let $\{r_n\}_1^\infty$ be a dense sequence in $\mathbf{R}$. Let $q(x) = \sum_{n=1}^\infty (n!)^{-1}|x - r_n|^{-1/2}$. Then $q \in L^1_{\mathrm{loc}}(\mathbf{R})$ but q is not square integrable over any interval of positive length. If g is continuous and nonzero at a point x_0, then $|g(x)| \geqq \epsilon > 0$ for some $\epsilon > 0$ and all x in some neighborhood of x_0. Hence $\int_{\mathbf{R}} |qg|^2\,dx = \infty$, so that (3) holds for this choice of q. It follows that

$$\mathfrak{D}(A_0) \cap \bigcup\{\mathfrak{D}(A(\sigma,\tau)) : \tau \in \mathbf{R}\backslash\{0\}, \sigma \in \mathbf{R}\} = \{0\}. \tag{4}$$

Finally let $\{\tau_n\}_1^\infty$, $\{\sigma_n\}_1^\infty$ be sequences in $\mathbf{R}$ satisfying $\tau_n \neq 0$ for all n, $\lim_{n\to\infty}\sigma_n = 1$, $\lim_{n\to\infty}\tau_n = 0$. Let $U_n = U(\sigma_n, \tau_n)$, $A_n = A(\sigma_n, \tau_n)$, $n \geqq 1$. Then since $U(\sigma, \tau; t)f$ is clearly a jointly continuous $\mathcal{H}$-valued function of σ, τ, t for each fixed $f \in \mathcal{H}$, it follows that (2) holds, and hence (1) holds also. This completes the proof in view of (4).

3. **Addition of skew-adjoint operators.** Let A, B, C be skew-adjoint operators on a Hilbert space $\mathcal{H}$. We say that C is the *generalized sum* or the *Lie sum* of A and B, and we write $C = A +_L B$, iff

$$\lim_{n\to\infty} (U(t/n)\, V(t/n))^n f = W(t)f$$

for all $t \in \mathbf{R}$, $f \in \mathcal{H}$ where U, V, W denote the (C_0) unitary groups generated by A, B, C respectively. This is the *right* definition from the point of view of infinite-dimensional Lie theory (cf. Goldstein [4]). $A +_L B$ (if it exists) is an extension of the closure $(A + B)^-$ of $A + B$ (defined on $\mathfrak{D}(A) \cap \mathfrak{D}(B)$), and $A +_L B$ equals $(A + B)^-$ if $A + B$ is essentially skew-adjoint. The basic properties of the Lie sum have been developed by Chernoff [1], [2].

EXAMPLE 2. *There exist skew-adjoint operators* A_n, B_n $(n = 0, 1, 2, \cdots)$ *on a Hilbert space* $\mathcal{H}$ *such that* $\lim_{n\to\infty} A_n = A_0$, $\lim_{n\to\infty} B_n = B_0$, $A_n +_L B_n$ *exists for all* $n \geqq 1$, *but* $A_0 +_L B_0$ *does not exist and* $A_n +_L B_n$ *does not converge to a skew-adjoint operator.*

The construction of Example 2 will be given in the next section.

The Trotter-Neveu-Kato approximation, which holds for nets as well as sequences (Seidman [8]), enables one to define a topology in a natural way on the set of all skew-adjoint operators on $\mathcal{H}$. Example 2 shows that generalized addition is not continuous with respect to this topology.

4. **Commutation of skew-adjoint operators.** Let A, B, C be skew-adjoint operators on a Hilbert space $\mathcal{H}$. We say that C is the *generalized commutator* or *Lie commutator* of A and B, and we write $C = [A, B]_L$, iff

$$\lim_{n\to\infty} \{U(t/n)\, V(t/n)\, U(-t/n)\, V(-t/n)\}^{n^2} f = W(t^2)f$$

for all $t \in \mathbf{R}$, $f \in \mathcal{H}$ where U, V, W denote the (C_0) unitary groups generated by A, B, C respectively. If the closure C of the restriction of $AB - BA$ to $\mathfrak{D}(AB) \cap \mathfrak{D}(BA) \cap \mathfrak{D}(B^2) \cap \mathfrak{D}(A^2)$ is skew-adjoint, then $[A, B]_L$ exists and equals C. This result was proved by Nelson [7, p. 111]; a similar result was proved independently by Goldstein [4].

EXAMPLE 3. *There exist skew-adjoint operators A, B with B a bounded operator such that the restriction of $AB - BA$ to $\mathfrak{D}(AB) \cap \mathfrak{D}(BA) \cap \mathfrak{D}(B^2) \cap \mathfrak{D}(A^2)$ has no skew-adjoint extension.*

EXAMPLE 4. *There exist skew-adjoint operators A, B with B a bounded operator such that $\mathfrak{D}(AB - BA)$ $(= \mathfrak{D}(AB) \cap \mathfrak{D}(BA)) = \{0\}$, but nevertheless $[A, B]_L$ exists (as a skew-adjoint operator).*

EXAMPLE 5. *There exist skew-adjoint operators A_n, B_n $(n = 0, 1, 2, \cdots)$ such that $\lim_{n\to\infty} A_n = A_0$, $\lim_{n\to\infty} B_n = B_0$, $[A_n, B_n]_L$ exists for $n \geqq 1$, but $[A_0, B_0]_L$ does not exist and $[A_n, B_n]_L$ does not converge to a skew-adjoint operator.*

CONSTRUCTION OF EXAMPLE 3. Let $\mathcal{H} = L^2(0, \infty)$. Let $A_0 f(x) = if''(x) - ix^2 f(x)$ for $f \in \mathfrak{D}(A_0) = C_c^\infty(0, \infty)$. Then the closure A of A_0 is skew-adjoint, has pure point spectrum, and its eigenvectors are the Hermite functions (see e.g. Dunford-Schwartz [3, Chapter XIII]).

Let $Bf(x) = i(x + 2)e^{-x}f(x)$ for $f \in \mathcal{H}$. Then B is a bounded skew-adjoint operator on $\mathcal{H}$ which leaves $C_c^\infty(0, \infty)$ invariant. Hence $C_c^\infty(0, \infty) \subset \mathfrak{D}(AB) \cap \mathfrak{D}(BA) \cap \mathfrak{D}(B^2) \cap \mathfrak{D}(A^2)$, and for all $f \in C_c^\infty(0, \infty)$ an elementary calculation shows that

$$(AB - BA)f = Lf$$

where $Lf(x) = -2(1 + x)e^{-x}f'(x) + xe^{-x}f(x)$ for $f \in \mathfrak{D}(L) = C_c^\infty(0, \infty)$. $-iL$ is symmetric and its adjoint is a restriction of the distributional differential operator $2i(1 + x)e^{-x}d/dx - ixe^{-x}I$. To compute the deficiency indices of $-iL$ we must solve $(-iL)^*f = \pm if$; the distributional solutions are of the form $C_1 f_+ + C_2 f_-$ where C_1, C_2 are constants and

$$f_\pm(x) = \{e^x(1 + x)^{-1}\}^{1/2} \exp\left\{\pm 2^{-1}\int_0^x e^t(1 + t)^{-1}\,dt\right\}.$$

Clearly $f_- \notin \mathcal{H}$ and $f_+ \in \mathcal{H}$; in fact it is easy to see that $f_+ \in \mathfrak{D}(L^*)$ so that the deficiency indices of $-iL$ are $(1, 0)$. Hence L is essentially maximal skew-symmetric with no skew-adjoint extension. {We remark that in fact $[A, B]_L$ exists and equals $\bar{L}$ in the sense of [4, Theorem 1], even though it does not exist in the sense of this section.}

CONSTRUCTION OF EXAMPLE 4. Let $\mathcal{H} = L^2(\mathbf{R})$. Let $\mathfrak{D}(A) = \{f \in \mathcal{H}$: f absolutely continuous, $f' \in \mathcal{H}\}$ and let $Af = f'$ for $f \in \mathfrak{D}(A)$. Let $\{r_n\}_1^\infty$ be a dense sequence in $\mathbf{R}$ and let

$$q(x) = \sum_x 2^{-n},$$

$\sum_x$ denoting the sum over all n such that $r_n < x$. Note that $0 < q(x) < 1$ for all $x \in \mathbf{R}$, and q is monotone increasing and hence differentiable a.e. Let B be the bounded skew-adjoint operator defined by $Bf(x) = iq(x)f(x)$, $f \in \mathcal{H}$. If U, V denote the (C_0) unitary groups generated by A, B respectively, then a straightforward calculation shows that

$$\{U(t/n)V(t/n)U(-t/n)V(-t/n)\}^{n^2} f(x)$$
$$= f(x) \exp \{it^2(q(x) - q(x + t/n))/(t/n)\}.$$

Hence

$$\lim_{n\to\infty} \{U(t/n)V(t/n)U(-t/n)V(-t/n)\}^{n^2} f = W(t^2)f$$

by the dominated convergence theorem, for all $t \in \mathbf{R}$, $f \in \mathcal{H}$, where

$$W(s)f(x) = f(x) \exp \{-isq'(x)\},$$

so that W is the (C_0) unitary group generated by C, where

$$Cf(x) = -iq'(x)f(x)$$

for $f \in \mathfrak{D}(C) = \{g \in \mathcal{H} : \int_{-\infty}^{\infty} |q'(x)g(x)|^2\, dx < \infty\}$. Thus $[A, B]_L$ exists and equals C. On the other hand, if $f \in \mathfrak{D}(AB - BA) = \mathfrak{D}(AB) \cap \mathfrak{D}(BA)$, then both f and qf are continuous on $\mathbf{R}$. Hence q is continuous at all points x_0 such that $f(x_0) \neq 0$. But q is discontinuous at each r_n, so that $f(r_n) = 0$ for all n. Hence by continuity $f \equiv 0$; i.e.,

$$\mathfrak{D}(AB - BA) = \{0\}.$$

Construction of Example 2. Let $\mathcal{H}$, A be as in Example 3. Let $A_0 = A$. Let M be the operator defined by $Mf(x) = f'(x)$ for $x > 0$, $f \in \mathfrak{D}(M) = \{f \in \mathcal{H} : f$ absolutely continuous, $f' \in \mathcal{H}$, $f(0) = 0\}$. Then $-iM$ is symmetric with deficiency indices $(1, 0)$ [**9**, p. 353], and integration by parts shows that $\mathfrak{D}(M) \supset \mathfrak{D}(A_0)$. It follows by [**6**, p. 287] that $B_0 = -A_0 - \epsilon M$ is skew-adjoint for sufficiently small $\epsilon > 0$. Choose and fix such an $\epsilon > 0$. Then $A_0 +_L B_0$ does not exist since M has no skew-adjoint extension. (These facts were established in [**1**].) Let $\{A_n\}_1^\infty$ be a sequence of bounded skew-adjoint operators converging to A_0; for instance, if $A_0 = \int_{-\infty}^{\infty} t\, dE(t)$ (by the spectral theorem), take $A_n = \int_{-n}^{n} t\, dE(t)$. Similarly, let $\{B_n\}_1^\infty$ be a sequence of bounded skew-adjoint operators converging to B_0. Then all the conditions of Example 2 hold. {We note that in fact $A_0 +_L B_0$ exists in the sense of generalized addition of (C_0) semigroup generators (rather than of (C_0) unitary group generators), and $A_n +_L B_n$ converges to $A_0 +_L B_0$ in the sense of convergence of (C_0) semigroup generators.}

CONSTRUCTION OF EXAMPLE 5. Let $A, B, \mathcal{H}$ be as in Example 3. Set $A_0 = A$, $B_0 = B$. Let $\{A_n\}_1^\infty$ [resp. $\{B_n\}_1^\infty$] be a sequence of bounded skew-adjoint operators converging to A_0 [resp. B_0]. The rest of the proof goes exactly as in the case of Example 2. {We remark that $[A_n, B_n]_L$ does converge to $\bar{L}$ in the sense of convergence of (C_0) semigroup generators, and $[A_0, B_0]_L$ exists and equals $\bar{L}$ in the sense of [4, Theorem 1], where $\bar{L}$ is as in Example 3.}

Example 5 shows that generalized commutation is not continuous in the topology on the set of all skew-adjoint operators (on a Hilbert space) defined above.

EXAMPLE 6. *There exist skew-adjoint operators A_0, B_0 such that $A_0 +_L B_0$ exists but $[A_0, B_0]_L$ does not exist; and there exist skew-adjoint operators A_0, B_0 such that $[A_0, B_0]_L$ exists but $A_0 +_L B_0$ does not exist.*

CONSTRUCTION. The first statement follows immediately from Example 3. To prove the latter statement, let $\mathcal{H}, A_0, B_0$ be as in the construction of Example 2. A simple calculation together with the result of Nelson [7] cited at the beginning of this section implies that $[A_0, B_0]_L$ exists and equals the skew-adjoint multiplication operator C_0 defined by $C_0 f(x) = -2\epsilon i x f(x)$ for $x > 0$ and $f \in \mathfrak{D}(C_0) = \{g \in \mathcal{H} : \int_0^\infty |xg(x)|^2\, dx < \infty\}$.

5. **Universal commutatability.** We call a skew-adjoint operator A on $\mathcal{H}$ *universally commutatable* iff $[A, B]_L$ exists for each skew-adjoint B on $\mathcal{H}$. We say that A is *universally commutatable in the classical sense* iff for all skew-adjoint B, $AB - BA$ is essentially skew-adjoint.

QUESTION. *Which skew-adjoint operators are universally commutatable?*

Chernoff [1], [2] has answered the corresponding question for universal addability. He showed that a skew-adjoint A is universally addable iff A is universally addable in the classical sense iff A is bounded. Example 3 shows that there are bounded skew-adjoint operators which are not universally commutatable (in either sense). We do not know if there are any universally commutatable operators other than those of the form $\lambda i I$, $\lambda \in \mathbf{R}$. It is clear however, that any unbounded skew-adjoint operator A cannot be universally commutatable in the classical sense. To see this, choose $x \notin \mathfrak{D}(A)$, and let $B = iP$ where P is the orthogonal projection onto the span of x. Then B is bounded and skew-adjoint, but

$$\mathfrak{D}(AB - BA) \subset \mathfrak{D}(AB) = \{x\}^\perp,$$

which is not dense in $\mathcal{H}$.

ACKNOWLEDGEMENT. The construction of Examples 2 and 5 was suggested to the writer by Paul R. Chernoff. Chernoff's suggestion yields much simpler examples than those given in an early draft of this paper. Finally, we wish to mention that a reading of Chernoff's thesis [1] provided much of the motivation and many of the ideas for the present paper.

REFERENCES

1. P. R. Chernoff, *Semigroup product formulas and addition of unbounded operators*. Ph.D. Thesis, Harvard University, Cambridge, Mass., 1968.

2. ———, *Semigroup product formulas and addition of unbounded operators,* Bull. Amer. Math. Soc. **76** (1970), 395–398. MR 41 #2457.

3. N. Dunford and J. T. Schwartz, *Linear operators*. II: *Spectral theory. Selfadjoint operators in Hilbert space,* Interscience, New York, 1963; Russian transl., "Mir", Moscow, 1966. MR 32 #6181.

4. J. A. Goldstein, *A Lie product formula for one parameter groups of isometries on Banach spaces,* Math. Ann. **186** (1970), 299–306.

5. ———, *Operator semigroups and abstract Cauchy problems,* Lecture Notes, Tulane University, New Orleans, Louisiana, 1970.

6. T. Kato, *Perturbation theory for linear operators,* Die Grundlehren der math. Wissenschaften, Band 132, Springer-Verlag, New York, 1966. MR 34 #3324.

7. E. Nelson, *Topics in dynamics*. I: *Flows,* Princeton Univ. Press, Princeton, N. J., 1970.

8. T. I. Seidman, *Approximation of operator semigroups,* J. Functional Analysis **5** (1970), 160–166.

9. K. Yosida, *Functional analysis,* Die Grundlehren der math. Wissenschaften, Band 123, Academic Press, New York; Springer-Verlag, Berlin, 1965. MR 31 #5054.

TULANE UNIVERSITY, NEW ORLEANS, LOUISIANA 70118