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THE DIRECT SCATTERING PROBLEM OF OBLIQUELY INCIDENT ELECTROMAGNETIC WAVES BY A PENETRABLE HOMOGENEOUS CYLINDER

DROSSOS GINTIDES AND LEONIDAS MINDRINOS

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ABSTRACT. In this paper we consider the direct scattering problem of obliquely incident time-harmonic electromagnetic plane waves by an infinitely long dielectric cylinder. We assume that the cylinder and the outer medium are homogeneous and isotropic. From the symmetry of the problem, Maxwell's equations are reduced to a system of two 2D Helmholtz equations in the cylinder and two 2D Helmholtz equations in the exterior domain where the fields are coupled on the boundary. We prove uniqueness and existence of this differential system by formulating an equivalent system of integral equations using the direct method. We transform this system into a Fredholm type system of boundary integral equations in a Sobolev space setting. To handle the hypersingular operators we take advantage of Maue's formula. Applying a collocation method we derive an efficient numerical scheme and provide accurate numerical results using as test cases transmission problems corresponding to analytic fields derived from fundamental solutions.

1. Introduction. An interesting area of electromagnetism for its applications and the arising theoretical problems is the scattering process from obliquely incident time-harmonic plane waves by an infinitely long cylinder. The basic waves in the propagation domain satisfy Maxwell's equations [**1, 3, 16, 18**] and, due to the symmetry of

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the problem, it is equivalent to finding two scalar fields satisfying a pair of two-dimensional Helmholtz equations with different wavenumbers. The complication appears in the boundary conditions. Even for the case of a perfect conductor, tangential derivatives appear in the boundary conditions which make the analysis more difficult. There are many studies providing analytical or numerical solutions [**2, 13, 19, 20, 21, 22, 24, 25**]. The proposed methods are based on specific geometries or well known numerical schemes without examining the well-posedness of the corresponding boundary value problem.

Recently, Wang and Nakamura [**23**] used a more elegant theoretical analysis to prove well-posedness of the problem based on the integral equation approach. They proved theoretical and numerical results for the case of the homogeneous impedance cylinder using integral equations. For the theoretical analysis they used properties of the Cauchy singular integrals and proved that the system derived is of Fredholm type with index zero. For the numerical results they applied a specific decomposition of the kernels and formulations using Hilbert's and Symm's integral operators. Considering trigonometric interpolation, they introduced an efficient numerical scheme.

The case for general dielectric cylinders is not considered yet; however, the same authors, in a later work [**17**], investigated a more complicated model also having a non-homogeneous part, in the sense that the permittivity and the permeability of the exterior medium are non-constants and smooth in a bounded domain surrounding the cylinder. The main theoretical analysis providing uniqueness and existence in non-homogeneous materials is much harder. For the well-posedness they used the Lax-Phillips method [**7**].

In this work, we examine the case of the infinite dielectric cylinder illuminated by a transverse magnetic polarized electromagnetic plane wave, known as oblique incidence. More precisely, in the second section starting from Maxwell's equations we initially describe the derivation of the mathematical model for the scattering process from obliquely incident time-harmonic plane waves for the case of infinite inhomogeneous cylinder. We assume that transmission conditions hold on the boundary. The boundary conditions involve normal and tangential derivatives of the fields.

In Section 3, we formulate the direct problem in differential form. We derive the Helmholtz equations and the exact form of the boundary conditions in the case of homogeneous cylinder. We prove that the problem is uniquely solvable using Green's formulas and Rellich's lemma. Considering the direct method, initially applied in transmission problems in [**5, 8, 9**], we formulate the problem into an equivalent system of integral equations. We show that this system is of Fredholm type in an appropriate Sobolev space setting. Due to uniqueness of the boundary value problem, existence follows from the Fredholm alternative. The system consists of compact, singular and hypersingular operators. We consider Maue's formula [**14**], as in the case of the normal derivative of the double layer potential, to reduce the hypersingularity of the tangential derivative of the double layer potential.

In the last section we investigate numerically the problem by a collocation method based on Kress's method for the two-dimensional integral equation with strongly singular operators [**10**]. We transform the system of integral equations to a linear system by parametrizing the operators and considering well-known quadrature rules. We derive accurate numerical results for the four fields, interior and exterior, and we numerically compute the far-field patterns of the two exterior fields computed for a specific boundary value problem. Namely, we consider boundary data corresponding to analytic fields derived from point sources, where the interior and exterior fields have singularities outside of their domain of consideration.

2. Formulation of the direct scattering problem for an inhomogeneous cylinder. We consider the scattering problem of an electromagnetic wave by a penetrable cylinder in $\mathbb{R}^3$. Let $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$. Then we model the cylinder as $\Omega_{\text{int}} = \{\mathbf{x} : (x, y) \in \Omega, z \in \mathbb{R}\}0$, where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary Γ. The cylinder Ω_{int} is oriented parallel to the z-axis and Ω is its horizontal cross section. We assume constant permittivity ϵ_0 and permeability μ_0 for the exterior domain $\Omega_{\text{ext}} := \mathbb{R}^3 \setminus \overline{\Omega}_{\text{int}}$. The interior domain Ω_{int} is characterized by the electric constants $\mu(\mathbf{x}) = \mu(x, y)$ and $\epsilon(\mathbf{x}) = \epsilon(x, y)$ for all $(x, y) \in \Omega$, $z \in \mathbb{R}$.

We define for $\mathbf{x} \in \Omega_{\text{ext}}$, $t \in \mathbb{R}$, the magnetic field $\mathbf{H}^{\text{ext}}(\mathbf{x}, t)$ and electric field $\mathbf{E}^{\text{ext}}(\mathbf{x}, t)$ and, equivalently, the interior fields $\mathbf{H}^{\text{int}}(\mathbf{x}, t)$

and $\mathbf{E}^{\text{int}}(\mathbf{x},t)$ for $\mathbf{x} \in \Omega_{\text{int}}$, $t \in \mathbb{R}$. Then, these fields satisfy the Maxwell's equations

$$\begin{aligned}
\nabla\times \mathbf{E}^{\text{ext}} + \mu_0 \frac{\partial \mathbf{H}^{\text{ext}}}{\partial t} &= 0, \qquad \mathbf{x} \in \Omega_{\text{ext}}, \\
\nabla\times \mathbf{H}^{\text{ext}} - \epsilon_0 \frac{\partial \mathbf{E}^{\text{ext}}}{\partial t} &= 0, \qquad \mathbf{x} \in \Omega_{\text{ext}}, \\
\nabla\times \mathbf{E}^{\text{int}} + \mu \frac{\partial \mathbf{H}^{\text{int}}}{\partial t} &= 0, \qquad \mathbf{x} \in \Omega_{\text{int}}, \\
\nabla\times \mathbf{H}^{\text{int}} - \epsilon \frac{\partial \mathbf{E}^{\text{int}}}{\partial t} &= 0, \qquad \mathbf{x} \in \Omega_{\text{int}}.
\end{aligned} \tag{2.1}$$

On the boundary Γ, we consider transmission conditions

$$\widehat{\boldsymbol{n}} \times \mathbf{E}^{\text{int}} = \widehat{\boldsymbol{n}} \times \mathbf{E}^{\text{ext}}, \qquad \widehat{\boldsymbol{n}} \times \mathbf{H}^{\text{int}} = \widehat{\boldsymbol{n}} \times \mathbf{H}^{\text{ext}}, \quad \mathbf{x} \in \Gamma,$$

where $\widehat{\boldsymbol{n}}$ is the outward normal vector, directed into Ω_{ext}.

In order to take advantage of the symmetry of the specific medium, we probe the cylinder with an incident transverse magnetic (TM) polarized electromagnetic plane wave, the so-called oblique incidence in the literature. An arbitrary time-harmonic incident electromagnetic plane wave has the form:

$$\begin{aligned}
\mathbf{E}^{\text{inc}}(\mathbf{x},t;\widehat{\boldsymbol{d}},\widehat{\boldsymbol{p}}) &= \frac{1}{k_0^2\sqrt{\epsilon_0}} \nabla\times\nabla\times(\widehat{\boldsymbol{p}}\, e^{ik_0 \mathbf{x}\cdot\widehat{\boldsymbol{d}}})e^{-i\omega t}, \\
\mathbf{H}^{\text{inc}}(\mathbf{x},t;\widehat{\boldsymbol{d}},\widehat{\boldsymbol{p}}) &= \frac{1}{ik_0\sqrt{\mu_0}} \nabla\times(\widehat{\boldsymbol{p}}\, e^{ik_0 \mathbf{x}\cdot\widehat{\boldsymbol{d}}})e^{-i\omega t},
\end{aligned}$$

where $\omega > 0$ is the frequency, $k_0 = \omega\sqrt{\mu_0\epsilon_0}$ is the wave number in the exterior domain, $\widehat{\boldsymbol{p}}$ is the polarization vector and $\widehat{\boldsymbol{d}}$ the vector describing the incident direction, satisfying $\widehat{\boldsymbol{d}} \perp \widehat{\boldsymbol{p}}$.

In the following, due to the linearity of the problem, we suppress the time-dependence and we consider the fields only as functions of the space variable $\mathbf{x}$. In order to describe the incident fields for the specific TM polarization, we define by θ the incident angle with respect to the negative z axis and by ϕ the polar angle of $\widehat{\boldsymbol{d}}$ (in spherical coordinates). Then $\widehat{\boldsymbol{d}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, -\cos\theta)$ and $\widehat{\boldsymbol{p}} = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$, assuming that $\theta \in (0,\pi/2)\cup(\pi/2,\pi)$.

Hence, we obtain

$$\mathbf{E}^{\rm inc}(\mathbf{x};\widehat{\boldsymbol{d}},\widehat{\boldsymbol{p}}) = \frac{1}{\sqrt{\epsilon_0}}\,\widehat{\boldsymbol{d}}\times\widehat{\boldsymbol{p}}\times\widehat{\boldsymbol{d}}\,e^{ik_0\mathbf{x}\cdot\widehat{\boldsymbol{d}}} = \frac{1}{\sqrt{\epsilon_0}}\,\widehat{\boldsymbol{p}}\,e^{ik_0\mathbf{x}\cdot\widehat{\boldsymbol{d}}},$$
$$\mathbf{H}^{\rm inc}(\mathbf{x};\widehat{\boldsymbol{d}},\widehat{\boldsymbol{p}}) = \frac{1}{\sqrt{\mu_0}}\widehat{\boldsymbol{d}}\times\widehat{\boldsymbol{p}}\,e^{ik_0\mathbf{x}\cdot\widehat{\boldsymbol{d}}} = \frac{1}{\sqrt{\mu_0}}(\sin\phi,-\cos\phi,0)\,e^{ik_0\mathbf{x}\cdot\widehat{\boldsymbol{d}}}.$$

Taking into account the cylindrical symmetry of the medium and the z-independence of the electric coefficients, we express the incident fields as separable functions of (x,y) and z. Thus, we define $\beta = k_0\cos\theta$ and $\kappa_0 = \sqrt{k_0^2-\beta^2} = k_0\sin\theta$, and it follows that the incident fields can be decomposed to

$$\begin{aligned}\mathbf{E}^{\rm inc}(\mathbf{x};\widehat{\boldsymbol{d}},\widehat{\boldsymbol{p}}) &= \mathbf{e}^{\rm inc}(x,y)\,e^{-i\beta z},\\ \mathbf{H}^{\rm inc}(\mathbf{x};\widehat{\boldsymbol{d}},\widehat{\boldsymbol{p}}) &= \mathbf{h}^{\rm inc}(x,y)\,e^{-i\beta z},\end{aligned}\tag{2.2}$$

where

$$\mathbf{e}^{\rm inc}(x,y) = \frac{1}{\sqrt{\epsilon_0}}\,\widehat{\boldsymbol{p}}\,e^{i\kappa_0(x\cos\phi+y\sin\phi)},$$
$$\mathbf{h}^{\rm inc}(x,y) = \frac{1}{\sqrt{\mu_0}}\,(\sin\phi,-\cos\phi,0)\,e^{i\kappa_0(x\cos\phi+y\sin\phi)}.$$

Now, we are in a position to transform equations (2.1) into a system of equations only for the z-component of the electric and magnetic fields. Firstly, we see that for the specific illumination of the form (2.2), using separation of variables, the scattered fields also take the form:

$$\mathbf{E}^{\rm sc}(\mathbf{x};\widehat{\boldsymbol{d}},\widehat{\boldsymbol{p}}) = \mathbf{e}^{\rm sc}(x,y)\,e^{-i\beta z},\quad \mathbf{x}\in\Omega_{\rm ext},$$
$$\mathbf{H}^{\rm sc}(\mathbf{x};\widehat{\boldsymbol{d}},\widehat{\boldsymbol{p}}) = \mathbf{h}^{\rm sc}(x,y)\,e^{-i\beta z},\quad \mathbf{x}\in\Omega_{\rm ext},$$

where $\mathbf{e}^{\rm sc} = (e_1^{sc},e_2^{sc},e_3^{sc})$ and $\mathbf{h}^{\rm sc} = (h_1^{sc},h_2^{sc},h_3^{sc})$. Then, the exterior fields are given by

$$\begin{aligned}\mathbf{E}^{\rm ext}(\mathbf{x};\widehat{\boldsymbol{d}},\widehat{\boldsymbol{p}}) &= \left(\mathbf{e}^{\rm sc}(x,y)+\mathbf{e}^{\rm inc}(x,y)\right)\,e^{-i\beta z}\\ &= \mathbf{e}^{\rm ext}(x,y)\,e^{-i\beta z}, && \mathbf{x}\in\Omega_{\rm ext},\\ \mathbf{H}^{\rm ext}(\mathbf{x};\widehat{\boldsymbol{d}},\widehat{\boldsymbol{p}}) &= \Big(\mathbf{h}^{\rm sc}(x,y)+\mathbf{h}^{\rm inc}(x,y)\Big)\,e^{-i\beta z}\\ &= \mathbf{e}^{\rm ext}(x,y)\,e^{-i\beta z}, && \mathbf{x}\in\Omega_{\rm ext}.\end{aligned}$$

Equivalently, the interior fields are represented by

$$\mathbf{E}^{\text{int}}(\mathbf{x};\widehat{\boldsymbol{d}},\widehat{\boldsymbol{p}}) = \mathbf{e}^{\text{int}}(x,y)\, e^{-i\beta z}, \quad \mathbf{x}\in\Omega_{\text{int}},$$
$$\mathbf{H}^{\text{int}}(\mathbf{x};\widehat{\boldsymbol{d}},\widehat{\boldsymbol{p}}) = \mathbf{h}^{\text{int}}(x,y)\, e^{-i\beta z}, \quad \mathbf{x}\in\Omega_{\text{int}},$$

where $\mathbf{e}^{\text{int}} = (e_1^{\text{int}}, e_2^{\text{int}}, e_3^{\text{int}})$ and $\mathbf{h}^{\text{int}} = (h_1^{\text{int}}, h_2^{\text{int}}, h_3^{\text{int}})$.

For any field of the form

$$\mathbf{E}(\mathbf{x};\widehat{\boldsymbol{d}},\widehat{\boldsymbol{p}}) = \mathbf{e}(x,y)\, e^{-i\beta z}, \quad \mathbf{x}\in\mathbb{R}^3,$$
$$\mathbf{H}(\mathbf{x};\widehat{\boldsymbol{d}},\widehat{\boldsymbol{p}}) = \mathbf{h}(x,y)\, e^{-i\beta z}, \quad \mathbf{x}\in\mathbb{R}^3,$$

we consider the Maxwell's equations in $\mathbb{R}^3$ for arbitrary ϵ,μ and $k^2 = \mu\epsilon\omega^2 - \beta^2$ (we remark here the space dependence of ϵ,μ). Then, following **[17]**, we obtain the relations

$$\begin{aligned}
e_1(x,y) &= -\frac{1}{k^2}\left(i\beta\frac{\partial e_3}{\partial x}(x,y) - i\mu\omega\frac{\partial h_3}{\partial y}(x,y)\right),\\
e_2(x,y) &= -\frac{1}{k^2}\left(i\beta\frac{\partial e_3}{\partial y}(x,y) + i\mu\omega\frac{\partial h_3}{\partial x}(x,y)\right),\\
h_1(x,y) &= -\frac{1}{k^2}\left(i\beta\frac{\partial h_3}{\partial x}(x,y) + i\epsilon\omega\frac{\partial e_3}{\partial y}(x,y)\right),\\
h_2(x,y) &= -\frac{1}{k^2}\left(i\beta\frac{\partial h_3}{\partial y}(x,y) - i\epsilon\omega\frac{\partial e_3}{\partial x}(x,y)\right).
\end{aligned}\tag{2.3}$$

Substituting (2.3) in (2.1), we have that the pair (e_3,h_3) satisfies the equations

$$\frac{k^2}{\epsilon\omega}\nabla\cdot\left(\frac{\epsilon\omega}{k^2}\nabla e_3\right) + \frac{k^2}{\epsilon\omega}J\,\nabla\left(\frac{\beta}{k^2}\right)\cdot\nabla h_3 + k^2 e_3 = 0,$$
$$\frac{k^2}{\mu\omega}\nabla\cdot\left(\frac{\mu\omega}{k^2}\nabla h_3\right) - \frac{k^2}{\mu\omega}J\,\nabla\left(\frac{\beta}{k^2}\right)\cdot\nabla e_3 + k^2 h_3 = 0,$$

where

$$\mathbf{J} = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}.$$

The interior and exterior domains are characterized by different wavenumbers, given by

$$k^2(\mathbf{x}) = \begin{cases} k^2_{\rm int}(\mathbf{x}) := \mu(x,y)\,\epsilon(x,y)\,\omega^2 - \beta^2, & \mathbf{x} \in \Omega_{\rm int}, \\ k^2_{\rm ext}(\mathbf{x}) := \mu_0\epsilon_0\omega^2 - \beta^2 = \kappa_0^2, & \mathbf{x} \in \Omega_{\rm ext}. \end{cases}$$

In this section, for completeness in the formulation of the direct problem, we keep the space dependence of $k_{\rm int}$. Later, we consider only the case of constant parameters. Here, we have to assume that $\mu(\mathbf{x})\epsilon(\mathbf{x}) > \epsilon_0\mu_0\cos\theta$ in order to have $\inf_{\mathbf{x}} k^2_{int}(\mathbf{x}) > 0$. Thus, the fields $e_3^{\rm ext}(x,y)$ and $h_3^{\rm ext}(x,y)$ satisfy

$$\Delta e_3^{\rm ext} + \kappa_0^2\, e_3^{\rm ext} = 0, \qquad \Delta h_3^{\rm ext} + \kappa_0^2\, h_3^{\rm ext} = 0, \quad \mathbf{x} \in \Omega_{\rm ext}, \tag{2.4}$$

and the interior fields
(2.5)

$$\begin{aligned} &\frac{k^2_{\rm int}(\mathbf{x})}{\epsilon(\mathbf{x})}\nabla\cdot\left(\frac{\epsilon(\mathbf{x})}{k^2_{\rm int}(\mathbf{x})}\nabla e_3^{\rm int}\right) \\ &\qquad + \frac{k^2_{\rm int}(\mathbf{x})}{\epsilon(\mathbf{x})\omega}\mathbf{J}\,\nabla\left(\frac{\beta}{k^2_{\rm int}(\mathbf{x})}\right)\cdot\nabla h_3^{\rm int} + k^2_{\rm int}(\mathbf{x})\, e_3^{\rm int} = 0, \quad \mathbf{x} \in \Omega_{\rm int}, \\ &\frac{k^2_{\rm int}(\mathbf{x})}{\mu(\mathbf{x})}\nabla\cdot\left(\frac{\mu(\mathbf{x})}{k^2_{\rm int}(\mathbf{x})}\nabla h_3^{\rm int}\right) \\ &\qquad - \frac{k^2_{\rm int}(\mathbf{x})}{\mu(\mathbf{x})\omega}\mathbf{J}\,\nabla\left(\frac{\beta}{k^2_{\rm int}(\mathbf{x})}\right)\cdot\nabla e_3^{\rm int} + k^2_{\rm int}(\mathbf{x})\, h_3^{\rm int} = 0, \quad \mathbf{x} \in \Omega_{\rm int}. \end{aligned}$$

Now, we are going to derive the exact form of the boundary conditions. We introduce the notation: $\mathbf{e}_t = \widehat{\boldsymbol{x}}\,e_1 + \widehat{\boldsymbol{y}}\,e_2$, $\mathbf{h}_t = \widehat{\boldsymbol{x}}\,h_1 + \widehat{\boldsymbol{y}}\,h_2$ and $\nabla_t = \widehat{\boldsymbol{x}}(\partial/\partial x) + \widehat{\boldsymbol{y}}(\partial/\partial y)$, where $\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}$ denote the unit vectors in $\mathbb{R}^2$. Let $(\widehat{\boldsymbol{n}}, \widehat{\boldsymbol{\tau}}, \widehat{\boldsymbol{z}})$ be a local coordinate system, where $\widehat{\boldsymbol{n}} = (n_1, n_2)$ is the outward normal vector and $\widehat{\boldsymbol{\tau}} = (-n_2, n_1)$ the outward tangent vector on Γ. Then, from (2.3), we obtain

$$\begin{aligned} \widehat{\boldsymbol{\tau}}\cdot\mathbf{e}_t &= -\frac{1}{k^2}\left(i\mu\omega\widehat{\boldsymbol{n}}\cdot\nabla_t h_3 + i\beta\widehat{\boldsymbol{\tau}}\cdot\nabla_t e_3\right), \\ \widehat{\boldsymbol{\tau}}\cdot\mathbf{h}_t &= -\frac{1}{k^2}\left(-i\epsilon\omega\widehat{\boldsymbol{n}}\cdot\nabla_t e_3 + i\beta\widehat{\boldsymbol{\tau}}\cdot\nabla_t h_3\right), \end{aligned} \tag{2.6}$$

using that $\widehat{\boldsymbol{\tau}}\cdot(\widehat{\boldsymbol{z}}\times\nabla_t\cdot) = \widehat{\boldsymbol{n}}\cdot\nabla_t\,\cdot$.

We observe, setting zero to the z-component of $\widehat{\boldsymbol{n}}$, $\widehat{\boldsymbol{\tau}}$ in $\mathbb{R}^3$, that

$$\begin{aligned}\widehat{\boldsymbol{n}} \times \mathbf{E} &= -e_3 \widehat{\boldsymbol{\tau}} + (n_1 e_2 - n_2 e_1)\, \widehat{\boldsymbol{z}},\\ \widehat{\boldsymbol{n}} \times \mathbf{H} &= -h_3 \widehat{\boldsymbol{\tau}} + (n_1 h_2 - n_2 h_1)\, \widehat{\boldsymbol{z}}.\end{aligned}$$

Then, from (2.3) and (2.6), we derive

$$\begin{aligned}\widehat{\boldsymbol{n}} \times \mathbf{E}^{\mathrm{ext}} &= -e_3^{\mathrm{ext}} \widehat{\boldsymbol{\tau}} + \widehat{\boldsymbol{\tau}} \cdot \mathbf{e}_t^{\mathrm{ext}}\, \widehat{\boldsymbol{z}},\\ \widehat{\boldsymbol{n}} \times \mathbf{H}^{\mathrm{ext}} &= -h_3^{\mathrm{ext}} \widehat{\boldsymbol{\tau}} + \widehat{\boldsymbol{\tau}} \cdot \mathbf{h}_t^{\mathrm{ext}}\, \widehat{\boldsymbol{z}},\end{aligned}$$

for the exterior fields, where $\mathbf{e}_t^{\mathrm{ext}} := \widehat{\boldsymbol{x}}\, e_1^{\mathrm{ext}} + \widehat{\boldsymbol{y}}\, e_2^{\mathrm{ext}}$, $\mathbf{h}_t^{\mathrm{ext}} := \widehat{\boldsymbol{x}}\, h_1^{\mathrm{ext}} + \widehat{\boldsymbol{y}}\, h_2^{\mathrm{ext}}$ and

$$\begin{aligned}\widehat{\boldsymbol{n}} \times \mathbf{E}^{\mathrm{int}} &= -e_3^{\mathrm{int}} \widehat{\boldsymbol{\tau}} + \widehat{\boldsymbol{\tau}} \cdot \mathbf{e}_t^{\mathrm{int}}\, \widehat{\boldsymbol{z}},\\ \widehat{\boldsymbol{n}} \times \mathbf{H}^{\mathrm{int}} &= -h_3^{\mathrm{int}} \widehat{\boldsymbol{\tau}} + \widehat{\boldsymbol{\tau}} \cdot \mathbf{h}_t^{\mathrm{int}}\, \widehat{\boldsymbol{z}}.\end{aligned}$$

for the interior fields, where $\mathbf{e}_t^{\mathrm{int}} := \widehat{\boldsymbol{x}}\, e_1^{int} + \widehat{\boldsymbol{y}}\, e_2^{\mathrm{int}}$, $\mathbf{h}_t^{\mathrm{int}} := \widehat{\boldsymbol{x}}\, h_1^{\mathrm{int}} + \widehat{\boldsymbol{y}}\, h_2^{\mathrm{int}}$.

Here, we observe that the tangential forms of the fields can be written in terms of $\widehat{\boldsymbol{\tau}}$ and $\widehat{\boldsymbol{z}}$, two linear independent vectors. Thus, the boundary condition

$$\widehat{\boldsymbol{n}} \times \mathbf{E}^{\mathrm{int}} = \widehat{\boldsymbol{n}} \times \mathbf{E}^{\mathrm{ext}}, \quad \mathbf{x} \in \Gamma,$$

is equivalent to the system

$$e_3^{\mathrm{int}} = e_3^{\mathrm{ext}}, \qquad \widehat{\boldsymbol{\tau}} \cdot \mathbf{e}_t^{\mathrm{int}} = \widehat{\boldsymbol{\tau}} \cdot \mathbf{e}_t^{\mathrm{ext}}, \quad \mathbf{x} \in \Gamma,$$

and equivalently for the magnetic fields

$$h_3^{\mathrm{int}} = h_3^{\mathrm{ext}}, \qquad \widehat{\boldsymbol{\tau}} \cdot \mathbf{h}_t^{\mathrm{int}} = \widehat{\boldsymbol{\tau}} \cdot \mathbf{h}_t^{\mathrm{ext}}, \quad \mathbf{x} \in \Gamma.$$

We define

$$\frac{\partial}{\partial n} = \widehat{\boldsymbol{n}} \cdot \nabla_t, \quad \frac{\partial}{\partial \tau} = \widehat{\boldsymbol{\tau}} \cdot \nabla_t,$$

and we rewrite the above boundary conditions as

(2.7)

$$\begin{aligned} e_3^{\mathrm{int}} &= e_3^{\mathrm{ext}}, & \mathbf{x} \in \Gamma,\\ \frac{\mu(\mathbf{x})}{k_{\mathrm{int}}^2(\mathbf{x})}\omega \frac{\partial h_3^{\mathrm{int}}}{\partial n} + \frac{\beta}{k_{\mathrm{int}}^2(\mathbf{x})} \frac{\partial e_3^{\mathrm{int}}}{\partial \tau} &= \frac{\mu_0}{\kappa_0^2}\omega \frac{\partial h_3^{\mathrm{ext}}}{\partial n} + \frac{\beta}{\kappa_0^2} \frac{\partial e_3^{\mathrm{ext}}}{\partial \tau}, & \mathbf{x} \in \Gamma,\end{aligned}$$

and
(2.8)

$$
\begin{aligned}
&h_3^{\text{int}} = h_3^{\text{ext}}, && \mathbf{x} \in \Gamma,\\
&\frac{\epsilon(\mathbf{x})}{k_{\text{int}}^2(\mathbf{x})}\omega\frac{\partial e_3^{\text{int}}}{\partial n} - \frac{\beta}{k_{\text{int}}^2(\mathbf{x})}\frac{\partial h_3^{\text{int}}}{\partial \tau} = \frac{\epsilon_0}{\kappa_0^2}\omega\frac{\partial e_3^{\text{ext}}}{\partial n} - \frac{\beta}{\kappa_0^2}\frac{\partial h_3^{\text{ext}}}{\partial \tau}, && \mathbf{x} \in \Gamma.
\end{aligned}
$$

To ensure that the scattered fields are outgoing, the components must satisfy in addition the radiation conditions in $\mathbb{R}^2$:

$$
\begin{aligned}
&\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial e_3^{\text{sc}}}{\partial r} - i\kappa_0 e_3^{\text{sc}}\right) = 0,\\
&\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial h_3^{\text{sc}}}{\partial r} - i\kappa_0 h_3^{\text{sc}}\right) = 0,
\end{aligned}
\tag{2.9}
$$

where $r = |(x,y)|$ uniformly over all directions.

Thus, the direct transmission problem for oblique incident wave, is to find the fields h_3^{int}, h_3^{sc}, e_3^{int} and e_3^{sc} which satisfy equations (2.4) and (2.5), the transmission conditions (2.7) and (2.8) and the radiation conditions (2.9).

We remark here that, since we consider TM polarized wave, see equation (2.2), the incident fields for $\mathbf{x} \in \Omega_{\text{ext}}$ are simplified to

$$
e_3^{\text{inc}}(x,y) = \frac{1}{\sqrt{\epsilon_0}}\sin\theta\, e^{i\kappa_0(x\cos\phi + y\sin\phi)}, \quad h_3^{\text{inc}}(x,y) = 0. \tag{2.10}
$$

3. The direct problem for a homogeneous cylinder using the integral equation method. From now on, $\mathbf{x} \in \mathbb{R}^2$. In this section, we consider the simplified version where $\mu(\mathbf{x}) = \mu_1$ and $\epsilon(\mathbf{x}) = \epsilon_1$ are constant in the interior domain. To simplify the following analysis, we set $\Omega_1 = \Omega \subset \mathbb{R}^2$, $\Omega_0 = \mathbb{R}^2 \setminus \Omega$ and

$$
\begin{aligned}
u_0(\mathbf{x}) &= e_3^{\text{sc}}(\mathbf{x}), \quad v_0(\mathbf{x}) = h_3^{\text{sc}}(\mathbf{x}), \quad \mathbf{x} \in \Omega_0,\\
u_1(\mathbf{x}) &= e_3^{\text{int}}(\mathbf{x}), \quad v_1(\mathbf{x}) = h_3^{\text{int}}(\mathbf{x}), \quad \mathbf{x} \in \Omega_1.
\end{aligned}
$$

In the following, $j = 0, 1$ counts for the exterior ($\mathbf{x} \in \Omega_0$) and interior domains ($\mathbf{x} \in \Omega_1$), respectively. Then, the direct scattering problem, presented in the previous section, is modified to

$$
\Delta u_j + \kappa_j^2 u_j = 0, \qquad \Delta v_j + \kappa_j^2 v_j = 0, \quad \mathbf{x} \in \Omega_j, \tag{3.1}
$$

for $j = 0, 1$ where $\kappa_1^2 = \mu_1 \epsilon_1 \omega^2 - \beta^2$, with boundary conditions

$$u_1 = u_0 + e_3^{\text{inc}}, \qquad \mathbf{x} \in \Gamma, \tag{3.2a}$$

$$\widetilde{\mu}_1 \omega \frac{\partial v_1}{\partial n} + \beta_1 \frac{\partial u_1}{\partial \tau} = \widetilde{\mu}_0 \omega \frac{\partial v_0}{\partial n} + \beta_0 \frac{\partial u_0}{\partial \tau} + \beta_0 \frac{\partial e_3^{\text{inc}}}{\partial \tau}, \qquad \mathbf{x} \in \Gamma, \tag{3.2b}$$

$$v_1 = v_0, \qquad \mathbf{x} \in \Gamma, \tag{3.2c}$$

$$\widetilde{\epsilon}_1 \omega \frac{\partial u_1}{\partial n} - \beta_1 \frac{\partial v_1}{\partial \tau} = \widetilde{\epsilon}_0 \omega \frac{\partial u_0}{\partial n} + \widetilde{\epsilon}_0 \omega \frac{\partial e_3^{\text{inc}}}{\partial n} - \beta_0 \frac{\partial v_0}{\partial \tau}, \qquad \mathbf{x} \in \Gamma, \tag{3.2d}$$

where $\widetilde{\mu}_j = \mu_j / \kappa_j^2$, $\widetilde{\epsilon}_j = \epsilon_j / \kappa_j^2$, $\beta_j = \beta / \kappa_j^2$, and the radiation conditions

$$\begin{aligned} \lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u_0}{\partial r} - i \kappa_0 u_0 \right) &= 0, \\ \lim_{r \to \infty} \sqrt{r} \left(\frac{\partial v_0}{\partial r} - i \kappa_0 v_0 \right) &= 0. \end{aligned} \tag{3.3}$$

Theorem 3.1. *If κ_1^2 is not an interior Dirichlet eigenvalue, then the problem* (3.1)–(3.3) *has at most one solution.*

Proof. It is sufficient to show that, if u_0, v_0, u_1, v_1 solve the homogeneous problem (3.1)–(3.3), that is, for $e_3^{\text{inc}} = 0$, then $u_0 = v_0 = 0$ in Ω_0 and $u_1 = v_1 = 0$ in Ω_1. Let S_r be a disk with radius r, boundary Γ_r, centered at the origin and containing Ω_1. We set $\Omega_r = S_r \setminus \overline{\Omega}_1$, see Figure 1.

The boundary conditions of the homogeneous problem read

$$\begin{aligned} u_1 &= u_0, && \mathbf{x} \in \Gamma, \\ \widetilde{\mu}_1 \frac{\partial v_1}{\partial n} - \widetilde{\mu}_0 \frac{\partial v_0}{\partial n} &= -\frac{\beta_1}{\omega} \frac{\partial u_1}{\partial \tau} + \frac{\beta_0}{\omega} \frac{\partial u_0}{\partial \tau}, && \mathbf{x} \in \Gamma, \\ v_1 &= v_0, && \mathbf{x} \in \Gamma, \\ \widetilde{\epsilon}_1 \frac{\partial u_1}{\partial n} - \widetilde{\epsilon}_0 \frac{\partial u_0}{\partial n} &= \frac{\beta_1}{\omega} \frac{\partial v_1}{\partial \tau} - \frac{\beta_0}{\omega} \frac{\partial v_0}{\partial \tau}, && \mathbf{x} \in \Gamma. \end{aligned} \tag{3.4}$$

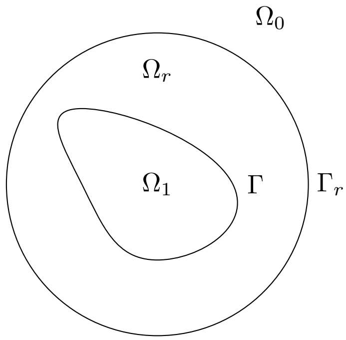

FIGURE 1. The set Ω_r.

We apply Green's first identity in Ω_1 and considering (3.1) we obtain

$$\begin{aligned}\widetilde{\epsilon}_1 \int_\Gamma u_1 \frac{\partial \overline{u}_1}{\partial n}\, ds &= \widetilde{\epsilon}_1 \int_{\Omega_1} \left(|\nabla u_1|^2 + u_1 \Delta \overline{u}_1\right)\, d\mathbf{x},\\ &= \widetilde{\epsilon}_1 \int_{\Omega_1} \left(|\nabla u_1|^2 - \kappa_1^2 |u_1|^2\right)\, d\mathbf{x},\\ \widetilde{\mu}_1 \int_\Gamma v_1 \frac{\partial \overline{v}_1}{\partial n}\, ds &= \widetilde{\mu}_1 \int_{\Omega_1} \left(|\nabla v_1|^2 + v_1 \Delta \overline{v}_1\right)\, d\mathbf{x}\\ &= \widetilde{\mu}_1 \int_{\Omega_1} \left(|\nabla v_1|^2 - \kappa_1^2 |v_1|^2\right)\, d\mathbf{x}.\end{aligned} \tag{3.5}$$

Similarly, Green's first identity in Ω_r, together with equations (3.4) and (3.5) gives

$$\begin{aligned}\widetilde{\epsilon}_0 \int_{\Gamma_r} u_0 \frac{\partial \overline{u}_0}{\partial n}\, ds &= \widetilde{\epsilon}_0 \int_{\Omega_r} \left(|\nabla u_0|^2 + u_0 \Delta \overline{u}_0\right)\, d\mathbf{x}\\ &\quad + \widetilde{\epsilon}_0 \int_\Gamma u_0 \frac{\partial \overline{u}_0}{\partial n}\, ds\\ &= \widetilde{\epsilon}_0 \int_{\Omega_r} \left(|\nabla u_0|^2 - \kappa_0^2 |u_0|^2\right)\, d\mathbf{x}\\ &\quad + \int_\Gamma u_0 \left(\widetilde{\epsilon}_1 \frac{\partial \overline{u}_1}{\partial n} - \frac{\beta_1}{\omega} \frac{\partial \overline{v}_1}{\partial \tau} + \frac{\beta_0}{\omega} \frac{\partial \overline{v}_0}{\partial \tau}\right) ds\end{aligned}$$

$$
\begin{aligned}
&= \widetilde{\epsilon}_0 \int_{\Omega_r} \left(|\nabla u_0|^2 - \kappa_0^2 |u_0|^2\right) \, d\mathbf{x} \\
&\quad + \widetilde{\epsilon}_1 \int_{\Omega_1} \left(|\nabla u_1|^2 - \kappa_1^2 |u_1|^2\right) \, d\mathbf{x} \\
&\quad - \frac{\beta_1}{\omega} \int_\Gamma u_1 \frac{\partial \overline{v}_1}{\partial \tau} \, ds + \frac{\beta_0}{\omega} \int_\Gamma u_0 \frac{\partial \overline{v}_0}{\partial \tau} \, ds
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{\mu}_0 \int_{\Gamma_r} v_0 \frac{\partial \overline{v}_0}{\partial n} \, ds &= \widetilde{\mu}_0 \int_{\Omega_r} \left(|\nabla v_0|^2 + v_0 \Delta \overline{v}_0\right) \, d\mathbf{x} \\
&\quad + \widetilde{\mu}_0 \int_\Gamma v_0 \frac{\partial \overline{v}_0}{\partial n} \, ds \\
&= \widetilde{\mu}_0 \int_{\Omega_r} \left(|\nabla v_0|^2 - \kappa_0^2 |v_0|^2\right) \, d\mathbf{x} \\
&\quad + \int_\Gamma v_0 \left(\widetilde{\mu}_1 \frac{\partial \overline{v}_1}{\partial n} - \frac{\beta_0}{\omega} \frac{\partial \overline{u}_0}{\partial \tau} + \frac{\beta_1}{\omega} \frac{\partial \overline{u}_1}{\partial \tau} \right) ds \\
&= \widetilde{\mu}_0 \int_{\Omega_r} \left(|\nabla v_0|^2 - \kappa_0^2 |v_0|^2\right) \, d\mathbf{x} \\
&\quad + \widetilde{\mu}_1 \int_{\Omega_0} \left(|\nabla v_1|^2 - \kappa_1^2 |v_1|^2\right) \, d\mathbf{x} - \frac{\beta_0}{\omega} \int_\Gamma v_0 \frac{\partial \overline{u}_0}{\partial \tau} \, ds \\
&\quad + \frac{\beta_1}{\omega} \int_\Gamma v_1 \frac{\partial \overline{u}_1}{\partial \tau} ds.
\end{aligned}
$$

We add the above two equations and, noting that

$$
- \int_\Gamma u_1 \frac{\partial \overline{v}_1}{\partial \tau} \, ds = \overline{\int_\Gamma v_1 \frac{\partial \overline{u}_1}{\partial \tau} \, ds}, \qquad \int_\Gamma u_0 \frac{\partial \overline{v}_0}{\partial \tau} \, ds = -\overline{\int_\Gamma v_0 \frac{\partial \overline{u}_0}{\partial \tau} \, ds},
$$

we obtain

$$
\Im \left(\widetilde{\epsilon}_0 \int_{\Gamma_r} u_0 \frac{\partial \overline{u}_0}{\partial n} \, ds + \widetilde{\mu}_0 \int_{\Gamma_r} v_0 \frac{\partial \overline{v}_0}{\partial n} \, ds \right) = 0,
$$

or, equivalently, using the radiation conditions (see **[23,** equation (2.12)])

$$
\lim_{r \to \infty} \int_{\Gamma_r} \left(\epsilon_0 |u_0|^2 + \tilde{\epsilon}_0 \left| \frac{\partial u_0}{\partial n} \right|^2 + \mu_0 |v_0|^2 + \tilde{\mu}_0 \left| \frac{\partial v_0}{\partial n} \right|^2 \right) ds = 0.
$$

Thus,

$$\lim_{r\to\infty}\int_{\Gamma_r}|u_0|^2 ds = \lim_{r\to\infty}\int_{\Gamma_r}|v_0|^2 ds = 0,$$

and by Rellich's lemma it follows that $u_0 = v_0 = 0$ in Ω_0. Hence, $u_0 = v_0 = 0$ in Γ and $u_1 = v_1 = 0$ in Γ from the boundary conditions. Then, $u_1 = v_1 = 0$ in Ω_1 follows from the unique solvability of the interior Dirichlet problem, given the assumption of the theorem. □

We define the fundamental solution of the Helmholtz equation in $\mathbb{R}^2$:

$$\Phi_j(\mathbf{x},\mathbf{y}) = \frac{i}{4}H_0^{(1)}(\kappa_j|\mathbf{x}-\mathbf{y}|), \quad \mathbf{x},\mathbf{y}\in\Omega_j,\ \mathbf{x}\neq\mathbf{y}, \tag{3.6}$$

where $H_0^{(1)}$ is the Hankel function of the first kind and zero order. For a continuous density f, we introduce the single- and double-layer potentials defined by

$$\begin{aligned} (S_j f)(\mathbf{x}) &= \int_\Gamma \Phi_j(\mathbf{x},\mathbf{y})f(\mathbf{y})\,ds(\mathbf{y}), && \mathbf{x}\in\Omega_j,\\ (D_j f)(\mathbf{x}) &= \int_\Gamma \frac{\partial\Phi_j}{\partial n(\mathbf{y})}(\mathbf{x},\mathbf{y})f(\mathbf{y})\,ds(\mathbf{y}), && \mathbf{x}\in\Omega_j, \end{aligned} \tag{3.7}$$

and their derivatives, normal and tangential as $\mathbf{x}\to\Gamma$, using the standard jump relations, see for example **[3, 6]**

(3.8)

$$\begin{aligned} \frac{\partial}{\partial n}(S_j f)(\mathbf{x}) &= \int_\Gamma \frac{\partial\Phi_j}{\partial n(\mathbf{x})}(\mathbf{x},\mathbf{y})f(\mathbf{y})\,ds(\mathbf{y}) \mp \frac{1}{2}f(\mathbf{x})\\ &:= (NS_j f)(\mathbf{x}) \mp \frac{1}{2}f(\mathbf{x}), && \mathbf{x}\in\Gamma,\\ \frac{\partial}{\partial n}(D_j f)(\mathbf{x}) &= \int_\Gamma \frac{\partial^2\Phi_j}{\partial n(\mathbf{x})\partial n(\mathbf{y})}(\mathbf{x},\mathbf{y})f(\mathbf{y})\,ds(\mathbf{y})\\ &:= (ND_j f)(\mathbf{x}), && \mathbf{x}\in\Gamma,\\ \frac{\partial}{\partial \tau}(S_j f)(\mathbf{x}) &= \int_\Gamma \frac{\partial\Phi_j}{\partial \tau(\mathbf{x})}(\mathbf{x},\mathbf{y})f(\mathbf{y})\,ds(\mathbf{y})\\ &:= (TS_j f)(\mathbf{x}), && \mathbf{x}\in\Gamma, \end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial\tau}(D_j f)(\mathbf{x}) &= \int_\Gamma \frac{\partial^2\Phi_j}{\partial\tau(\mathbf{x})\partial n(\mathbf{y})}(\mathbf{x},\mathbf{y})f(\mathbf{y})\,ds(\mathbf{y}) \pm \frac{1}{2}\frac{\partial f}{\partial\tau}(\mathbf{x})\\ &:= (TD_j f)(\mathbf{x}) \pm \frac{1}{2}\frac{\partial f}{\partial\tau}(\mathbf{x}), \qquad \mathbf{x}\in\Gamma,\end{aligned}$$

where the upper (lower) sign indicates the limits obtained by approaching the boundary Γ from Ω_0 (Ω_1). This means when $j=0$, $(j=1)$. For the last equation, we assume f to be continuously differentiable. We presented the jump relations for continuous densities only for simplicity, since in the following proof we use a Sobolev space setting.

Here, we have to mention the continuity of the single-layer potentials in $\mathbb{R}^2$ and the discontinuity of the double-layer potentials $(\pm\frac{1}{2}f)$. All the integrals are well defined and particularly the potentials S_j, D_j and NS_j have weakly singular kernels (logarithmic singularity), the potentials TS_j have Cauchy type singularity (of order $1/|\mathbf{x}-\mathbf{y}|$) and the potentials ND_j, TD_j are hypersingular (of order $1/|\mathbf{x}-\mathbf{y}|^2$).

Theorem 3.2. *If κ_1^2 is not an interior Dirichlet eigenvalue and κ_0^2 is not an interior Dirichlet and Neumann eigenvalue, then the problem* (3.1)–(3.3) *has a unique solution.*

Proof. We apply the direct method, see for instance **[9]**, to transform the problem into a system of integral equations. We consider Green's second theorem in the interior domain

$$\begin{aligned}-u_1(\mathbf{x}) &= \int_\Gamma \frac{\partial\Phi_1}{\partial n(\mathbf{y})}(\mathbf{x},\mathbf{y})u_1(\mathbf{y})\,ds(\mathbf{y})\\ &\quad - \int_\Gamma \Phi_1(\mathbf{x},\mathbf{y})\frac{\partial u_1}{\partial n(\mathbf{y})}(\mathbf{y})\,ds(\mathbf{y}),\\ &= (D_1u_1)(\mathbf{x}) - (S_1\partial_\eta u_1)(\mathbf{x}), \quad \mathbf{x}\in\Omega_1,\end{aligned} \tag{3.9}$$

similarly

$$-v_1(\mathbf{x}) = (D_1v_1)(\mathbf{x}) - (S_1\partial_\eta v_1)(\mathbf{x}), \quad \mathbf{x}\in\Omega_1,$$

and in the exterior domain

$$\begin{aligned}u_0(\mathbf{x}) &= (D_0u_0)(\mathbf{x}) - (S_0\partial_\eta u_0)(\mathbf{x}), \quad \mathbf{x}\in\Omega_0,\\ v_0(\mathbf{x}) &= (D_0v_0)(\mathbf{x}) - (S_0\partial_\eta v_0)(\mathbf{x}), \quad \mathbf{x}\in\Omega_0.\end{aligned} \tag{3.10}$$

Letting $\mathbf{x} \to \Gamma$, in the above formulas and taking the normal and the tangential derivatives on Γ, we obtain

$$(3.11a) \quad \left(NS_j \pm \tfrac{1}{2}I\right)\partial_\eta u_j = ND_j u_j, \qquad \left(NS_j \pm \tfrac{1}{2}I\right)\partial_\eta v_j = ND_j v_j,$$

$$(3.11b) \quad \left(D_j \mp \tfrac{1}{2}I\right) u_j = S_j \partial_\eta u_j, \qquad \left(D_j \mp \tfrac{1}{2}I\right) v_j = S_j \partial_\eta v_j,$$

$$(3.11c) \quad TD_j u_j - TS_j \partial_\eta u_j = \pm\tfrac{1}{2}\partial_\tau u_j, \quad TD_j v_j - TS_j \partial_\eta v_j = \pm\tfrac{1}{2}\partial_\tau v_j.$$

Combining the relations in (3.11b) for $j = 0$ with the boundary conditions (3.2d) and (3.2b), respectively, we have

$$(3.12) \quad \begin{aligned} \left(D_0 - \frac{1}{2}I\right)u_0 &= \frac{\widetilde{\epsilon}_1}{\widetilde{\epsilon}_0} S_0 \partial_\eta u_1 - \frac{\beta_1}{\widetilde{\epsilon}_0 \omega} S_0 \partial_\tau v_1 \\ &\quad + \frac{\beta_0}{\widetilde{\epsilon}_0 \omega} S_0 \partial_\tau v_0 - S_0 \partial_\eta e_3^{\mathrm{inc}}, \\ \left(D_0 - \frac{1}{2}I\right)v_0 &= \frac{\widetilde{\mu}_1}{\widetilde{\mu}_0} S_0 \partial_\eta v_1 + \frac{\beta_1}{\widetilde{\mu}_0 \omega} S_0 \partial_\tau u_1 - \frac{\beta_0}{\widetilde{\mu}_0 \omega} S_0 \partial_\tau u_0 - \frac{\beta_0}{\widetilde{\mu}_0 \omega} S_0 \partial_\tau e_3^{\mathrm{inc}}. \end{aligned}$$

We define

$$(3.13) \quad K_j := \left(NS_j \pm \tfrac{1}{2}I\right)^{-1} ND_j, \qquad L_j := 2(TD_j - TS_j K_j).$$

The operator K_0 is well defined since the integral equation (3.11a) for $j = 0$ corresponds to the solution of the interior Neumann problem and we assumed κ_0^2 not to be an interior eigenvalue. Similarly, K_1 is well defined if κ_1^2 is not an interior Dirichlet eigenvalue.

Then, the system of equations (3.12), using (3.11), is transformed to

$$\begin{aligned} \left(D_0 - \frac{1}{2}I\right)u_0 - \frac{\widetilde{\epsilon}_1}{\widetilde{\epsilon}_0} S_0 K_1 u_1 - \frac{\beta_1}{\widetilde{\epsilon}_0 \omega} S_0 L_1 v_1 - \frac{\beta_0}{\widetilde{\epsilon}_0 \omega} S_0 L_0 v_0 \\ = -S_0 \partial_\eta e_3^{\mathrm{inc}}, \\ \left(D_0 - \frac{1}{2}I\right)v_0 - \frac{\widetilde{\mu}_1}{\widetilde{\mu}_0} S_0 K_1 v_1 + \frac{\beta_1}{\widetilde{\mu}_0 \omega} S_0 L_1 u_1 + \frac{\beta_0}{\widetilde{\mu}_0 \omega} S_0 L_0 u_0 \\ = -\frac{\beta_0}{\widetilde{\mu}_0 \omega} S_0 \partial_\tau e_3^{\mathrm{inc}}. \end{aligned}$$

We consider now equations (3.2a) and (3.2c), to obtain

$$\begin{aligned}\left(D_0 - \frac{1}{2}I\right)u_0 - \frac{\widetilde{\epsilon}_1}{\widetilde{\epsilon}_0}S_0K_1u_0 - \frac{\beta_1}{\widetilde{\epsilon}_0\omega}S_0L_1v_0 - \frac{\beta_0}{\widetilde{\epsilon}_0\omega}S_0L_0v_0 \\ = -S_0\partial_\eta e_3^{\mathrm{inc}} + \frac{\widetilde{\epsilon}_1}{\widetilde{\epsilon}_0}S_0K_1e_3^{\mathrm{inc}}, \\ \left(D_0 - \frac{1}{2}I\right)v_0 - \frac{\widetilde{\mu}_1}{\widetilde{\mu}_0}S_0K_1v_0 + \frac{\beta_1}{\widetilde{\mu}_0\omega}S_0L_1u_0 + \frac{\beta_0}{\widetilde{\mu}_0\omega}S_0L_0u_0 \\ = -\frac{\beta_0}{\widetilde{\mu}_0\omega}S_0\partial_\tau e_3^{\mathrm{inc}} - \frac{\beta_1}{\widetilde{\mu}_0\omega}S_0L_1e_3^{\mathrm{inc}}.\end{aligned}$$

The above system in compact form reads

$$(\mathbf{D} + \mathbf{K})\,\mathbf{u} = \mathbf{b}, \tag{3.14}$$

where

$$\begin{aligned}\mathbf{D} &= \begin{pmatrix} D_0 - 1/2I & 0 \\ 0 & D_0 - 1/2I \end{pmatrix}, \\ \mathbf{K} &= \begin{pmatrix} -(\widetilde{\epsilon}_1/\widetilde{\epsilon}_0)S_0K_1 & -(1/\widetilde{\epsilon}_0\omega)S_0(\beta_1L_1 + \beta_0L_0) \\ (1/\widetilde{\mu}_0\omega)S_0(\beta_1L_1 + \beta_0L_0) & -(\widetilde{\mu}_1/\widetilde{\mu}_0)S_0K_1 \end{pmatrix}, \\ \mathbf{u} &= \begin{pmatrix} u_0|_\Gamma \\ v_0|_\Gamma \end{pmatrix}, \\ \mathbf{b} &= \begin{pmatrix} -S_0\partial_\eta + (\widetilde{\epsilon}_1/\widetilde{\epsilon}_0)S_0K_1 \\ -(1/\widetilde{\mu}_0\omega)S_0(\beta_0\partial_\tau + \beta_1L_1) \end{pmatrix} e_3^{\mathrm{inc}}.\end{aligned}$$

We assume that Γ is of class $C^{2,\alpha}$, $0 < \alpha \leq 1$. We know that $D_0 : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is compact; thus, $(D_0 - 1/2I)^{-1} : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is bounded, if κ_0^2 is not an interior Dirichlet eigenvalue. Then, $\mathbf{D} : (H^{-1/2}(\Gamma))^2 \to (H^{-1/2}(\Gamma))^2$ is bounded and (3.14) is transformed to

$$(\mathbf{I} + \mathbf{D}^{-1}\mathbf{K})\,\mathbf{u} = \mathbf{D}^{-1}\mathbf{b}. \tag{3.15}$$

First, we show that $\mathbf{K}$ is compact. We recall that

$$\begin{aligned}S_0K_1 &= S_0\left(NS_1 - \tfrac{1}{2}I\right)^{-1}ND_1, \\ S_0(\beta_1L_1+\beta_0L_0) &= 2S_0\left(\beta_1TD_1+\beta_0TD_0-\beta_0TS_0\left(NS_0 + \tfrac{1}{2}I\right)^{-1}ND_0\right)\end{aligned}$$

$$- 2\beta_1 S_0 T S_1 \left(NS_1 - \tfrac{1}{2}I\right)^{-1} ND_1,$$

and the following properties, see **[4, 5]**, $S_0 : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is compact, ND_j, $TD_j : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ are bounded, $TS_j : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ are bounded and $(NS_j \pm \frac{1}{2}I)^{-1} : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ are bounded due to compactness of $NS_j : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$. Then, the operators

$$S_0 K_1,\ S_0(\beta_1 L_1 + \beta_0 L_0) : H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma),$$

are compact. Hence, $\mathbf{K} : (H^{1/2}(\Gamma))^2 \to (H^{-1/2}(\Gamma))^2$ is also compact resulting in the compactness of $\mathbf{D}^{-1}\mathbf{K} : (H^{1/2}(\Gamma))^2 \to (H^{-1/2}(\Gamma))^2$.

Next we prove the uniqueness of solutions of equation (3.15). Solvability follows from the Fredholm alternative theorem. Let $(\phi_0, \psi_0)^T$ be the solution of the homogeneous form of (3.15). Then, the potentials

$$\begin{aligned} u_{0,0}(\mathbf{x}) &= (D_0\phi_0)(\mathbf{x}) - (S_0\partial_\eta\phi_0)(\mathbf{x}), \quad \mathbf{x} \in \Omega_0, \\ v_{0,0}(\mathbf{x}) &= (D_0\psi_0)(\mathbf{x}) - (S_0\partial_\eta\psi_0)(\mathbf{x}), \quad \mathbf{x} \in \Omega_0, \end{aligned}$$

and

$$\begin{aligned} u_{1,1}(\mathbf{x}) &= -(D_1\phi_1)(\mathbf{x}) + (S_1\partial_\eta\phi_1)(\mathbf{x}), \quad \mathbf{x} \in \Omega_1, \\ v_{1,1}(\mathbf{x}) &= -(D_1\psi_1)(\mathbf{x}) + (S_1\partial_\eta\psi_1)(\mathbf{x}), \quad \mathbf{x} \in \Omega_1, \end{aligned}$$

solve the homogeneous form of the problem (3.1)–(3.3). From Theorem 3.1, we have that $u_{0,0} = v_{0,0} = 0$ in Ω_0 and $u_{1,1} = v_{1,1} = 0$ in Ω_1 where the densities ϕ_1, ψ_1 depend on the solution of the homogeneous case.

We construct

$$\begin{aligned} u_{0,1}(\mathbf{x}) &= (D_0\phi_0)(\mathbf{x}) - (S_0\partial_\eta\phi_0)(\mathbf{x}), \quad \mathbf{x} \in \Omega_1, \\ v_{0,1}(\mathbf{x}) &= (D_0\psi_0)(\mathbf{x}) - (S_0\partial_\eta\psi_0)(\mathbf{x}), \quad \mathbf{x} \in \Omega_1, \end{aligned}$$

and

$$\begin{aligned} u_{1,0}(\mathbf{x}) &= -(D_1\phi_1)(\mathbf{x}) + (S_1\partial_\eta\phi_1)(\mathbf{x}), \quad \mathbf{x} \in \Omega_0, \\ v_{1,0}(\mathbf{x}) &= -(D_1\psi_1)(\mathbf{x}) + (S_1\partial_\eta\psi_1)(\mathbf{x}), \quad \mathbf{x} \in \Omega_0. \end{aligned}$$

Considering the jump relations, at the boundary Γ we obtain

$$\begin{aligned} u_{0,0} - u_{0,1} &= \phi_0, \quad u_{1,1} - u_{1,0} = \phi_1, \\ v_{0,0} - v_{0,1} &= \psi_0, \quad v_{1,1} - v_{1,0} = \psi_1. \end{aligned} \tag{3.16}$$

Since $u_{0,0} = u_{1,1} = v_{0,0} = v_{1,1} = 0$, $\phi_0 = \phi_1$ and $\psi_0 = \psi_1$, on Γ, we find

$$u_{0,1} = u_{1,0}, \qquad v_{0,1} = v_{1,0}, \quad \mathbf{x} \in \Gamma.$$

Similarly, we can rewrite the other two boundary conditions of (3.4) for those fields taking the differences of the normal and tangential derivatives as $\mathbf{x} \to \Gamma$. Thus, we see that $u_{0,1}$, $v_{0,1}$, $u_{1,0}$ and $v_{1,0}$ solve the homogeneous problem, but with κ_1 and κ_0 interchanged and from Theorem 3.1, we also get $u_{0,1} = v_{0,1} = 0$, on Γ and hence $\phi_0 = \psi_0 = 0$ from (3.16). $\square$

In order to handle the hypersingularity of the operators TD_j we work in a similar way as Mitzner **[15]** derived the Maue's formula **[14]** of the hypersingular operator ND_j, namely

$$(ND_j f)(\mathbf{x}) = \left(TS_j \frac{\partial f}{\partial \tau} \right)(\mathbf{x}) + \kappa_j^2 \, \widehat{\boldsymbol{n}}(\mathbf{x}) \cdot (S_j \, \widehat{\boldsymbol{n}} f)(\mathbf{x}), \quad \mathbf{x} \in \Gamma. \tag{3.17}$$

This transformation reduces the hypersingularity to singularity of Cauchy type (first term) and to a weak singularity (second term).

Theorem 3.3. *Let $f \in C^{1,\alpha}(\Gamma)$. The hypersingular operator TD_j can be transformed to*

$$(TD_j f)(\boldsymbol{x}) = -\left(NS_j \frac{\partial f}{\partial \tau} \right)(\boldsymbol{x}) + \kappa_j^2 \, \widehat{\boldsymbol{\tau}}(\boldsymbol{x}) \cdot (S_j \, \widehat{\boldsymbol{n}} f)(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma. \tag{3.18}$$

Proof. We recall equation (3.9). Applying Green's first theorem to $\mathbf{v} \cdot \nabla u_1, \mathbf{v}$ arbitrary constant vector, and Φ_1, using that

$$\Delta \Phi_1(\mathbf{x}, \mathbf{y}) + \kappa_1^2 \Phi_1(\mathbf{x}, \mathbf{y}) = -\delta(\mathbf{x} - \mathbf{y}),$$

yields

$$\begin{aligned} &\int_{\Omega_1} \nabla_y (\mathbf{v} \cdot \nabla_y u_1(\mathbf{y})) \cdot \nabla_y \Phi_1(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} \\ &\qquad - \kappa_1^2 \int_{\Omega_1} \Phi_1(\mathbf{x}, \mathbf{y}) \, \mathbf{v} \cdot \nabla_y u_1(\mathbf{y}) \, d\mathbf{y} \\ &\qquad = \mathbf{v} \cdot \nabla_x u_1(\mathbf{x}) + \int_\Gamma \frac{\partial \Phi_1}{\partial n(\mathbf{y})}(\mathbf{x}, \mathbf{y}) \mathbf{v} \cdot \nabla_y u_1(\mathbf{y}) \, ds(\mathbf{y}). \end{aligned} \tag{3.19}$$

The first integral can be transformed to

$$\begin{aligned}\int_{\Omega_1} \nabla(\mathbf{v}\cdot\nabla u_1)\cdot\nabla\Phi_1\,d\mathbf{y} \\ &= -\int_{\Omega_1}\big(\kappa_1^2 u_1\mathbf{v} + \nabla\times(\mathbf{v}\times\nabla u_1)\big)\cdot\nabla\Phi_1\,d\mathbf{y} \\ &= -\kappa_1^2\mathbf{v}\cdot\int_{\Omega_1} u_1\nabla\Phi_1\,d\mathbf{y} \\ &\quad - \mathbf{v}\cdot\int_\Gamma(\widehat{\boldsymbol{n}}\times\nabla\Phi_1)\times\nabla u_1\,ds(\mathbf{y}).\end{aligned}$$

Then, (3.19) reads

$$\mathbf{v}\cdot\left(\kappa_1^2\int_{\Omega_1}\nabla_y(\Phi_1\,u_1)d\mathbf{y} + \int_\Gamma(\widehat{\boldsymbol{n}}\times\nabla_y\Phi_1)\times\nabla_y u_1\,ds(\mathbf{y}) + \int_\Gamma\frac{\partial\Phi_1}{\partial n_y}\nabla_y u_1\,ds(\mathbf{y})\right) = -\mathbf{v}\cdot\nabla_x u_1.$$

Using some vector identities and suppressing the inner products with $\mathbf{v}$ (holds for any vector), we end up to

$$-\nabla_x u_1 = \int_\Gamma\left(-\nabla_y\Phi_1\times(\widehat{\boldsymbol{n}}_y\times\nabla_y u_1) + \kappa_1^2\Phi_1 u_1\widehat{\boldsymbol{n}}_y + \nabla_y\Phi_1\frac{\partial u_1}{\partial n_y}\right)ds(\mathbf{y}),$$

for $\mathbf{x}\in\Omega_1$. We multiply this equation with $\widehat{\boldsymbol{n}}(\mathbf{x})$ (inner product) and consider the limit as $\mathbf{x}$ approaches the boundary Γ from inside and the corresponding jump relations. We obtain **[15]**

(3.20)

$$\begin{aligned}-\frac{1}{2}\frac{\partial}{\partial n}u_1 \\ &= \int_\Gamma\big((\widehat{\boldsymbol{n}}_x\times\nabla_x\Phi_1)\cdot(\widehat{\boldsymbol{n}}_y\times\nabla_y u_1) + \kappa_1^2\Phi_1 u_1(\widehat{\boldsymbol{n}}_x\cdot\widehat{\boldsymbol{n}}_y)\big)\,ds(\mathbf{y}) \\ &\quad - \int_\Gamma\frac{\partial\Phi_1}{\partial n_x}\frac{\partial u_1}{\partial n_y}\,ds(\mathbf{y}),\quad \mathbf{x}\in\Gamma.\end{aligned}$$

Now, equating the above equation and the normal derivative of (3.9) as $\mathbf{x}\to\Gamma$, we obtain (3.17) in $\mathbb{R}^2$.

We take the tangential derivative of (3.9), and, considering the jump relations, we get

$$-\frac{\partial}{\partial \tau} u_1(\mathbf{x}) = (TD_1 u_1)(\mathbf{x}) - \frac{1}{2}\frac{\partial u_1}{\partial \tau}(\mathbf{x}) - \left(TS_1 \frac{\partial u_1}{\partial n}\right)(\mathbf{x}), \ \ \mathbf{x} \in \Gamma.$$

We replace $\widehat{\boldsymbol{n}}(\mathbf{x})$ by $\widehat{\boldsymbol{\tau}}(\mathbf{x})$ in (3.20) (considering the appropriate jump relations) and restricting ourselves in $\mathbb{R}^2$, we have

$$\begin{aligned} -\frac{1}{2}\frac{\partial}{\partial \tau} u_1 = \int_\Gamma \big(-(\widehat{\boldsymbol{n}}_x \cdot \nabla_x \Phi_1)(\widehat{\boldsymbol{\tau}}_y \cdot \nabla_y u_1) + \kappa_1^2 \Phi_1 u_1 (\widehat{\boldsymbol{\tau}}_x \cdot \widehat{\boldsymbol{n}}_y)\big) \, ds(\mathbf{y}) \\ - \int_\Gamma \frac{\partial \Phi_1}{\partial \tau_x}\frac{\partial u_1}{\partial n_y} \, ds(\mathbf{y}), \quad \mathbf{x} \in \Gamma. \end{aligned}$$

Observing the last two equations, we obtain (3.18), the equivalent of the Maue's formula for the tangential derivative of the double-layer potential which also reduces the hypersingularity of the potential. $\square$

4. Numerical results. In this section, we present numerical examples by implementing the proposed method. We use quadrature rules to integrate the singularities considering trigonometric interpolation. Regarding the convergence and the error analysis of the quadrature formulas, we refer the reader to **[11]** for the weakly singular operators and to **[10]** for the hypersingular. We solve the system of integral equations considering these rules by the Nyström method.

We assume the following parametrization for the boundary:

$$\Gamma = \{\mathbf{z}(t) = (z_1(t), z_2(t)) : t \in [0, 2\pi]\},$$

where $\mathbf{z} : \mathbb{R} \to \mathbb{R}^2$ is a C^2-smooth, 2π-periodic and counterclockwise oriented parametrization. We assume, in addition, that $\mathbf{z}$ is injective in $[0, 2\pi)$, that is, $\mathbf{z}'(t) \neq 0$ for all $t \in [0, 2\pi]$.

Now, we transform the operators in (3.7) and their derivatives, see (3.8), into their parametric forms

$$\begin{aligned} (S_j \psi)(t) &= \int_0^{2\pi} M^{S_j}(t,s) \psi(s) \, ds, \\ (D_j \psi)(t) &= \int_0^{2\pi} M^{D_j}(t,s) \psi(s) \, ds, \end{aligned}$$

$$(NS_j\psi)(t) = \int_0^{2\pi} M^{NS_j}(t,s)\psi(s)\,ds,$$
$$(TS_j\psi)(t) = \int_0^{2\pi} M^{TS_j}(t,s)\psi(s)\,ds,$$

and the special forms

(4.2a)
$$(ND_j\psi)(t) = \frac{1}{|\mathbf{z}'(t)|}\int_0^{2\pi}\left[\frac{1}{4\pi}\cot\left(\frac{s-t}{2}\right)\frac{\partial\psi}{\partial s}(s) - M^{ND_j}(t,s)\psi(s)\right]ds$$
$$+\kappa_j^2\int_0^{2\pi}(\widehat{\boldsymbol{n}}(t)\cdot\widehat{\boldsymbol{n}}(s))M^{S_j}(t,s)\psi(s)\,ds,$$

(4.2b)
$$(TD_j\psi)(t) = \frac{1}{|\mathbf{z}'(t)|}\int_0^{2\pi} M^{TD_j}(t,s)\psi(s)\,ds$$
$$+\kappa_j^2\int_0^{2\pi}(\widehat{\boldsymbol{\tau}}(t)\cdot\widehat{\boldsymbol{n}}(s))M^{S_j}(t,s)\psi(s)\,ds,$$

for $t\in[0,2\pi]$, $j=0,1$ and $\psi(t)=f(\mathbf{z}(t))$, $\widehat{\boldsymbol{n}}(t)=\widehat{\boldsymbol{n}}(\mathbf{z}(t))$, $\widehat{\boldsymbol{\tau}}(t)=\widehat{\boldsymbol{\tau}}(\mathbf{z}(t))$, where

(4.3)
$$M^{S_j}(t,s) = \frac{i}{4}H_0^{(1)}(\kappa_j|\mathbf{r}(t,s)|)|\mathbf{z}'(s)|,$$
$$M^{D_j}(t,s) = \frac{i\kappa_j}{4}\frac{\widehat{\boldsymbol{n}}(s)\cdot\mathbf{r}(t,s)}{|\mathbf{r}(t,s)|}H_1^{(1)}(\kappa_j|\mathbf{r}(t,s)|)|\mathbf{z}'(s)|,$$
$$M^{NS_j}(t,s) = -\frac{i\kappa_j}{4}\frac{\widehat{\boldsymbol{n}}(t)\cdot\mathbf{r}(t,s)}{|\mathbf{r}(t,s)|}H_1^{(1)}(\kappa_j|\mathbf{r}(t,s)|)|\mathbf{z}'(s)|,$$
$$M^{TS_j}(t,s) = -\frac{i\kappa_j}{4}\frac{\widehat{\boldsymbol{\tau}}(t)\cdot\mathbf{r}(t,s)}{|\mathbf{r}(t,s)|}H_1^{(1)}(\kappa_j|\mathbf{r}(t,s)|)|\mathbf{z}'(s)|,$$
$$M^{ND_j}(t,s) = \frac{i}{4}M(t,s)\left[\kappa_j^2 H_0^{(1)}(\kappa_j|\mathbf{r}(t,s)|) - 2\kappa_j\frac{H_1^{(1)}(\kappa_j|\mathbf{r}(t,s)|)}{|\mathbf{r}(t,s)|}\right]$$
$$+\frac{i\kappa_j}{4}\frac{\mathbf{z}'(t)\cdot\mathbf{z}'(s)}{|\mathbf{r}(t,s)|}H_1^{(1)}(\kappa_j|\mathbf{r}(t,s)|)+\frac{1}{8\pi}\sin^{-2}\left(\frac{t-s}{2}\right),$$
$$M^{TD_j}(t,s) = \frac{i}{4}M^J(t,s)\left[\kappa_j^2 H_0^{(1)}(\kappa_j|\mathbf{r}(t,s)|) - 2\kappa_j\frac{H_1^{(1)}(\kappa_j|\mathbf{r}(t,s)|)}{|\mathbf{r}(t,s)|}\right]$$

$$+\frac{i\kappa_j}{4}\frac{\mathbf{J}\,\mathbf{z}'(t)\cdot\mathbf{z}'(s)}{|\mathbf{r}(t,s)|}H_1^{(1)}(\kappa_j|\mathbf{r}(t,s)|),$$

with $\mathbf{r}(t,s)=\mathbf{z}(t)-\mathbf{z}(s)$, and

$$M(t,s)=\frac{(\mathbf{z}'(t)\cdot\mathbf{r}(t,s))(\mathbf{z}'(s)\cdot\mathbf{r}(t,s))}{|\mathbf{r}(t,s)|^2},$$
$$M^J(t,s)=\frac{(\mathbf{J}\,\mathbf{z}'(t)\cdot\mathbf{r}(t,s))(\mathbf{z}'(s)\cdot\mathbf{r}(t,s))}{|\mathbf{r}(t,s)|^2}.$$

Here, we have used the formulas $H_1^{(1)}(t)=-H_0^{(1)'}(t)$ and $H_1^{(1)'}(t)=H_0^{(1)}(t)-(1/t)H_1^{(1)}(t)$. The form in equation (4.2a) is based on (3.17), derived by Kress **[10]** and improved in **[12]**. The derivation of (4.2b) is easier. Namely, we define $\partial_{t_J}:=\mathbf{J}\,\mathbf{z}'\partial_z$ and

$$M^{TD_j}(t,s):=\frac{\partial^2}{\partial t_J\partial s}M^{S_j}(t,s).$$

Then

$$\begin{aligned}-\left(NS_j\frac{\partial f}{\partial\tau}\right)(\mathbf{z}(t))&=-\int_0^{2\pi}\frac{\mathbf{J}\,\mathbf{z}'(t)}{|\mathbf{z}'(t)|}\frac{\partial}{\partial z(t)}M^{S_j}(t,s)\\&\quad\cdot\frac{\mathbf{z}'(s)}{|\mathbf{z}'(s)|}\frac{\partial\psi}{\partial z(s)}(s)|\mathbf{z}'(s)|\,ds\\&=-\frac{1}{|\mathbf{z}'(t)|}\int_0^{2\pi}\frac{\partial}{\partial t_J}M^{S_j}(t,s)\frac{\partial}{\partial s}\psi(s)\,ds\\&=\frac{1}{|\mathbf{z}'(t)|}\int_0^{2\pi}\frac{\partial^2}{\partial t_J\partial s}M^{S_j}(t,s)\psi(s)\,ds,\end{aligned}$$

and (4.2b) follows by simply adding the parametrized form of the second term in the right-hand side of (3.18). The kernels in (4.3) admit the decomposition

$$M^k(t,s)=M_1^k(t,s)\ln\left(4\sin^2\left(\frac{t-s}{2}\right)\right)+M_2^k(t,s),$$

for $k=S_j$, D_j, NS_j, ND_j, TD_j where M_1^k and M_2^k are analytic, due to logarithmic singularity of the functions at $t=s$. The case of M^{TS_j} has to be treated differently because of the Cauchy type singularity of the kernel as $t = s$. Thus, we split the kernel as

$$M^{TS_j}(t,s) = M_1^{TS_j}(t,s) \ln\left(4\sin^2\left(\frac{t-s}{2}\right)\right) + \frac{1}{4\pi}\cot\left(\frac{s-t}{2}\right) + M_2^{TS_j}(t,s).$$

The kernels M_1^k are defined for $t \neq s$ by, see **[10]**

$$\begin{aligned}
M_1^{S_j}(t,s) &= -\frac{1}{4\pi} J_0(\kappa_j |\mathbf{r}(t,s)|)|\mathbf{z}'(s)|,\\
M_1^{D_j}(t,s) &= -\frac{\kappa_j}{4\pi} \frac{\widehat{\boldsymbol{n}}(s)\cdot \mathbf{r}(t,s)}{|\mathbf{r}(t,s)|} J_1(\kappa_j |\mathbf{r}(t,s)|)|\mathbf{z}'(s)|,\\
M_1^{NS_j}(t,s) &= \frac{\kappa_j}{4\pi} \frac{\widehat{\boldsymbol{n}}(t)\cdot \mathbf{r}(t,s)}{|\mathbf{r}(t,s)|} J_1(\kappa_j |\mathbf{r}(t,s)|)|\mathbf{z}'(s)|,\\
M_1^{TS_j}(t,s) &= \frac{\kappa_j}{4\pi} \frac{\widehat{\boldsymbol{\tau}}(t)\cdot \mathbf{r}(t,s)}{|\mathbf{r}(t,s)|} J_1(\kappa_j |\mathbf{r}(t,s)|)|\mathbf{z}'(s)|,\\
M_1^{ND_j}(t,s) &= -\frac{1}{4\pi} M(t,s)\left[\kappa_j^2 J_0(\kappa_j |\mathbf{r}(t,s)|) - 2\kappa_j \frac{J_1(\kappa_j |\mathbf{r}(t,s)|)}{|\mathbf{r}(t,s)|}\right]\\
&\quad - \frac{\kappa_j}{4\pi} \frac{\mathbf{z}'(t)\cdot \mathbf{z}'(s)}{|\mathbf{r}(t,s)|} J_1(\kappa_j |\mathbf{r}(t,s)|),\\
M_1^{TD_j}(t,s) &= -\frac{1}{4\pi} M^J(t,s)\left[\kappa_j^2 J_0(\kappa_j |\mathbf{r}(t,s)|) - 2\kappa_j \frac{J_1(\kappa_j |\mathbf{r}(t,s)|)}{|\mathbf{r}(t,s)|}\right]\\
&\quad - \frac{\kappa_j}{4\pi} \frac{\mathbf{J}\,\mathbf{z}'(t)\cdot \mathbf{z}'(s)}{|\mathbf{r}(t,s)|} J_1(\kappa_j |\mathbf{r}(t,s)|),
\end{aligned}$$

with diagonal terms

$$\begin{aligned}
M_1^{S_j}(t,t) &= -\frac{1}{4\pi}|\mathbf{z}'(t)|, & M_1^{D_j}(t,t) &= 0,\\
M_1^{NS_j}(t,t) &= 0, & M_1^{TS_j}(t,t) &= 0,\\
M_1^{ND_j}(t,t) &= -\frac{\kappa_j^2}{8\pi}|\mathbf{z}'(t)|^2, & M_1^{TD_j}(t,t) &= 0,
\end{aligned}$$

where J_0 and J_1 are the Bessel functions of order zero and one, respectively. The kernels M_2^k, for $t \neq s$, are given by

$$M_2^k(t,s) = M^k(t,s) - M_1^k(t,s)\ln\left(4\sin^2\left(\frac{t-s}{2}\right)\right)$$

and

$$M_2^{TS_j}(t,s) = M^{TS_j}(t,s) - M_1^{TS_j}(t,s)\ln\left(4\sin^2\left(\frac{t-s}{2}\right)\right) - \frac{1}{4\pi}\cot\left(\frac{s-t}{2}\right),$$

with diagonal terms

$$\begin{aligned}
M_2^{S_j}(t,t) &= \left[\frac{i}{4} - \frac{C}{2\pi} - \frac{1}{2\pi}\ln\left(\frac{\kappa_j}{2}|\mathbf{z}'(t)|\right)\right]|\mathbf{z}'(t)|,\\
M_2^{D_j}(t,t) &= \frac{1}{4\pi}\frac{\widehat{\boldsymbol{n}}(t)\cdot\mathbf{z}''(t)}{|\mathbf{z}'(t)|},\\
M_2^{NS_j}(t,t) &= \frac{1}{4\pi}\frac{\widehat{\boldsymbol{n}}(t)\cdot\mathbf{z}''(t)}{|\mathbf{z}'(t)|},\\
M_2^{TS_j}(t,t) &= -\frac{1}{4\pi}\frac{\widehat{\boldsymbol{\tau}}(t)\cdot\mathbf{z}''(t)}{|\mathbf{z}'(t)|},\\
M_2^{ND_j}(t,t) &= \left[\pi i - 1 - 2C - 2\ln\left(\frac{\kappa_j}{2}|\mathbf{z}'(t)|\right)\right]\frac{\kappa_j^2}{8\pi}|\mathbf{z}'(t)|^2 + \frac{1}{24\pi}\\
&\quad + \frac{(\mathbf{z}'(t)\cdot\mathbf{z}''(t))^2}{4\pi|\mathbf{z}'(t)|^4} - \frac{\mathbf{z}'(t)\cdot\mathbf{z}'''(t)}{12\pi|\mathbf{z}'(t)|^2} - \frac{|\mathbf{z}''(t)|^2}{8\pi|\mathbf{z}'(t)|^2},\\
M_2^{TD_j}(t,t) &= -\frac{(\mathbf{z}'(t)\cdot\mathbf{z}''(t))(\mathbf{J}\,\mathbf{z}'(t)\cdot\mathbf{z}''(t))}{4\pi|\mathbf{z}'(t)|^4} + \frac{\mathbf{J}\,\mathbf{z}'(t)\cdot\mathbf{z}'''(t)}{12\pi|\mathbf{z}'(t)|^2},
\end{aligned}$$

where C is the Euler's constant. For the last approximation, we used the same arguments as in the case of $M_2^{ND_j}$.

Considering the equidistant points $t_j = j\pi/n$, $j = 0,\dots,2n-1$, we use the trapezoidal rule to approximate the operators with smooth kernel

$$\int_0^{2\pi}\psi(s)\,ds \approx \frac{\pi}{n}\sum_{j=0}^{2n-1}\psi(t_j),$$

and the following quadrature rules for the singular kernels

$$\int_0^{2\pi} \ln\left(4\sin^2\left(\frac{t-s}{2}\right)\right)\psi(s)\,ds \approx \sum_{j=0}^{2n-1} R_j^{(n)}(t)\psi(t_j),$$

$$\frac{1}{4\pi}\int_0^{2\pi} \cot\left(\frac{s-t}{2}\right)\frac{\partial}{\partial s}\psi(s)\,ds \approx \sum_{j=0}^{2n-1} T_j^{(n)}(t)\psi(t_j),$$

$$\int_0^{2\pi} \cot\left(\frac{s-t}{2}\right)\psi(s)\,ds \approx \sum_{j=0}^{2n-1} S_j^{(n)}(t)\psi(t_j),$$

with weights

$$R_j^{(n)}(t) = -\frac{2\pi}{n}\sum_{m=1}^{n-1}\frac{1}{m}\cos\left(m(t-t_j)\right) - \frac{\pi}{n^2}\cos(n(t-t_j)),$$

$$T_j^{(n)}(t) = -\frac{1}{2n}\sum_{m=1}^{n-1} m\cos\left(m(t-t_j)\right) - \frac{1}{4}\cos(n(t-t_j)),$$

$$S_j^{(n)}(t) = \frac{\pi}{n}\left[1-(-1)^j\cos(nt)\right]\cot\left(\frac{t_j-t}{2}\right), \quad t \neq t_j.$$

Then, the system (3.14), or similarly (3.15), considering the above parametric forms of the integral operators and the quadrature rules, is transformed to a linear system by applying the Nyström method.

To illustrate the efficiency of our method, we consider two different cases. In the first example, motivated by **[23]**, we construct a model where the scattered fields can be analytically computed and in the second one we consider the scattering of obliquely incident waves.

In both examples, the parametrization of the obstacle is given by

$$\mathbf{z}(t) = (2\cos t + 1.5\cos 2t - 1,\ 2.5\sin t), \quad t \in [0, 2\pi].$$

4.1. Example with analytic solution. We consider four arbitrary points $\mathbf{z}_1$, $\mathbf{z}_2 \in \Omega_1$ and $\mathbf{z}_3$, $\mathbf{z}_4 \in \Omega_0$, and we define the boundary functions f_k, $k = 1, 2, 3, 4$ by

$$f_1 = H_0^{(1)}(\kappa_1|\mathbf{r}_3(\mathbf{x})|) - H_0^{(1)}(\kappa_0|\mathbf{r}_1(\mathbf{x})|),$$

$$
\begin{aligned}
f_2 &= -\widetilde{\mu}_1\omega\kappa_1 H_1^{(1)}(\kappa_1|\mathbf{r}_4(\mathbf{x})|)\frac{\widehat{\boldsymbol{n}}(\mathbf{x})\cdot\mathbf{r}_4(\mathbf{x})}{|\mathbf{r}_4(\mathbf{x})|}\\
&\quad - \beta_1\kappa_1 H_1^{(1)}(\kappa_1|\mathbf{r}_3(\mathbf{x})|)\frac{\widehat{\boldsymbol{\tau}}(\mathbf{x})\cdot\mathbf{r}_3(\mathbf{x})}{|\mathbf{r}_3(\mathbf{x})|}\\
&\quad + \widetilde{\mu}_0\omega\kappa_0 H_1^{(1)}(\kappa_0|\mathbf{r}_2(\mathbf{x})|)\frac{\widehat{\boldsymbol{n}}(\mathbf{x})\cdot\mathbf{r}_2(\mathbf{x})}{|\mathbf{r}_2(\mathbf{x})|}\\
&\quad + \beta_0\kappa_0 H_1^{(1)}(\kappa_0|\mathbf{r}_1(\mathbf{x})|)\frac{\widehat{\boldsymbol{\tau}}(\mathbf{x})\cdot\mathbf{r}_1(\mathbf{x})}{|\mathbf{r}_1(\mathbf{x})|},\\
f_3 &= H_0^{(1)}(\kappa_1|\mathbf{r}_4(\mathbf{x})|) - H_0^{(1)}(\kappa_0|\mathbf{r}_2(\mathbf{x})|),\\
f_4 &= -\widetilde{\epsilon}_1\omega\kappa_1 H_1^{(1)}(\kappa_1|\mathbf{r}_3(\mathbf{x})|)\frac{\widehat{\boldsymbol{n}}(\mathbf{x})\cdot\mathbf{r}_3(\mathbf{x})}{|\mathbf{r}_3(\mathbf{x})|}\\
&\quad + \beta_1\kappa_1 H_1^{(1)}(\kappa_1|\mathbf{r}_4(\mathbf{x})|)\frac{\widehat{\boldsymbol{\tau}}(\mathbf{x})\cdot\mathbf{r}_4(\mathbf{x})}{|\mathbf{r}_4(\mathbf{x})|}\\
&\quad + \widetilde{\epsilon}_0\omega\kappa_0 H_1^{(1)}(\kappa_0|\mathbf{r}_1(\mathbf{x})|)\frac{\widehat{\boldsymbol{n}}(\mathbf{x})\cdot\mathbf{r}_1(\mathbf{x})}{|\mathbf{r}_1(\mathbf{x})|}\\
&\quad - \beta_0\kappa_0 H_1^{(1)}(\kappa_0|\mathbf{r}_2(\mathbf{x})|)\frac{\widehat{\boldsymbol{\tau}}(\mathbf{x})\cdot\mathbf{r}_2(\mathbf{x})}{|\mathbf{r}_2(\mathbf{x})|},
\end{aligned}
$$

where $\mathbf{r}_k(\mathbf{x}) = \mathbf{x} - \mathbf{z}_k$. Then, the fields

$$
\begin{aligned}
u_0(\mathbf{x}) &= H_0^{(1)}(\kappa_0|\mathbf{x}-\mathbf{z}_1|), \quad v_0(\mathbf{x}) = H_0^{(1)}(\kappa_0|\mathbf{x}-\mathbf{z}_2|), \quad \mathbf{x}\in\Omega_0,\\
u_1(\mathbf{x}) &= H_0^{(1)}(\kappa_1|\mathbf{x}-\mathbf{z}_3|), \quad v_1(\mathbf{x}) = H_0^{(1)}(\kappa_1|\mathbf{x}-\mathbf{z}_4|), \quad \mathbf{x}\in\Omega_1,
\end{aligned}
\tag{4.4}
$$

solve the following problem

$$
\begin{aligned}
\Delta u_0 + \kappa_0^2\, u_0 = 0, \quad \Delta v_0 + \kappa_0^2\, v_0 = 0, \quad \mathbf{x}\in\Omega_0,\\
\Delta u_1 + \kappa_1^2\, u_1 = 0, \quad \Delta v_1 + \kappa_1^2\, v_1 = 0, \quad \mathbf{x}\in\Omega_1,
\end{aligned}
$$

with boundary conditions

$$
\begin{aligned}
u_1 &= u_0 + f_1, && \mathbf{x}\in\Gamma,\\
\widetilde{\mu}_1\omega\frac{\partial v_1}{\partial n} + \beta_1\frac{\partial u_1}{\partial \tau} &= \widetilde{\mu}_0\omega\frac{\partial v_0}{\partial n} + \beta_0\frac{\partial u_0}{\partial \tau} + f_2, && \mathbf{x}\in\Gamma,\\
v_1 &= v_0 + f_3, && \mathbf{x}\in\Gamma,\\
\widetilde{\epsilon}_1\omega\frac{\partial u_1}{\partial n} - \beta_1\frac{\partial v_1}{\partial \tau} &= \widetilde{\epsilon}_0\omega\frac{\partial u_0}{\partial n} - \beta_0\frac{\partial v_0}{\partial \tau} + f_4, && \mathbf{x}\in\Gamma,
\end{aligned}
$$

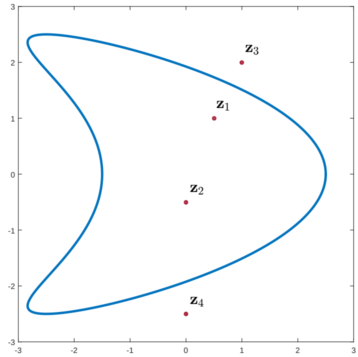

FIGURE 2. The parametrization of the boundary Γ and the source points.

and the radiation conditions

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u_0}{\partial r} - i\kappa_0 u_0\right) = 0, \qquad \lim_{r\to\infty} \sqrt{r}\left(\frac{\partial v_0}{\partial r} - i\kappa_0 v_0\right) = 0.$$

For this problem, we can again derive a system as (3.14), where now $\mathbf{b}$ is replaced by

$$\mathbf{b}_f = \begin{pmatrix} -(1/\widetilde{\epsilon}_0\omega)S_0 f_4 + (\widetilde{\epsilon}_1/\widetilde{\epsilon}_0)S_0 K_1 f_1 + (\beta_1/\widetilde{\epsilon}_0\omega)S_0 L_1 f_3 \\ -(1/\widetilde{\mu}_0\omega)S_0 f_2 + (\widetilde{\mu}_1/\widetilde{\mu}_0)S_0 K_1 f_3 - (\beta_1/\widetilde{\mu})_0\omega S_0 L_1 f_1 \end{pmatrix}.$$

Given (4.4) and the asymptotic behavior of the Hankel function [**4**], we know that the far field patterns of u_0 and v_0 are given by

$$\begin{aligned} u_0^\infty(\widehat{\boldsymbol{x}}) &= \frac{-4ie^{i\pi/4}}{\sqrt{8\pi\kappa_0}} e^{-i\kappa_0\widehat{\boldsymbol{x}}\cdot\mathbf{z}_1}, \quad \widehat{\boldsymbol{x}} \in S, \\ v_0^\infty(\widehat{\boldsymbol{x}}) &= \frac{-4ie^{i\pi/4}}{\sqrt{8\pi\kappa_0}} e^{-i\kappa_0\widehat{\boldsymbol{x}}\cdot\mathbf{z}_2}, \quad \widehat{\boldsymbol{x}} \in S, \end{aligned} \tag{4.5}$$

where S is the unit ball. Numerically, the far field patterns are given

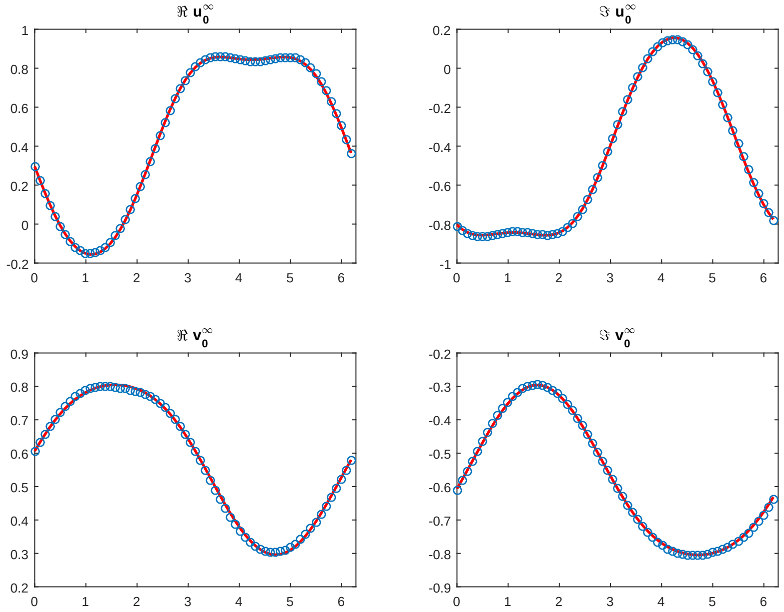

FIGURE 3. The far field patterns: Reconstructed (blue open circles) and exact (red solid line).

by

$$
\begin{aligned}
u_0^\infty(\widehat{\boldsymbol{x}}) &= \frac{e^{i\pi/4}}{\sqrt{8\pi\kappa_0}} \int_0^{2\pi} e^{-i\kappa_0 \widehat{\boldsymbol{x}}\cdot \mathbf{z}(s)} \left[-i\kappa_0 (\widehat{\boldsymbol{x}} \cdot \widehat{\boldsymbol{n}}(s))\varphi_0(s) \right. \\
&\quad \left. -(K_0\varphi_0)(s)\right] |\mathbf{z}'(s)|| \, ds, \\
v_0^\infty(\widehat{\boldsymbol{x}}) &= \frac{e^{i\pi/4}}{\sqrt{8\pi\kappa_0}} \int_0^{2\pi} e^{-i\kappa_0 \widehat{\boldsymbol{x}}\cdot \mathbf{z}(s)} \left[-i\kappa_0 (\widehat{\boldsymbol{x}} \cdot \widehat{\boldsymbol{n}}(s))\psi_0(s) \right. \\
&\quad \left. -(K_0\psi_0)(s)\right] |\mathbf{z}'(s)| \, ds,
\end{aligned}
\tag{4.6}
$$

where $\boldsymbol{\varphi} := (\varphi_0,\, \psi_0)^T$ solves

$$
(\mathbf{D} + \mathbf{K})\, \boldsymbol{\varphi} = \mathbf{b}_f .
$$

	u_0^∞			v_0^∞		
n \ t	0	$\pi/2$	π	0	$\pi/2$	π
32	0.006470	0.009056	0.003944	0.004714	0.007460	0.001767
64	0.003157	0.004729	0.001820	0.002249	0.004288	0.000601
128	0.001581	0.002374	0.000908	0.001125	0.002161	0.000292

TABLE 1. Absolute errors of the far field patterns of u_0 and v_0 for different orders n at discrete points t.

Here, we have used the representations (3.10) for the exterior fields and the asymptotics of the Hankel function. The operator K_0 is given by (3.13).

We consider the points $\mathbf{z}_1 = (0.5,\, 1)$ and $\mathbf{z}_2 = (0,\, -0.5)$ in Ω_1 and the points $\mathbf{z}_3 = (1,\, 2)$ and $\mathbf{z}_4 = (0,\, -2.5)$ in Ω_0, see Figure 2. We set $\omega = 1$ and $n = 32$. The exact values (4.5) and the reconstructed (4.6) for $(\epsilon_1, \mu_1) = (3,\, 2)$ and $(\epsilon_0, \mu_0) = (1,\, 1)$ are presented in Figure 3. The results are presented for $\theta = \pi/3$. In Table 1, we provide the absolute errors of the far field patterns for different values of n and t.

As a general comment, we could say that the reconstructions are accurate and illustrate the feasibility of the proposed method. However, the convergence is slower compared to the impedance cylinder case **[23]**. The main reason is the complexity of the matrix $\mathbf{K}$ involving the product of four operators: $S_0 T S_j (N S_j + 1/2 I)^{-1} N D_j$, $j = 0, 1$ resulting in an increase of the condition number. As $\theta \to \pi/2$, the results improve considerably.

4.2. Example with oblique incidence. In this example, we consider the usual obliquely incident (TM) polarized electromagnetic plane wave, resulting in the forms (2.10). We keep the same values for all parameters as in the previous example. We restrict the computations of the fields to the rectangular domain $[-5,\, 5]^2$, and we consider a two-dimensional uniform-space discretization, namely, $\mathbf{x}_{kj} = (-5 + k\delta, -5 + j\delta)$, where $\delta = 10/(2m-1)$, for $k, j = 0, \ldots, 2m-1$. We use $m = 128$.

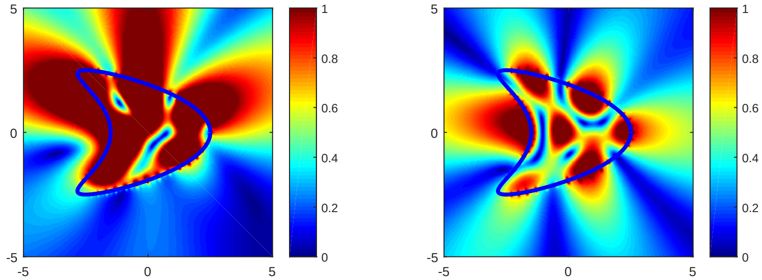

FIGURE 4. The norms of the electric fields u_0 and u_1 (left) and of the magnetic fields v_0 and v_1 (right) for $\phi = \pi/2$.

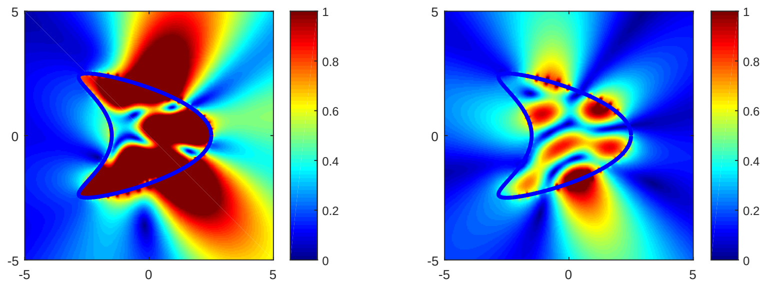

FIGURE 5. The norms of the electric fields u_0 and u_1 (left) and of the magnetic fields v_0 and v_1 (right) for $\phi = \pi/9$.

The values of the norms of the scattered electric and magnetic fields $|u_0|$, $|v_0|$ and the interior electric and magnetic fields $|u_1|$, $|v_1|$ are presented in Figures 4 and 5 for different values of the polar angle ϕ, which in $\mathbb{R}^2$ corresponds to the incident direction $(\cos\phi, \sin\phi) \in S$.

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REFERENCES

1. F. Cakoni and D. Colton, *Qualitative methods in inverse scattering theory*, Springer-Verlag, Berlin, 2006.

2. A.C. Cangellaris and R. Lee, *Finite element analysis of electromagnetic scattering from inhomogeneous cylinders at oblique incidence*, IEEE Trans. Ant. Prop. **39** (1991), 645–650.

3. D. Colton and R. Kress, *Integral equation methods in scattering theory*, Pure Appl. Math., John Wiley & Sons Inc., New York, 1983.

4. ______, *Inverse acoustic and electromagnetic scattering theory*, 2nd edition, Appl. Math. Sci. **93**, Springer-Verlag, Berlin, 1998.

5. M. Costabel and E. Stephan, *A direct boundary integral equation method for transmission problems*, J. Math. Anal. Appl. **106** (1985), 367–413.

6. R. Courant and D. Hilbert, *Methods of mathematical physics*, volume 2, Wiley-Interscience, New York, 1962.

7. V. Isakov, *Inverse problems for partial differential equations*, Appl. Math. Sci. **127**, Springer, New York, 2006.

8. R. Kittappa and R.E. Kleinman, *Acoustic scattering by penetrable homogeneous objects*, J. Math. Phys. **16** (1975), 421–432.

9. R.E. Kleinman and P.A. Martin, *On single integral equations for the transmission problem of acoustics*, SIAM J. Appl. Math. **48** (1988), 307–325.

10. R. Kress, *On the numerical solution of a hypersingular integral equation in scattering theory*, J. Comp. Appl. Math. **61** (1995), 345–360.

11. ______, *Linear integral equations*, 2nd edition, Springer Verlag, Berlin, 1999.

12. ______, *A collocation method for a hypersingular boundary integral equation via trigonometric differentiation*, J. Integral Equations Appl. **26** (2014), 197–213.

13. M. Lucido, G. Panariello and F. Schettiho, *Scattering by polygonal cross-section dielectric cylinders at oblique incidence*, IEEE Trans. Ant. Prop. **58** (2010), 540–551.

14. A.W. Maue, *Über die formulierung eines allgemeinen beugungsproblems durch eine integralgleichung*, Z. Phys. **126** (1949), 601–618.

15. K.M. Mitzner, *Acoustic scattering from an interface between media of greatly different density*, J. Math. Phys. **7** (1966), 2053–2060.

16. P. Monk, *Finite element methods for maxwells equations*, Oxford University Press, Oxford, 2003.

17. G. Nakamura and H. Wang, *The direct electromagnetic scattering problem from an imperfectly conducting cylinder at oblique incidence*, J. Math. Anal. Appl. **397** (2013), 142–155.

18. J.C. Nédélec, *Acoustic and electromagnetic equations*, Springer-Verlag, New York, 2001.

19. R.G. Rojas, *Scattering by an inhomogeneous dielectric/ferrite cylinder of arbitrary cross-section shape-oblique incidence case*, IEEE Trans. Ant. Prop. **36** (1988), 238–246.

20. J.L. Tsalamengas, *Exponentially converging nyström methods applied to the integral-integrodifferential equations of oblique scattering/ hybrid wave propagation in presence of composite dielectric cylinders of arbitrary cross section*, IEEE Trans. Ant. Prop. **55** (2007), 3239–3250.

21. N.L. Tsitsas, E.G. Alivizatos, H.T. Anastassiu and D.I. Kaklamani, *Optimization of the method of auxiliary sources (mas) for oblique incidence scattering by an infinite dielectric cylinder*, Electr. Engin. **89** (2007), 353–361.

22. J.R. Wait, *Scattering of a plane wave from a circular dielectric cylinder at oblique incidence*, Canadian J. Phys. **33** (1955), 189–195.

23. H. Wang and G. Nakamura, *The integral equation method for electromagnetic scattering problem at oblique incidence*, Appl. Num. Math. **62** (2012), 860–873.

24. J. Yan, R.K. Gordon and A.A. Kishk, *Electromagnetic scattering from impedance elliptic cylinders using finite difference method*, Electromagnetism **15** (1995), 157–173.

25. H.A. Yousif and A.Z. Elsherbeni, *Oblique incidence scattering from two eccentric cylinders*, J. Electr. Waves Appl. **11** (1997), 1273–1288.

Department of Mathematics, National Technical University of Athens, Greece
Email address: **dgindi@math.ntua.gr**

Computational Science Center, University of Vienna, Austria
Email address: **leonidas.mindrinos@univie.ac.at**