

COMPARISON THEOREMS FOR SECOND ORDER DELAY DIFFERENTIAL EQUATIONS¹

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1. **Introduction.** Consider the delay differential equations

$$(1.1) \quad x''(t) + M(t)x(t) + N(t)x(t - \Delta(t)) = 0,$$

$$(1.2) \quad x''(t) + m(t)x(t) + n(t)x(t - \Delta(t)) = 0,$$

where $M(t)$, $N(t)$, $m(t)$, $n(t)$, and $\Delta(t) \geq 0$ are defined and continuous on $[0, B)$, $B \leq +\infty$. In case $n \equiv 0 \equiv N$, the classical Sturm comparison theorem says that whenever $M(t) \geq m(t)$ every solution of (1.1) must have a zero between consecutive zeros of a nontrivial solution of (1.2). The equation

$$(1.3) \quad x''(t) + \frac{1}{2}x(t) - \frac{1}{2}x(t - \pi) = 0$$

shows that such a theorem is in general not possible for equations (1.1), and (1.2), for (1.3) has both oscillatory and nonoscillatory solutions, viz. $x(t) = \sin t$ and $x(t) \equiv 1$.

Initial value problems for delay equations of the above type are posed in the following way (see [6, Chapter I, §2]). Given a function $\varphi(t)$ continuous on the initial set

$$I = \{t - \Delta(t) : t - \Delta(t) \leq 0\} \cup \{0\}$$

and given a real number r , one seeks a solution $x(t)$ such that

$$(1.4) \quad x(t - \Delta(t)) \equiv \varphi(t - \Delta(t)), \quad t - \Delta(t) \in I,$$

$$(1.5) \quad x'(0+) = r.$$

In this paper we obtain results which allow us to estimate the first zero of a solution of (1.1) corresponding to initial functions which do not change sign on the initial set in terms of the first positive zero of nontrivial solutions of (1.2) corresponding to the identically zero initial function. As important applications of these results we obtain uniqueness theorems for solutions of boundary value problems (BVP's) for nonlinear delay differential equations.

The following example serves to illustrate our results.

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Consider the delay differential equation without previous history

$$(1.6) \quad x''(t) - x(t) + n(t)x(t/2) = 0, \quad 0 \leq t \leq \pi,$$

where

$$n(t) = \frac{4 \sin t}{(2(1 - \cos t))^{1/2}}, \quad n(0) = 4.$$

One easily verifies that $x(t) = \sin t$ is a solution of (1.6). Applying our results (Corollary 3.5) we may conclude that for every continuous function $N(t)$ with $N(t) \geq n(t)$, every solution of

$$(1.7) \quad x''(t) - x(t) + N(t)x(t/2) = 0$$

must have a zero on $(0, \pi]$.

On the other hand, it also follows from our results (Theorem 3.2), that if in (1.7) $0 \leq N(t) \leq 1$, then every nontrivial solution $x(t)$ of (1.7) with $x(0) = 0$ can have no other zeros.

Much of the qualitative theory of linear delay differential equations has been collected by Norkin in [6] and in the survey papers [4], [5]. We refer the interested reader to these sources for the various aspects of this theory.

2. Preliminary results. In this section we gather some terminology and results from [2] and [3] which we shall need to prove our comparison theorems.

Let T be a positive real number and let R denote the real line. Let $f(t, x, y)$ be a real valued function defined and continuous on $[0, T] \times R^2$. Further let $\Delta(t) \geq 0$ be continuous on $[0, T]$ and let $\tau = \min_{0 \leq t \leq T} (t - \Delta(t))$. We consider the nonlinear delay differential equation

$$(2.1) \quad x''(t) = f(t, x(t), x(t - \Delta(t))), \quad 0 \leq t \leq T,$$

together with the boundary conditions

$$(2.2) \quad x(t) \equiv \varphi(t), \quad \tau \leq t \leq 0,$$

$$(2.3) \quad x(T) = A,$$

where $\varphi(t)$ is a continuous real valued function with domain $[\tau, 0]$ and A is a real number.

A function $\alpha(t) \in C[\tau, T] \cap C^2[0, T]$ is called a lower solution of (2.1)-(2.3) in case

$$(2.4) \quad \begin{aligned} \alpha(t) &\leq \varphi(t), & \tau \leq t \leq 0, \\ \alpha(T) &\leq A, \end{aligned}$$

and

$$(2.5) \quad \alpha''(t) \geq f(t, \alpha(t), \alpha(t - \Delta(t))), \quad 0 \leq t \leq T.$$

A function $\beta(t) \in C[\tau, T] \cap C^2[0, T]$ is called an upper solution of (2.1)–(2.3) provided that

$$(2.6) \quad \beta(t) \geq \varphi(t), \quad \tau \leq t \leq 0, \quad \beta(T) \geq A,$$

and

$$(2.7) \quad \beta''(t) \leq f(t, \beta(t), \beta(t - \Delta(t))), \quad 0 \leq t \leq T.$$

The following result is a special case of Theorem 9 of [3]; we refer to this paper for a proof.

THEOREM 2.1. *Suppose f is such that*

$$(2.8) \quad f(t, x, y) \geq f(t, x, \bar{y}),$$

for all $(t, x, y), (t, x, \bar{y}) \in [0, T] \times R^2$ with $y \leq \bar{y}$.

Then the BVP (2.1)–(2.3) has a solution $x(t) \in C[\tau, T] \cap C^2[0, T]$ if and only if there exists a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ of (2.1)–(2.3) such that $\alpha(t) \leq \beta(t), 0 \leq t \leq T$. Furthermore if such lower and upper solutions exist, then a solution $x(t)$ of (2.1)–(2.3) exists such that $\alpha(t) \leq x(t) \leq \beta(t), \tau \leq t \leq T$.

3. Linear second order equations. Consider now the linear equation

$$(3.1) \quad x''(t) + p(t)x(t) + q(t)x(t - \Delta(t)) = 0,$$

where $p(t), q(t)$ and $\Delta(t) \geq 0$ are continuous on the interval $[0, B]$. Let $\delta = \inf_{0 \leq t < B} (t - \Delta(t))$. Further assume that $\varphi(t)$ is continuous on $(\delta, 0]$. Using standard methods (see [6, pp. 19–25]) one may establish the following existence-uniqueness theorem.

THEOREM 3.1. *There exists a unique solution $x(t) \in C(\delta, B) \cap C^2[0, B]$ of (2.1) satisfying the initial conditions (1.4), (1.5) provided that*

$$\sup_{\delta < t \leq 0} |\varphi(t)| < \infty.$$

We shall now establish comparison theorems between solutions of certain initial value problems of (1.1) and (1.2). The following additional assumptions are made:

$$(3.2) \quad M(t) \geq m(t), \quad N(t) \geq n(t) \geq 0.$$

In what follows we shall denote by $X(t, \varphi, r)$ the solution of (1.1),

satisfying the initial conditions (1.4), (1.5) and by $x(t, \varphi, r)$ the corresponding solution of (1.2).

THEOREM 3.2. *Assume that $\varphi(t)$ does not change sign on $(\delta, 0]$ and that $\varphi(0) \neq 0$. Further let there exist r such that $X(t, \varphi, r)$ has no zeros on $[0, B)$. Then $x(t, 0, s)$ has no zeros on $(0, B)$ for any $s \in R (s \neq 0)$.*

PROOF. It follows from the linearity and homogeneity of equation (1.2) and from Theorem 3.1 that

$$x(t, 0, s) = s x(t, 0, 1).$$

Assuming that the conclusion of the theorem does not hold we conclude that there exists $t_1 > 0$ such that $x(t_1, 0, 1) = 0$ and $x(t, 0, 1) > 0$ for $0 < t < t_1$. Since (1.1) is also linear and homogeneous we may assume without loss of generality that $\varphi(t) \geq 0, \delta < t \leq 0$, and hence that $X(t, \varphi, r) > 0, 0 \leq t < B$, for some r .

Let

$$u = \min_{0 \leq t \leq t_1} X(t, \varphi, r) \quad \text{and} \quad v = \max_{0 \leq t \leq t_1} x(t, 0, 1)$$

and $w = \min \{1, u/v\}$. The solution $w x(t, 0, 1) = x(t, 0, w)$ of (1.2) has the property that

$$x(t, 0, w) \leq w \cdot v \leq (u/v) \cdot v = u$$

and hence we conclude that

$$x(t, 0, w) \leq X(t, \varphi, r), \quad 0 \leq t \leq t_1.$$

Consider now equation (1.2) together with the boundary conditions

$$(3.3) \quad x(t) \equiv 0, \quad \delta < t \leq 0, \quad x(t_1) = X(t_1, \varphi, r).$$

It follows from (3.2) that $x(t, 0, w)$ is a lower solution and $X(t, \varphi, r)$ is an upper solution of this BVP; by Theorem 2.1 we may therefore conclude that (1.2) has a solution $x(t)$ satisfying (3.3) and

$$x(t, 0, w) \leq x(t) \leq X(t, \varphi, r).$$

But $x(t)$ must be of the form $x(t, 0, s)$ for some $s \geq w$, contradicting that $x(t_1) = X(t_1, \varphi, r)$.

THEOREM 3.3. *Assume that $\varphi(t)$ does not change sign on $(\delta, 0]$ and that $\varphi(0) = 0$. Further let there exist $r \neq 0$ ($r > 0$ if $\varphi(t) \geq 0, r < 0$ if $\varphi(t) \leq 0$) such that $X(t, \varphi, r)$ has no zeros on $(0, B)$. Then $x(t, 0, s)$ has no zeros on $(0, B)$ for any $s \in R, s \neq 0$.*

PROOF. The proof is similar to the proof of the previous theorem.

COROLLARY 3.4. *Assume that*

$$x(t_1, 0, 1) = 0, \quad x(t, 0, 1) > 0, \quad 0 < t < t_1 < B.$$

Then $X(t, \varphi, r)$ must vanish on $(0, t_1]$ provided that $\varphi(t)$ does not change sign on $(\delta, 0]$ and

- (i) $\varphi(0) \neq 0$, r arbitrary, or
- (ii) $\varphi(0) = 0$ and $r > 0$ if $\varphi(t) \geq 0$, or $r < 0$ if $\varphi(t) \leq 0$.

The following examples illustrate that if the condition that φ does not change sign in Theorem 3.2 or the condition that φ and r have the same sign in Theorem 3.3 are dropped these results no longer remain valid.

Consider the equation

$$(3.4) \quad x''(t) + x(t) + x(t - (t + 1)) = 0.$$

Let $\varphi(t)$ be a function continuous on $[-1, 0]$ with $\varphi(-1) = -1$, $\varphi(0) = 1$. The general solution corresponding to this initial function φ is given by

$$x(t) = C_1 \sin t + 1.$$

Choosing C_1 such that $|C_1| < 1$, $x(t)$ never vanishes on $[0, \infty)$. The general solution of (3.4) with respect to any continuous nonpositive function $\varphi(t)$ on $[-1, 0]$ with $\varphi(-1) = -1$, $\varphi(0) = 0$, is given by

$$x(t) = C_1 \sin t - \cos t + 1.$$

For any $C_1 \geq 0$, $x(t) > 0$, $0 < t \leq \pi$. On the other hand, every solution of

$$x''(t) + x(t) = 0$$

has at least one zero on $[0, \pi]$.

Consider next the equation

$$(3.5) \quad x''(t) + x(t) + tx(t - (t + 1)) = 0.$$

Let $\varphi(t)$ be a nonnegative continuous initial function defined on $[-1, 0]$ with

$$\varphi(-1) = 1 = \varphi(0).$$

The solution $x(t)$ corresponding to the initial function $\varphi(t)$ with $x'(0) = r$ is given by

$$x(t) = (1 + r) \sin t + \cos t - t.$$

This function has a zero on $[0, \pi]$ for any r , it, however, never vanishes on $[\pi, 2\pi]$, for $r = 0$, for example. We may therefore con-

clude that the comparison theorems obtained are best possible for general delays $\Delta(t)$ (compare (3.5) with $x'' + x = 0$).

For delay differential equations without previous history (i.e., $t - \Delta(t) \geq 0$ for $t \geq 0$) the initial set I reduces to the singleton set $\{0\}$. Corollary 3.4 in this case yields a comparison between the first zeros of all solutions of (1.2) and (1.7). We have:

COROLLARY 3.5. *Assume that $t - \Delta(t) \geq 0$ for $t \geq 0$ and that*

$$x(t_1, 0, 1) = 0, \quad x(t, 0, 1) > 0, \quad 0 < t < t_1 < B.$$

Then $X(t, \varphi, r)$ must have a zero on $(0, t_1]$ for any choice of the constants φ and r .

Again one may construct examples to show that even in this case it is not possible, in general, to conclude that $X(t, \varphi, r)$ ($\varphi \neq 0$ or $r \neq 0$) must have a zero between the second and third zero of $x(t, 0, 1)$.

One property that distinguishes homogeneous linear second order ordinary differential equations from homogeneous linear second order delay differential equations is the fact that solutions of the latter may have multiple zeros. It is the case (see [6, Chapter II, §2]) that if $x_1(t)$ and $x_2(t)$ are linearly independent solutions on $[0, t_1]$ of (3.1) such that

$$(3.6) \quad \begin{aligned} x(t - \Delta(t)) &\equiv x(0)\varphi(t - \Delta(t)), & t - \Delta(t) &\leq 0, \\ \varphi(0) &= 1, \end{aligned}$$

then there exists a homogeneous linear second order ordinary differential equation defined for $0 \leq t \leq t_1$ whose solution space has $x_1(t)$ and $x_2(t)$ as a basis if and only if solutions of (3.1) satisfying (3.6) have no multiple zeros on $[0, t_1]$.

It is therefore of interest to obtain criteria for the absence of multiple zeros.

As a corollary to our comparison theorems we obtain the following result.

COROLLARY 3.6. *Consider equation (3.1) with $q(t) \geq 0$. Let $\varphi(t)$ be a nonnegative initial function with $\varphi(0) = 1$. Let $x(t)$ be a nontrivial solution of (3.1) satisfying (3.6). Then the first zero of $x(t)$ on $(0, B)$ cannot be a multiple zero.*

PROOF. It follows from Theorem 3.1 that if $x(t)$ is a nontrivial solution of (3.1), (3.6) having a zero on $(0, B)$ then it must have a smallest zero on $(0, B)$. Assume that $x(t)$, a nontrivial solution of (3.1), (3.6), is

such that it has a multiple zero as its first zero, say at $t_0 > 0$. And assume for definiteness that $x(t) > 0, t \in (0, t_0)$. Consider the BVP

$$(3.7) \quad \begin{aligned} y'' + p(t)y &= 0, \\ y(\epsilon) &= x(\epsilon), \quad y(t_0) = 0, \quad 0 < \epsilon < t_0. \end{aligned}$$

One easily verifies that $x(t)$ is an upper solution and that $y \equiv 0$ is a lower solution of (3.7). Hence by Theorem 2.1 there exists a solution $y(t)$ of (3.7) such that $0 \leqq y(t) \leqq x(t), \epsilon \leqq t \leqq t_0$. This, however, implies that $y(t_0) = 0 = y'(t_0)$, contradicting the fact that $y(\epsilon) > 0$.

In a similar manner one may verify the following result.

COROLLARY 3.7. *Let $x(t)$ be a solution of (3.1) with $q(t) \geqq 0$, corresponding to a nonnegative (nonpositive) initial function $\varphi(t)$ with $\varphi(0) = 0$, such that $x'(0+) > 0$ (< 0). Then the first zero of $x(t)$ on $(0, B)$ cannot be a multiple zero.*

Again one may construct examples to illustrate that the second zero of such a solution may be a multiple zero, and in case $\varphi(t)$ changes sign on the initial set, the first zero may be of multiplicity greater than one.

4. Uniqueness of solution of BVP's. In this section we shall employ the results developed in §3 to obtain uniqueness of solutions of the BVP (2.1)-(2.3). The results obtained may be considered to be extensions of some of the uniqueness theorems obtained by Willett [7] and Bailey, Shampine and Waltman [1] for ordinary differential equations.

We shall make the following assumption concerning initial value problems of the nonlinear second order equation (2.1)

$$(4.1) \quad \text{For every real number } r \text{ there exists a unique solution of (2.1), (2.2) such that } x'(0+) = r, \text{ defined on some right hand neighborhood of } 0.$$

We furthermore assume that there exist continuous functions $p(t)$ and $q(t)$ defined on $[0, T]$ with $q(t) \geqq 0$ and that

$$(4.2) \quad f(t, x, y) - f(t, \bar{x}, \bar{y}) \geqq -p(t)(x - \bar{x}) - q(t)(y - \bar{y})$$

for any $x, \bar{x}, y, \bar{y} \in R$ with $x \geqq \bar{x}, y \geqq \bar{y}$.

THEOREM 4.1. *Assume that (4.1) and (4.2) hold. Further let the solution $x(t, 0, r), r \neq 0$, of*

$$(4.3) \quad x'' + p(t)x(t) + q(t)x(t - \Delta(t)) = 0,$$

$$(4.4) \quad x \equiv 0, \quad t \leqq 0,$$

$$(4.5) \quad x'(0+) = r,$$

be such that $x(t, 0, r) \neq 0$, for $0 < t \leq T$. Then the BVP (2.1)-(2.3) has at most one solution.

PROOF. Assume (2.1)-(2.3) has two solutions, say $x(t)$ and $y(t)$. By (3.1) there exists t_1 , $0 < t_1 \leq T$, such that

$$x(t_1) = y(t_1), \quad x(t) > y(t), \quad 0 < t < t_1.$$

Let $v(t) = x(t) - y(t)$; this function then has the following properties:

(i) $v(t) \equiv 0$, $t \leq 0$.

(ii) $v(t_1) = 0$, $v(t) > 0$, $0 < t < t_1$.

(iii) $v''(t) + p(t)v(t) + q(t)v(t - \Delta(t)) \geq 0$, $0 \leq t \leq t_1$.

Let $r > 0$ be chosen large enough such that the solution $x(t, 0, r)$ of (4.3)-(4.5) satisfies

$$x(t, 0, r) \geq v(t), \quad 0 \leq t \leq t_1.$$

By hypothesis $x(t_1, 0, r) > 0$ and therefore $x(t, 0, r)$ is an upper solution of the BVP

$$(4.6) \quad \begin{aligned} x''(t) + p(t)x(t) + q(t)x(t - \Delta(t)) &= 0, \\ x(t) &\equiv 0, \quad t \leq 0, \quad x(t_1) = 0. \end{aligned}$$

By (iii) $v(t)$ is a lower solution of (4.6). Therefore by Theorem 2.1 there exists a solution $z(t)$ of (4.6) such that

$$v(t) \leq z(t) \leq x(t, 0, r).$$

Since $v(t)$ is positive on $(0, t_1)$ it follows that $z(t)$ is a nontrivial solution of (4.3)-(4.5) having a zero at $t = t_1$. This contradiction establishes the theorem.

This theorem is particularly useful for estimating intervals of uniqueness for BVP's for differential difference equations (or delay equations with $\Delta(t) \geq \delta > 0$) for in these cases the initial value problem (4.3)-(4.5) may be solved by the methods of steps (see [6, Chapter I, §5]).

REMARK. If we assume that $\Delta(t) > 0$, $0 \leq t \leq T$, then the previous theorem remains valid without assuming hypothesis (4.1).

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