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Generalized $\mathcal{L}$-geodesic and monotonicity of the generalized reduced volume in the Ricci flow

By

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Abstract

Suppose M is a complete n-dimensional manifold, $n \geq 2$, with a metric $\overline{g}_{ij}(x,t)$ that evolves by the Ricci flow $\partial_t \overline{g}_{ij} = -2\overline{R}_{ij}$ in $M \times (0,T)$. For any $0 < p < 1$, $(p_0, t_0) \in M \times (0,T)$, $q \in M$, we define the $\mathcal{L}_p$-length between p_0 and q, $\mathcal{L}_p$-geodesic, the generalized reduced distance l_p and the generalized reduced volume $\widetilde{V}_p(\tau)$, $\tau = t_0 - t$, corresponding to the $\mathcal{L}_p$-geodesic at the point p_0 at time t_0. Under the condition $\overline{R}_{ij} \geq -c_1 \overline{g}_{ij}$ on $M \times (0, t_0)$ for some constant $c_1 > 0$, we will prove the existence of a $\mathcal{L}_p$-geodesic which minimize the $\mathcal{L}_p(q, \overline{\tau})$-length between p_0 and q for any $\overline{\tau} > 0$. This result for the case $p = 1/2$ was mentioned and used many times by G. Perelman but no proof of it was given in Perelman's papers on Ricci flow. Let $g(\tau) = \overline{g}(t_0 - \tau)$ and let $\widetilde{V}_p^{\overline{\tau}}(\tau)$ be the rescaled generalized reduced volume. Suppose M also has nonnegative curvature operator with respect to the metric $\overline{g}(t)$ for any $t \in (0,T)$ and when $1/2 < p < 1$, M has uniformly bounded scalar curvature on $(0,T)$. Let $0 < c < 1$ and let $\tau_0 = \min((2(1-p))^{-1/(2p-1)}, t_0)$. For any $1/2 \leq p < 1$ we prove that there exists a constant $A_0 \geq 0$ with $A_0 = 0$ for $p = 1/2$ such that $e^{-A_0\tau}\widetilde{V}_p(\tau)$ is a monotone decreasing function in $(0, \overline{\tau}_1)$ where $\overline{\tau}_1 = (1-c)\tau_0$ if $1/2 < p < 1$ and $\overline{\tau}_1 = t_0$ if $p = 1/2$. When $(M, \overline{g})$ is an ancient κ-solution of the Ricci flow, we will prove a monotonicity property of the rescaled generalized volume $\widetilde{V}_p^{\overline{\tau}}(\tau)$ with respect to $\overline{\tau}$ for any $1/2 \leq p < 1$. When $p = 1/2$, the $\mathcal{L}_p$-length, $\mathcal{L}_p$-geodesic, the l_p function and $\widetilde{V}_p(\tau)$ are equal to the $\mathcal{L}$-length, $\mathcal{L}$-geodesic, the reduced distance l and the reduced volume $\widetilde{V}(\tau)$ introduced by Perelman in his papers on Ricci flow. We will also prove a result on the reduced distance l and the reduced volume $\widetilde{V}$ which was used by Perelman without proof in [18].

0. Introduction

Recently there is a lot of study on the Ricci flow on manifold by R. Hamilton [5]–[10], S. Y. Hsu [11]–[15], G. Perelman [18], [19], W. X. Shi [20], [21],

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L. F. Wu [23], [24], and others. We refer the readers to the lecture notes by B. Chow [2] and the book [3] by B. Chow and D. Knopf on the basics of Ricci flow and the papers [18], [19], of G. Perelman for the most recent results on Ricci flow.

In the paper [5] R. Hamilton proved that if M is a compact manifold with a metric $\overline{g}$ that evolves by the Ricci flow

$$\frac{\partial}{\partial t}\overline{g}_{ij} = -2\overline{R}_{ij} \tag{0.1}$$

where $\overline{R}_{ij}$ is the Ricci curvature of $\overline{g}$ and $\overline{g}_{ij}(x,0)=\overline{g}_{ij}(x)$ is a metric of strictly positive Ricci curvature, then the evolving metric will converge modulo scaling to a metric of constant positive curvature. Similiar result was obtained by R. Hamilton [6] for compact 4-dimensional manifolds with positive curvature operator. Harnack inequality for the Ricci flow was proved by R. Hamilton in [8].

Short time existence of solutions of the Ricci flow on complete non-compact Riemannian manifold with bounded curvature was proved by W. X. Shi [20]. Global existence and uniqueness of solutions of the Ricci flow on non-compact manifold $\mathbb{R}^2$ was obtained by S. Y. Hsu in [11]. Asymptotic behaviour of solutions of the Ricci flow equation on $\mathbb{R}^2$ was proved by by S. Y. Hsu in [12], [13], [14].

In [18], [19], G. Perelman introduced the concept of $\mathcal{L}$-length, $\mathcal{L}$-geodesic, the reduced distance l and the reduced volume $\widetilde{V}(\tau)$ for Ricci flow on complete manifolds with postive bounded curvature operator. G. Perelman found that these are very useful tools in studying Ricci flow on manifolds. He used these tools to proved many new properties for the Ricci flow in [18], [19]. These included the non-local collapsing theorem and the asymptotic convergence of a subsequence of the rescaled solution of an ancient κ-solution to a soliton solution of the Ricci flow on complete manifold with postive bounded curvature operator. Recently R. Ye ([25], [26]) extended the concept of $\mathcal{L}$-geodesic, the reduced distance l and the reduced volume $\widetilde{V}(\tau)$ to manifolds with a lower bound on the Ricci curvature.

In this paper we will generalize the notion of $\mathcal{L}$-length, $\mathcal{L}$-geodesic, the reduced distance l and the reduced volume $\widetilde{V}(\tau)$ of G. Perelman. For any $0<p<1$, $\tau>0$, we will define the $\mathcal{L}_p(q,\tau)$-length, $L_p(q,\tau)$-length, $\mathcal{L}_p$-exponential map, $\mathcal{L}_p$-geodesic, and reduced volume $\widetilde{V}_p(\tau)$ and prove various properties of them in this paper. When $p=1/2$, $\mathcal{L}_p(q,\tau)$, $L_p(q,\tau)$, $\mathcal{L}_p$-exponential map, $\widetilde{V}_p(\tau)$ are equal to the $\mathcal{L}(q,\tau)$, $L(q,\tau)$, $\mathcal{L}$-exponential map, and $\widetilde{V}(\tau)$ defined by G. Perelman in [18].

For any $q\in M$, $\overline{\tau}>0$, we will prove the existence of a $\mathcal{L}_p$-geodesic which minimize the $\mathcal{L}_p(q,\overline{\tau})$-length. This result for the case $p=1/2$ was mentioned and used many times in G. Perelman's paper on Ricci flow [18], [19], but no proof of it is given in his papers. There is also no detail proof of this important result in the recent book of J.W. Morgan and G. Tian [17] and the paper of H. D. Cao and X. P. Zhu [4] on Ricci flow. My result is new and answers in affirmative the existence of such $\mathcal{L}$-geodesic minimizer for the $L_p(q,\tau)$-length which is crucial to the proof of many results in [18], [19]. We also prove that for any $\mathcal{L}_p(q,\overline{\tau})$-length minimizing $\mathcal{L}_p$-geodesic there does not exists any $\mathcal{L}_p$-conjugate points along the curve.

One remarkable property of the reduced volume $\widetilde{V}(\tau)$, $\tau = t_0 - t$, with respect to any point $(p_0,t_0) \in M \times (0,T)$ proved by G. Perelman is that it is a monotone decreasing function of $\tau \in (0,t_0)$. Surprisingly in this paper we find that the generalized reduced volume $\widetilde{V}_p(\tau)$ with respect to any point $(p_0,t_0) \in M \times (0,T)$ also has similar monotonicity property. Suppose M is complete and has nonnegative curvature operator with respect to the metric $\overline{g}(t)$ for any $t \in (0,T)$ and when $1/2 < p < 1$, M has uniformly bounded scalar curvature on $(0,T)$. Let $0 < c < 1$ and let

$$\tau_0 = \min((2(1-p))^{-\frac{1}{2p-1}}, t_0). \tag{0.2}$$

For any $1/2 \le p < 1$ we prove that there exists a constant $A_0 \ge 0$ with $A_0 = 0$ for $p = 1/2$ such that $e^{-A_0\tau}\widetilde{V}_p(\tau)$ is a monotone decreasing function in $(0,\overline{\tau}_1)$ where $\overline{\tau}_1 = (1-c)\tau_0$ if $1/2 < p < 1$ and $\overline{\tau}_1 = t_0$ if $p = 1/2$. Since the proof of monotonicity uses Hamilton's Harnack inequality which is valid when the manifold has non-negative curvature, the condition of non-negative curvature is necessary for the argument to hold.

Suppose $(M,\overline{g})$ is an ancient κ-solution of the Ricci flow and $g(\tau) = \overline{g}(t_0 - \tau)$ for some constant $t_0 < 0$. Let $\overline{\tau}_0 > 0$ for $1/2 < p < 1$ and $\overline{\tau}_0 = 0$ for $p = 1/2$. When $1/2 < p < 1$, suppose also that $(M, g(\tau))$ is compact. Let $\widetilde{V}_p^{\overline{\tau}}(\rho)$ be the rescaled generalized volume. Then for any $1/2 \le p < 1$ there exist constants $A_0 \ge 0$, $A_1 \ge 0$, $A_2 \ge 0$, $c_2 > 0$, such that $e^{-W(\overline{\tau},\rho)}\widetilde{V}_p^{\overline{\tau}}(\rho)$ is a monotone decreasing function of $\overline{\tau} > \overline{\tau}_0$ for any ρ satisfying

$$0 < \rho \le \left(\frac{1}{2(1-p)}\right)^{\frac{1}{2p-1}} \tag{0.3}$$

where $W(\overline{\tau},\rho) = (A_0\rho + A_1\rho^{2p} + A_2\rho^{2p-3}e^{2c_2\overline{\tau}\rho})\overline{\tau}$. Moreover

$$\lim_{\overline{\tau}\to 0^+} \widetilde{V}_p^{\overline{\tau}}(\rho) = \left(\frac{\sqrt{\pi}}{1-p}\right)^n \rho^{\frac{(1-p)n}{2}}. \tag{0.4}$$

Note that when $p = 1/2$, one can take $A_0 = A_1 = A_2 = 0$ and the result reduces to Perelman's monotonicity property for ancient κ-solution of the Ricci flow [18].

When $(M,\overline{g})$ is an ancient κ-solution of the Ricci flow in $(-\infty,0)$ with uniformly bounded nonnegative curvature operator, then for any $t_0 < 0$, $p_0 \in M$, $0 < p < 1$, $\tau_2 > \tau_1 > 0$ we prove the existence of $\{q_i\}_{i=\infty}^{\infty} \subset M$ and $\{\overline{\tau}_i\}_{i=1}^{\infty}$, $\overline{\tau}_i \to \infty$ as $i \to \infty$, such that the rescaled l_p function $l_p^{\overline{\tau}_i}(q,\tau)$ converges uniformly on $B_0(q_i,r) \times [\tau_1,\tau_2]$ as $i \to \infty$ for any $r > 0$ where $B_0(q_i,r)$ is the geodesic ball of radius $r > 0$ with respect to the metric $\overline{g}(t_0)$.

We will also prove a result on the reduced distance l and the reduced volume $\widetilde{V}(\tau)$ used by Perelman without proof in [18]. Suppose $(M,\overline{g})$ is an ancient

κ-solution of the Ricci flow with uniformly bounded nonnegative curvature operator such that $\overline{g}(t)$ is not a flat metric for any $t<0$. If $\widetilde{V}(\tau_1)=\widetilde{V}(\tau_2)$ for some $\tau_2>\tau_1>0$, we prove that the reduced distance $l\in C^\infty(M\times[\tau_1,\tau_2])$ and $g(\tau)=\overline{g}(t_0-\tau)$ is a shrinking soliton in $M\times[\tau_1,\tau_2]$.

The plan of the paper is as follows. In Section 1 we will use a modification of the technique of [18] to prove the first variation formula for the $\mathcal{L}_p(q,\overline{\tau})$-length. We will also prove the existence of a $\mathcal{L}_p(q,\overline{\tau})$-geodesic minimizer for the $L_p(q,\overline{\tau})$-length. We will prove various properties of the $\mathcal{L}_p$-exponential map and $\mathcal{L}_p$ cut locus in Section 2. In Section 3 we will prove the second variation formula for the $L_p(q,\tau)$-length. We will prove various properties of the $L_p(q,\tau)$-length, the generalized reduced distance l_p and the generalized reduced volume $\widetilde{V}_p(\tau)$. In Section 4 we will prove the monotonicity property the generalized reduced volume $\widetilde{V}_p(\tau)$. In Section 5 we will prove the monotonicity property of the rescaled generalized reduced volume $\widetilde{V}_p^{\overline{\tau}}(\tau)$ with respect to $\overline{\tau}$. In Section 6 we will prove a result on the the reduced distance and the reduced volume used by Perelman without proof in [18].

We first start with a definition. Let $(M,\overline{g})$ be a Riemannian manifold with the metric $\overline{g}$ evolving by the Ricci flow (0.1) in $M\times(0,T)$. Let $(p_0,t_0)\in M\times(0,T)$. For any $0<t<t_0$, let $\tau=t_0-t$ and

$$g(\tau)=\overline{g}(t_0-\tau). \tag{0.5}$$

Let $R(q,\tau)$, $R_{ij}(q,\tau)$, $R(X_1,X_2)X_3(q,\tau)$ and $Rm(q,\tau)$ be the scalar curvature, Ricci curvature, curvature and Riemannian curvature of M at q with respect to the metric $g(\tau)$ and $X_1,X_2,X_3\in T_qM$. For any $0<p<1$, $p_0,q\in M$, $\overline{\tau}\in(0,t_0)$, and piecewise differentiable curve $\gamma:[0,\overline{\tau}]\to M$ joining p_0 and q with $\gamma(0)=p_0$ and $\gamma(\overline{\tau})=q$, we define the $\mathcal{L}_p^{p_0}(q,\overline{\tau})$-length of the curve γ between p_0 and q by

$$\mathcal{L}_p^{p_0}(q,\gamma,\overline{\tau})=\int_0^{\overline{\tau}}\tau^p(R(\gamma(\tau),\tau)+|\gamma'(\tau)|^2)\,d\tau$$

where $|\gamma'(\tau)|=|\gamma'(\tau)|_{g(\tau)}$. Let $\mathcal{F}^{p_0}(q,\overline{\tau})$ be the family of all piecewise differentiable curves $\gamma:[0,\overline{\tau}]\to M$ satisfying $\gamma(0)=p_0$ and $\gamma(\overline{\tau})=q$,

$$L_p^{p_0}(q,\overline{\tau})=\inf_{\gamma\in\mathcal{F}^{p_0}(q,\overline{\tau})}\mathcal{L}_p^{p_0}(q,\gamma,\overline{\tau}),$$

and let

$$l_p^{p_0}(q,\tau)=(1-p)\frac{L_p^{p_0}(q,\tau)}{\tau^{1-p}} \tag{0.6}$$

be the generalized reduced distance. Let

$$\widetilde{V}_p^{p_0}(\tau)=\int_M\tau^{-(1-p)n}e^{-l_p^{p_0}(q,\tau)}dV_{g(\tau)}$$

be the generalized reduced volume corresponding to the $\mathcal{L}_p^{p_0}(\cdot,\tau)$-length with respect to (p_0,t_0). Then $l^{p_0}(q,\tau)=l_{\frac{1}{2}}^{p_0}(q,\tau)$ and $\widetilde{V}^{p_0}(\tau)=\widetilde{V}_{\frac{1}{2}}^{p_0}(\tau)$ are the reduced length and reduced volume of Perelman [18]. Let $L^{p_0}(q,\tau)=L^{p_0}_{\frac{1}{2}}(q,\tau)$. When there is no ambiguity, we will drop the superscript p_0.

Let $q_0\in M$ and $0<\tau_0<t_0$. For any $0<p<1$, $q\in M$, $\overline{\tau}\in(\tau_0,t_0)$, and piecewise differentiable curve $\gamma:[\tau_0,\overline{\tau}]\to M$ joining q_0 and q with $\gamma(\tau_0)=q_0$ and $\gamma(\overline{\tau})=q$, we define the $\mathcal{L}^{q_0}_{\tau_0,p}(q,\overline{\tau})$-length of the curve γ between q_0 and q by

$$\mathcal{L}^{q_0}_{\tau_0,p}(q,\gamma,\overline{\tau})=\int_{\tau_0}^{\overline{\tau}}\tau^p(R(\gamma(\tau),\tau)+|\gamma'(\tau)|^2)\,d\tau.$$

Let $\mathcal{F}^{q_0}_{\tau_0}(q,\overline{\tau})$ be the family of all piecewise differentiable curves $\gamma:[\tau_0,\overline{\tau}]\to M$ satisfying $\gamma(\tau_0)=q_0$ and $\gamma(\overline{\tau})=q$ and let

$$L^{q_0}_{\tau_0,p}(q,\overline{\tau})=\inf_{\gamma\in\mathcal{F}^{q_0}_{\tau_0}(q,\overline{\tau})}\mathcal{L}^{q_0}_{\tau_0,p}(q,\gamma,\overline{\tau}).$$

For any $r>0$, $0\le\tau\le t_0$, we let $B_\tau(q,r)$ be the geodesic ball of radius r in M around the point q with respect to the metric $g(\tau)$. For any $v\in T_{p_0}M$, we let

$$\mathcal{B}(v,r)=\{v'\in T_{p_0}M:|v-v'|_{g(p_0,0)}<r\}.$$

We also let $d_\tau(q_1,q_2)=d_{g(\tau)}(q_1,q_2)$ be the distance between q_1 and q_2 with respect to the metric $g(\tau)$. For any $0<\tau<t_0$ and measurable set $E\subset M$ with respect to the metric $g(\tau)$, we let $m_\tau(E)$ be the measure of E with respect to the metric $g(\tau)$. We let $dV_{g(\tau)}(q)=\sqrt{g(q,\tau)}dq$ be the volume form of the metric $g(\tau)$.

Let $\kappa>0$. A Ricci flow $(M,\overline{g})$ is said to be κ-noncollapsing at the point (q',t') on the scale $r_0>0$ [18] if $\forall 0<r\le r_0$,

$$\mathrm{Vol}_{\overline{g}(t')}(B_{\overline{g}(t')}(q',r))\ge\kappa r^n$$

holds whenever

$$|\overline{\mathrm{Rm}}|(q,t)\le r^{-2}\quad\forall d_{\overline{g}(t')}(q',q)<r,t'-r^2\le t\le t'$$

holds where $B_{\overline{g}(t')}(q',r)$ is the geodesic ball of radius r in M around the point q' with respect to the metric $\overline{g}(t')$. A Ricci flow $(M,\overline{g})$ is said to be an ancient κ-solution if it is a solution of the Ricci flow in $M\times(-\infty,0]$ such that for each $t\le 0$ the metric $\overline{g}(t)$ is not a flat metric, $(M,\overline{g}(t))$ is a complete manifold of nonnegative and uniformly bounded curvature, and $(M,\overline{g}(t))$ is κ-noncollapsing on all scales at all points of $M\times(-\infty,0]$.

We will assume that M is complete with respect to $\overline{g}(t)$ for any $0<t<T$ for the rest of the paper. Unless stated otherwise we will fix the point (p_0,t_0) and consider the $\mathcal{L}_p(q,\tau)$, $l_p(q,\tau)$, etc. all with respect to this fixed reference point. We also associate the product manifold $M\times(0,t_0)$ with the product metric $g\,dx^2\oplus d\tau^2$.

1. Existence of $\mathcal{L}_p(q,\overline{\tau})$-geodesic minimizer

In this section we will use the technique of [18] to prove the first variation formula for $\mathcal{L}_p(q,\gamma,\overline{\tau})$ for any curve $\gamma : [0,\overline{\tau}] \to M$ joining p_0 and q with $\gamma(0) = p_0$ and $\gamma(\overline{\tau}) = q$. We will prove the non-trivial fact that the $L_p(q,\overline{\tau})$ length can be realized by some $\mathcal{L}_p$-geodesic on M. We will let $\langle\cdot,\cdot\rangle_{g(\tau)}$ be the inner product with respect to the metric $g(\tau)$. When there is no ambiguity, we will write $\langle\cdot,\cdot\rangle$ for $\langle\cdot,\cdot\rangle_{g(\tau)}$.

Lemma 1.1. *Let $\gamma \in \mathcal{F}(q,\overline{\tau})$ and let $Y : [0,\overline{\tau}] \to TM$ be a vector field along γ with $Y(0) = 0$. Suppose γ is differentiable on $[0,\overline{\tau}]$. Then*

$$\begin{aligned}\delta_Y\mathcal{L}_p(q,\gamma,\overline{\tau}) = {}& 2\overline{\tau}^p\langle X(\overline{\tau}),Y(\overline{\tau})\rangle\\ &+\int_0^{\overline{\tau}}\tau^p\left\langle Y,\nabla R-\frac{2p}{\tau}X-2\nabla_XX-4Ric(X,\cdot)\right\rangle d\tau\end{aligned} \tag{1.1}$$

where $X = X(\tau) = \gamma'(\tau)$ and the inner product in the integral is evaluated at τ.

Proof. Let $f : [0,\overline{\tau}]\times(-\varepsilon,\varepsilon) \to M$ be a variation of γ with respect to the vector field Y such that $f(0,z) = p_0$ for all $z \in (-\varepsilon,\varepsilon)$. Then

$$\begin{aligned}&\frac{d}{dz}\mathcal{L}_p(f(\overline{\tau},z),f(\cdot,z),\overline{\tau})\\ =&\frac{d}{dz}\int_0^{\overline{\tau}}\tau^p(R(f(\tau,z),\tau)+|\nabla_\tau f|^2)\,d\tau\\ =&\int_0^{\overline{\tau}}\tau^p(\langle\nabla_zf,\nabla R\rangle+2\langle\nabla_\tau f,\nabla_z\nabla_\tau f\rangle)\,d\tau\\ =&\int_0^{\overline{\tau}}\tau^p(\langle\nabla_zf,\nabla R\rangle+2\langle\nabla_\tau f,\nabla_\tau\nabla_z f\rangle)\,d\tau \qquad (1.2)\\ =&\int_0^{\overline{\tau}}\tau^p\bigg(\langle\nabla_zf,\nabla R\rangle+2\frac{d}{d\tau}\langle\nabla_\tau f,\nabla_zf\rangle-2\langle\nabla_\tau\nabla_\tau f,\nabla_zf\rangle\\ &\qquad -4\mathrm{Ric}(\nabla_\tau f,\nabla_zf)\bigg)\,d\tau\\ =&2\overline{\tau}^p\langle\nabla_\tau f(\overline{\tau},z),\nabla_zf(\overline{\tau},z)\rangle\\ &+\int_0^{\overline{\tau}}\tau^p\left\langle\nabla_zf,\nabla R-\frac{2p}{\tau}\nabla_\tau f-2\nabla_\tau\nabla_\tau f-4\mathrm{Ric}(\nabla_\tau f,\cdot)\right\rangle d\tau. \qquad (1.3)\end{aligned}$$

By putting $z = 0$ in (1.3), (1.1) follows. □

Let $s_0 = t_0^{1-p}$. For any $0 \le \overline{s} < s_0$, $p_0 \in M$, $\widetilde{\gamma} \in \mathcal{F}^{p_0}(q,\overline{s})$, $0 \le s \le \overline{s}$, let $\widetilde{R}(q,s) = R(q,s^{\frac{1}{1-p}})$,

$$\widetilde{\mathcal{L}}_p^{p_0}(q,\widetilde{\gamma},\overline{s}) = \frac{1}{1-p}\int_0^{\overline{s}}(s^{\frac{2p}{1-p}}\widetilde{R}(\widetilde{\gamma}(s),s)+(1-p)^2|\widetilde{\gamma}'(s)|^2)\,ds$$

where $|\widetilde{\gamma}'(s)| = |\widetilde{\gamma}'(s)|_{g(s^{1/(1-p)})}$ and let

$$\widetilde{L}_p^{p_0}(q,\overline{s}) = \inf_{\widetilde{\gamma}\in\mathcal{F}^{p_0}(q,\overline{s})} \widetilde{\mathcal{L}}_p^{p_0}(q,\widetilde{\gamma},\overline{s}).$$

Then by direct computation,

$$\mathcal{L}_p^{p_0}(q,\gamma,\overline{\tau}) = \widetilde{\mathcal{L}}_p^{p_0}(q,\widetilde{\gamma},\overline{\tau}^{1-p}) \quad \forall \gamma \in \mathcal{F}^{p_0}(q,\overline{\tau}) \tag{1.4}$$

where

$$\widetilde{\gamma}(s) = \gamma(\tau), s = \tau^{1-p}. \tag{1.5}$$

Hence

$$L_p^{p_0}(q,\overline{\tau}) = \widetilde{L}_p^{p_0}(q,\overline{\tau}^{1-p}). \tag{1.6}$$

We will now let $\widetilde{g}(q,s) = g(q,\tau)$, $\widetilde{R}(q,s) = R(q,\tau)$, $\widetilde{\mathrm{Ric}}(q,s) = \mathrm{Ric}(q,\tau)$, and $\widetilde{\Gamma}_{ij}^r(q,s) = \Gamma_{ij}^r(q,s)$ where $s = \tau^{1-p}$ for the rest of the paper. When there is no ambiguity, we will drop the superscript p_0.

Lemma 1.2. *Let $\widetilde{\gamma} \in \mathcal{F}(q,\overline{s})$ and let $\widetilde{Y} : [0,\overline{s}] \to TM$ be a vector field along $\widetilde{\gamma}$ with $\widetilde{Y}(0) = 0$. Suppose $\widetilde{\gamma}$ is differentiable on $[0,\overline{s}]$. Then*

$$\begin{aligned}
&\delta_{\widetilde{Y}}\widetilde{\mathcal{L}}_p(q,\widetilde{\gamma},\overline{s})\\
&= \frac{1}{1-p}\int_0^{\overline{s}} (s^{\frac{2p}{1-p}}\widetilde{Y}(\widetilde{R}) + 2(1-p)^2\langle \widetilde{X}, \nabla_{\widetilde{X}}\widetilde{Y}\rangle)\,ds\\
&= 2(1-p)\langle \widetilde{X}(\overline{s}), \widetilde{Y}(\overline{s})\rangle\\
&\quad + \frac{1}{1-p}\int_0^{\overline{s}} \langle \widetilde{Y}, s^{\frac{2p}{1-p}}\nabla\widetilde{R} - 2(1-p)^2\nabla_{\widetilde{X}}\widetilde{X} - 4(1-p)s^{\frac{p}{1-p}}\widetilde{Ric}(\widetilde{X},\cdot)\rangle\,ds
\end{aligned} \tag{1.7}$$

where $\widetilde{X}(s) = \widetilde{\gamma}'(s)$, $\widetilde{R}(s) = \widetilde{R}(\widetilde{\gamma}(s),s)$, $\widetilde{Ric}(s) = \widetilde{Ric}(\widetilde{\gamma}(s),s)$ and $\nabla = \nabla^{g(\tau)}$ with $s = \tau^{1-p}$.

Proof. Let $\widetilde{f} : [0,\overline{s}] \times (-\varepsilon,\varepsilon) \to M$ be a variation of $\widetilde{\gamma}$ with respect to the vector field $\widetilde{Y}$ such that $\widetilde{f}(0,z) = p_0$ for all $z \in (-\varepsilon,\varepsilon)$. Since

$$\frac{d}{ds}\widetilde{g}_{ij}(q,s) = \frac{d}{d\tau}g_{ij}(q,\tau)\cdot\frac{d\tau}{ds} = \frac{2}{1-p}s^{\frac{p}{1-p}}\widetilde{R}_{ij}(q,s) \tag{1.8}$$

where $s = \tau^{1-p}$, we have

$$
\begin{aligned}
&\frac{d}{dz}\widetilde{\mathcal{L}}_p(\widetilde{f}(\overline{s},z),\widetilde{f}(\cdot,z),\overline{s})\\
&=\frac{1}{1-p}\int_0^{\overline{s}}(s^{\frac{2p}{1-p}}\langle\nabla_z\widetilde{f},\nabla\widetilde{R}\rangle+2(1-p)^2\langle\nabla_s\widetilde{f},\nabla_z\nabla_s\widetilde{f}\rangle)\,ds\\
&=\frac{1}{1-p}\int_0^{\overline{s}}(s^{\frac{2p}{1-p}}\langle\nabla_z\widetilde{f},\nabla\widetilde{R}\rangle+2(1-p)^2\langle\nabla_s\widetilde{f},\nabla_s\nabla_z\widetilde{f}\rangle)\,ds\\
&=\frac{1}{1-p}\int_0^{\overline{s}}\left\{s^{\frac{2p}{1-p}}\langle\nabla_z\widetilde{f},\nabla\widetilde{R}\rangle+2(1-p)^2\left[\frac{d}{ds}\langle\nabla_s\widetilde{f},\nabla_z\widetilde{f}\rangle\right.\right.\\
&\left.\left.-\langle\nabla_s\nabla_s\widetilde{f},\nabla_z\widetilde{f}\rangle-\frac{2}{1-p}s^{\frac{p}{1-p}}\widetilde{\mathrm{Ric}}(\nabla_s\widetilde{f},\nabla_z\widetilde{f})\right]\right\}ds\\
&=2(1-p)\langle\nabla_s\widetilde{f}(\overline{s},z),\nabla_z\widetilde{f}(\overline{s},z)\rangle\\
&+\frac{1}{1-p}\int_0^{\overline{s}}\langle\nabla_z\widetilde{f},s^{\frac{2p}{1-p}}\nabla\widetilde{R}-2(1-p)^2\nabla_s\nabla_s\widetilde{f}-4(1-p)s^{\frac{p}{1-p}}\widetilde{\mathrm{Ric}}(\nabla_s\widetilde{f},\cdot)\rangle\,ds.
\end{aligned}
\tag{1.9}
$$

Putting $z=0$ in (1.9) we get (1.7) and the lemma follows. $\square$

From Lemma 1.1 and Lemma 1.2 it is natural to define the following. We say that a curve $\widetilde{\gamma}\in\mathcal{F}(q,\overline{s})$ is a $\widetilde{\mathcal{L}}_p$-geodesic at $s\in(0,\overline{s})$ if it satisfies

$$
\nabla_{\widetilde{X}}\widetilde{X}-\frac{1}{2(1-p)^2}s^{\frac{2p}{1-p}}\nabla\widetilde{R}+\frac{2}{1-p}s^{\frac{p}{1-p}}\widetilde{\mathrm{Ric}}(\widetilde{X},\cdot)=0 \tag{1.10}
$$

at s where $\widetilde{X}(s)=\widetilde{\gamma}'(s)$ and $\nabla=\nabla^{g(\tau)}$ with $\tau=s^{\frac{1}{1-p}}$. We say that it is a $\widetilde{\mathcal{L}}_p$-geodesic in $(0,\overline{s})$ if it satisfies (1.10) in $(0,\overline{s})$. Similarly we say a curve $\gamma\in\mathcal{F}(q,\overline{\tau})$ is a $\mathcal{L}_p$-geodesic at $\tau\in(0,\overline{\tau})$ if it satisfies

$$
\nabla_X X-\frac{1}{2}\nabla R+\frac{p}{\tau}X+2\mathrm{Ric}(X,\cdot)=0 \tag{1.11}
$$

at τ where $X(\tau)=\gamma'(\tau)$. We say that γ is a $\mathcal{L}_p$-geodesic in $(0,\overline{\tau})$ if it satisfies (1.11) in $(0,\overline{\tau})$. Note that when $p=1/2$, the $\mathcal{L}_p$-geodesic is equal to the $\mathcal{L}$-geodesic defined by Perelman in [18]. By direct computation we have the following lemma.

Lemma 1.3. *$\gamma\in\mathcal{F}(q,\overline{\tau})$ is a $\mathcal{L}_p$-geodesic at $\tau\in(0,\overline{\tau})$ if and only if $\widetilde{\gamma}\in\mathcal{F}(q,\overline{s})$ is a $\widetilde{\mathcal{L}}_p$-geodesic at $s\in(0,\overline{s})$ where γ, $\widetilde{\gamma}$, s and τ are related by* (1.5) *and $\overline{s}=\overline{\tau}^{1-p}$. Moreover*

$$
\widetilde{\gamma}'(0)=\frac{1}{1-p}\lim_{\tau\to 0}\tau^p\gamma'(\tau). \tag{1.12}
$$

Lemma 1.4. *For any $\widetilde{v}\in T_{p_0}M$, there exists a unique solution $\widetilde{\gamma}(s)=\widetilde{\gamma}_{\widetilde{v}}(s)=\widetilde{\gamma}(s;\widetilde{v})$ of* (1.10) *in $(0,s_0)$ with*

$$
\begin{cases}\widetilde{\gamma}(0)=p_0\\ \widetilde{\gamma}'(0)=\widetilde{v}\end{cases} \tag{1.13}
$$

for some constant $s_0 \in (0, t_0^{1-p}]$ *where* $(0, s_0)$ *is the maximal interval of existence of the solution. If* $s_0 < t_0^{1-p}$*, then*

$$\lim_{s \to s_0^-} d_0(p_0, \widetilde{\gamma}(s)) = \infty. \tag{1.14}$$

If the Ricci curvature of M *is uniformly bounded on* $(0, t_0]$*, then* $s_0 = t_0^{1-p}$.

Proof. Uniqueness of solution of (1.10) satisfying (1.13) follows by standard O.D.E. theory. Hence we only need to prove existence of solution of (1.10) satisfying (1.13). We will use a continuity argument similar to that of section 17 of [16] to prove the existence of solution of (1.10) satisfying (1.13). We first observe that by standard O.D.E. theory there exists a constant $s_0' \in (0, t_0^{1-p})$ such that (1.10), (1.13), has a unique solution $\widetilde{\gamma}(s)$ in $(0, s_0')$. Let $(0, s_0)$ be the maximum interval of existence of solution $\widetilde{\gamma}(s)$ of (1.10) and (1.13). Then $s_0 \le t_0^{1-p}$. If $s_0 = t_0^{1-p}$, we are done. So we suppose that $s_0 < t_0^{1-p}$. We claim that (1.14) holds. Suppose not. Then there exist constants $s_1 \in (0, s_0)$ and $C_1 > 0$ such that

$$d_0(p_0, \widetilde{\gamma}(s)) \le C_1 \quad \forall s_1 \le s < s_0. \tag{1.15}$$

Let

$$r_0 = \sup_{0 \le s < s_0} d_0(p_0, \widetilde{\gamma}(s)).$$

By (1.15) $r_0 < \infty$. Since $\overline{B_0(p_0, r_0)} \times [0, s_0^{1/(1-p)}]$ is compact in $(q, \tau) \in M \times [0, t_0)$ when M is equipped with the metric $g(0)$, there exists a constant $K_1 > 0$ such that

$$|R| + |\nabla R| + |\text{Ric}| \le K_1 \tag{1.16}$$

on $(q, \tau) \in \overline{B_0(p_0, r_0)} \times [0, s_0^{1/(1-p)}]$. Then by (1.10) and (1.16),

$$\begin{aligned}
\left|\frac{d}{ds}|\widetilde{X}|^2\right| &= \left|2\langle \widetilde{X}, \nabla_{\widetilde{X}} \widetilde{X}\rangle + \frac{2}{1-p} s^{\frac{p}{1-p}} \widetilde{\text{Ric}}(\widetilde{X}, \widetilde{X})\right| \\
&= \left|\left\langle \widetilde{X}, \frac{1}{(1-p)^2} s^{\frac{2p}{1-p}} \nabla \widetilde{R} - \frac{4}{1-p} s^{\frac{p}{1-p}} \widetilde{\text{Ric}}(\widetilde{X}, \cdot)\right\rangle + \frac{2}{1-p} s^{\frac{p}{1-p}} \widetilde{\text{Ric}}(\widetilde{X}, \widetilde{X})\right| \\
&\le \left|\frac{1}{(1-p)^2} s^{\frac{2p}{1-p}} \langle \widetilde{X}, \nabla \widetilde{R}\rangle\right| + \frac{2}{1-p} s^{\frac{p}{1-p}} \left|\widetilde{\text{Ric}}(\widetilde{X}, \widetilde{X})\right| \\
&\le A_1 K_1 (s^{\frac{2p}{1-p}} |\widetilde{X}| + s^{\frac{p}{1-p}} |\widetilde{X}|^2) \\
&\le C_1 (2 s^{\frac{p}{1-p}} |\widetilde{X}|^2 + (s^{\frac{3p}{1-p}}/4)) \quad \forall 0 \le s \le s_0.
\end{aligned}$$

where $A_1 = \max((1-p)^{-2}, 2(1-p)^{-1})$ and $C_1 = A_1K_1$. Hence $\forall 0 \le s \le s_0$,

$$\begin{cases} \dfrac{d}{ds}\left(e^{-2C_1(1-p)s^{\frac{1}{1-p}}}|\widetilde{X}|^2\right) \le (C_1/4)e^{-2C_1(1-p)s^{\frac{1}{1-p}}}s^{\frac{3p}{1-p}} \\ \dfrac{d}{ds}\left(e^{2C_1(1-p)s^{\frac{1}{1-p}}}|\widetilde{X}|^2\right) \ge -(C_1/4)e^{2C_1(1-p)s^{\frac{1}{1-p}}}s^{\frac{3p}{1-p}} \end{cases}$$

(1.17)

$$\Rightarrow \quad e^{-C_2s^{\frac{1}{1-p}}}(|\widetilde{v}|^2 - C_3e^{C_2s^{\frac{1}{1-p}}}s^{\frac{1+2p}{1-p}}) \le |\widetilde{X}(s)|^2 \le e^{C_2s^{\frac{1}{1-p}}}(|\widetilde{v}|^2 + C_3s^{\frac{1+2p}{1-p}})$$

where $C_2 = 2(1-p)C_1$ and $C_3 = C_2/(8(1+2p))$. By (1.17) and standard O.D.E. theory there exists a constant $\varepsilon_0 \in (0, t_0^{1-p} - s_0]$ such that we can extend $\widetilde{\gamma}$ to a solution of (1.10), (1.13), on $(0, s_0 + \varepsilon_0)$. This contradicts the maximality of s_0. Thus (1.14) holds.

If the Ricci curvature of M is uniformly bounded on $M \times (0, t_0]$, then by the local estimates for the solutions of Ricci flow [20] and a similar argument as before we will get a contradiction if $s_0 < t_0^{1-p}$. Hence $s_0 = t_0^{1-p}$ and the lemma follows. □

By Lemma 1.3, Lemma 1.4, and (1.12), we have

Corollary 1.1. *For any $v \in T_{p_0}M$, there exists a unique solution $\gamma(\tau) = \gamma_v(\tau) = \gamma(\tau; v)$ of (1.11) in $(0, \tau_0)$ with*

$$\begin{cases} \gamma(0) = p_0 \\ \lim_{\tau \to 0} \tau^p \gamma'(\tau) = v. \end{cases} \tag{1.18}$$

for some constant $\tau_0 \in (0, t_0^{1-p}]$ where $(0, \tau_0)$ is the maximal interval of existence of the solution. If $\tau_0 < t_0^{1-p}$, then

$$\lim_{\tau \to \tau_0^-} d_0(p_0, \gamma(\tau)) = \infty. \tag{1.19}$$

If the Ricci curvature of M is uniformly bounded on $[0, t_0)$, then $\tau_0 = t_0$.

We will now prove that the $L_p(q, \overline{\tau})$-length can be realized by some $\mathcal{L}_p(q, \overline{\tau})$-geodesic in M. We first recall a lemma of [25]:

Lemma 1.5 (Lemma 2.1 of [25]). *If there exists a constant $c_1 > 0$ such that*

$$Ric(q, \tau) \ge -c_1 g(q, \tau) \qquad \text{on } M \times [0, \overline{\tau}], \tag{1.20}$$

then

$$e^{-2c_1\tau}g(0) \le g(\tau) \le e^{2c_1(\overline{\tau}-\tau)}g(\overline{\tau}) \quad \text{on } M \times [0, \overline{\tau}].$$

If there exists a constant $c_2 > 0$ such that

$$Ric(q, \tau) \le c_2 g(q, \tau) \quad \text{on } M \times [0, \overline{\tau}], \tag{1.21}$$

then

$$e^{2c_2(\tau-\overline{\tau})}g(\overline{\tau}) \le g(\tau) \le e^{2c_2\tau}g(0) \quad \text{on } M \times [0, \overline{\tau}].$$

Lemma 1.6. *Suppose there exists a constant $c_1 > 0$ such that* (1.20) *holds. Then for any $\gamma \in \mathcal{F}(q,\overline{\tau})$,*

(1.22)
$$\mathcal{L}_p(q,\gamma,\overline{\tau}) \ge -\frac{c_1 n}{p+1}\overline{\tau}^{p+1} + \frac{(1-p)e^{-2c_1\overline{\tau}}}{\tau_2^{1-p}-\tau_1^{1-p}} d_0(\gamma(\tau_1),\gamma(\tau_2))^2 \quad \forall 0 \le \tau_1 \le \tau_2 \le \overline{\tau}.$$

Proof. By Lemma 1.5 and the Hölder inequality,

$$\begin{aligned}\left(e^{-c_1\overline{\tau}}\int_{\tau_1}^{\tau_2} |\gamma'(\tau)|_{g(0)}\,d\tau\right)^2 &\le \left(\int_{\tau_1}^{\tau_2} |\gamma'(\tau)|d\tau\right)^2\\ &\le \int_{\tau_1}^{\tau_2} \tau^p|\gamma'(\tau)|^2 d\tau \cdot \int_{\tau_1}^{\tau_2} \tau^{-p}d\tau\\ &= \frac{\tau_2^{1-p}-\tau_1^{1-p}}{1-p}\int_0^{\overline{\tau}} \tau^p|\gamma'(\tau)|^2 d\tau \quad \forall 0 \le \tau_1 \le \tau_2 \le \overline{\tau}. \qquad (1.23)\end{aligned}$$

By (1.20),

$$\int_0^{\overline{\tau}} \tau^p R(\gamma(\tau),\tau)d\tau \ge -\frac{c_1 n}{p+1}\overline{\tau}^{p+1}. \qquad (1.24)$$

By (1.23) and (1.24) the lemma follows. □

Lemma 1.7. *Let $r_0 > 0$. Suppose there exists a constant $c_2 > 0$ such that* (1.21) *holds in $B_0(p_0,r_0)\times[0,\overline{\tau}]$. Then*

$$L_p(q,\tau) \le \frac{c_2 n}{p+1}\tau^{p+1} + \frac{e^{2c_2\tau}}{p+1}\frac{d_0(p_0,q)^2}{\tau^{1-p}} \quad \forall q \in B_0(p_0,r_0), 0 < \tau \le \overline{\tau}. \qquad (1.25)$$

Proof. Let $q \in B_0(p_0,r_0)$, $\tau \in (0,\overline{\tau}]$, and let $\gamma : [0,\tau] \to M$ be a minimizing geodesic joining p_0 and q with respect to the metric $g(0)$ with $|\gamma'|_{g(0)} = d_0(p_0,q)/\tau$ on $[0,\tau]$. Then by Lemma 1.5,

$$\begin{aligned}L_p(q,\tau) \le \mathcal{L}_p(q,\gamma,\tau) &\le \frac{c_2 n}{p+1}\tau^{p+1} + e^{2c_2\tau}\int_0^\tau \rho^p|\gamma'(\rho)|_{g(0)}^2 d\rho\\ &\le \frac{c_2 n}{p+1}\tau^{p+1} + \frac{e^{2c_2\tau}}{p+1}\frac{d_0(p_0,q)^2}{\tau^{1-p}}.\end{aligned}$$

and (1.25) follows. □

Lemma 1.8. *Let $\widetilde{\gamma} : [0,\overline{s}] \to M$ be a continuous curve satisfying*

$$\int_0^{\overline{s}} |\widetilde{\gamma}'(s)|^2\,ds < \infty \qquad (1.26)$$

where $|\widetilde{\gamma}'(s)| = |\widetilde{\gamma}'(s)|_{\widetilde{g}(s)}$ and let $\widetilde{Y}(s) \not\equiv 0$ be a smooth vector field along $\widetilde{\gamma}$. Then there exists a variation $f : [0,\overline{s}]\times[-\varepsilon,\varepsilon] \to M$ of $\widetilde{\gamma}$ with respect to $\widetilde{Y}(s)$ and a constant $C > 0$ such that

$$\left|\frac{\partial f}{\partial s}(s,z)\right|^2 + \left|\nabla_z\left(\frac{\partial f}{\partial s}\right)(s,z)\right|^2 \le C(|\widetilde{\gamma}'(s)|^2 + |\nabla_{\widetilde{X}}\widetilde{Y}(s)|^2) \qquad (1.27)$$

for any $|z| \le \varepsilon$ *and a.e.* $s \in (0, \overline{s})$ *where* $\widetilde{X}(s) = \widetilde{\gamma}'(s)$ *and*

$$\lim_{z \to 0} \int_0^{\overline{s}} \left\langle \frac{\partial f}{\partial s}(s,z), \nabla_z \left(\frac{\partial f}{\partial s} \right)(s,z) \right\rangle ds = \int_0^{\overline{s}} \langle \widetilde{X}, \nabla_{\widetilde{X}} \widetilde{Y} \rangle \, ds. \tag{1.28}$$

Proof. For any $s \in [0, \overline{s}]$, let $\beta(z,s) = \beta(z, \widetilde{\gamma}(s), \widetilde{Y}(s))$ be the geodesic with respect to the metric $\widetilde{g}(s)$ which satisfies

$$\begin{cases} \beta(0,s) = \widetilde{\gamma}(s) \\ \dfrac{\partial \beta}{\partial z}(0,s) = \widetilde{Y}(s). \end{cases} \tag{1.29}$$

Let $f(s,z) = \beta(z, \widetilde{\gamma}(s), \widetilde{Y}(s))$. By the same argument as the proof of Proposition 2.2 of Chapter 9 of [1] there exists a constant $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ f is well defined on $[0, \overline{s}] \times [-\varepsilon, \varepsilon]$ and is a variation of $\widetilde{\gamma}$ with respect to $\widetilde{Y}(s)$. We claim that there exists a constant $0 < \varepsilon < \varepsilon_0$ to be determined later such that f satisfies (1.27) and (1.28). Since $f([0, \overline{s}] \times [-\varepsilon, \varepsilon])$ is compact, there exists a finite family of co-ordinate charts $\{(U_i, \phi_i)\}_{i=1}^{i_0}$ such that $f([0, \overline{s}] \times [-\varepsilon, \varepsilon]) \subset \cup_{i=1}^{i_0} U_i$. Without loss of generality we may assume that $f([0, \overline{s}] \times [-\varepsilon, \varepsilon]) \subset U_1$. We write $\beta = (\beta^1, \beta^2, \dots, \beta^n)$, $f = (f^1, f^2, \dots, f^n)$, $\widetilde{\gamma}(s) = (a^1(s), a^2(s), \dots, a^n(s))$, and $\widetilde{Y}(s) = b^i(s) \partial / \partial x_i$ in the local coordinates (U_1, ϕ_1). Then

$$\frac{\partial^2 \beta^k}{\partial z^2} + \frac{\partial \beta^i}{\partial z} \frac{\partial \beta^j}{\partial z} \widetilde{\Gamma}_{ij}^k (\beta(z,s), s) = 0 \quad \forall |z| \le \varepsilon, 0 \le s \le \overline{s}, k = 1, 2, \dots, n. \tag{1.30}$$

Let

$$E(z,s) = \widetilde{g}_{kl}(\beta(z,s), s) \frac{\partial \beta^k}{\partial z} \frac{\partial \beta^l}{\partial z}.$$

By (1.29),

$$E(0,s) = |\widetilde{Y}(s)|^2 \le \max_{0 \le s \le \overline{s}} |\widetilde{Y}(s)|^2 = C_1 \quad \text{(say)} \tag{1.31}$$

By (1.30),

$$\begin{aligned} \left| \frac{\partial E}{\partial z} \right| &= \left| 2 \widetilde{g}_{kl} \frac{\partial \beta^l}{\partial z} \frac{\partial^2 \beta^k}{\partial z^2} + \frac{\partial \widetilde{g}_{kl}}{\partial x_m} \frac{\partial \beta^m}{\partial z} \frac{\partial \beta^k}{\partial z} \frac{\partial \beta^l}{\partial z} \right| \\ &\le 2 \left| \widetilde{g}_{kl} \frac{\partial \beta^l}{\partial z} \frac{\partial \beta^i}{\partial z} \frac{\partial \beta^j}{\partial z} \widetilde{\Gamma}_{ij}^k (\beta(z,s), s) \right| + \left| \frac{\partial \widetilde{g}_{kl}}{\partial x_m} \frac{\partial \beta^m}{\partial z} \frac{\partial \beta^k}{\partial z} \frac{\partial \beta^l}{\partial z} \right| \\ &\le C_2^2 E^{\frac{3}{2}} \qquad \qquad \forall |z| \le \varepsilon, 0 \le s \le \overline{s} \end{aligned} \tag{1.32}$$

for some constant $C_2 > 0$. Let $\varepsilon = \min(1/(C_1^{\frac{1}{2}} C_2^2), \varepsilon_0)$. Then by (1.31) and (1.32),

$$\left|\frac{1}{\sqrt{E(0,s)}}-\frac{1}{\sqrt{E(z,s)}}\right|\le\frac{C_2^2}{2}|z|\le\frac{C_2^2}{2}\varepsilon \qquad \forall|z|\le\varepsilon, 0\le s\le\overline{s}$$

$$\Rightarrow\ \frac{1}{\sqrt{E(z,s)}}\ge\frac{1}{\sqrt{E(0,s)}}-\frac{C_2^2}{2}\varepsilon\ge\frac{1}{\sqrt{C_1}}-\frac{C_2^2}{2}\varepsilon\ge\frac{1}{2\sqrt{C_1}}\ \forall|z|\le\varepsilon, 0\le s\le\overline{s}$$

(1.33)

$$\Rightarrow\ E(z,s)\le 4C_1 \qquad \forall|z|\le\varepsilon, 0\le s\le\overline{s}.$$

Let

$$w(z,s)=\frac{\partial\beta}{\partial s}$$

and let

$$F(z,s)=|w|^2+|\nabla_z w|^2.$$

By (1.26),

$$\frac{da^i}{ds}\in L^2(0,\overline{s})\subset L^1(0,\overline{s})\quad\forall i=1,2,\dots,n.$$

By standard theory on analysis for any $i=1,2,\dots,n$, there exists a set $E_i\subset[0,\overline{s}]$ of measure zero such that da^i/ds is continuous on $(0,\overline{s})\setminus E_i$. Let $E_0=\cup_{i=1}^n E_i$ and $A_0=(0,\overline{s})\setminus E_0$. Then $|E_0|=0$ and da^i/ds is continuous on A_0 for all $i=1,2,\dots,n$.

We write $\nabla_{\widetilde{X}}\widetilde{Y}(s)=c^i(s)\partial/\partial x_i$ in local coordinates. Then

$$c^k(s)=\frac{db^k}{ds}+b^j\frac{da^i}{ds}\widetilde{\Gamma}_{ij}^k(\widetilde{\gamma}(s),s)\quad\forall 0\le s\le\overline{s}, k=1,2,\dots,n. \tag{1.34}$$

By (1.26) and (1.34) $\nabla_{\widetilde{X}}\widetilde{Y}(s)\in L^2(0,\overline{s})$ and $\nabla_{\widetilde{X}}\widetilde{Y}(s)$ is continuous on A_0. Differentiating (1.30) and (1.29) with respect to $s\in A_0$,

(1.35)

$$0=\nabla_s\nabla_z\frac{\partial\beta}{\partial z}=\nabla_z\nabla_s\frac{\partial\beta}{\partial z}+\widetilde{R}\left(\frac{\partial\beta}{\partial s},\frac{\partial\beta}{\partial z}\right)\frac{\partial\beta}{\partial z}=\nabla_z\nabla_z w+\widetilde{R}\left(w,\frac{\partial\beta}{\partial z}\right)\frac{\partial\beta}{\partial z}$$

holds for any $|z|\le\varepsilon, s\in A_0, k=1,2,\dots,n$ and

$$\begin{cases} w(0,s)=\widetilde{\gamma}'(s) & \forall s\in A_0\\ \nabla_z w(0,s)=\nabla_{\widetilde{X}}\widetilde{Y}(s) & \forall s\in A_0.\end{cases} \tag{1.36}$$

Then by (1.33), (1.35), (1.36), and Hölder's inequality,

$$\begin{aligned}\left|\frac{\partial F}{\partial z}\right| &= 2\left|\langle w,\nabla_z w\rangle+\langle\nabla_z w,\nabla_z\nabla_z w\rangle\right|\\ &\le F+2\left|\langle\nabla_z w,\widetilde{R}\left(w,\frac{\partial\beta}{\partial z}\right)\frac{\partial\beta}{\partial z}\rangle\right|\\ &\le F+|\nabla_z w|^2+\left|\widetilde{R}\left(w,\frac{\partial\beta}{\partial z}\right)\frac{\partial\beta}{\partial z}\right|^2\\ &\le C_3F\quad\forall|z|\le\varepsilon, s\in A_0\\ \Rightarrow\quad F(z,s)&\le e^{C_3z}F(0,s)\le e^{C_3\varepsilon}(|\widetilde{\gamma}'(s)|^2+|\nabla_{\widetilde{X}}\widetilde{Y}|^2)\end{aligned}$$

holds for any $|z| \le \varepsilon$ and $s \in A_0$ where $C_3 > 0$ is a constant. Hence (1.27) follows. By (1.27) and the Cauchy-Schwarz inequality,

$$\left|\left\langle \frac{\partial f}{\partial s}(s,z), \nabla_z\left(\frac{\partial f}{\partial s}\right)(s,z)\right\rangle\right| \le C(|\widetilde{\gamma}'(s)|^2 + |\nabla_{\widetilde{X}}\widetilde{Y}|^2) \quad \forall |z| \le \varepsilon, s \in A_0. \tag{1.37}$$

Since both $\gamma'(s)$ and $\nabla_{\widetilde{X}}\widetilde{Y}$ are continuous at s for any $s \in A_0$, by (1.35), (1.36), and the continuous dependence of solutions of O.D.E. on initial data, $\nabla_z w$ is continuous at (z, s) for any $s \in A_0$ and $|z| \le \varepsilon$. Hence

$$\nabla_z\left(\frac{\partial f}{\partial s}\right)(s,z) = \nabla_s\left(\frac{\partial f}{\partial z}\right)(s,z) \quad \forall |z| \le \varepsilon, s \in A_0. \tag{1.38}$$

Since by (1.38)

$$\left\langle \frac{\partial f}{\partial s}(s,z), \nabla_z\left(\frac{\partial f}{\partial s}\right)(s,z)\right\rangle \to \langle \widetilde{X}, \nabla_{\widetilde{X}}\widetilde{Y}\rangle \quad \text{as } z \to 0 \quad \forall s \in A_0,$$

by (1.37) and the Lebesgue dominated convergence theorem (1.28) follows. □

Theorem 1.1. *Let $\overline{\tau} \in (0, t_0)$ and $\overline{s} = \overline{\tau}^{1-p}$. Suppose (M, g) satisfies (1.20) for some constant $c_1 > 0$. Then for any $q \in M$, there exists a $\widetilde{\mathcal{L}}_p$-geodesic $\widetilde{\gamma} \in C^1([0, \overline{s}]) \cap C^\infty((0, \overline{s}])$ such that $\widetilde{\gamma}(0) = p_0$, $\widetilde{\gamma}(\overline{s}) = q$, and*

$$\widetilde{L}_p(q, \overline{s}) = \widetilde{\mathcal{L}}_p(q, \widetilde{\gamma}, \overline{s}). \tag{1.39}$$

Proof. Choose a sequence of curves $\{\widetilde{\gamma}_i\}_{i=1}^\infty \subset \mathcal{F}(q, \overline{s})$ such that

$$\widetilde{\mathcal{L}}_p(q, \widetilde{\gamma}_i, \overline{s}) \le \widetilde{L}_p(q, \overline{s}) + 1 \quad \forall i \in \mathbb{Z}^+ \tag{1.40}$$

and

$$\widetilde{L}_p(q, \overline{s}) = \lim_{i \to \infty} \widetilde{\mathcal{L}}_p(q, \widetilde{\gamma}_i, \overline{s}). \tag{1.41}$$

Let $\gamma_i(\tau) = \widetilde{\gamma}_i(s)$, $\tau = s^{1/(1-p)}$. By Lemma 1.6, (1.4), (1.6) and (1.41) there exist constants $K = K(\overline{\tau}, L_p(q, \overline{\tau})) > 0$ and $C_1 > 0$ independent of $i \in \mathbb{Z}^+$ such that

$$\begin{cases} d_0(p_0, \widetilde{\gamma}_i(s)) \le K \quad \forall 0 \le s \le \overline{s}, i \in \mathbb{Z}^+ \\ d_0(\widetilde{\gamma}_i(s), \widetilde{\gamma}_i(s')) \le C_1|s - s'|^{1/2} \quad \forall s, s' \in [0, \overline{s}], i \in \mathbb{Z}^+. \end{cases} \tag{1.42}$$

Hence the sequence of curves $\{\widetilde{\gamma}_i\}_{i=1}^\infty$ are uniformly Hölder continuous on $[0, \overline{s}]$. Since M is complete with respect to $g(0)$, $\overline{B_0(p_0, K)}$ is compact. By the Ascoli Theorem there exists a continuous curve $\widetilde{\gamma} : [0, \overline{s}] \to \overline{B_0(p_0, K)}$ such that $\widetilde{\gamma}_i$ converges uniformly to $\widetilde{\gamma}$ on $[0, \overline{s}]$ as $i \to \infty$. Then $\widetilde{\gamma}(0) = p_0$ and $\widetilde{\gamma}(\overline{s}) = q$. Letting $i \to \infty$ in (1.42),

$$\begin{cases} d_0(p_0, \widetilde{\gamma}(s)) \le K & \forall 0 \le s \le \overline{s} \\ d_0(\widetilde{\gamma}(s), \widetilde{\gamma}(s')) \le C_1|s - s'|^{1/2} & \forall s, s' \in [0, \overline{s}]. \end{cases}$$

Hence $\widetilde{\gamma}$ is uniformly Hölder continuous on $[0,\overline{s}]$. By Fatou's Lemma and Lemma 1.5,

$$\begin{aligned} &\widetilde{\mathcal{L}}_p(q,\widetilde{\gamma},\overline{s}) \le \lim_{i\to\infty} \widetilde{\mathcal{L}}_p(q,\widetilde{\gamma}_i,\overline{s}) = \widetilde{L}_p(q,\overline{s}) \\ \Rightarrow\quad &\widetilde{L}_p(q,\overline{s}) = \widetilde{\mathcal{L}}_p(q,\widetilde{\gamma},\overline{s}) \quad \text{and} \quad |\widetilde{\gamma}'|_{g(0)} \in L^2(0,\overline{s}). \end{aligned} \tag{1.43}$$

We now claim that $\widetilde{\gamma} \in C^\infty([0,\overline{s}])$. Since $\overline{B_0(p_0,K)}$ is compact, there exists a finite family of co-ordinate charts $\{(\phi_k, B_0(q_k,r_k)\}_{k=1}^{k_0}$ such that $\overline{B_0(p_0,K)} \subset \cup_{k=1}^{k_0} B_0(q_k,r_k)$. Let

$$I_k = I_k(\widetilde{\gamma}) = \{s \in [0,\overline{s}] : \widetilde{\gamma}(s) \in B_0(q_k,r_k)\} \quad \forall k = 1,2,\dots,k_0.$$

Then I_k is relatively open with respect to the interval $[0,\overline{s}]$ for all $k = 1,2,\dots,k_0$ and $[0,\overline{s}] = \cup_{k=1}^{k_0} I_k$. For any $k = 1,2,\dots,k_0$, $s \in I_k$, we write

$$\phi_k(\widetilde{\gamma}(s)) = (a_k^1(s), a_k^2(s),\dots,a_k^n(s))$$

in the local coordinates $(\phi_k, B_0(q_k,r_k))$. When there is no ambiguity, we will drop the subscript k. To prove the claim we fix one $k \in \{1,2,\dots,k_0\}$. Then

$$\widetilde{X}(s) = \widetilde{\gamma}'(s) = \frac{da^i}{ds}\frac{\partial}{\partial x_i} \quad \text{in } I_k.$$

By (1.43),

$$\widetilde{g}_{ij}(\widetilde{\gamma}(s),0)\frac{da^i}{ds}\frac{da^j}{ds} \in L^1(I_k) \quad \Rightarrow \quad \frac{da^i}{ds} \in L^2(I_k) \quad \forall i = 1,2,\dots,n. \tag{1.44}$$

Let

$$\widetilde{Y}(s) = b^j(s)\frac{\partial}{\partial x_j}$$

be a smooth vector field along $\widetilde{\gamma}$ such that $\widetilde{Y}(s) = 0$ for any $s \notin I_k$. Since $\widetilde{\gamma}$ is a minimizer of $\widetilde{L}_p(q,\overline{s})$ and by (1.43) $\widetilde{\gamma}$ satisfies (1.26), by Lemma 1.8, Lebesgue dominated convergence theorem and an argument similar to the proof of Lemma 1.2,

$$\begin{aligned} &\int_0^{\overline{s}} (s^{\frac{2p}{1-p}}\widetilde{Y}(\widetilde{R}) + 2(1-p)^2\langle \widetilde{X}, \nabla_{\widetilde{X}}\widetilde{Y}\rangle)\,ds = 0 \\ \Rightarrow\quad &\int_{I_k}\left\{s^{\frac{2p}{1-p}} b^j \frac{\partial \widetilde{R}}{\partial x^j} + 2(1-p)^2 \widetilde{g}_{lr}\frac{da^l}{ds}\left(\frac{db^r}{ds} + b^j\frac{da^i}{ds}\widetilde{\Gamma}_{ij}^r\right)\right\} ds = 0 \end{aligned} \tag{1.45}$$

$$\begin{aligned} \Rightarrow\quad &\left|\int_{I_k}\widetilde{g}_{lr}\frac{da^l}{ds}\left(\frac{db^r}{ds} + b^j\frac{da^i}{ds}\widetilde{\Gamma}_{ij}^r\right) ds\right| \le C\left|\int_{I_k} s^{\frac{2p}{1-p}} b^j\frac{\partial\widetilde{R}}{\partial x^j}\,ds\right| \\ \Rightarrow\quad &\left|\int_{I_k}\widetilde{g}_{lr}\frac{da^l}{ds}\frac{db^r}{ds}\,ds\right| \le \left|\int_{I_k}\widetilde{g}_{lr}\frac{da^l}{ds}b^j\frac{da^i}{ds}\widetilde{\Gamma}_{ij}^r\,ds\right| + C\sum_{j=1}^n\int_{I_k}|b^j|\,ds. \end{aligned} \tag{1.46}$$

Since by (1.44) and the Cauchy-Schwarz inequality,

$$\left|\int_{I_k}\widetilde{g}_{lr}\frac{da^l}{ds}b^j\frac{da^i}{ds}\widetilde{\Gamma}_{ij}^r\,ds\right|\le C\max_{1\le j\le n}\|b^j\|_\infty\int_J\widetilde{g}_{lr}\frac{da^l}{ds}\frac{da^r}{ds}\,ds$$

where $J=\cup_{j=1}^n\text{supp }b^j$, by (1.46) we have

$$\left|\int_{I_k}\widetilde{g}_{lr}\frac{da^l}{ds}\frac{db^r}{ds}\,ds\right|\le C\max_{1\le j\le n}\|b^j\|_\infty\int_J\widetilde{g}_{lr}\frac{da^l}{ds}\frac{da^r}{ds}\,ds+C\sum_{j=1}^n\int_{I_k}|b^j|\,ds. \tag{1.47}$$

We now choose $\phi\in C^\infty(\mathbb{R})$, $0\le\phi\le 1$, such that $\phi(s)=1$ for all $s\le 0$ and $\phi(s)=0$ for all $s\ge 1$. For any $h>0$ let $\phi_h(s)=\phi(s/h)$. Since

$$\int_{\mathbb{R}}\phi'(s)\,ds=-1,$$

by (1.44) and standard theory in analysis [22] there exists a set $E_k\subset I_k$ of measure zero such that

$$\begin{cases}\displaystyle\lim_{h\to 0^+}\frac{1}{h}\int_{s'-h}^{s'}\frac{da^1}{ds}(s)\phi'((s'-s)/h)\,ds=-\frac{da^1}{ds}(s')\quad\forall s'\in I_k\setminus E_k\\ \displaystyle\lim_{h\to 0^+}\frac{1}{h}\int_{s'}^{s'+h}\frac{da^1}{ds}(s)\phi'((s-s')/h)\,ds=-\frac{da^1}{ds}(s')\quad\forall s'\in I_k\setminus E_k.\end{cases} \tag{1.48}$$

Let $s_0'\in I_k\setminus E_k$. Without loss of generality we may assume that $s_0'\ne 0,\overline{s}$. By continuity of $\widetilde{\gamma}$ there exists a constant $0<\varepsilon\le s_0'/2$ such that

$$\widetilde{\gamma}(s)\in B_0(q_k,r_k)\quad\forall s\in I_\varepsilon(s_0')=(s_0'-\varepsilon,s_0'+\varepsilon)\subset I_k.$$

Let $s_1,s_2\in I_{\varepsilon/2}(s_0')\setminus E_k$ be such that $s_1<s_2$ and let $\varepsilon_1=(s_2-s_1)/3$. Putting

$$b^r(s)=\widetilde{g}^{r1}(\widetilde{\gamma}(s),s)\phi_{h_2}(s-s_2)\phi_{h_1}(s_1-s)\quad\forall r=1,2,\dots,n$$

in (1.47) where $0<h_1,h_2\le\varepsilon_1$ we get

$$
\begin{aligned}
&\left|\frac{1}{h_2}\int_{s_2}^{s_2+h_2}\frac{da^1}{ds}(s)\phi'((s-s_2)/h_2)\,ds-\frac{1}{h_1}\int_{s_1-h_1}^{s_1}\frac{da^1}{ds}(s)\phi'((s_1-s)/h_1)\,ds\right|\\
&\le C\int_{s_1-h_1}^{s_2+h_2}\widetilde{g}_{lr}\frac{da^l}{ds}\frac{da^r}{ds}\,ds+C(s_2-s_1+h_1+h_2)\\
&\quad+\left|\int_{s_1-h_1}^{s_2+h_2}\widetilde{g}_{lr}\frac{d\widetilde{g}^{r1}}{ds}\frac{da^l}{ds}\phi_{h_2}(s-s_2)\phi_{h_1}(s_1-s)\,ds\right|\\
&\le C\int_{s_1-h_1}^{s_2+h_2}\widetilde{g}_{lr}\frac{da^l}{ds}\frac{da^r}{ds}\,ds+C(s_2-s_1+h_1+h_2)\\
&\quad+\left|\int_{s_1-h_1}^{s_2+h_2}\widetilde{g}_{lr}\left(\frac{\partial\widetilde{g}^{r1}}{\partial s}+\frac{\partial\widetilde{g}^{r1}}{\partial x_j}\frac{da^j}{ds}\right)\frac{da^l}{ds}\phi_{h_2}(s-s_2)\phi_{h_1}(s_1-s)\,ds\right|\\
&\le C\int_{s_1-h_1}^{s_2+h_2}\widetilde{g}_{lr}\frac{da^l}{ds}\frac{da^r}{ds}\,ds+C(s_2-s_1+h_1+h_2)\\
&\quad+\frac{2}{1-p}\left|\int_{s_1-h_1}^{s_2+h_2}s^{\frac{p}{1-p}}\widetilde{g}_{rl}\widetilde{g}^{ri}\widetilde{R}_{ij}\widetilde{g}^{j1}\frac{da^l}{ds}\phi_{h_2}(s-s_2)\phi_{h_1}(s_1-s)\,ds\right|\quad\text{(by (1.8))}\\
&\le C\int_{s_1-h_1}^{s_2+h_2}\widetilde{g}_{lr}\frac{da^l}{ds}\frac{da^r}{ds}\,ds+C(s_2-s_1+h_1+h_2)\\
&\quad+C(s_2-s_1+h_1+h_2)^{1/2}\left(\int_{s_1-h_1}^{s_2+h_2}\widetilde{g}_{lr}\frac{da^1}{ds}\frac{da^r}{ds}\,ds\right)^{1/2}
\end{aligned}
$$

(1.49)

$$
\le C\int_{s_1-h_1}^{s_2+h_2}\widetilde{g}_{lr}\frac{da^l}{ds}\frac{da^r}{ds}\,ds+C(s_2-s_1+h_1+h_2).
$$

Letting $h_1\to 0$ in (1.49), by (1.26), (1.44), and (1.48),

$$
\left|\frac{da^1}{ds}(s_1)\right|\le C\quad\forall s_1\in I_{\varepsilon/2}(s_0')\setminus E_k. \tag{1.50}
$$

for some constant $C>0$. Hence letting $h_1,h_2\to 0$ in (1.49), by (1.26), (1.48), and (1.50) we have

$$
\left|\frac{da^1}{ds}(s_2)-\frac{da^1}{ds}(s_1)\right|\le C|s_2-s_1|\quad\forall s_1,s_2\in I_{\varepsilon/2}(s_0')\setminus E_k. \tag{1.51}
$$

We now choose $0\le\eta\in C_0^\infty(\mathbb{R})$ such that $\eta(s)=0$ for any $|s|\ge 1$ and $\int_{\mathbb{R}}\eta\,ds=1$. For any $h>0$, $s_0\in I_k$, let $\eta_h(s)=\eta(s/h)/h$ and

$$
a^1*\eta_h(s_0)=\int_{\mathbb{R}}a^1(s_0-s)\eta_h(s)\,ds.
$$

For any $s_0\in I_{\varepsilon/2}(s_0')$, we choose a sequence $\{s_{0,i}\}_{i=1}^\infty\subset I_{\varepsilon/2}(s_0')\setminus E_k$ such that $\lim_{i\to\infty}s_{0,i}=s_0$. By (1.51) $\{da^1(s_{0,i})/ds\}_{i=1}^\infty$ is a Cauchy sequence. Hence

$$
\lim_{i\to\infty}\frac{da^1}{ds}(s_{0,i})
$$

exists. Let

$$f(s_0)=\lim_{i\to\infty}\frac{da^1}{ds}(s_{0,i})\quad\forall s_0\in I_\varepsilon(s_0').$$

By (1.51) f is well defined on $I_{\varepsilon/2}(s_0')$. We now claim that $a^1\in C^1(I_{\varepsilon/2}(s_0'))$ with $da^1/ds=f$ on $I_{\varepsilon/2}(s_0')$. To prove the claim we observe that by (1.51)

$$\begin{aligned}&\left|\frac{d}{ds}a^1*\eta_h(s_0)-f(s_0)\right|\\ &\quad=\left|\int_{|\rho|\le h}\frac{da^1}{ds}(s_0-\rho)\eta_h(\rho)\,d\rho-f(s_0)\right|\\ &\quad\le\left|\int_{|\rho|\le h}\left(\frac{da^1}{ds}(s_0-\rho)-\frac{da^1}{ds}(s_{0,i})\right)\eta_h(\rho)\,d\rho\right|+\left|\frac{da^1}{ds}(s_{0,i})-f(s_0)\right|\\ &\quad\le C\int_{|\rho|\le h}|s_0-\rho-s_{0,i}|\eta_h(\rho)\,d\rho+\left|\frac{da^1}{ds}(s_{0,i})-f(s_0)\right|\\ &\quad\le C(|s_0-s_{0,i}|+h)+\left|\frac{da^1}{ds}(s_{0,i})-f(s_0)\right|.\end{aligned}$$

Letting first $i\to\infty$ and then $h\to0$ in the above inequality, we get that $d(a^1*\eta_h)/ds$ converges uniformly to f on $I_{\varepsilon/2}(s_0')$ as $h\to0$. Since $a^1*\eta_h$ converges uniformly to a^1 on $I_{\varepsilon/2}(s_0')$ as $h\to0$. Hence $a^1\in C^1(I_\varepsilon(s_0'))$ with $da^1/ds=f$ on $I_{\varepsilon/2}(s_0')$. Since s_0' is arbitrary, $a^1\in C^1([0,\overline{s}])$. By a similar argument $a^l\in C^1([0,\overline{s}])$ for any $l=1,2,\dots,n$. Hence $\widetilde{\gamma}\in C^1([0,\overline{s}])$ and there exists a constant $C>0$ such that

(1.52)
$$\left|\frac{da_k^l}{ds}(s_2)-\frac{da_k^l}{ds}(s_1)\right|\le C|s_2-s_1|\quad\forall s_1,s_2\in I_k,k=1,2,\dots,k_0,l=1,2,\dots,n.$$

Thus $d^2a_k^l(s)/ds^2$ exists a.e. $s\in I_k$ with

$$\frac{d^2a_k^l}{ds^2}\in L^\infty(I_k)\quad\forall k=1,2,\dots,k_0,l=1,2,\dots,n.$$

By (1.45) for any $b^j\in C_0^\infty(I_k)$, $j=1,2,\dots,n$,

$$\int_{I_k}\left\{s^{2p/(1-p)}\frac{\partial\widetilde{R}}{\partial x_j}+2(1-p)^2\left(-\frac{d}{ds}\left(\widetilde{g}_{jl}\frac{da^l}{ds}\right)+\widetilde{g}_{lr}\frac{da^l}{ds}\frac{da^i}{ds}\widetilde{\Gamma}_{ij}^r\right)\right\}b^j\,ds=0$$

(1.53)
$$\Rightarrow 2(1-p)^2\left(-\frac{d}{ds}\left(\widetilde{g}_{jl}\frac{da^l}{ds}\right)+\widetilde{g}_{lr}\frac{da^l}{ds}\frac{da^i}{ds}\widetilde{\Gamma}_{ij}^r\right)+s^{2p/(1-p)}\frac{\partial\widetilde{R}}{\partial x_j}=0$$

for a.e. $s\in I_k$. Hence

$$2(1-p)^2\left(-\widetilde{g}_{jl}\frac{d^2a^l}{ds^2}-\frac{d\widetilde{g}_{jl}}{ds}\frac{da^l}{ds}+\widetilde{g}_{lr}\frac{da^l}{ds}\frac{da^i}{ds}\widetilde{\Gamma}_{ij}^r\right)+s^{2p/(1-p)}\frac{\partial\widetilde{R}}{\partial x_j}=0$$

for a.e. $s \in I_k$. Thus

$$\frac{d^2 a^m}{ds^2} = -\widetilde{g}^{mj}\frac{d\widetilde{g}_{jl}}{ds}\frac{da^l}{ds} + \widetilde{g}_{lr}\widetilde{g}^{mj}\frac{da^l}{ds}\frac{da^i}{ds}\widetilde{\Gamma}^r_{ij} + \frac{1}{(1-p)^2}s^{2p/(1-p)}\widetilde{g}^{mj}\frac{\partial \widetilde{R}}{\partial x_j} \tag{1.54}$$

a.e. $s \in I_k$. By (1.52) and (1.54) there exists a set E_k' of measure zero such that

$$\left|\frac{d^2 a_k^j}{ds^2}(s_2) - \frac{d^2 a_k^j}{ds^2}(s_1)\right| \le C|s_2 - s_1| \tag{1.55}$$

for any $s_1, s_2 \in I_\varepsilon(s_0')\backslash E_k'$, $k = 1, 2, \dots, k_0$ and $j = 1, 2, \dots, n$. By an argument similar to the proof of $\widetilde{\gamma} \in C^1([0, \overline{s}])$ but with (1.55) replacing (1.51) in the proof we get that $\widetilde{\gamma} \in C^2([0, \overline{s}])$ and

$$\left|\frac{d^2 a_k^j}{ds^2}(s_2) - \frac{d^2 a_k^j}{ds^2}(s_1)\right| \le C|s_2 - s_1| \quad \forall s_1, s_2 \in I_k, k = 1, 2, \dots, k_0, j = 1, 2, \dots, n$$

for some constant $C > 0$. Hence by (1.53),

$$2(1-p)^2\left(-\frac{d}{ds}\left(\widetilde{g}_{jl}\frac{da^l}{ds}\right) + \widetilde{g}_{lr}\frac{da^l}{ds}\frac{da^i}{ds}\widetilde{\Gamma}^r_{ij}\right) + s^{2p/(1-p)}\frac{\partial \widetilde{R}}{\partial x_j} = 0 \quad \text{in } I_k$$

for any $k = 1, 2, \dots, k_0, j = 1, 2, \dots, n$. Thus $\widetilde{\gamma}$ satisfies (1.10) in $(0, \overline{s})$. Then by standard O.D.E. theory $\widetilde{\gamma} \in C^\infty((0, \overline{s}])$. Hence $\widetilde{\gamma}$ is a $\widetilde{\mathcal{L}}_p$-geodesic and the theorem follows. □

By (1.4), (1.5), (1.6) and Theorem 1.1 we have

Theorem 1.2. *Suppose (M, g) satisfies (1.20) in $[0, \overline{\tau}]$ for some constant $c_1 > 0$. Then for any $q \in M$, there exists a $\mathcal{L}_p$-geodesic $\gamma \in C([0, \overline{\tau}]) \cap C^\infty((0, \overline{\tau}])$ satisfying (1.18) for some $v \in T_{p_0}M$ such that $\gamma(\overline{\tau}) = q$ and*

$$L_p(q, \overline{\tau}) = \mathcal{L}_p(q, \gamma, \overline{\tau}).$$

By putting $p = 1/2$ in Theorem 1.2 we obtain a result which is mentioned and used without proof in Perelman's paper [18], [19].

Corollary 1.2. *Suppose (M, g) satisfies (1.20) in $[0, \overline{\tau}]$ for some constant $c_1 > 0$. Then for any $q \in M$, there exists a $\mathcal{L}$-geodesic $\gamma \in C([0, \overline{\tau}]) \cap C^\infty((0, \overline{\tau}])$ satisfying (1.18) with $p = 1/2$ for some $v \in T_{p_0}M$ such that $\gamma(\overline{\tau}) = q$ and*

$$L(q, \overline{\tau}) = \mathcal{L}(q, \gamma, \overline{\tau}).$$

By an argument similar to the proof of Theorem 1.1 and Theorem 1.2 we have

Theorem 1.3. *Suppose (M, g) satisfies (1.20) in $[0, \overline{\tau}]$ for some constant $c_1 > 0$. Let $q_0 \in M$ and let $\tau_0 \in (0, \overline{\tau})$. Then for any $q \in M$, there exists a $\mathcal{L}_p$-geodesic $\gamma \in C^\infty([\tau_0, \overline{\tau}])$ satisfying $\gamma(\tau_0) = q_0$, $\gamma(\overline{\tau}) = q$, and*

$$L_p^{q_0,\tau_0}(q, \overline{\tau}) = \mathcal{L}_p^{q_0,\tau_0}(q, \gamma, \overline{\tau}).$$

Theorem 1.4. *Let $t_0 > 0$, $s_0 = t_0^{1-p}$, and let g and $\overline{g}$ be related by (0.5). Suppose $(M, \overline{g})$ satisfies (1.20) in $(0, t_0)$ for some constant $c_1 > 0$. Then for any $\overline{s} \in (0, s_0)$, $q \in M$, $\widetilde{L}_p^{p_0}(q, \overline{s})$ is locally Lipschitz in p_0 with respect to the metric $g(0) = \overline{g}(t_0)$.*

Proof. Let $r_0 > 0$, $\overline{s} \in (0, s_0)$, $\overline{\tau} = \overline{s}^{\frac{1}{1-p}}$, $\overline{p}_0 \in M$, and let $p_1, p_2 \in B_0(\overline{p}_0, r_0)$. By Theorem 1.1 for each $i = 1, 2$, there exists a $\widetilde{\mathcal{L}}_p^{p_i}(q, \overline{s})$-length minimizing $\widetilde{\mathcal{L}}_p$-geodesic $\widetilde{\gamma}_i : [0, \overline{s}] \to M$ such that $\widetilde{\gamma}_i(0) = p_i$ and $\widetilde{\gamma}_i(\overline{s}) = q$. Let $\widetilde{\gamma} : [0, d] \to M$ be a normalized minimizing geodesic with respect to the metric $g(0)$ with $\widetilde{\gamma}(0) = p_1$, $\widetilde{\gamma}(d) = p_2$, $|\widetilde{\gamma}'| = |\widetilde{\gamma}'|_{g(0)} = 1$ on $[0, d]$ with $d = d_0(p_1, p_2)$. Then $\widetilde{\gamma}([0, d]) \subset B_0(\overline{p}_0, 3r_0)$. Let $r_1 = 3r_0 + 2d_0(\overline{p}_0, q)$ and let

$$K_0 = \sup_{B_0(\overline{p}_0, r_1) \times [0, \overline{\tau}]} (|R| + |\mathrm{Ric}|).$$

Let $\gamma_i(\tau) = \widetilde{\gamma}_i(s)$ with $s = \tau^{1-p}$, $i = 1, 2$. For $i = 1, 2$, let $d_i = d_0(p_i, q)$ and let $\overline{\gamma}_i : [0, d_i] \to M$ be a normalized minimizing geodesic with respect to the metric $g(0)$ with $\overline{\gamma}_i(0) = p_i$, $\overline{\gamma}_i(d_i) = q$. Then $d_i < r_0 + d_0(\overline{p}_0, q)$ and $\overline{\gamma}_i([0, d_i]) \subset B_0(p_0, r_1)$ for $i = 1, 2$. Hence by Lemma 1.6 and the proof of Lemma 1.7, there exist constants $A_1 = A_1(\overline{s}, r_1, K_0) > 0$ and $r_2 = r_2(\overline{s}, r_1, K_0) \ge r_1$ such that

$$\begin{cases} \widetilde{L}_p^{p_i}(q, \overline{s}) \le A_1 & \forall i = 1, 2 \\ d_0(p_i, \widetilde{\gamma}_i(s)) < r_2 & \forall 0 \le s \le \overline{s}, i = 1, 2. \end{cases}$$

Let

$$K_1 = \sup_{B_0(p_0, 2r_2) \times [0, \overline{\tau}]} (|R| + |R_t| + |\nabla R| + |\mathrm{Ric}|).$$

We now assume that $d = d_0(p_1, p_2) < \min(1, \overline{s}/4)$. Let

$$\beta(s) = \begin{cases} \widetilde{\gamma}(s) & \text{if } 0 \le s < d \\ \widetilde{\gamma}_2(s - d) & \text{if } d \le s \le \overline{s} - d \\ \widetilde{\gamma}_2(2s - \overline{s}) & \text{if } \overline{s} - d < s \le \overline{s}. \end{cases}$$

Then

$$
\begin{aligned}
\widetilde{L}_p^{p_1}(q,\overline{s}) &\le \widetilde{\mathcal{L}}_p^{p_1}(q,\beta,\overline{s})\\
&= \frac{1}{1-p}\int_0^d (s^{\frac{2p}{1-p}}\widetilde{R}(\widetilde{\gamma}(s),s)+(1-p)^2|\widetilde{\gamma}'(s)|^2)\,ds\\
&\quad+\frac{1}{1-p}\int_d^{\overline{s}-d}(s^{\frac{2p}{1-p}}\widetilde{R}(\widetilde{\gamma}_2(s'),s)+(1-p)^2|\widetilde{\gamma}_2'(s')|^2_{\widetilde{g}(s)})\,ds\\
&\quad+\frac{1}{1-p}\int_{\overline{s}-d}^{\overline{s}}(s^{\frac{2p}{1-p}}\widetilde{R}(\widetilde{\gamma}_2(s''),s)+4(1-p)^2|\widetilde{\gamma}_2'(s'')|^2_{\widetilde{g}(s)})\,ds\\
&= I_1+I_2+I_3 \qquad (1.56)
\end{aligned}
$$

where $s'=s-d$ and $s''=2s-\overline{s}$. By Lemma 1.5,

(1.57)

$$
I_1\le\frac{K_1}{1+p}d^{\frac{1+p}{1-p}}+(1-p)e^{2K_1d^{\frac{1}{1-p}}}d\le\frac{K_1}{1+p}d^{\frac{1+p}{1-p}}+(1-p)e^{2K_1}d,
$$

$$
\begin{aligned}
I_2&\le\frac{1}{1-p}\int_0^{\overline{s}-2d}((w+d)^{\frac{2p}{1-p}}\widetilde{R}(\widetilde{\gamma}_2(w),w+d)+(1-p)^2e^{2K_1d^{\frac{1}{1-p}}}|\widetilde{\gamma}_2'(w)|^2)\,dw\\
&=\frac{1}{1-p}\int_0^{\overline{s}-2d}(w^{\frac{2p}{1-p}}\widetilde{R}(\widetilde{\gamma}_2(w),w)+(1-p)^2|\widetilde{\gamma}_2'(w)|^2)\,dw\\
&\quad+\frac{1}{1-p}\int_0^{\overline{s}-2d}[(w+d)^{\frac{2p}{1-p}}-w^{\frac{2p}{1-p}}]\widetilde{R}(\widetilde{\gamma}_2(w),w+d)\,dw\\
&\quad+\frac{1}{1-p}\int_0^{\overline{s}-2d}w^{\frac{2p}{1-p}}(\widetilde{R}(\widetilde{\gamma}_2(w),w+d)-\widetilde{R}(\widetilde{\gamma}_2(w),w))\,dw\\
&\quad+(1-p)(e^{2K_1d^{\frac{1}{1-p}}}-1)\int_0^{\overline{s}-2d}|\widetilde{\gamma}_2'(w)|^2\,dw\\
&\le\frac{1}{1-p}\int_0^{\overline{s}-2d}(w^{\frac{2p}{1-p}}\widetilde{R}(\widetilde{\gamma}_2(w),w)+(1-p)^2|\widetilde{\gamma}_2'(w)|^2)\,dw+C_1'd \qquad (1.58)
\end{aligned}
$$

for some constant $C_1'>0$ and

$$
\begin{aligned}
I_3&\le\frac{1}{1-p}K_1\overline{s}^{\frac{2p}{1-p}}d+2(1-p)e^{2K_1d^{\frac{1}{1-p}}}\int_{\overline{s}-2d}^{\overline{s}}|\widetilde{\gamma}_2'(s)|^2\,ds\\
&\le\frac{1}{1-p}\int_{\overline{s}-2d}^{\overline{s}}(s^{\frac{2p}{1-p}}\widetilde{R}(\widetilde{\gamma}_2(s),s)+(1-p)^2|\widetilde{\gamma}_2'(s)|^2)\,ds+C_1''\int_{\overline{s}-2d}^{\overline{s}}|\widetilde{\gamma}_2'(s)|^2\,ds\\
&\quad+\frac{3K_1}{1-p}\overline{s}^{\frac{2p}{1-p}}d. \qquad (1.59)
\end{aligned}
$$

Hence by (1.56), (1.57), (1.58) and (1.59),

$$\widetilde{L}_p^{p_1}(q,\overline{s}) \le \widetilde{L}_p^{p_2}(q,\overline{s}) + C_1''\int_{\overline{s}-2d}^{\overline{s}} |\widetilde{\gamma}_2'(s)|^2\,ds + C_2'd \tag{1.60}$$

for some constant $C_2' > 0$. Let $\widetilde{v}_i = \widetilde{\gamma}_i'(0)$ for $i = 1,2$. By the same argument as the proof of Lemma 1.4, (1.17) holds in $(0,\overline{s})$ for some constant $C_2 > 0$, $C_3 > 0$, depending only on K_1 with $\widetilde{X}(s)$ and $\widetilde{v}$, being replaced by $\widetilde{\gamma}_i'(s)$ and $\widetilde{v}_i = \widetilde{\gamma}_i'(0)$, for $i = 1,2$. By (1.17) there exist constants $C_4 > 0$, $C_5 > 0$ and $C_6 > 0$ such that

$$C_4|\widetilde{v}_i|^2 - C_5 \le |\widetilde{\gamma}_i'(s)|^2 \le C_6(1+|\widetilde{v}_i|^2) \quad \forall 0 \le s \le \overline{s}, i = 1,2. \tag{1.61}$$

By (1.61),

$$\begin{aligned}
C_4\overline{s}|\widetilde{v}_2|^2 &= C_4\int_0^{\overline{s}} |\widetilde{v}_2|^2\,ds \le C_5\overline{s} + \int_0^{\overline{s}} |\widetilde{\gamma}_2'(s)|^2\,ds\\
&\le C_5\overline{s} + \frac{1}{1-p}\left[\widetilde{L}_p^{p_2}(q,\overline{s}) - \frac{1}{1-p}\int_0^{\overline{s}} s^{\frac{2p}{1-p}}\widetilde{R}(\widetilde{\gamma}_2(s),s)\,ds\right]\\
&\le C_5\overline{s} + \frac{1}{1-p}\left(\widetilde{L}_p^{p_2}(q,\overline{s}) + \frac{K_1}{1+p}\overline{s}^{\frac{1+p}{1-p}}\right)\\
&\le C_5\overline{s} + +\frac{1}{1-p}\left(A_1 + \frac{K_1}{1+p}\overline{s}^{\frac{1+p}{1-p}}\right) = C_7 \quad \text{(say)}.
\end{aligned} \tag{1.62}$$

By (1.61) and (1.62),

$$\int_{\overline{s}-2d}^{\overline{s}} |\gamma_2'(s)|^2\,ds \le 2C_6(1+|\widetilde{v}_2|^2)d \le 2C_6(1+(C_7/(C_4\overline{s}))d. \tag{1.63}$$

By (1.60) and (1.63) there exists a constant $C_8 = C_8 > 0$ such that

$$\widetilde{L}_p^{p_1}(q,\overline{s}) \le \widetilde{L}_p^{p_2}(q,\overline{s}) + C_8 d.$$

Interchanging the role of p_1 and p_2 in the above inequality,

$$\widetilde{L}_p^{p_2}(q,\overline{s}) \le \widetilde{L}_p^{p_1}(q,\overline{s}) + C_8 d.$$

Hence

$$|\widetilde{L}_p^{p_1}(q,\overline{s}) - \widetilde{L}_p^{p_2}(q,\overline{s})| \le C_8 d_0(p_1,p_2)$$

for any $p_1, p_2 \in B_0(\overline{p}_0, r_0)$ and $d_0(p_1,p_2) > \min(1,\overline{s}/4)$ and the theorem follows. □

By (1.6) and Theorem 1.4 we have the following theorem.

Theorem 1.5. *Let $t_0 > 0$ and let g and $\overline{g}$ be related by (0.5). Suppose $(M,\overline{g})$ satisfies (1.20) in $(0,t_0)$ for some constant $c_1 > 0$. Then for any $\overline{\tau} \in (0,t_0)$, $q \in M$, $L_p^{p_0}(q,\overline{\tau})$ is locally Lipschitz in p_0 with respect to the metric $g(0) = \overline{g}(t_0)$.*

2. Properties of the $\mathcal{L}_p$-exponential map and $\mathcal{L}_p$ cut locus

In this section we will generalize the $\mathcal{L}$-exponential map of Perelman [18] and define the $\mathcal{L}_p$-exponential map corresponding to the $\mathcal{L}_p$-geodesic curve. We will also derive some elementary properties of the $\mathcal{L}_p$-exponential map.

We first start will a definition. Let $\overline{\tau} > 0$. For any $v \in T_{p_0}M$ let $\widetilde{v} = v/(1-p)$. By Lemma 1.4 there exists a unique solution $\widetilde{\gamma}_{\widetilde{v}}(s) = \widetilde{\gamma}(s;\widetilde{v})$ of (1.10), (1.13), in $(0,s_0)$ for some $s_0 > 0$. Let $\gamma_v(\tau) = \gamma(\tau;v) = \widetilde{\gamma}(s;\widetilde{v})$ where s and τ are related by (1.5). Then γ_v is the unique solution of (1.11), (1.18), in $(0,\tau_0)$ where $\tau_0 = s_0^{1/(1-p)}$. Similar to [25] for any $\overline{\tau} > 0$ we let

$$U_p(\overline{\tau}) = \{v \in T_{p_0}M : \gamma_v \text{ exists on } (0,\tau_0) \text{ for some } \tau_0 > \overline{\tau}\}.$$

Then $U_p(\tau_2) \subset U_p(\tau_1)$ for any $0 < \tau_1 < \tau_2 < t_0$. We define the $\mathcal{L}_p$-exponential map $\mathcal{L}_p\text{-}\exp_{p_0}^{\overline{\tau}} : U_p(\overline{\tau}) \to M$ by

$$\mathcal{L}_p\text{-}\exp_{p_0}^{\overline{\tau}}(v) = \gamma_v(\overline{\tau}) = \widetilde{\gamma}_{\widetilde{v}}(\overline{\tau}^{1-p}).$$

By O.D.E. theory and the equivalence of the O.D.E. (1.10), (1.13), and (1.11), (1.18), through the transformation (1.5), $U_p(\overline{\tau})$ is open in $T_{p_0}M$. Note that by Corollary 1.1 if (M,g) has uniformly bounded Ricci curvature on $M \times (0,t_0)$, then $U_p(\overline{\tau}) = T_{p_0}M$ for any $0 \le \overline{\tau} < t_0$.

Let $q_0 \in M$, $\tau_0 \in (0,t_0)$ and $v \in T_{q_0}M$. By an argument similar to the proof of Lemma 1.4 and Corollary 1.1 there exists a unique solution $\gamma_{\tau_0,v}^{q_0}(\tau) = \gamma_{\tau_0}^{q_0}(\tau;v)$ of (1.11) in (τ_0,τ_1) for some $\tau_1 > \tau_0$ such that

$$\begin{cases} \gamma_{\tau_0}^{q_0}(\tau_0;v) = q_0 \\ \tau_0^p \gamma_{\tau_0}^{q_0}{}'(\tau_0;v) = v. \end{cases}$$

For any $\overline{\tau} > \tau_0$, let

$$U_{\tau_0,p}^{q_0}(\overline{\tau}) = \{v \in T_{q_0}M : \gamma_{\tau_0,v}^{q_0} \text{ exists on } (\tau_0,\tau_1) \text{ for some } \tau_1 > \overline{\tau}\}.$$

We define the $\mathcal{L}_{\tau_0,p}^{q_0}$-exponential map $\mathcal{L}_{\tau_0,p}^{q_0}\text{-}\exp^{\overline{\tau}} : U_{\tau_0,p}^{q_0}(\overline{\tau}) \to M$ by

$$\mathcal{L}_{\tau_0,p}^{q_0}\text{-}\exp^{\overline{\tau}}(v) = \gamma_{\tau_0}^{q_0}(\overline{\tau};v).$$

Lemma 2.1. *Suppose (M,g) satisfies (1.20) in $[0,t_0)$ for some constant $c_1 > 0$. Then for any $r_0 > 0$ and $m_0 > 0$, there exists a constant $s_1 \in (0,t_0^{1-p})$ such that for any $\widetilde{v} \in T_{p_0}M$ satisfying $|\widetilde{v}|_{g(p_0,0)} \le m_0$ there exists a unique $\widetilde{\mathcal{L}}_p$-geodesic $\widetilde{\gamma} = \widetilde{\gamma}_{\widetilde{v}} : [0,s_1] \to M$ satisfying (1.13) and*

$$\widetilde{\gamma}(s) \in B_0(p_0,r_0) \quad \forall 0 \le s \le s_1. \tag{2.1}$$

Hence $\mathcal{B}(0,(1-p)m_0) \subset U_p(\tau)$ for any $0 < \tau \le \tau_1$ where $\tau_1 = s_1^{\frac{1}{1-p}}$ and

$$\bigcup_{0<\tau<t_0} U_p(\tau) = T_{p_0}M.$$

Proof. We will use an argument similar to the proof of Proposition 2.5 of [25] to prove the lemma. Let $\widetilde{v} \in T_{p_0}M$ satisfy $|\widetilde{v}|_{g(p_0,0)} \le m_0$. Since M is complete, $\overline{B_0(p_0,r_0)}$ is compact. Then there exists a constant $K_1 > 0$ such that (1.16) holds for any $(q,\tau) \in \overline{B_0(p_0,r_0)} \times [0,t_0/2]$. Let $C_2 = C_2(K_1) > 0$ and $C_3 > 0$ be as in the proof of Lemma 1.4. Let

$$s_1' = \min(1, (t_0/2)^{1-p}, e^{-c_1-(C_2/2)}(m_0^2 + C_3)^{-1/2} r_0/2)$$

and $s_1 = s_1'/2$. By Lemma 1.4 there exists a maximal interval $[0,\overline{s})$ such that there exists a unique $\widetilde{\mathcal{L}}_p$-geodesic $\widetilde{\gamma} : [0,\overline{s}) \to M$ which satisfies (1.13). We claim that $\overline{s} \ge s_1'$. Suppose not. Then $\overline{s} < s_1'$. Let

$$s_0 = \sup\{s' \le \overline{s} : \widetilde{\gamma}(s) \in B_0(p_0,r_0) \quad \forall 0 \le s \le s'\}.$$

Suppose $s_0 < \overline{s}$. By the same argument as the proof of Lemma 1.4 (1.17) holds in $(0,s_0)$. Hence by (1.17) and Lemma 1.5,

$$\begin{aligned} e^{-c_1} d_0(p_0,\widetilde{\gamma}(s_0)) &\le e^{-c_1}\int_0^{s_0} |\widetilde{\gamma}'(s)|_{g(0)}\,ds \le \int_0^{s_0} |\widetilde{\gamma}'(s)|\,ds \\ &\le e^{C_2/2}(m_0^2 + C_3)^{1/2} s_0 \\ \Rightarrow \qquad d_0(p_0,\widetilde{\gamma}(s_0)) &\le e^{c_1+(C_2/2)}(m_0^2 + C_3)^{1/2} s_1' < r_0. \end{aligned}$$

By continuity there exists $s_2 \in (s_0,\overline{s}]$ such that

$$d_0(p_0,\widetilde{\gamma}(s)) < r_0 \quad \forall 0 \le s \le s_2.$$

This contradicts the choice of s_0. Hence $s_0 = \overline{s}$. Then (1.17) holds on $[0,\overline{s}]$. Thus by (1.17) we can extend $\widetilde{\gamma}$ to a solution of (1.10), (1.13), in $(0,\overline{s}+\delta)$ for some $\delta \in (0, s_1' - \overline{s})$. Contradiction arises. Hence $\overline{s} \ge s_1'$ and the lemma follows. $\square$

Theorem 2.1. *Suppose (M,g) satisfies (1.20) in $[0,t_0)$ for some constant $c_1 > 0$. Then there exists a constant $\overline{\tau}_0 \in (0,t_0)$ such that for any $0 < \overline{\tau} \le \overline{\tau}_0$ there exist a constant $r_1 > 0$ and an open set $O_1 \subset M$ with $p_0 \in O_1$ such that $\mathcal{L}_p$-$exp_{p_0}^{\overline{\tau}}\big|_{\mathcal{B}(0,r_1)} : \mathcal{B}(0,r_1) \to O_1$ is a diffeomorphism.*

Proof. We will use a modification of the proof of Proposition 2.6 of [25] to prove the lemma. Let $(\phi_0, B_0(p_0,r_0))$ be a local normal co-ordinate chart around p_0. By Lemma 2.1 there exists a constant $s_1 \in (0, t_0^{1-p})$ such that $\mathcal{B}(0,1) \subset U_p(\tau)$ for any $0 < \tau \le \tau_1 = s_1^{\frac{1}{1-p}}$ and (2.1) holds for any $\widetilde{\mathcal{L}}_p$-geodesic which satisfies (1.13) with $|\widetilde{v}| \le 1$. By the inverse function theorem it suffices to check that the kernel of $d(\mathcal{L}_p\text{-}\exp_{p_0}^{\overline{\tau}})_0$ is equal to zero for sufficiently small $\overline{\tau}$. Suppose not. Then there exists $\overline{\tau} \in (0,\overline{\tau}_0)$ and $0 \ne v_0 \in T_0(T_{p_0}M) = T_{p_0}M$ such that

$$d(\mathcal{L}_p\text{-}\exp_{p_0}^{\overline{\tau}})_0(v_0) = 0 \tag{2.2}$$

where $\overline{\tau}_0 \in (0,\tau_1)$ is some constant to be determined later in the proof. Let $\widetilde{v}_0 = v_0/(1-p)$ and $\overline{s} = \overline{\tau}^{1-p}$. By rescaling v_0 if necessary we may assume without loss of generality that $|\widetilde{v}_0|_{g(p_0,0)} = 1$. Then $|v_0|_{g(p_0,0)} = 1-p$. For any $0 \le z \le 1$, let

$$h(s,z) = \widetilde{\gamma}(s; z\widetilde{v}_0)$$

be the solution of (1.10), (1.13), in $[0,s_1]$ with $\widetilde{v}$ being replaced by $z\widetilde{v}_0$ given by Lemma 2.1. Then h is a variation of $\widetilde{\gamma}(s;\widetilde{v}_0)$ with $h(s,1) = \widetilde{\gamma}(s;\widetilde{v}_0)$. Let

$$Y(s,z) = \frac{\partial h}{\partial z}(s,z), \overline{Y}(s) = Y(s,0) \text{ and } \widetilde{X}(s) = \frac{\partial h}{\partial s}(s,0).$$

We write $\widetilde{X}(s) = a^i(s)\partial/\partial x_i$ and $\overline{Y}(s) = b^j(s)\partial/\partial x_j$ in the local co-odinates $(\phi_0, B_0(p_0,r_0))$. By (2.2),

$$0 = \left.\frac{d}{dz}\right|_{z=0} \mathcal{L}_p\text{-}\exp_{p_0}^{\overline{\tau}}(zv_0) = \left.\frac{d}{dz}\right|_{z=0} \widetilde{\gamma}(\overline{s}; z\widetilde{v}_0) = Y(\overline{s},0) = \overline{Y}(\overline{s})$$

$$\Rightarrow \quad b^j(\overline{s}) = 0 \quad \forall j = 1,2,\dots,n. \tag{2.3}$$

Note that

$$\begin{aligned}
& h(0,z) = \widetilde{\gamma}(0; z\widetilde{v}_0) = p_0 \quad \forall 0 \le z \le 1 \\
\Rightarrow \quad & Y(0,z) = \frac{\partial h}{\partial z}(0,z) = 0 \quad \forall 0 \le z \le 1 \\
\Rightarrow \quad & b^j(0)\frac{\partial}{\partial x_j} = \overline{Y}(0) = 0 \\
\Rightarrow \quad & b^j(0) = 0 \quad \forall j = 1,2,\dots,n
\end{aligned} \tag{2.4}$$

and

$$\nabla_s\overline{Y}(0) = \frac{\partial^2 h}{\partial s\partial z}(0,0) = \frac{\partial^2 h}{\partial z\partial s}(0,0) = \left.\frac{\partial^2}{\partial z\partial s}\right|_{(0,0)} \widetilde{\gamma}(s; z\widetilde{v}_0) = \left.\frac{\partial}{\partial z}\right|_{z=0} (z\widetilde{v}_0) = \widetilde{v}_0. \tag{2.5}$$

By an argument similar to the proof of Lemma 1.4 (1.17) holds for some constants $C_2 > 0$, $C_3 > 0$, with $\widetilde{v} = 0$. Hence there exists a constant $C > 0$ such that

$$|\widetilde{X}(s)| \le C \quad \forall 0 \le s \le s_1.$$

Since $\widetilde{\gamma}$ satisfies (1.10),

$$\begin{aligned}
& 0 = \nabla_s\left(\frac{\partial h}{\partial s}\right) - \frac{1}{2(1-p)^2} s^{\frac{2p}{1-p}} \nabla\widetilde{R} + \frac{2}{1-p} s^{\frac{p}{1-p}} \widetilde{\mathrm{Ric}}\left(\frac{\partial h}{\partial s}, \cdot\right) \\
\Rightarrow \quad & 0 = \nabla_z\nabla_s\left(\frac{\partial h}{\partial s}\right) - \frac{1}{2(1-p)^2} s^{\frac{2p}{1-p}} \nabla_z(\nabla\widetilde{R}) + \frac{2}{1-p} s^{\frac{p}{1-p}} \nabla_z\left(\widetilde{\mathrm{Ric}}\left(\frac{\partial h}{\partial s}, \cdot\right)\right) \\
\Rightarrow \quad & 0 = \nabla_s\nabla_s Y(s,z) + \widetilde{R}\left(\frac{\partial h}{\partial z}, \frac{\partial h}{\partial s}\right)\frac{\partial h}{\partial s} - \frac{1}{2(1-p)^2} s^{\frac{2p}{1-p}} \nabla_z(\nabla\widetilde{R}) \\
& \quad + \frac{2}{1-p} s^{\frac{p}{1-p}} (\nabla_z\widetilde{\mathrm{Ric}})\left(\frac{\partial h}{\partial s}, \cdot\right) + \frac{2}{1-p} s^{\frac{p}{1-p}} \widetilde{\mathrm{Ric}}(\nabla_s Y, \cdot)
\end{aligned} \tag{2.6}$$

where $\widetilde{R}(X_1,X_2)X_3(q,s)=R(X_1,X_2)X_3(q,\tau)$ for any $X_1,X_2,X_3\in T_qM$ with $s=\tau^{1-p}$. Putting $z=0$ we get

$$
\begin{aligned}
0=&\nabla_s\nabla_s\overline{Y}(s)+\widetilde{R}(\overline{Y},\widetilde{X})\widetilde{X}-\frac{1}{2(1-p)^2}s^{\frac{2p}{1-p}}\nabla_{\overline{Y}(s)}(\nabla\widetilde{R}(h(s,0),s))\\
&+\frac{2}{1-p}s^{\frac{p}{1-p}}(\nabla_{\overline{Y}(s)}\widetilde{\mathrm{Ric}})(\widetilde{X},\cdot)+\frac{2}{1-p}s^{\frac{p}{1-p}}\widetilde{\mathrm{Ric}}(\nabla_sY,\cdot)\quad\text{ in }(0,s_1)\\
=&\frac{d^2b^k}{ds^2}+\left(b^j\frac{db^i}{ds}+b^i\frac{db^j}{ds}\right)\widetilde{\Gamma}_{ij}^k+b^ib^jb^m\frac{\partial\widetilde{\Gamma}_{ij}^k}{\partial x^m}+b^m\left(\frac{db^r}{ds}+b^ib^j\widetilde{\Gamma}_{ij}^r\right)\widetilde{\Gamma}_{mr}^k\\
&+g^{jk}b^i\left\langle\widetilde{R}\left(\frac{\partial}{\partial x_i},\widetilde{X}\right)\widetilde{X},\frac{\partial}{\partial x_j}\right\rangle-\frac{1}{2(1-p)^2}s^{\frac{2p}{1-p}}g^{kj}b^i\left\langle\nabla_i(\nabla\widetilde{R}),\frac{\partial}{\partial x_j}\right\rangle\\
&+\frac{2}{1-p}s^{\frac{p}{1-p}}g^{jk}a^ib^m\nabla_m\widetilde{R}_{ij}+\frac{2}{1-p}s^{\frac{p}{1-p}}g^{jk}\left(\frac{db^i}{ds}+b^mb^r\widetilde{\Gamma}_{mr}^i\right)\widetilde{R}_{ij}
\end{aligned}
\tag{2.7}
$$

holds in $(0,s_1)$ for all $k=1,2,\dots,n$. By (2.5),

$$
g_{ij}(p_0,0)\frac{db^i}{ds}(0)\frac{db^j}{ds}(0)=1.
$$

Hence

$$
\begin{aligned}
&\lambda_1\sum_{k=1}^n\left(\frac{db^k}{ds}(0)\right)^2\le g_{ij}(p_0,0)\frac{db^i}{ds}(0)\frac{\partial b^j}{\partial s}(0)\le\lambda_2\sum_{k=1}^n\left(\frac{db^k}{ds}(0)\right)^2\\
\Rightarrow\quad&\lambda_1\sum_{k=1}^n\left(\frac{db^k}{ds}(0)\right)^2\le 1\le\lambda_2\sum_{k=1}^n\left(\frac{db^k}{ds}(0)\right)^2
\end{aligned}
\tag{2.8}
$$

for some constant $\lambda_2>\lambda_1>0$ depending only on $g_{ij}(p_0,0)$. Let

$$
E=\sum_{k=1}^n\left\{(b^k)^2+\left(\frac{db^k}{ds}\right)^2\right\}
$$

and

$$
s_2=\sup\{0<s_1'\le s_1:|b^i(s)|\le 1\quad\forall 0\le s\le s_1',i=1,2,\dots,n\}.
$$

By (2.4) $s_2>0$. Then by (2.7) and (2.8),

$$
\begin{aligned}
\left|\frac{dE}{ds}\right| &= 2\left|\sum_{k=1}^{n}\left(b^k\frac{db^k}{ds}+\frac{db^k}{ds}\frac{d^2b^k}{ds^2}\right)\right| \\
&\le \sum_{k=1}^{n}\left\{2|b^k|\left|\frac{db^k}{ds}\right| + C\left|\frac{db^k}{ds}\right|\left[\sum_{i=1}^{n}\left(|b^i| + \left|\frac{db^i}{ds}\right|\right) + \sum_{i,j,m=1}^{n}|b^ib^jb^m|\right.\right. \\
&\qquad \left.\left. + \sum_{i,j=1}^{n}\left(|b^ib^j| + |b^i|\left|\frac{db^j}{ds}\right|\right)\right]\right\} \\
&\le C_4E \quad \forall 0\le s\le s_2 \\
\Rightarrow \qquad & \frac{d}{ds}(e^{-C_4s}E) \le 0 \quad \forall 0\le s\le s_2
\end{aligned}
$$

$$
\text{(2.9)} \qquad \Rightarrow \qquad E(s) \le e^{C_4s}E(0) \le e^{C_4s_1}/\lambda_1 = C_5 \quad \forall 0\le s\le s_2.
$$

for some constants $C_1 > 0$ and $C_4 > 0$ independent of s_2. We claim that

$$
s_2 \ge \min(1/(2\sqrt{C_5}), s_1).
$$

Suppose not. Then $s_2 < \min(1/(2\sqrt{C_5}), s_1)$. By (2.4) and (2.9),

$$
|b^i(s)| = \left|\int_0^s \frac{db^i}{ds}\,ds\right| \le \sqrt{C_5}s \le \frac{1}{2} \quad \forall 0\le s\le s_2, i=1,2,\dots,n.
$$

Then by continuity there exists a constant $\delta > 0$ such that

$$
|b^i(s)| \le 1 \quad \forall 0\le s\le s_2+\delta, i=1,2,\dots,n.
$$

This contradicts the maximality of s_2. Hence $s_2 \ge \min(1/(2\sqrt{C_5}), s_1)$. Let

$$
s_3 = \min(1/(2\sqrt{C_5}), s_1).
$$

Then (2.9) holds for all $0\le s\le s_3$. By (2.8) there exists $i_0 \in \{1,2,\dots,n\}$ such that

$$
\left|\frac{db^{i_0}}{ds}(0)\right| \ge \frac{1}{\sqrt{n\lambda_2}}.
$$

By replacing $\widetilde{v}_0$ by $-\widetilde{v}_0$ if necessary and permutating the indices we may assume without loss of generality that

$$
\frac{db^1}{ds}(0) \ge \frac{1}{\sqrt{n\lambda_2}}. \tag{2.10}
$$

By (2.7) and (2.9),

$$
\frac{d^2b^1}{ds^2}(s) + C_6 \ge 0 \quad \forall 0\le s\le s_3 \tag{2.11}
$$

for some constant $C_6 > 0$. Let

$$
\overline{s}_0 = \min(s_1, (2\sqrt{C_5})^{-1}, (2\sqrt{n\lambda_2}C_6)^{-1}),
$$

$\overline{\tau}_0 = \overline{s}_0^{1/(1-p)}$, and $s_4 = \sup\{s' \le \overline{s} : db^1(s)/ds > 0 \quad \forall 0 \le s < s'\}$. Then $s_4 \le \overline{s} \le \overline{s}_0$. Integrating (2.11), by (2.10) we have

$$\frac{db^1}{ds}(s) \ge \frac{db^1}{ds}(0) - C_6 s \ge \frac{1}{\sqrt{n\lambda_2}} - C_6\overline{s}_0 \ge \frac{1}{2\sqrt{n\lambda_2}} > 0 \quad \forall 0 \le s \le s_4. \tag{2.12}$$

Suppose $s_4 < \overline{s}$. Then $s_4 < \overline{s}_0$. By (2.12) and continuity there exists $\delta \in (0, \overline{s}-s_4)$ such that $db^1(s)/ds > 0$ on $(0, s_4+\delta)$. This contradicts the maximality of s_4. Hence $s_4 = \overline{s}$. Integrating (2.12) over $(0,\overline{s})$,

$$b^1(\overline{s}) > b^1(0) = 0.$$

This contradicts (2.3). Hence no such v_0 exists. Thus $\ker(d(\mathcal{L}_p\text{-}\exp_{p_0}^{\overline{\tau}})_0) = 0$ for any $0 < \overline{\tau} \le \overline{\tau}_0$ and the theorem follows. $\square$

By the proof of Theorem 2.1 it is natural to define the following:

Definition 2.1. Let $\widetilde{\gamma}(s)$ be a $\widetilde{\mathcal{L}}_p$-geodesic in (s_1, s_2). We say that a vector field $\widetilde{Y}(s)$ along $\widetilde{\gamma}$ is a $\widetilde{\mathcal{L}}_p$-Jacobi field in (s_1, s_2) if $\widetilde{Y}(s)$ satisfies

$$\nabla_s\nabla_s\widetilde{Y} + \widetilde{R}(\widetilde{Y},\widetilde{\gamma}')\widetilde{\gamma}' - \frac{1}{2(1-p)^2}s^{\frac{2p}{1-p}}\nabla_{\widetilde{Y}}(\nabla\widetilde{R}) + \frac{2}{1-p}s^{\frac{p}{1-p}}\nabla_{\widetilde{Y}}(\widetilde{\mathrm{Ric}}(\widetilde{\gamma}',\cdot)) = 0 \tag{2.13}$$

in (s_1, s_2) along $\widetilde{\gamma}$.

Definition 2.2. Let $\gamma(\tau)$ be a $\mathcal{L}_p$-geodesic in (τ_1, τ_2) and let $\widetilde{\gamma}(s)$ be given by (1.5). We say that a vector field $Y(\tau)$ along γ is a $\mathcal{L}_p$-Jacobi field in (τ_1, τ_2) if $\widetilde{Y}(s) = Y(s^{1-p})$ is a $\widetilde{\mathcal{L}}_p$-Jacobi field in (s_1, s_2) in (s_1, s_2) along $\widetilde{\gamma}$ where $s_i = \tau_i^{1-p}$, $i = 1, 2$.

Definition 2.3. Let $\widetilde{\gamma}(s)$ be a $\widetilde{\mathcal{L}}_p$-geodesic on $[0, \overline{s}]$. For any $0 \le s_0 < s_1 \le \overline{s}$, we say that $\widetilde{\gamma}(s_1)$ is $\widetilde{\mathcal{L}}_p$-conjugate to $\widetilde{\gamma}(s_0)$ along $\widetilde{\gamma}|_{[s_0,s_1]}$ if there exists a $\widetilde{\mathcal{L}}_p$-Jacobi field $\widetilde{Y}(s) \not\equiv 0$ along $\widetilde{\gamma}|_{[s_0,s_1]}$ such that $\widetilde{Y}(s_0) = \widetilde{Y}(s_1) = 0$.

Definition 2.4. Let $\gamma(\tau)$ be a $\mathcal{L}_p$-geodesic on $[0, \overline{\tau}]$. For any $0 \le \tau_0 < \tau_1 \le \overline{\tau}$, we say that $\gamma(\tau_1)$ is $\mathcal{L}_p$-conjugate to $\gamma(\tau_0)$ along $\gamma|_{[\tau_0,\tau_1]}$ if $\widetilde{\gamma}(s_1)$ is $\widetilde{\mathcal{L}}_p$-conjugate to $\widetilde{\gamma}(s_0)$ along $\widetilde{\gamma}|_{[s_0,s_1]}$ where $\widetilde{\gamma}(s)$ is given by (1.5) with $s = \tau^{1-p}$ and $s_i = \tau_i^{1-p}$ for $i = 0, 1$.

Theorem 2.2. *Let $0 < \overline{\tau} < t_0$, $v, w \in T_{p_0}M$, and let $\gamma = \gamma(\tau; v) : [0, \overline{\tau}] \to M$ be a $\mathcal{L}_p$-geodesic which satisfies* (1.18). *Let $\widetilde{\gamma}(s) = \widetilde{\gamma}(s; \widetilde{v})$ be given by* (1.5) *with $s = \tau^{1-p}$ where $\widetilde{v} = v/(1-p)$. Suppose $Y(\tau)$ is a $\mathcal{L}_p$-Jacobi field along γ with $Y(0) = 0$ and $\nabla_s\widetilde{Y}(0) = w/(1-p)$ where $\widetilde{Y}(s) = Y(\tau)$ with $s = \tau^{1-p}$. Then*

$$Y(\tau) = d(\mathcal{L}_p\text{-}exp_{p_0}^{\tau})_v(w) \quad \forall 0 \le \tau \le \overline{\tau}. \tag{2.14}$$

Proof. Let $\alpha : (-\varepsilon_1, \varepsilon_1) \to T_{p_0}M$ be a curve in $T_{p_0}M$ such that $\alpha(0) = v$, $\alpha'(0) = w$. Let $\overline{s} = \overline{\tau}^{1-p}$. By continuous dependence of solutions of O.D.E. on the initial data there exist $\varepsilon \in (0, \varepsilon_1)$ such that for any $\rho \in (-\varepsilon, \varepsilon)$ there exists a unique solution $\widetilde{\gamma}(s; \alpha(\rho)/(1-p))$ of (1.10) on $[0, \overline{s}]$ which satisfies (1.13) with $\widetilde{v}$ being replaced by $\alpha(\rho)/(1-p)$. For any $|\rho| < \varepsilon$, let $\gamma(\tau; \alpha(\rho)) = \widetilde{\gamma}(s; \alpha(\rho)/(1-p))$ with $s = \tau^{1-p}$. Then $\gamma(\tau; \alpha(\rho))$ is the $\mathcal{L}_p$-geodesic on $[0, \overline{\tau}]$ which satisfies (1.18) with v being replaced by $\alpha(\rho)$. Let

$$h(\tau, \rho) = \mathcal{L}_p\text{-}\exp^{\tau}_{p_0}(\alpha(\rho)) = \gamma(\tau; \alpha(\rho)) \quad \forall 0 \le \tau \le \overline{\tau}, |\rho| < \varepsilon$$

and

$$\widetilde{Y_1}(s) = Y_1(\tau) = \frac{\partial h}{\partial \rho}(\tau, 0) \quad \forall 0 \le \tau \le \overline{\tau}, s = \tau^{1-p}.$$

Then

$$Y_1(\tau) = d(\mathcal{L}_p\text{-}\exp^{\tau}_{p_0})_v(w) \quad \forall 0 \le \tau \le \overline{\tau}.$$

Since $h(0, \rho) = p_0 \ \forall \rho \in (-\varepsilon, \varepsilon)$,

$$Y_1(0) = 0 = Y(0) \quad \Rightarrow \quad \widetilde{Y}_1(0) = \widetilde{Y}(0) = 0. \tag{2.15}$$

Then

$$\begin{aligned} \nabla_s \widetilde{Y}_1(0) &= \frac{\partial^2 h}{\partial s \partial \rho}(0,0) = \frac{\partial}{\partial \rho}\bigg|_{\rho=0} \frac{\partial h}{\partial s}(0, \rho) = \frac{\partial}{\partial \rho}\bigg|_{\rho=0} \widetilde{\gamma}'(0; \alpha(\rho)/(1-p)) \\ &= \frac{\partial}{\partial \rho}\bigg|_{\rho=0} \frac{\alpha(\rho)}{1-p} = \frac{\alpha'(0)}{1-p} = \frac{w}{1-p} = \nabla_s \widetilde{Y}(0). \end{aligned} \tag{2.16}$$

By an argument similar to the proof of Theorem 2.1 $\widetilde{Y}_1(s)$ is a $\widetilde{\mathcal{L}}_p$-Jacobi field along $\widetilde{\gamma}$. Since both $\widetilde{Y}$ and $\widetilde{Y}_1$ satisfies (2.13), (2.15), and (2.16) on $[0, \overline{s}]$, by uniqueness of O.D.E. $\widetilde{Y} \equiv \widetilde{Y}_1$ on $[0, \overline{s}]$. Hence $Y(\tau)$ satisfies (2.14) and the theorem follows. □

By Theorem 2.2 and an argument similar to the proof of Proposition 3.9 in Chapter 5 of [1] we have

Theorem 2.3. *Let $0 < \overline{\tau} < t_0$, $v \in T_{p_0}M$, and let $\gamma = \gamma(\tau; v) : [0, \overline{\tau}] \to M$ be a $\mathcal{L}_p$-geodesic which satisfies* (1.18)*. If $\gamma(\overline{\tau})$ is not $\mathcal{L}_p$-conjugate to p_0, then for any $V_0 \in T_{\gamma(\overline{\tau})}M$ there exists a $\mathcal{L}_p$-Jacobi field $Y(\tau)$ along γ with $Y(0) = 0$ and $Y(\overline{\tau}) = V_0$.*

Definition 2.5 (cf. Definition 4 of [25])**.** For any $\overline{\tau} \in (0, t_0)$, we define the injectivity domain $\Omega_p(\overline{\tau})$ at time $\overline{\tau}$ by

$$\begin{aligned} \Omega_p(\overline{\tau}) = \{q \in M : \exists \text{ a unique } &\mathcal{L}_p(q, \overline{\tau})\text{-length minimizing } \mathcal{L}_p\text{-geodesic} \\ &\gamma : [0, \overline{\tau}] \to M \text{ such that } \gamma(0) = p_0, \gamma(\overline{\tau}) = q, \text{ and } q \text{ is not} \\ &\mathcal{L}_p\text{-conjugate to } p_0 \text{ along } \gamma\}. \end{aligned}$$

and we define the cut locus $C_p(\overline{\tau})$ at time $\overline{\tau}$ by $C_p(\overline{\tau}) = M \setminus \Omega(\overline{\tau})$.

Definition 2.6. For any $q_0 \in M$, $0 < \tau_0 < \overline{\tau} < t_0$, we define the $\mathcal{L}^{q_0}_{\tau_0,p}$-injectivity domain $\Omega^{q_0}_{\tau_0,p}(\overline{\tau})$ at time $\overline{\tau}$ by

$$\begin{aligned}\Omega^{q_0}_{\tau_0,p}(\overline{\tau}) = \{q \in M :\ & \exists \text{ a unique } \mathcal{L}^{q_0}_{\tau_0,p}(q,\overline{\tau})\text{-length minimizing } \mathcal{L}_p\text{-geodesic}\\ & \gamma : [\tau_0,\overline{\tau}] \to M \text{ such that } \gamma(\tau_0) = q_0, \gamma(\overline{\tau}) = q, \text{ and } q \text{ is not}\\ & \mathcal{L}_p\text{-conjugate to } q_0 \text{ along } \gamma\}\end{aligned}$$

and we define the $\mathcal{L}^{q_0}_{\tau_0,p}$-cut locus $C^{q_0}_{\tau_0,p}(\overline{\tau})$ at time $\overline{\tau}$ by $C^{q_0}_{\tau_0,p}(\overline{\tau}) = M \setminus \Omega^{q_0}_{\tau_0,p}(\overline{\tau})$.

By the theory of ordinary Riemannian geometry and a similar argument as the discussion on p. 513 of [25] $L_p(q,\tau)$ is a smooth function in $\cup_{\tau'>0}\Omega_p(\tau') \times \{\tau'\}$ and $L^{q_0}_{\tau_0,p}(q,\tau)$ is a smooth function in $\cup_{\tau'>\tau_0}\Omega^{q_0}_{\tau_0,p}(\tau') \times \{\tau'\}$.

Lemma 2.2. *Let $\overline{\tau} \in (0,t_0)$. Suppose (M,g) satisfies* (1.20) *for some constant $c_1 > 0$. Then for any $0 < \rho \le \overline{\tau}$, $L_p(\cdot,\rho)$ is locally Lipschitz in M with respect to the metric $g(\rho)$.*

Proof. We will use a modification of the proof of Proposition 2.12 of [25] and the proof of Theorem 1.4 to prove the lemma. Let $0 < \rho \le \overline{\tau}$, $r_0 > 0$, and let $q_1, q_2 \in B_0(p_0, r_0)$. By Theorem 1.2 for $i = 1,2$, there exists $\mathcal{L}_p(q_i,\rho)$-length minimizing $\mathcal{L}_p$-geodesic γ_i, with $\gamma_i(0) = p_0$ and $\gamma_i(\rho) = q_i$. Let $\gamma : [0,d] \to M$ be a normalized minimizing geodesic with respect to the metric $g(0)$ with $\gamma(0) = q_1$, $\gamma(d) = q_2$, $|\gamma'| = |\gamma'|_{g(0)} = 1$ on $[0,d]$ where $d = d_0(q_1,q_2)$. Then $\gamma([0,d]) \subset B_0(p_0, 2r_0)$. Let

$$K_0 = \sup_{B_0(p_0,2r_0)\times[0,\overline{\tau}]} (|R| + |\mathrm{Ric}|).$$

Then $K_0 < \infty$. For $i = 1,2$, let $d_i = d_0(p_0,q_i)$ and let $\overline{\gamma}_i : [0,d_i] \to M$ be a normalized minimizing geodesic with respect to the metric $g(0)$ with $\overline{\gamma}_i(0) = p_0$, $\overline{\gamma}_i(d_i) = q_i$. Then $d_i < r_0$ and $\overline{\gamma}_i([0,d_i]) \subset B_0(p_0,2r_0)$ for $i = 1,2$. By Lemma 1.6 and the proof of Lemma 1.7, there exist constants $A_1 = A_1(\overline{\tau}, r_0, K_0) > 0$ and $r_1 = r_1(\overline{\tau}, r_0, K_0) \ge 2r_0$ such that

$$\begin{cases} L_p(q_i,\rho) \le A_1 & \forall i = 1,2\\ d_0(p_0,\gamma_i(\tau)) < r_1 & \forall 0 \le \tau \le \rho, i = 1,2.\end{cases}$$

Let

$$K_1 = \sup_{B_0(p_0,2r_1)\times[0,\overline{\tau}]} (|R| + |\nabla R| + |\mathrm{Ric}|)$$

and let $\widetilde{\gamma}_i(s) = \gamma_i(\tau)$ with $s = \tau^{1-p}$ for $i = 1,2$. Then $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$, are $\widetilde{\mathcal{L}}_p$-geodesics. We assume now $d = d_0(q_1,q_2) < \rho/4$. Similar to the proof of Proposition 2.12 of [25] we let

$$\beta(\tau) = \begin{cases} \gamma_1(\tau) & \text{if } 0 \le \tau \le \rho - 2d\\ \gamma_1(2\tau - \rho + 2d) & \text{if } \rho - 2d \le \tau \le \rho - d\\ \gamma(\tau - \rho + d) & \text{if } \rho - d \le \tau \le \rho.\end{cases}$$

Then by Lemma 1.5,

$$
\begin{aligned}
&L_p(q_2,\rho)\\
&\le \mathcal{L}_p(q_2,\beta,\rho)\\
&\le L_p(q_1,\rho)-\int_{\rho-2d}^{\rho}\tau^pR(\gamma_1(\tau),\tau)\,d\tau+\int_{\rho-2d}^{\rho-d}\tau^p[R(\gamma_1(\tau'),\tau)+4|\gamma_1'(\tau')|_{g(\tau)}^2]\,d\tau\\
&\quad+\int_{\rho-d}^{\rho}\tau^p[R(\gamma(\tau''),\tau)+|\gamma'(\tau'')|_{g(\tau)}^2]\,d\tau\\
&\le L_p(q_1,\rho)+(p+1)^{-1}[2K_1(\rho^{p+1}-(\rho-2d)^{p+1})+e^{2K_1\rho}(\rho^{p+1}-(\rho-d)^{p+1})]\\
&\quad+2e^{2K_1\rho}\int_{\rho-2d}^{\rho}\tau^p|\gamma_1'(\tau)|^2\,d\tau. \qquad (2.17)
\end{aligned}
$$

where $\tau'=2\tau-\rho+2d$ and $\tau''=\tau-\rho+d$. By the same argument as the proof of Lemma 1.4, (1.17) holds in $(0,\overline{s}_\rho)$, $\overline{s}_\rho=\rho^{1-p}$, for some constant $C_2>0$, $C_3>0$, depending only on K_1 with $\widetilde{X}(s)$ and $\widetilde{v}$, being replaced by $\widetilde{\gamma}_i'(s)$ and $\widetilde{v}_i=\widetilde{\gamma}_i'(0)$, for $i=1,2$. Since $\widetilde{\gamma}_i'(s)=\tau^p\gamma_i'(\tau)/(1-p)$, by (1.17) there exist constants $C_4>0$, $C_5>0$ and $C_6>0$ such that

$$
C_4|v_i|^2-C_5\le\tau^{2p}|\gamma_i'(\tau)|^2\le C_6(1+|v_i|^2)\quad\forall 0\le\tau\le\rho, i=1,2. \tag{2.18}
$$

By (2.18),

$$
\begin{aligned}
C_4\rho|v_i|^2=C_4\int_0^\rho|v_i|^2\,d\tau&\le C_5\rho+\int_0^\rho\tau^{2p}|\gamma_i'(\tau)|^2\,d\tau\\
&\le C_5\rho+\rho^p\left(L_p(q_i,\rho)+K_1\int_0^\rho\tau^p\,d\tau\right)\\
&\le C_5\rho+\rho^p(A_1+(K_1\rho^{p+1}/(p+1)))=C_7\quad\forall i=1,2. \qquad (2.19)
\end{aligned}
$$

By (2.18) and (2.19),

$$
\begin{aligned}
\int_{\rho-2d}^{\rho}\tau^p|\gamma_i'(\tau)|^2\,d\tau&\le C_6(1+|v_i|^2)\int_{\rho-2d}^{\rho}\tau^{-p}\,d\tau\le 2C_6(1+|v_i|^2)d/(\rho-2d)^p\\
&\le 4dC_6(1+(C_7/C_4\rho))/\rho^p\quad\forall i=1,2. \qquad (2.20)
\end{aligned}
$$

By (2.17) and (2.20) there exists a constant $C_8=C_8(\rho,\overline{\tau},r_0)>0$ such that

$$
L_p(q_2,\rho)\le L_p(q_1,\rho)+C_8d_0(q_1,q_2)\quad\forall q_1,q_2\in B_0(p_0,r_0), d_0(q_1,q_2)<\rho/4.
$$

Hence by Lemma 1.5,

$$
L_p(q_2,\rho)\le L_p(q_1,\rho)+C_9d_\rho(q_1,q_2)\quad\forall q_1,q_2\in B_0(p_0,r_0), d_0(q_1,q_2)<\rho/4
$$

for some constant $C_9>0$. Interchanging the role of q_1 and q_2 in the above inequality,

$$
L_p(q_1,\rho)\le L_p(q_2,\rho)+C_9d_\rho(q_1,q_2)\quad\forall q_1,q_2\in B_0(p_0,r_0), d_0(q_1,q_2)<\rho/4.
$$

Hence

$$|L_p(q_1,\rho)-L_p(q_2,\rho)|\le C_9 d_\rho(q_1,q_2)\quad \forall q_1,q_2\in B_0(p_0,r_0), d_0(q_1,q_2)<\rho/4$$

and the lemma follows. □

By an argument similar to the proof of Proposition 2.13 of [25] we have

Lemma 2.3. *Let $\overline{\tau}\in(0,t_0)$. Suppose (M,g) satisfies (1.20) in $(0,\overline{\tau})$ for some constant $c_1>0$. Then for any $q\in M$, $L_p(q,\cdot)$ is locally Lipschitz in $(0,\overline{\tau}]$.*

Lemma 2.4. *Suppose (M,g) satisfies (1.20) in $(0,t_0)$ for some constant $c_1>0$. Then $\Omega_p(\overline{\tau})$ is open in M for any $\overline{\tau}\in(0,t_0)$ and*

$$\cup_{0<\tau<t_0}\Omega_p(\tau)\times\{\tau\}$$

is open in $M\times(0,t_0)$ with respect to the product metric $g\,dx^2\oplus d\tau^2$. Hence $C_p(\overline{\tau})$ is close in M for any $\overline{\tau}\in(0,t_0)$ and $\cup_{0<\tau<t_0}C_p(\tau)\times\{\tau\}$ is closed in $M\times(0,t_0)$ with respect to the product metric $g\,dx^2\oplus d\tau^2$.

Proof. Let $\overline{\tau}\in(0,t_0)$ and $q\in\Omega_p(\overline{\tau})$. Let $\gamma(\tau;v)$ be the minimizing $\mathcal{L}_p(q,\overline{\tau})$-geodesic given by Theorem 1.2 which satisfies (1.11) and (1.18) for some $v\in T_{p_0}M$. Since q is not $\mathcal{L}_p$-conjugate to p_0, by Theorem 2.2,

$$\ker(d(\mathcal{L}_p\text{-}\exp_{p_0}^{\overline{\tau}})_v)=0\quad\Rightarrow\quad\det(d(\mathcal{L}_p\text{-}\exp_{p_0}^{\overline{\tau}})_v)\ne 0.$$

By the inverse function theorem there exist $\varepsilon\in(0,\min(\overline{\tau},t_0-\overline{\tau})/2)$, $\mathcal{B}(v,r_0)$ and an open neighbourhood $O(q)$ of $(q,\overline{\tau})$ in $M\times(0,t_0)$ such that the map

$$\phi:\mathcal{B}(v,r_0)\times(\overline{\tau}-\varepsilon,\overline{\tau}+\varepsilon)\to O(q)$$

given by $\phi(v,\tau)=(\mathcal{L}_p\text{-}\exp_{p_0}^{\tau}(v),\tau)$ is a differeomorphism for any $\tau\in(\overline{\tau}-\varepsilon,\overline{\tau}+\varepsilon)$ and

$$\det(d(\mathcal{L}_p\text{-}\exp_{p_0}^{\tau})_{v'})\ne 0\quad\forall|\tau-\overline{\tau}|<\varepsilon, v'\in\mathcal{B}(v,r_0). \tag{2.21}$$

We claim that there exists $B_0(q,r_1)\times\{\overline{\tau}\}\subset O(q)$ such that $B_0(q,r_1)\subset\Omega_p(\overline{\tau})$. Suppose not. Then there exists a sequence of points $\{q_i\}_{i=1}^\infty$, $q_i\notin\Omega_p(\overline{\tau})$ $\forall i\in\mathcal{Z}^+$, such that $q_i\to q$ as $i\to\infty$. By the proof of Lemma 2.2, there exist a constant $C_1>0$ and $i_0\in\mathbb{Z}^+$ such that

$$\begin{aligned}&L_p(q_i,\overline{\tau})\le L_p(q,\overline{\tau})+C_1d_0(q_i,q)\quad\forall i\ge i_0\\ \Rightarrow\quad&\exists C_2>0\text{ such that }L_p(q_i,\overline{\tau})\le C_2\quad\forall i\in\mathbb{Z}^+.\end{aligned}\tag{2.22}$$

Now by Theorem 1.2 for any $i=1,2,\dots,$ either

(i) q_i is $\mathcal{L}_p$-conjugate to p_0 along some $\mathcal{L}_p(q_i,\overline{\tau})$-length minimizing $\mathcal{L}_p$-geodesic $\gamma(\cdot;v_i)$ satisfying $\gamma(0;v_i)=p$ and $\gamma(\overline{\tau};v_i)=q_i$

or

(ii) there exists two $\mathcal{L}_p(q_i,\overline{\tau})$-length minimizing $\mathcal{L}_p$-geodesics $\gamma(\cdot;v_i)$ and $\gamma(\cdot;v_i')$ satisfying $\gamma(0;v_i)=\gamma(0;v_i')=p, \gamma(\overline{\tau};v_i)=\gamma(\overline{\tau};v_i')=q_i,$ with $v_i\ne v_i'$

where $\gamma(\cdot;w)$ stands for the solution of (1.11) and (1.18) with v being replaced by w. Then either (i) or (ii) holds for infinitely many $i\in\mathbb{Z}^+$. We now divide the proof into two cases:

Case 1: (i) holds for infinitely many $i\in\mathbb{Z}^+$.

Without loss of generality we may assume that (i) holds for all $i\in\mathbb{Z}^+$. Then by Theorem 2.2,

(2.23)
$$\ker(d(\mathcal{L}_p\text{-}\exp_{p_0}^{\overline{\tau}})_{v_i})\ne 0\quad\forall i\in\mathbb{Z}^+\quad\Rightarrow\quad\det(d(\mathcal{L}_p\text{-}\exp_{p_0}^{\overline{\tau}})_{v_i})=0\quad\forall i\in\mathbb{Z}^+.$$

By (2.22) and an argument similar to the proof of (2.19) there exists a constant $r_2>0$ such that

$$|v_i|\le r_2\quad\forall i=1,2,\dots. \tag{2.24}$$

Since the closed ball $\overline{\mathcal{B}(0,r_2)}$ is compact in $T_{p_0}M$ with respect to the metric $g(p_0,0)$, the sequence $\{v_i\}_{i=1}^\infty$ has a subsequence which we may assume without loss of generality to be the sequence itself such that $v_i\to v_0$ as $i\to\infty$ for some $v_0\in T_{p_0}M$. By continuous dependence of solutions of O.D.E. on the initial data $\gamma(\tau;v_i)$ will converge uniformly to a $\mathcal{L}_p$-geodesic $\gamma(\tau;v_0)$ on $[0,\overline{\tau}]$ as $i\to\infty$. By Fatou's Lemma and (2.22),

$$L_p(q,\overline{\tau})\le\mathcal{L}_p(q,\gamma(\cdot;v_0),\overline{\tau})\le\lim_{i\to\infty}\mathcal{L}_p(q_i,\gamma_i(\cdot;v_i),\overline{\tau})=\lim_{i\to\infty}L_p(q_i,\overline{\tau})\le L_p(q,\overline{\tau}).$$

Hence

$$L_p(q,\overline{\tau})=\mathcal{L}_p(q,\gamma(\cdot;v_0),\overline{\tau}).$$

Thus $\gamma(\cdot;v_0)$ is a minimizing $\mathcal{L}_p(q,\overline{\tau})$-geodesic. Since $q\in\Omega(\overline{\tau})$, the minimizing $\mathcal{L}_p(q,\overline{\tau})$-geodesic is unique. Hence $v_0=v$. Letting $i\to\infty$ in (2.23),

$$\det(d(\mathcal{L}_p\text{-}\exp_{p_0}^{\overline{\tau}})_v)=0.$$

This contradicts (2.21). Hence case 1 does not hold.

Case 2: (ii) holds for infinitely many $i\in\mathbb{Z}^+$.

Without loss of generality we may assume that (ii) holds for all $i\in\mathbb{Z}^+$. By the same argument as case 1 there exists $r_2>0$ such that $v_i,v_i'\in\overline{\mathcal{B}(0,r_2)}$ for all $i\in\mathbb{Z}^+$. Then as in case 1 by choosing a subsequence if necessary we may assume without loss of generality that $v_i\to v$ and $v_i'\to v$ as $i\to\infty$. Then there exists $i_0\in\mathbb{Z}^+$ such that $v_i,v_i'\in\mathcal{B}(v,r_0)$ for all $i\ge i_0$. Since the map ϕ is a differeomorphism,

$$\mathcal{L}_p\text{-}\exp_{p_0}^{\overline{\tau}}(v_i)=\mathcal{L}_p\text{-}\exp_{p_0}^{\overline{\tau}}(v_i')=q_i\quad\forall i\ge i_0\qquad\Rightarrow\quad v_i=v_i'\quad\forall i\ge i_0.$$

Contradiction arise. Hence case 2 does not hold. Thus no such sequence $\{q_i\}_{i=1}^{\infty}$ exists. Hence there exists $B_0(q,r_1)\times\{\overline{\tau}\}\subset O(q)$ such that $B_0(q,r_1)\subset\Omega_p(\overline{\tau})$. Therefore $\Omega_p(\overline{\tau})$ is open. By a similar argument $\cup_{0<\tau<t_0}\Omega_p(\tau)\times\{\tau\}$ is open in $M\times(0,t_0)$ and the lemma follows. □

By Theorem 1.2, Lemma 2.4, and an argument similar to the proof of Proposition 2.16 of [25] but with Lemma 2.2 replacing Proposition 2.12 in the proof there we have

Lemma 2.5. *Suppose (M,g) satisfies (1.20) in $(0,t_0)$ for some constant $c_1>0$. Then $C_p(\overline{\tau})$ is a closed set of measure zero for any $\overline{\tau}\in(0,t_0)$.*

Lemma 2.6. *Suppose (M,g) satisfies (1.20) in $(0,t_0)$ for some constant $c_1>0$. Then $\cup_{0<\tau<t_0}C_p(\tau)\times\{\tau\}$ is a closed set of measure zero in $M\times(0,t_0)$ with respect to the product metric $g\,dx^2\oplus d\tau^2$.*

Proof. This result for the case $p=1/2$ is stated in [25]. We will give a proof of it here for any $0<p<1$. By Lemma 2.4 we know that $\cup_{0<\tau<t_0}C_p(\tau)\times\{\tau\}$ is closed in $M\times(0,t_0)$. It suffices to show that $\cup_{\tau_1\le\tau\le\tau_2}(C_p(\tau)\cap B_0(p_0,r_0))\times\{\tau\}$ has measure zero in $M\times(0,t_0)$ for any $0<\tau_1<\tau_2<t_0$ and $r_0>0$.

Let $\overline{\tau}\in[\tau_1,\tau_2]$, $\delta>0$, $\Omega_p(\tau,r_0)=\Omega_p(\tau)\cap B_0(p_0,r_0)$,

$$D_p(r_0)=\cup_{0<\tau<t_0}\Omega_p(\tau,r_0)\times\{\tau\},$$

and

$$C_p(\tau_1,\tau_2,r_0)=\cup_{\tau_1\le\tau\le\tau_2}(C_p(\tau)\cap B_0(p_0,r_0))\times\{\tau\}.$$

We choose a compact set $K(\overline{\tau})\subset\Omega_p(\overline{\tau},r_0)$ such that

$$m_{\overline{\tau}}(\Omega(\overline{\tau},r_0)\setminus K(\overline{\tau}))<\delta.$$

Then by Lemma 2.5,

$$m_{\overline{\tau}}(B_0(p_0,r_0)\setminus K(\overline{\tau}))<\delta. \tag{2.25}$$

Since $D_p(r_0)$ is open, for any $q\in K(\overline{\tau})$ there exists $\varepsilon_q>0$ and an open ball $O_{\overline{\tau}}(q)\subset\Omega(\overline{\tau})$ containing q such that $O_{\overline{\tau}}(q)\times[\overline{\tau}-\varepsilon_q,\overline{\tau}+\varepsilon_q]\subset D_p(r_0)$. Since $K(\overline{\tau})$ is compact,

$$K(\overline{\tau})\subset\cup_{i=1}^{n(\overline{\tau})}O_{\overline{\tau}}(q_i)$$

for some $q_1,\dots,q_{n(\overline{\tau})}\in K(\overline{\tau})$. Let $\varepsilon_{\overline{\tau}}=\min_{1\le i\le n_0}\varepsilon_{q_i}$. Since $[\tau_1,\tau_2]$ is compact, there exists $\tau_1\le\overline{\tau}_1<\overline{\tau}_2<\cdots<\overline{\tau}_{k_0}\le\tau_2$ such that

$$[\tau_1,\tau_2]\subset\cup_{k=1}^{k_0}(\overline{\tau}_k-\varepsilon_k,\overline{\tau}_k+\varepsilon_k)$$

where $\varepsilon_k=\varepsilon_{\overline{\tau}_k}$ for all $k=1,2,\dots,k_0$. Let $I_1=(\overline{\tau}_1-\varepsilon_1,\overline{\tau}_1+\varepsilon_1)\cap[\tau_1,\tau_2]$ and $I_k=((\overline{\tau}_k-\varepsilon_k,\overline{\tau}_k+\varepsilon_k)\setminus\cup_{j=1}^{k-1}I_j)\cap[\tau_1,\tau_2]$ for all $k=2,3,\dots,k_0$. Then

$$[\tau_1,\tau_2]=\cup_{k=1}^{k_0}I_k$$

For any $k = 1, 2, \dots, k_0$, let $E_k = (\cup_{i=1}^{n_k} O_{\overline{\tau}_k}(q_i)) \times I_k$ where $n_k = n(\overline{\tau}_k)$. Then $\cup_{k=1}^{k_0} E_k \subset D_p(r_0)$ and

$$
\begin{aligned}
C_p(\tau_1, \tau_2, r_0) &\subset (B_0(p_0, r_0) \times [\tau_1, \tau_2]) \setminus \cup_{k=1}^{k_0} E_k \\
&\subset \cup_{k=1}^{k_0} (B_0(p_0, r_0) \setminus K(\overline{\tau}_k)) \times I_k \\
\Rightarrow \quad m(C_p(\tau_1, \tau_2, r_0)) &\le m\left(\cup_{k=1}^{k_0} (B_0(p_0, r_0) \setminus K(\overline{\tau}_k)) \times I_k\right) \\
&\le \sum_{k=1}^{k_0} m\left((B_0(p_0, r_0) \setminus K(\overline{\tau}_k)) \times I_k\right). \qquad (2.26)
\end{aligned}
$$

Let $C_1 = \sup_{\overline{B_0(p_0,r_0)} \times [\tau_1,\tau_2]} R(q, \tau)$. Then $C_1 < \infty$. Note that by [9] the volume form $\sqrt{g}$ of M satisfies

$$
\left|\frac{d\sqrt{g}}{d\tau}\right| = |R\sqrt{g}| \le C_1\sqrt{g} \quad \text{in } \overline{B_0(p_0, r_0)} \times [\tau_1, \tau_2] \qquad (2.27)
$$

$$
\Rightarrow \quad \sqrt{g}(q, \tau) \le e^{C_1 \varepsilon_k} \sqrt{g}(q, \overline{\tau}_k) \quad \forall q \in \overline{B_0(p_0, r_0)}, \tau \in I_k, k = 1, 2, \dots, n. \qquad (2.28)
$$

By (2.25), (2.26), and (2.28),

$$
\begin{aligned}
m(C_p(\tau_1, \tau_2, r_0)) &\le \sum_{k=1}^{k_0} \left\{ e^{C_1 \varepsilon_k} |I_k| m_{\overline{\tau}_k}(B_0(p_0, r_0) \setminus K(\overline{\tau}_k)) \right\} \\
&\le e^{C_1} \delta \sum_{k=1}^{k_0} |I_k| \le e^{C_1}(\tau_2 - \tau_1)\delta.
\end{aligned}
$$

Letting $\delta \to 0$,

$$
m(C_p(\tau_1, \tau_2, r_0)) = 0
$$

and the lemma follows. □

By Lemma 2.2 and Lemma 2.3 we have

Lemma 2.7. *Suppose (M, g) satisfies (1.20) in $(0, t_0)$ for some constant $c_1 > 0$. Then $|\nabla L_p(q, \tau)|$ and $\partial L_p(q, \tau)/\partial \tau$ are locally bounded measurable functions on $M \times (0, t_0)$.*

3. Second variation formula for the $\mathcal{L}_p(q, \tau)$-length

In this section we will prove the second variation formula for $L_p(q, \overline{\tau})$. We will prove various properties of the $L_p(q, \tau)$-length, the generalized reduced distance l_p, and the generalized reduced volume $\widetilde{V}_p(\tau)$. We will now assume that (M, g) satisfies (1.20) in $(0, t_0)$ for some constant $c_1 > 0$ for the rest of the paper. For any $\overline{\tau} \in (0, t_0)$, let

$$
\begin{aligned}
U_p'(\overline{\tau}) = \{ &v \in U_p(\overline{\tau}) : \gamma_v(\overline{\tau}) \in \Omega_p(\overline{\tau}) \text{ where } \gamma_v(\cdot) = \gamma(\cdot; v) : [0, \overline{\tau}] \to M \text{ is the} \\
&\mathcal{L}_p\text{-geodesic that satisfies (1.18)}\}.
\end{aligned}
$$

Note that by the definition of $\Omega_p(\overline{\tau})$ and Theorem 1.2,

$$\mathcal{L}_p\text{-}\exp_{p_0}^{\overline{\tau}}\big|_{U_p'(\overline{\tau})} : U_p'(\overline{\tau}) \to \Omega_p(\overline{\tau})$$

is a diffeomorphism. For any $v \in U_p(\overline{\tau})$, let $J_p(v, \overline{\tau})$ be the Jacobian of the $\mathcal{L}_p$-$\exp_{p_0}^{\overline{\tau}}$ map at v. Let

$$\Omega_p = \cup_{0<\tau<t_0} \Omega_p(\tau) \times \{\tau\}.$$

By the same argument as the discussion on p. 518 of [25] $L_p(q, \tau)$ is a smooth function in Ω_p. If $\tau \in (0, t_0)$ and $q \in \Omega_p(\tau)$, then there exists a unique $\mathcal{L}_p(q, \tau)$-length minimizing $\mathcal{L}_p$-geodesic γ satisfying $\gamma(0) = p_0$, $\gamma(\tau) = q$, such that q is not $\mathcal{L}_p$-conjugate to p_0. Then by Lemma 1.1,

$$\nabla L_p(q, \tau) = 2\tau^p \gamma'(\tau). \tag{3.1}$$

Lemma 3.1. *Let $(q, \overline{\tau}) \in \Omega_p$ and let γ be a $\mathcal{L}_p$-geodesic satisfying $\gamma(0) = p_0$, $\gamma(\overline{\tau}) = q$, which minimizes the $\mathcal{L}_p(q, \overline{\tau})$-length. Suppose Y is as in Lemma 1.1. Then*

$$\begin{aligned} &\delta_Y^2 L_p(q, \overline{\tau}) \\ &\quad \le 2\overline{\tau}^p \langle X(\overline{\tau}), \nabla_Y Y(\overline{\tau})\rangle \\ &\qquad + \int_0^{\overline{\tau}} \tau^p \{Hess_R(Y, Y) + 2\langle R(Y, X)Y, X\rangle + 2|\nabla_X Y|^2 + 2\nabla_X Ric(Y, Y) \\ &\qquad - 4\nabla_Y Ric(X, Y)\}\, d\tau \end{aligned} \tag{3.2}$$

where $X = X(\tau) = \gamma'(\tau)$.

Proof. We will use a modification of the argument of section 7 of [18] to prove the theorem. Let $f : [0, \overline{\tau}] \times (-\varepsilon, \varepsilon) \to M$ be as in the proof of Lemma 1.1. Since

$$\begin{cases} L_p(f(\overline{\tau}, z), \overline{\tau}) \le \mathcal{L}_p(f(\overline{\tau}, z), f(\cdot, z), \overline{\tau}) & \forall |z| < \varepsilon \\ L_p(f(\overline{\tau}, 0), \overline{\tau}) = \mathcal{L}_p(f(\overline{\tau}, 0), f(\cdot, 0), \overline{\tau}), \end{cases}$$

differentiating (1.2) with respect to z and putting $z = 0$,

$$\begin{aligned} &\delta_Y^2 L_p(q, \overline{\tau}) \\ &\quad = \frac{d^2}{dz^2}\bigg|_{z=0} L_p(f(\overline{\tau}, z), \overline{\tau}) \le \frac{d^2}{dz^2}\bigg|_{z=0} \mathcal{L}_p(f(\overline{\tau}, z), f(\cdot, z), \overline{\tau}) \\ &\quad \le \int_0^{\overline{\tau}} \tau^p (Y(Y(R)) + 2|\nabla_X Y|^2 + 2\langle X, \nabla_Y \nabla_X Y\rangle)\, d\tau \\ &\quad \le \int_0^{\overline{\tau}} \tau^p \{Y(Y(R)) + 2|\nabla_X Y|^2 + 2\langle X, \nabla_X \nabla_Y Y\rangle + 2\langle R(Y, X)Y, X\rangle\}\, d\tau. \end{aligned} \tag{3.3}$$

Since

$$\begin{aligned}\langle\nabla_Y Y, X\rangle &= Y(\langle Y, X\rangle) - \langle Y, \nabla_Y X\rangle = Y(\langle Y, X\rangle) - \langle Y, \nabla_X Y\rangle\\ &= Y(\langle Y, X\rangle) - \frac{1}{2}X(\langle Y, Y\rangle),\end{aligned}$$

we have

$$\begin{aligned}&\frac{d}{d\tau}\langle\nabla_Y Y, X\rangle\\ &\quad= X(\langle\nabla_Y Y, X\rangle) + \frac{\partial}{\partial\tau}\langle\nabla_Y Y, X\rangle\\ &\quad= \langle\nabla_X\nabla_Y Y, X\rangle + \langle\nabla_Y Y, \nabla_X X\rangle + Y(\frac{\partial g}{\partial\tau}(Y, X)) - \frac{1}{2}X(\frac{\partial g}{\partial\tau}(Y, Y))\\ &\quad= \langle\nabla_X\nabla_Y Y, X\rangle + \langle\nabla_Y Y, \nabla_X X\rangle + 2Y(\mathrm{Ric}(Y, X)) - X(\mathrm{Ric}(Y, Y))\\ &\quad= \langle\nabla_X\nabla_Y Y, X\rangle + \langle\nabla_Y Y, \nabla_X X\rangle + 2\mathrm{Ric}(\nabla_Y Y, X) + 2\nabla_Y\mathrm{Ric}(X, Y)\\ &\qquad - (\nabla_X\mathrm{Ric})(Y, Y).\end{aligned}\tag{3.4}$$

Note that (3.4) is stated in section 7 of [18] but there is no proof of it in [18]. We refer the reader to [16] for another proof of (3.4) by B. Klein and J. Lott. By (3.4),

$$\begin{aligned}&2\int_0^{\overline{\tau}}\tau^p\langle X, \nabla_X\nabla_Y Y\rangle\, d\tau\\ &\quad= 2\int_0^{\overline{\tau}}\tau^p\left\{\frac{d}{d\tau}\langle X, \nabla_Y Y\rangle - \langle\nabla_X X, \nabla_Y Y\rangle - 2\mathrm{Ric}(\nabla_Y Y, X)\right.\\ &\qquad\left. - 2\nabla_Y\mathrm{Ric}(X, Y) + (\nabla_X\mathrm{Ric})(Y, Y)\right\} d\tau\\ &\quad= 2\overline{\tau}^p\langle X(\overline{\tau}), \nabla_Y Y(\overline{\tau})\rangle\\ &\qquad - 2\int_0^{\overline{\tau}}\tau^p\left\{\frac{p}{\tau}\langle\nabla_Y Y, X\rangle + \left\langle\nabla_Y Y, \frac{1}{2}\nabla R - \frac{p}{\tau}X - 2\mathrm{Ric}(X, \cdot)\right\rangle\right.\\ &\qquad\left. + 2\mathrm{Ric}(\nabla_Y Y, X) + 2\nabla_Y\mathrm{Ric}(X, Y) - \nabla_X\mathrm{Ric}(Y, Y)\right\} d\tau\\ &\quad= 2\overline{\tau}^p\langle X(\overline{\tau}), \nabla_Y Y(\overline{\tau})\rangle\\ &\qquad + \int_0^{\overline{\tau}}\tau^p\{-(\nabla_Y Y)R - 4\nabla_Y\mathrm{Ric}(X, Y) + 2\nabla_X\mathrm{Ric}(Y, Y)\}\, d\tau.\end{aligned}\tag{3.5}$$

By (3.3) and (3.5), (3.2) follows. □

Lemma 3.2. *Let $(q, \overline{\tau}) \in \Omega_p$ and let γ, X, be as in Lemma 3.1. Let $b > (1-p)/2$ be a constant and let $Y(\tau)$ be a vector field along γ such that $|Y(\overline{\tau})| = 1$ and $Y(\tau)$ solves the O.D.E.*

$$\nabla_X Y = -Ric(Y, \cdot) + \frac{b}{\tau}Y \quad \text{in } (0, \overline{\tau}).\tag{3.6}$$

Then

$$\begin{aligned} &Hess_{L_p(q,\overline{\tau})}(Y(\overline{\tau}),Y(\overline{\tau}))\\ &\quad\le -2\overline{\tau}^p Ric(q,\overline{\tau})(Y(\overline{\tau}),Y(\overline{\tau}))+\frac{2b^2}{(p+2b-1)\overline{\tau}^{1-p}}\\ &\qquad+(2p-1)\int_0^{\overline{\tau}}\tau^{p-1}Ric(Y,Y)\,d\tau-\int_0^{\overline{\tau}}\tau^pH(X,Y)\,d\tau \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \Delta L_p(q,\overline{\tau})\le& -2\overline{\tau}^pR(q,\overline{\tau})+\frac{2nb^2}{(p+2b-1)\overline{\tau}^{1-p}}+\frac{2p-1}{\overline{\tau}^{2b}}\int_0^{\overline{\tau}}\tau^{p+2b-1}R\,d\tau\\ &-\frac{1}{\overline{\tau}^{2b}}\int_0^{\overline{\tau}}\tau^{p+2b}H(X)\,d\tau \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} &H(X,Y)\\ &\quad=-Hess_R(Y,Y)-2\langle R(Y,X)Y,X\rangle-4(\nabla_X Ric(Y,Y)-\nabla_Y Ric(Y,X))\\ &\qquad-2Ric_\tau(Y,Y)+2|Ric(Y,\cdot)|^2-\frac{1}{\tau}Ric(Y,Y) \end{aligned}$$

and

$$H(X)=-R_\tau-\frac{1}{\tau}R-2\langle X,\nabla R\rangle+2Ric(X,X) \tag{3.9}$$

is the Hamilton's expressions for the matrix Harnack inequality and the trace Harnack inequality respectively (with time equal to $-\tau$).

Proof. We will use a modification of the argument of section 7 of [18] to prove the lemma. By (3.6),

$$\begin{aligned} &\frac{d}{d\tau}|Y|^2=2\,\mathrm{Ric}(Y,Y)+2\langle\nabla_XY,Y\rangle=\frac{2b}{\tau}|Y|^2\quad\forall 0\le\tau\le\overline{\tau}\\ &\Rightarrow\quad |Y(\tau)|^2=\left(\frac{\tau}{\overline{\tau}}\right)^{2b}\quad\forall 0\le\tau\le\overline{\tau}\quad\text{and}\quad Y(0)=0. \end{aligned} \tag{3.10}$$

Let

$$\begin{aligned} I_1=\int_0^{\overline{\tau}}\tau^p\{&\mathrm{Hess}_R(Y,Y)+2\langle R(Y,X)Y,X\rangle+2|\nabla_XY|^2+2\nabla_X\mathrm{Ric}(Y,Y)\\ &-4\nabla_Y\mathrm{Ric}(X,Y)\}\,d\tau. \end{aligned}$$

Then

$$I_1=-\int_0^{\overline{\tau}}\tau^pH(X,Y)\,d\tau-I_2 \tag{3.11}$$

where

$$
\begin{aligned}
I_2 = \int_0^{\overline{\tau}} \tau^p \bigg\{ & 2\,\mathrm{Ric}_\tau(Y,Y) - 2|\mathrm{Ric}(Y,\cdot)|^2 + \frac{1}{\tau}\mathrm{Ric}(Y,Y) \\
& + 2\nabla_X \mathrm{Ric}(Y,Y) - 2|\nabla_X Y|^2 \bigg\}\, d\tau.
\end{aligned}
\tag{3.12}
$$

By p. 17 of [18] and (3.6),

$$
\begin{aligned}
\frac{d}{d\tau}\mathrm{Ric}(Y,Y) &= \mathrm{Ric}_\tau(Y,Y) + \nabla_X \mathrm{Ric}(Y,Y) + 2\,\mathrm{Ric}(\nabla_X Y, Y) \\
&= \mathrm{Ric}_\tau(Y,Y) + \nabla_X \mathrm{Ric}(Y,Y) - 2|\mathrm{Ric}(Y,\cdot)|^2 + \frac{2b}{\tau}\mathrm{Ric}(Y,Y).
\end{aligned}
\tag{3.13}
$$

By (3.6), (3.10), (3.12), and (3.13), I_2 is equal to

$$
\begin{aligned}
\int_0^{\overline{\tau}} \tau^p \bigg\{ & 2\frac{d}{d\tau}\mathrm{Ric}(Y,Y) - 2\nabla_X \mathrm{Ric}(Y,Y) + 4|\mathrm{Ric}(Y,\cdot)|^2 - \frac{4b}{\tau}\mathrm{Ric}(Y,Y) \\
& - 2|\mathrm{Ric}(Y,\cdot)|^2 + \frac{1}{\tau}\mathrm{Ric}(Y,Y) + 2\nabla_X \mathrm{Ric}(Y,Y) \\
& - 2\left[|\mathrm{Ric}(Y,\cdot)|^2 - \frac{2b}{\tau}\mathrm{Ric}(Y,Y) + \frac{b^2}{\tau^2}|Y|^2 \right] \bigg\}\, d\tau
\end{aligned}
$$

Hence

$$
\begin{aligned}
I_2 = {} & 2\overline{\tau}^p \mathrm{Ric}(q,\overline{\tau})(Y(\overline{\tau}), Y(\overline{\tau})) - \frac{2b^2}{(p+2b-1)\overline{\tau}^{1-p}} \\
& + (1-2p)\int_0^{\overline{\tau}} \tau^{p-1}\mathrm{Ric}(Y,Y)\, d\tau.
\end{aligned}
\tag{3.14}
$$

By Lemma 1.1,

$$
\delta_Y L_p(q,\overline{\tau}) = 2\overline{\tau}^p \langle X, Y\rangle \quad \Rightarrow \quad \delta_{\nabla_Y Y} L_p(q,\overline{\tau}) = 2\overline{\tau}^p \langle X, \nabla_Y Y\rangle.
\tag{3.15}
$$

By (3.11), (3.14), (3.15), and Lemma 3.1, (3.7) follows. Let $\{V_i\}_{i=1}^n$ be an orthonormal basis of $T_{\gamma(\overline{\tau})}M$ with respect to the metric $g(\gamma(\overline{\tau}),\overline{\tau})$. For any $i = 1, 2, \ldots, n$, let $\widetilde{Y}_i$ the solution of (3.6) with $\widetilde{Y}_i(\overline{\tau}) = V_i$. By an argument similar to the proof of (3.10),

$$
\langle \widetilde{Y}_i(\tau), \widetilde{Y}_j(\tau)\rangle = \left(\frac{\tau}{\overline{\tau}}\right)^{2b} \delta_{ij} \quad \forall i, j = 1, 2, \ldots, n.
$$

Let $e_i = \widetilde{Y}_i/|\widetilde{Y}_i|$. Then $\widetilde{Y}_i(\tau) = (\tau/\overline{\tau})^b e_i(\tau)$. By putting $Y = \widetilde{Y}_i$ in (3.7) and summing over $i = 1, 2, \ldots, n$, by an argument similar to the derivation of (7.10) of [18] in [16], we get (3.8) and the lemma follows. □

Lemma 3.3. *Let $(q,\overline{\tau}) \in \Omega_p$ and let γ, X, be as in Lemma 3.1. Suppose $Y(\tau)$ is a $\mathcal{L}_p$-Jacobi field along γ with $Y(0) = 0$. Then*

$$
Hess_{L_p(q,\overline{\tau})}(Y(\overline{\tau}), Y(\overline{\tau})) = 2\overline{\tau}^p \langle \nabla_{Y(\overline{\tau})} X(\overline{\tau}), Y(\overline{\tau})\rangle.
$$

Proof. Let $\alpha : (-\varepsilon, \varepsilon) \to M$ be a curve in M such that $\alpha(0) = q$, $\alpha'(0) = Y(\overline{\tau})$. Since $\Omega_p(\overline{\tau})$ is open, by choosing $\varepsilon > 0$ sufficiently small we may assume without loss of generality that $\alpha(-\varepsilon, \varepsilon) \subset \Omega_p(\overline{\tau})$. Then for any $z \in (-\varepsilon, \varepsilon)$ there exists a unique $\mathcal{L}_p(\alpha(z), \overline{\tau})$-length minimizing geodesic $\gamma_z : [0, \overline{\tau}] \to M$ which satisfies $\gamma_z(0) = p_0$ and $\gamma_z(\overline{\tau}) = \alpha(z)$. Let $f : [0, \overline{\tau}] \times (-\varepsilon, \varepsilon) \to M$ be given by $f(\tau, z) = \gamma_z(\tau)$ and let

$$\overline{Y}(\tau) = \frac{\partial f}{\partial z}(\tau, 0).$$

Then $\overline{Y}$ is a $\mathcal{L}_p$-Jacobi field along γ with $\overline{Y}(0) = 0$ and $\overline{Y}(\overline{\tau}) = Y(\overline{\tau})$. By uniqueness of solution of the O.D.E. for $\mathcal{L}_p$-Jacobi field, $\overline{Y}(\tau) = Y(\tau)$ for any $0 \le \tau \le \overline{\tau}$. By Lemma 1.1, (1.3) holds. Since

$$L_p(f(\overline{\tau}, z), \overline{\tau}) = \mathcal{L}_p(f(\overline{\tau}, z), f(\cdot, z), \overline{\tau}) \quad \forall |z| < \varepsilon,$$

differentiating (1.3) with respect to z,

$$\begin{aligned}
&\frac{d^2}{dz^2} L_p(f(\overline{\tau}, z), \overline{\tau})\\
&\quad = \frac{d^2}{dz^2} \mathcal{L}_p(f(\overline{\tau}, z), f(\cdot, z), \overline{\tau})\\
&\quad = 2\overline{\tau}^p \langle \nabla_z \nabla_\tau f(\overline{\tau}, z), \nabla_z f(\overline{\tau}, z)\rangle + 2\overline{\tau}^p \langle \nabla_\tau f(\overline{\tau}, z), \nabla_z \nabla_z f(\overline{\tau}, z)\rangle\\
&\qquad + \int_0^{\overline{\tau}} \tau^p \langle \nabla_z \nabla_z f, \nabla R - \frac{2p}{\tau} \nabla_\tau f - 2\nabla_\tau \nabla_\tau f - 4\mathrm{Ric}(\nabla_\tau f, \cdot)\rangle \, d\tau\\
&\qquad + \int_0^{\overline{\tau}} \tau^p \langle \nabla_z f, \nabla_z \{\nabla R - (2p/\tau) \nabla_\tau f - 2\nabla_\tau \nabla_\tau f - 4\mathrm{Ric}(\nabla_\tau f, \cdot)\}\rangle \, d\tau
\end{aligned}$$

(3.16)

$$\quad = 2\overline{\tau}^p \langle \nabla_z \nabla_\tau f(\overline{\tau}, z), \nabla_z f(\overline{\tau}, z)\rangle + 2\overline{\tau}^p \langle \nabla_\tau f(\overline{\tau}, s), \nabla_z \nabla_z f(\overline{\tau}, z)\rangle + I_1 + I_2.$$

Note that since γ is a $\mathcal{L}_p$-geodesic, I_1 vanishes when $z = 0$. Since $Y(\tau)$ is a $\mathcal{L}_p$-Jacobi field along γ, by the derivation of the $\mathcal{L}_p$-Jacobi equation in the proof of Theorem 2.1, I_2 also vanishes when $z = 0$. Hence by putting $z = 0$ in (3.16), by (3.15),

$$\begin{aligned}
&\delta_Y^2 L_p(q, \overline{\tau}) = \delta_{\nabla_Y Y} L_p(q, \overline{\tau}) + 2\overline{\tau}^p \langle \nabla_Y X, Y\rangle\\
\Rightarrow\quad &\mathrm{Hess}_{L_p(q, \overline{\tau})}(Y(\overline{\tau}), Y(\overline{\tau})) = \delta_Y^2 L_p(q, \overline{\tau}) - \delta_{\nabla_Y Y} L_p(q, \overline{\tau}) = 2\overline{\tau}^p \langle \nabla_Y X, Y\rangle.
\end{aligned}$$

□

Lemma 3.4. *Let $\overline{\tau} \in (0, t_0)$, $v \in U_p'(\overline{\tau})$, and let $b > (1 - p)/2$ be a constant. Then*

$$\frac{d}{d\tau} \log J_p(v, \overline{\tau})$$

(3.17)

$$\quad \le \frac{b^2 n}{(p + 2b - 1)\overline{\tau}} + \frac{2p - 1}{2\overline{\tau}^{p+2b}} \int_0^{\overline{\tau}} \tau^{p+2b-1} R \, d\tau - \frac{1}{2\overline{\tau}^{p+2b}} \int_0^{\overline{\tau}} \tau^{p+2b} H(X) \, d\tau$$

where the integration is along the $\mathcal{L}_p$-geodesic $\gamma_v(\tau)$ which satisfies (1.11) *and* (1.18), $X(\tau) = \gamma_v'(\tau)$, *and $H(X)$ is given by* (3.9).

Proof. We will use a modification of the proof of a similar result for the case $p = b = 1/2$ in [18] to prove the lemma. Let $\gamma_v : [0, \overline{\tau}] \to M$ be the unique $\mathcal{L}_p$-geodesic which satisfies (1.11) and (1.18). Let $\{V_i\}_{i=1}^n$ be an orthonormal basis of $T_{\gamma_v(\overline{\tau})}M$ with respect to the metric $g(\gamma_v(\overline{\tau}), \overline{\tau})$. By Theorem 2.3 for any $i = 1, 2, \dots, n$, there exists a $\mathcal{L}_p$-Jacobi field $Y_i(\tau)$ along γ_v with $Y_i(0) = 0$ and $Y_i(\overline{\tau}) = V_i$. Then by Lemma 3.3,

$$\begin{aligned} \frac{d}{d\tau}|Y_i|^2(\overline{\tau}) &= 2\,\mathrm{Ric}(Y_i(\overline{\tau}), Y_i(\overline{\tau})) + 2\langle \nabla_{X(\overline{\tau})} Y_i(\overline{\tau}), Y_i(\overline{\tau})\rangle \\ &= 2\,\mathrm{Ric}(Y_i(\overline{\tau}), Y_i(\overline{\tau})) + \frac{1}{\overline{\tau}^p}\mathrm{Hess}_{L_p(q,\overline{\tau})}(Y_i(\overline{\tau}), Y_i(\overline{\tau})) \quad \forall i = 1, 2, \dots, n. \end{aligned} \tag{3.18}$$

For any $i = 1, 2, \dots, n$, let $\widetilde{Y}_i(\tau)$ be the solution of (3.6) with $\widetilde{Y}_i(\overline{\tau}) = V_i$. Then by Lemma 3.2 and (3.18),

$$\begin{aligned} &\frac{d}{d\tau}|Y_i|^2(\overline{\tau}) \\ &\le \frac{2b^2}{(p+2b-1)\overline{\tau}} + \frac{2p-1}{\overline{\tau}^p}\int_0^{\overline{\tau}} \tau^{p-1}\mathrm{Ric}(\widetilde{Y}_i, \widetilde{Y}_i)\,d\tau - \frac{1}{\overline{\tau}^p}\int_0^{\overline{\tau}} \tau^p H(X, \widetilde{Y}_i)\,d\tau. \end{aligned} \tag{3.19}$$

Summing (3.19) over $i = 1, 2, \dots, n$, similar to the proof of Lemma 3.2 we have

$$\begin{aligned} &\sum_{i=1}^n \frac{d}{d\tau}|Y_i|^2(\overline{\tau}) \\ &\le \frac{2nb^2}{(p+2b-1)\overline{\tau}} + \frac{2p-1}{\overline{\tau}^{p+2b}}\int_0^{\overline{\tau}} \tau^{p+2b-1}R\,d\tau - \frac{1}{\overline{\tau}^{p+2b}}\int_0^{\overline{\tau}} \tau^{p+2b}H(X)\,d\tau. \end{aligned} \tag{3.20}$$

Now

$$\frac{d}{d\tau}\log J_p(v, \overline{\tau}) = \frac{1}{2|Y_i(\overline{\tau})|^2}\sum_{i=1}^n \frac{d}{d\tau}|Y_i|^2(\overline{\tau}) = \frac{1}{2}\sum_{i=1}^n \frac{d}{d\tau}|Y_i|^2(\overline{\tau}). \tag{3.21}$$

Hence by (3.20) and (3.21) the lemma follows. □

By putting $b = (1-p)$ in (3.17) we have

Corollary 3.1. *Let $\overline{\tau} \in (0, t_0)$ and $v \in U_p'(\overline{\tau})$. Then*

$$\frac{d}{d\tau}\log J_p(v, \overline{\tau}) \le \frac{(1-p)n}{\overline{\tau}} + \frac{2p-1}{2\overline{\tau}^{2-p}}\int_0^{\overline{\tau}} \tau^{1-p}R\,d\tau - \frac{1}{2\overline{\tau}^{2-p}}\int_0^{\overline{\tau}} \tau^{2-p}H(X)\,d\tau \tag{3.22}$$

where the integration is along the $\mathcal{L}_p$-geodesic $\gamma_v(\tau)$ which satisfies (1.11) *and* (1.18), $X(\tau) = \gamma_v'(\tau)$, *and $H(X)$ is given by* (3.9).

Lemma 3.5. *Let $q \in M$ and let $\widetilde{\gamma} : [0, \overline{s}] \to M$ be a $\widetilde{\mathcal{L}}_p$-geodesic satisfying $\widetilde{\gamma}(0) = p_0$ and $\widetilde{\gamma}(\overline{s}) = q$. Suppose there exists $s_0 \in (0, \overline{s})$ such that $\widetilde{\gamma}(s_0)$ is $\widetilde{\mathcal{L}}_p$-conjugate to p_0 along $\widetilde{\gamma}$. Then there exists a vector field $\widetilde{Y}_1$ along $\widetilde{\gamma}$ such that*

$$\delta^2_{\widetilde{Y}_1} \widetilde{\mathcal{L}}_p(q, \widetilde{\gamma}, \overline{s}) < 0.$$

Proof. Let $\widetilde{X}(s) = \widetilde{\gamma}'(s)$. Since $\widetilde{\gamma}(s_0)$ is $\widetilde{\mathcal{L}}_p$-conjugate to p_0 along $\widetilde{\gamma}$, there exists a $\widetilde{\mathcal{L}}_p$-Jacobi field $\widetilde{Y} : [0, s_0] \to M$ along $\widetilde{\gamma}|_{[0,s_0]}$, $\widetilde{Y} \not\equiv 0$, such that $\widetilde{Y}(0) = 0$ and $\widetilde{Y}(s_0) = 0$. Since $\widetilde{Y} \not\equiv 0$, $\nabla_{\widetilde{X}} \widetilde{Y}(s_0) \neq 0$. Let W be a parallel vector field along $\widetilde{\gamma}$ with respect to the metric $\widetilde{g}(s_0) = g(s_0^{1/(1-p)})$ such that $W(s_0) = \nabla_{\widetilde{X}} \widetilde{Y}(s_0)$. Let

$$\widetilde{Y}_0(s) = \begin{cases} \widetilde{Y}(s) & \forall 0 \le s \le s_0 \\ 0 & \forall s_0 < s \le \overline{s}. \end{cases}$$

Let $h \in (0, \min(s_0, \overline{s} - s_0)/2)$ be a constant to be determined later. We choose $\phi \in C^\infty(\mathbb{R})$, $0 \le \phi \le 1$ on $[0, \overline{s}]$, such that $\phi(s) = 0$ for all $|s - s_0| \ge h$ and $\phi(s_0) = 1$. Let $\widetilde{Y}_1(s) = \widetilde{Y}_0(s) + \lambda \phi W(s)$ where $\lambda \in \mathbb{R}$ is some constant to determined later. Let $\widetilde{f} : [0, \overline{s}] \times (-\varepsilon, \varepsilon)$ be a variation of $\widetilde{\gamma}$ with respect to $\widetilde{Y}_1$ such that $\widetilde{f}(0, z) = p_0$ on $(-\varepsilon, \varepsilon)$ and $\widetilde{f}(s, 0) = \widetilde{\gamma}(s)$ for any $0 \le s \le \overline{s}$ given by Proposition 2.2 of Chapter 9 of [1]. By the same argument as the proof of Lemma 1.2 (1.9) holds. Differentiating (1.9) with respect to z, by the same argument as the proof of (2.6),

$$\begin{aligned}
&\frac{d^2}{dz^2} \widetilde{\mathcal{L}}_p(\widetilde{f}(\overline{s}, z), \widetilde{f}(\cdot, z), \overline{s}) \\
&= \frac{1}{1-p} \int_0^{\overline{s}} \langle \nabla_z \nabla_z \widetilde{f}, s^{\frac{2p}{1-p}} \nabla \widetilde{R} - 2(1-p)^2 \nabla_s \nabla_s \widetilde{f} - 4(1-p) s^{\frac{p}{1-p}} \widetilde{\mathrm{Ric}}(\nabla_s \widetilde{f}, \cdot) \rangle \, ds \\
&\quad - 2(1-p) \int_0^{\overline{s}} \left\langle \nabla_z \widetilde{f}, \nabla_z \left\{ \nabla_s \nabla_s \widetilde{f} - \frac{1}{2(1-p)^2} s^{\frac{2p}{1-p}} \nabla \widetilde{R} \right.\right. \\
&\quad \left.\left. + \frac{2}{1-p} s^{\frac{p}{1-p}} \widetilde{\mathrm{Ric}}(\nabla_s \widetilde{f}, \cdot) \right\} \right\rangle ds \\
&= I_1(z) - 2(1-p) I_2(z) \qquad (3.23)
\end{aligned}$$

where $I_1(z)$ is equal to

$$\frac{1}{1-p} \int_0^{\overline{s}} \langle \nabla_z \nabla_z \widetilde{f}, s^{\frac{2p}{1-p}} \nabla \widetilde{R} - 2(1-p)^2 \nabla_s \nabla_s \widetilde{f} - 4(1-p) s^{\frac{p}{1-p}} \widetilde{\mathrm{Ric}}(\nabla_s \widetilde{f}, \cdot) \rangle \, ds$$

and

$$I_2(z) = \int_0^{\overline{s}} \left\langle \nabla_z \widetilde{f}, \nabla_s \nabla_s \nabla_z \widetilde{f} + \widetilde{R}(\nabla_z \widetilde{f}, \nabla_s \widetilde{f}) \nabla_s \widetilde{f} - \frac{1}{2(1-p)^2} s^{\frac{2p}{1-p}} \nabla_z (\nabla \widetilde{R}) \right.$$
$$\left. + \frac{2}{1-p} s^{\frac{p}{1-p}} \nabla_z \left(\widetilde{\mathrm{Ric}}(\nabla_s \widetilde{f}, \cdot) \right) \right\rangle ds.$$

Since $\widetilde{\gamma}$ is a $\widetilde{\mathcal{L}}_p$-geodesic,

$$I_1(0) = 0. \tag{3.24}$$

Since $\widetilde{Y}$ is a $\widetilde{\mathcal{L}}_p$-Jacobi field on $[0, s_0]$,

$$\begin{aligned}
I_2(0) & \\
&= \int_0^{\overline{s}} \left\langle \widetilde{Y}_0 + \lambda\phi W, \nabla_{\widetilde{X}} \nabla_{\widetilde{X}} (\widetilde{Y}_0 + \lambda\phi W) + \widetilde{R}(\widetilde{Y}_0 + \lambda\phi W, \widetilde{X})\widetilde{X} \right. \\
&\quad \left. - \frac{1}{2(1-p)^2} s^{\frac{2p}{1-p}} \nabla_{\widetilde{Y}_0 + \lambda\phi W} (\nabla \widetilde{R}) + \frac{2}{1-p} s^{\frac{p}{1-p}} \nabla_{\widetilde{Y}_0 + \lambda\phi W} (\widetilde{\mathrm{Ric}}(\widetilde{X}, \cdot)) \right\rangle ds \\
&= \lambda \int_0^{\overline{s}} \left\langle \widetilde{Y}_0 + \lambda\phi W, \nabla_{\widetilde{X}} \nabla_{\widetilde{X}} (\phi W) + \phi \widetilde{R}(W, \widetilde{X})\widetilde{X} \right. \\
&\quad \left. - \frac{1}{2(1-p)^2} s^{\frac{2p}{1-p}} \phi \nabla_W (\nabla \widetilde{R}) + \frac{2}{1-p} s^{\frac{p}{1-p}} \phi \nabla_W (\widetilde{\mathrm{Ric}}(\widetilde{X}, \cdot)) \right\rangle ds \\
&= \lambda I_{2,1} + \lambda^2 I_{2,2}
\end{aligned} \tag{3.25}$$

where

$$\begin{aligned}
I_{2,1} = \int_0^{s_0} & \left\langle \widetilde{Y}, \nabla_{\widetilde{X}} \nabla_{\widetilde{X}} (\phi W) + \phi \widetilde{R}(W, \widetilde{X})\widetilde{X} - \frac{1}{2(1-p)^2} s^{\frac{2p}{1-p}} \phi \nabla_W (\nabla \widetilde{R}) \right. \\
& \left. + \frac{2}{1-p} s^{\frac{p}{1-p}} \phi \nabla_W (\widetilde{\mathrm{Ric}}(\widetilde{X}, \cdot)) \right\rangle ds
\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}
I_{2,2} = \int_0^{\overline{s}} & \left\langle \phi W, \nabla_{\widetilde{X}} \nabla_{\widetilde{X}} (\phi W) + \phi \widetilde{R}(W, \widetilde{X})\widetilde{X} - \frac{1}{2(1-p)^2} s^{\frac{2p}{1-p}} \phi \nabla_W (\nabla \widetilde{R}) \right. \\
& \left. + \frac{2}{1-p} s^{\frac{p}{1-p}} \phi \nabla_W (\widetilde{\mathrm{Ric}}(\widetilde{X}, \cdot)) \right\rangle ds.
\end{aligned} \tag{3.27}$$

Now by (1.8),

$$
\begin{aligned}
&\int_0^{s_0}\langle\widetilde{Y},\nabla_{\widetilde{X}}\nabla_{\widetilde{X}}(\phi W)\rangle\,ds\\
&\quad=\int_0^{s_0}\bigg\{\frac{d}{ds}\langle\widetilde{Y},\nabla_{\widetilde{X}}(\phi W)\rangle-\langle\nabla_{\widetilde{X}}\widetilde{Y},\nabla_{\widetilde{X}}(\phi W)\rangle\\
&\qquad-\frac{2}{1-p}s^{\frac{p}{1-p}}\widetilde{\mathrm{Ric}}(\widetilde{Y},\nabla_{\widetilde{X}}(\phi W))\bigg\}\,ds\\
&\quad=-\int_0^{s_0}\bigg\{\langle\nabla_{\widetilde{X}}\widetilde{Y},\nabla_{\widetilde{X}}(\phi W)\rangle+\frac{2}{1-p}s^{\frac{p}{1-p}}\widetilde{\mathrm{Ric}}(\widetilde{Y},\nabla_{\widetilde{X}}(\phi W))\bigg\}\,ds\\
&\quad=-\int_0^{s_0}\bigg\{\frac{d}{ds}\langle\nabla_{\widetilde{X}}\widetilde{Y},\phi W\rangle-\langle\nabla_{\widetilde{X}}\nabla_{\widetilde{X}}\widetilde{Y},\phi W\rangle-\frac{2}{1-p}s^{\frac{p}{1-p}}\phi\widetilde{\mathrm{Ric}}(\nabla_{\widetilde{X}}\widetilde{Y},W)\\
&\qquad+\frac{2}{1-p}s^{\frac{p}{1-p}}\widetilde{\mathrm{Ric}}(\widetilde{Y},\nabla_{\widetilde{X}}(\phi W))\bigg\}\,ds\\
&\quad=-|\nabla_{\widetilde{X}}\widetilde{Y}(s_0)|^2+\int_0^{s_0}\bigg\{\phi\langle\nabla_{\widetilde{X}}\nabla_{\widetilde{X}}\widetilde{Y},W\rangle+\frac{2}{1-p}s^{\frac{p}{1-p}}\phi\,\widetilde{\mathrm{Ric}}(\nabla_{\widetilde{X}}\widetilde{Y},W)\\
&\qquad-\frac{2}{1-p}s^{\frac{p}{1-p}}\widetilde{\mathrm{Ric}}(\widetilde{Y},\nabla_{\widetilde{X}}(\phi W))\bigg\}\,ds.
\end{aligned}\tag{3.28}
$$

Since

$$
\begin{aligned}
&s^{\frac{p}{1-p}}\widetilde{\mathrm{Ric}}(\widetilde{Y},\nabla_{\widetilde{X}}(\phi W))\\
&\quad=\frac{d}{ds}(s^{\frac{p}{1-p}}\widetilde{\mathrm{Ric}}(\widetilde{Y},\phi W))-\frac{p}{1-p}s^{\frac{2p-1}{1-p}}\widetilde{\mathrm{Ric}}(\widetilde{Y},\phi W)-s^{\frac{p}{1-p}}\frac{\partial}{\partial s}(\widetilde{\mathrm{Ric}})(\widetilde{Y},\phi W)\\
&\qquad-s^{\frac{p}{1-p}}\nabla_{\widetilde{X}}(\widetilde{\mathrm{Ric}})(\widetilde{Y},\phi W)-s^{\frac{p}{1-p}}\widetilde{\mathrm{Ric}}(\nabla_{\widetilde{X}}\widetilde{Y},\phi W)
\end{aligned}\tag{3.29}
$$

By (3.26), (3.28), and (3.29),

$$
I_{2,1}=-|\nabla_{\widetilde{X}}\widetilde{Y}(s_0)|^2+\int_{s_0-h}^{s_0}\phi G(W,\widetilde{X},\widetilde{Y})\,ds\tag{3.30}
$$

where

$$
\begin{aligned}
&G(W,\widetilde{X},\widetilde{Y})\\
&\quad=\left\langle\widetilde{Y},\widetilde{R}(W,\widetilde{X})\widetilde{X}-\frac{1}{2(1-p)^2}s^{\frac{2p}{1-p}}\nabla_W(\nabla\widetilde{R})+\frac{2}{1-p}s^{\frac{p}{1-p}}\nabla_W(\widetilde{\mathrm{Ric}}(\widetilde{X},\cdot))\right\rangle\\
&\qquad+\langle\nabla_{\widetilde{X}}\nabla_{\widetilde{X}}\widetilde{Y},W\rangle+\frac{2}{1-p}s^{\frac{p}{1-p}}\widetilde{\mathrm{Ric}}(\nabla_{\widetilde{X}}\widetilde{Y},W)\\
&\qquad+\frac{2}{1-p}\bigg\{\frac{p}{1-p}s^{\frac{2p-1}{1-p}}\widetilde{\mathrm{Ric}}(\widetilde{Y},W)+s^{\frac{p}{1-p}}\frac{\partial}{\partial s}(\widetilde{\mathrm{Ric}})(\widetilde{Y},W)\\
&\qquad+s^{\frac{p}{1-p}}\nabla_{\widetilde{X}}(\widetilde{\mathrm{Ric}})(\widetilde{Y},W)+s^{\frac{p}{1-p}}\widetilde{\mathrm{Ric}}(\nabla_{\widetilde{X}}\widetilde{Y},W)\bigg\}.
\end{aligned}
$$

Let $C_1 = 1 + \max_{0 \le s \le \overline{s}} |G(W, \widetilde{X}, \widetilde{Y})|$ and let

$$h = \frac{|\nabla_{\widetilde{X}} \widetilde{Y}(s_0)|^2}{2C_1}.$$

Then by (3.30),

$$I_{2,1} \le -\frac{1}{2} |\nabla_{\widetilde{X}} \widetilde{Y}(s_0)|^2. \tag{3.31}$$

We now choose $\lambda < 0$ such that $0 > \lambda > -|\nabla_{\widetilde{X}} \widetilde{Y}(s_0)|^2 / [4(1 + |I_{2,2}|)]$. Then by putting $z = 0$ in (3.23), by (3.24), (3.25), and (3.31),

$$\delta^2_{\widetilde{Y}_1} \widetilde{\mathcal{L}}_p(q, \widetilde{\gamma}, \overline{s}) \le |\lambda|(1-p)(-|\nabla_{\widetilde{X}} \widetilde{Y}(s_0)|^2 + 2\lambda I_{2,2}) \le \lambda(1-p)|\nabla_{\widetilde{X}} \widetilde{Y}(s_0)|^2/2 < 0$$

and the lemma follows. □

As a consequence of Lemma 3.5 and the equivalence of the $\mathcal{L}_p$-geodesic and $\widetilde{\mathcal{L}}_p$-geodesic by relations (1.4), (1.5), we have

Corollary 3.2. *Let $q \in M$ and let $\gamma : [0, \overline{\tau}] \to M$ be a $\mathcal{L}_p$-geodesic satisfying $\gamma(0) = p_0$ and $\gamma(\overline{\tau}) = q$. Suppose there exists $\tau_0 \in (0, \overline{\tau})$ such that $\gamma(\tau_0)$ is $\mathcal{L}_p$-conjugate to p_0 along γ. Then there exists a vector field Y_1 along γ such that*

$$\delta^2_{Y_1} \mathcal{L}_p(q, \gamma, \overline{\tau}) < 0.$$

By an argument similar to the proof of Lemma 3.5 and Corollary 3.2 we have

Lemma 3.6. *Let $q \in M$ and let $\gamma : [0, \overline{\tau}] \to M$ be a $\mathcal{L}_p$-geodesic satisfying $\gamma(0) = p_0$ and $\gamma(\overline{\tau}) = q$. Suppose there exists $\tau_0 \in (0, \overline{\tau})$ such that q is $\mathcal{L}_p$-conjugate to $\gamma(\tau_0)$ along $\gamma|_{[\tau_0, \overline{\tau}]}$. Then there exists a vector field Y_1 along γ such that*

$$\delta^2_{Y_1} \mathcal{L}_p(q, \gamma, \overline{\tau}) < 0.$$

Corollary 3.3.

$$U'_p(\tau_2) \subset U'_p(\tau_1) \quad \forall 0 < \tau_1 < \tau_2 < t_0.$$

Proof. Let $0 < \tau_1 < \tau_2 < t_0$. Let $v \in U'_p(\tau_2)$. Then $\gamma_v(\tau_2) \in \Omega_p(\tau_2)$ where $\gamma_v(\cdot) = \gamma(\cdot; v) : [0, \tau_2] \to M$ is the $\mathcal{L}_p$-geodesic that satisfies (1.18). By the definition of $\Omega_p(\tau_2)$, $\gamma(\cdot; v)$ is the unique $\mathcal{L}_p(\gamma_v(\tau_2), \tau_2)$-length minimizing $\mathcal{L}_p$-geodesic joining p_0 and $\gamma_v(\tau_2)$ and $\gamma_v(\tau_2)$ is not $\mathcal{L}_p$-conjugate to p_0 along γ_v. By an argument similar to the proof of Proposition 2.2 of chapter 13 of [1], $\gamma(\cdot; v)|_{[0, \tau_1]}$ is the unique $\mathcal{L}_p(\gamma_v(\tau_1), \tau_1)$-length minimizing $\mathcal{L}_p$-geodesic joining p_0 and $\gamma_v(\tau_1)$. Suppose $\gamma_v(\tau_1)$ is $\mathcal{L}_p$-conjugate to p_0 along $\gamma_v(\cdot)|_{[0, \tau_1]}$. Then by Corollary 3.2 γ_v is not a $\mathcal{L}_p(\gamma_v(\tau_2), \tau_2)$-length minimizing $\mathcal{L}_p$-geodesic.

Contradiction arises. Hence $\gamma_v(\tau_1)$ is not $\mathcal{L}_p$-conjugate to p_0 along $\gamma(\cdot;v)|_{[0,\tau_1]}$. Thus $v \in U_p'(\tau_1)$ and the lemma follows. $\square$

By Lemma 3.6 and an argument similar to the proof of Corollary 3.3 we have

Corollary 3.4. *Let $\overline{\tau} \in (0,t_0)$ and $q \in M$. Suppose $\gamma : [0,\overline{\tau}] \to M$ is the $\mathcal{L}_p(q,\overline{\tau})$-length minimizing $\mathcal{L}_p$-geodesic which satisfies $\gamma(0) = p_0$, $\gamma(\overline{\tau}) = q$, given by Theorem* 1.2. *Then $q \in \Omega_{\tau_1,p}^{\gamma(\tau_1)}(\overline{\tau})$ for any $0 < \tau_1 < \overline{\tau}$.*

Corollary 3.5. *By Corollary* 3.4 *and an argument similar to the proof of Proposition* 2.15 *of* [25], (3.2), (3.7), (3.8), (3.17), (3.22), *etc. in this section holds in $M \times (0,t_0)$ in the barrier sense of Perelman* [18].

4. Monotonicity property of the generalized reduced volume $\widetilde{V}_p(\tau)$

In this section we will prove the monotonicity property of the generalized reduced volume $\widetilde{V}_p(\tau)$ for $1/2 \le p < 1$. We first start with a lemma.

Lemma 4.1. *Suppose M has nonnegative curvature operator in $(0,T)$. Then for any $\overline{\tau} \in (0,t_0)$, $v \in U_p'(\overline{\tau})$, there exists a constant $C_1 = C_1(v,\overline{\tau}) > 0$ such that*

$$\frac{d}{d\tau}\left(\tau^{-(1-p)n}e^{-C_1\tau}J_p(v,\tau)\right) \le 0 \quad \forall 0 < \tau \le \overline{\tau} \tag{4.1}$$

and

$$\lim_{\tau\to 0^+}\tau^{-(1-p)n}J_p(v,\tau) = (1-p)^{-n}. \tag{4.2}$$

Hence

$$\tau^{-(1-p)n}e^{-C_1\tau}J_p(v,\tau) \le (1-p)^{-n} \quad \forall 0 < \tau \le \overline{\tau}. \tag{4.3}$$

If M also has uniformly bounded scalar curvature on $(0,T)$, then we can take

$$C_1 = \left(\frac{(2p-1)_+}{2(2-p)} + \frac{t_0}{2(2-p)(t_0-\overline{\tau})}\right)\|R\|_{L^\infty}. \tag{4.4}$$

Proof. Let $v \in U_p'(\overline{\tau})$ and let $\gamma_v : [0,\overline{\tau}] \to M$ be the unique $\mathcal{L}_p$-geodesic which satisfies (1.18). We extend γ_v to a $\mathcal{L}_p$-geodesic on $[0,\overline{\tau}+\varepsilon)$ for some constant $\varepsilon \in (0,t_0-\overline{\tau})$. Let

$$r_1 = \sup_{0\le\tau\le\overline{\tau}+\varepsilon} d_0(p_0,\gamma_v(\tau))$$

and let

$$C_1 = \left\{\frac{(2p-1)_+}{2(2-p)} + \frac{\overline{\tau}+\varepsilon}{2\varepsilon(2-p)}\right\}\sup_{\substack{q\,\in\, B_0(p_0,\,r_1)\\ 0\,\le\,\tau\,\le\,\overline{\tau}+\varepsilon}}|R(q,\tau)|.$$

If M also has uniformly bounded scalar curvature on $(0,T)$, by Corollary 1.1 we can choose $\varepsilon = t_0 - \overline{\tau}$ and let C_1 be given by (4.4). Since M is complete with respect to the metric $g(\tau)$ for any $\tau \in (0, t_0)$, $\overline{B_0(p_0, r_1)} \times [0, \overline{\tau}]$ is compact. Hence $C_1 < \infty$. Let $H(X)$ be given by (3.9). Since M has nonnegative curvature operator in $(0,T)$, as observed by Perelman [18] by Hamilton's Harnack inequality for the solutions of Ricci flow [8],

$$H(X(\tau)) \ge -\left(\frac{1}{\tau} + \frac{1}{\overline{\tau}+\varepsilon-\tau}\right)R(\gamma(\tau),\tau) \ge -\frac{\overline{\tau}+\varepsilon}{\tau(\overline{\tau}+\varepsilon-\tau)}R(\gamma(\tau),\tau) \tag{4.5}$$

for any $0 < \tau \le \overline{\tau}$. Since $v \in U_p'(\overline{\tau})$, by Corollary 3.3 $v \in U_p'(\tau)$ for any $0 < \tau \le \overline{\tau}$. Hence by Corollary 3.1 and (4.5),

$$\begin{aligned}
&\frac{d}{d\tau}\log J_p(v,\tau)\\
&\quad\le \frac{(1-p)n}{\tau} + \frac{2p-1}{2\tau^{2-p}}\int_0^\tau \rho^{1-p}R\,d\rho - \frac{1}{2\tau^{2-p}}\int_0^\tau \rho^{2-p}H(X)\,d\rho\\
&\quad\le \frac{(1-p)n}{\tau} + \frac{2p-1}{2\tau^{2-p}}\int_0^\tau \rho^{1-p}R\,d\rho + \frac{\overline{\tau}+\varepsilon}{2\tau^{2-p}}\int_0^\tau \frac{\rho^{1-p}}{\overline{\tau}+\varepsilon-\rho}R\,d\rho\\
&\quad\le \frac{(1-p)n}{\tau} + C_1 \quad \forall 0 < \tau \le \overline{\tau}.
\end{aligned}$$

Thus

$$\frac{d}{d\tau}\log\left(\tau^{-(1-p)n}e^{-C_1\tau}J_p(v,\tau)\right) \le 0 \quad \forall 0 < \tau \le \overline{\tau}$$

and (4.1) follows. Let $\widetilde{\gamma}_{\widetilde{v}}(s) = \gamma_v(\tau)$ and let $\widetilde{J}_p(v,s) = J_p(v,\tau)$ where $\widetilde{v} = v/(1-p)$ and $s = \tau^{1-p}$. Then $\widetilde{\gamma}_{\widetilde{v}}(s)$ satisfies (1.10) and (1.13) in $(0,\overline{s})$ where $\overline{s} = \overline{\tau}^{1-p}$. We write

$$\widetilde{\gamma}_{\widetilde{v}}(s) = (\widetilde{\gamma}_{\widetilde{v}}^1(s), \widetilde{\gamma}_{\widetilde{v}}^2(s), \dots, \widetilde{\gamma}_{\widetilde{v}}^n(s))$$

and

$$v = (v^1, v^2, \dots, v^n)$$

in the normal coordinate system around p_0 with respect to the metric $g(p_0,0)$. Differentiating (1.13) with respect to v^j, $j = 1,2,\dots,n$,

$$\begin{cases}
\dfrac{\partial\widetilde{\gamma}_{\widetilde{v}}^i}{\partial v^j}(0) = 0 & \forall i,j = 1,2,\dots,n\\[2ex]
\dfrac{d}{ds}\left(\dfrac{\partial\widetilde{\gamma}_{\widetilde{v}}^i}{\partial v^j}\right)(0) = \dfrac{\delta_j^i}{1-p} & \forall i,j = 1,2,\dots,n.
\end{cases}$$

Hence there exists $s_0 \in (0,\overline{s})$ and functions $\varepsilon_j^i(s)$ such that $\varepsilon_j^i(s) \to 0$ as $s \to 0$ for all $i,j = 1,2,\dots,n$ and

$$\frac{\partial\widetilde{\gamma}_{\widetilde{v}}^i}{\partial v^j}(s) = \frac{\delta_j^i + \varepsilon_j^i(s)}{1-p}s \quad \forall i,j = 1,2,\dots,n. \tag{4.6}$$

Since $\sqrt{g(p_0,0)}=1$ in the normal coordinates around p_0, by (4.6)

$$\begin{aligned}\widetilde{J}_p(v,s)&=\sqrt{g(p_0,s^{\frac{1}{1-p}})}\det\left(\frac{\partial\widetilde{\gamma}^i_{\widetilde{v}}}{\partial v^j}(s)\right)\\&=\frac{s^n}{(1-p)^n}\sqrt{g(p_0,s^{\frac{1}{1-p}})}\det(\delta^i_j+\varepsilon^i_j(s))\\\Rightarrow\quad\lim_{s\to0^+}s^{-n}\widetilde{J}_p(v,s)&=\frac{1}{(1-p)^n}\lim_{s\to0^+}\det(\delta^i_j+\varepsilon^i_j(s))=(1-p)^{-n}\end{aligned}$$

and (4.2) follows. By (4.1) and (4.2), (4.3) follows. □

Lemma 4.2. *Let $\overline{\tau}\in(0,t_0)$ and $v\in U_p'(\overline{\tau})$. Suppose $\gamma_v:[0,\overline{\tau}]\to M$ is the unique $\mathcal{L}_p$-geodesic that satisfies* (1.18)*. Then*

$$lim_{\tau\to0^+}l_p(\gamma_v(\tau),\tau)=|v|^2. \tag{4.7}$$

Proof. Let $\overline{s}=\overline{\tau}^{1-p}$. Let $r_1>0$, $\widetilde{v}$, and $\widetilde{\gamma}_{\widetilde{v}}$ be as in the proof of Lemma 4.1 and let

$$K_1=\sup_{\substack{q\in B_0(p_0,r_1)\\0\le\tau\le\overline{\tau}}}(|R|+|\nabla R|+|\mathrm{Ric}|).$$

Now

$$\begin{aligned}&l_p(\gamma_v(\tau),\tau)\\&=\frac{1-p}{\tau^{1-p}}\int_0^\tau\rho^pR(\gamma_v(\rho),\rho)\,d\rho+\frac{1-p}{\tau^{1-p}}\int_0^\tau\rho^p|\gamma_v'(\rho)|^2\,d\rho=I_1(\tau)+I_2(\tau)\end{aligned}\tag{4.8}$$

where

$$|I_1(\tau)|\le\frac{1-p}{1+p}K_1\tau^{2p}\to0\qquad\text{as }\tau\to0. \tag{4.9}$$

By the same argument as the proof of Lemma 1.4, there exist constants $C_2>0$, $C_3>0$, such that (1.17) holds on $[0,\overline{s}]$. Then by (1.17),

$$\begin{aligned}\int_0^\tau\frac{e^{-C_2\rho}}{\rho^p}(|v|^2-C_3'e^{C_2\rho}\rho^{1+2p})\,d\rho&\le\int_0^\tau\rho^p|\gamma_v'(\rho)|^2\,d\rho\\&\le\int_0^\tau\frac{e^{C_2\rho}}{\rho^p}(|v|^2+C_3'\rho^{1+2p})\,d\rho\end{aligned}\tag{4.10}$$

where $C_3'=(1-p)^2C_3$. Now

$$\int_0^\tau\frac{e^{C_2\rho}}{\rho^p}(|v|^2+C_3'\rho^{1+2p})\,d\rho\le e^{C_2\tau}\left(\frac{\tau^{1-p}}{1-p}|v|^2+C_3'\frac{\tau^{2+p}}{2+p}\right)\tag{4.11}$$

and

$$\int_0^\tau\frac{e^{-C_2\rho}}{\rho^p}(|v|^2-C_3'e^{C_2\rho}\rho^{1+2p})\,d\rho\ge\frac{\tau^{1-p}}{1-p}e^{-C_2\tau}|v|^2-C_3'\frac{\tau^{2+p}}{2+p}\tag{4.12}$$

By (4.10), (4.11), and (4.12),

$$
\begin{aligned}
e^{-C_2\tau}|v|^2 - C_3'\frac{1-p}{2+p}\tau^{1+2p} &\le \frac{1-p}{\tau^{1-p}}\int_0^\tau \rho^p|\gamma_v'(\rho)|^2\,d\rho \\
&\le e^{C_2\tau}\left(|v|^2 + C_3'\frac{1-p}{2+p}\tau^{1+2p}\right).
\end{aligned}
\tag{4.13}
$$

Letting $\tau \to 0$ in (4.13),

$$
\lim_{\tau\to 0^+} I_2(\tau) = |v|^2. \tag{4.14}
$$

By (4.8), (4.9), and (4.14), we get (4.7). □

Theorem 4.1. *Suppose M has nonnegative curvature operator with respect to the metric $g(\tau)$ for any $\tau \in [0,T)$. Suppose M also has uniformly bounded scalar curvature on $M \times (0,T)$ when $1/2 < p < 1$. Let $A_0 = 0$ if $p = 1/2$. For any $1/2 < p < 1$ and $0 < c < 1$, let*

$$
A_0 = \left(\frac{(2p-1)_+}{2(2-p)} + \frac{1}{2(2-p)c}\right)\|R\|_{L^\infty} \tag{4.15}
$$

and let τ_0 be given by (0.2). Then

$$
e^{-A_0\tau_2}\widetilde{V}_p(\tau_2) \le e^{-A_0\tau_1}\widetilde{V}_p(\tau_1) \le (\sqrt{\pi}/(1-p))^n \tag{4.16}
$$

for any $0 < \tau_1 \le \tau_2 < \overline{\tau}_1$ and $1/2 \le p < 1$ where $\overline{\tau}_1 = (1-c)\tau_0$ if $1/2 < p < 1$ and $\overline{\tau}_1 = t_0$ if $p = 1/2$.

Proof. Let $p \in [1/2, 1)$, $0 < \tau_2 < \overline{\tau}_1$, and $v \in U_p'(\tau_2)$. Let $\gamma = \gamma_v : [0,\tau_2] \to M$ be the unique $\mathcal{L}_p$-geodesic that satisfies (1.18) and let $X(\tau) = \gamma'(\tau)$. By Corollary 3.3 and its proof, $v \in U_p'(\tau)$ for any $0 < \tau \le \tau_2$ and $\gamma|_{[0,\tau]}$ is the unique $\mathcal{L}_p(\gamma_v(\tau),\tau)$-length minimizing $\mathcal{L}_p$-geodesic between p_0 and $\gamma_v(\tau)$ for any $0 < \tau \le \tau_2$. Hence $L_p(\gamma(\tau),\tau) = \mathcal{L}_p(\gamma(\tau),\gamma,\tau)$ and

$$
\frac{dL_p}{d\tau}(\gamma(\tau),\tau) = \frac{d}{d\tau}\mathcal{L}_p(\gamma(\tau),\gamma,\tau) = \tau^p(R(\gamma(\tau),\tau) + |X(\tau)|^2) \quad \forall 0 < \tau \le \tau_2. \tag{4.17}
$$

When there is no ambiguity we will write R, X, L_p, and l_p for $R(\gamma(\tau),\tau)$, $X(\tau)$, $L_p(\gamma(\tau),\tau)$, and $l_p(\gamma(\tau),\tau)$. Then

$$
\begin{aligned}
\frac{dl_p}{d\tau}(\gamma(\tau),\tau) &= \frac{(1-p)}{\tau^{1-p}}\frac{dL_p}{d\tau}(\gamma(\tau),\tau) - \frac{(1-p)^2}{\tau^{2-p}}L_p(\gamma(\tau),\tau) \\
&= \frac{(1-p)}{\tau^{2-p}}[\tau^{p+1}(R + |X|^2) - (1-p)L_p] \quad \forall 0 < \tau \le \tau_2.
\end{aligned}
\tag{4.18}
$$

Now by (1.11),

$$
\begin{aligned}
&\frac{d}{d\tau}(R+|X|^2)\\
&\quad = R_\tau + \langle X, \nabla R\rangle + 2\langle X, \nabla_X X\rangle + 2\operatorname{Ric}(X,X)\\
&\quad = R_\tau + \langle X, \nabla R\rangle + \langle X, \nabla R - \frac{2p}{\tau}X - 4\operatorname{Ric}(X,\cdot)\rangle + 2\operatorname{Ric}(X,X)\\
&\quad = R_\tau + 2\langle X, \nabla R\rangle - 2\operatorname{Ric}(X,X) - \frac{2p}{\tau}|X|^2\\
&\quad = -H(X) - \frac{2p}{\tau}(R+|X|^2) + \frac{2p-1}{\tau}R
\end{aligned}
\tag{4.19}
$$

where $H(X)$ is given by (3.9). Hence

$$
\begin{aligned}
&\frac{d}{d\tau}[\tau^{p+1}(R+|X|^2)]\\
&\quad = \tau^{p+1}\left\{\frac{d}{d\tau}(R+|X|^2) + \frac{(p+1)}{\tau}(R+|X|^2)\right\}\\
&\quad = \tau^{p+1}\left\{-H(X) - \frac{2p}{\tau}(R+|X|^2) + \frac{2p-1}{\tau}R + \frac{(p+1)}{\tau}(R+|X|^2)\right\}\\
&\quad = -\tau^{p+1}H(X) + (1-p)\tau^p(R+|X|^2) + (2p-1)\tau^p R \quad \forall 0<\tau\le\tau_2.
\end{aligned}
\tag{4.20}
$$

Since

$$
\lim_{\tau\to 0}\tau^{p+1}|X|^2 = 0 \tag{4.21}
$$

by (1.18), integrating (4.20) over $(0,\tau)$,

$$
\begin{aligned}
&\tau^{p+1}(R+|X|^2) = -\int_0^\tau \rho^{p+1}H(X)\,d\rho + (1-p)L_p + (2p-1)\int_0^\tau \rho^p R\,d\rho\\
&\Rightarrow\quad (1-p)L_p - \tau^{p+1}(R+|X|^2) = \int_0^\tau \rho^{p+1}H(X)\,d\rho - (2p-1)\int_0^\tau \rho^p R\,d\rho
\end{aligned}
\tag{4.22}
$$

for any $0<\tau\le\tau_2$. Let

$$
Z_p(v,\tau) = \tau^{-(1-p)n}e^{-l_p(\gamma_v(\tau),\tau)}e^{-A_0\tau}J_p(v,\tau).
$$

By Corollary 3.1, (4.18), and (4.22), $\forall 0<\tau\le\tau_2$,

$$
\begin{aligned}
&\frac{d}{d\tau}\log Z_p(v,\tau)\\
&\quad = \frac{2p-1}{2\tau^{2-p}}\int_0^\tau \rho^{1-p}R\,d\rho - \frac{1}{2\tau^{2-p}}\int_0^\tau \rho^{2-p}H(X)\,d\rho - A_0\\
&\qquad + \frac{(1-p)}{\tau^{2-p}}\left\{\int_0^\tau \rho^{p+1}H(X)\,d\rho - (2p-1)\int_0^\tau \rho^p R\,d\rho\right\}\\
&\quad \le \frac{2p-1}{2\tau^{2-p}}\int_0^\tau \rho^{1-p}R\,d\rho - A_0 - \frac{1}{2\tau^{2-p}}\int_0^\tau \rho^{2-p}(1-2(1-p)\rho^{2p-1})H(X)\,d\rho.
\end{aligned}
\tag{4.23}
$$

When $p = 1/2$, the right hand side of (4.23) is ≤ 0. When $1/2 < p < 1$, by Corollary 1.1 we can extend γ to a $\mathcal{L}_p$-geodesic on $(0,(1-c)\tau_0)$. Since $\tau_2 < (1-c)\tau_0$, by the Hamilton's Harnack inequality [8] and an argument similar to the proof of (4.5),

$$H(X(\tau)) \ge -\frac{1}{c\tau}R(\gamma(\gamma(\tau),\tau) \quad \forall 0 < \tau < (1-c)\tau_0 \tag{4.24}$$

By (4.24) when $1/2 < p < 1$, the right hand side of (4.23) is bounded above by

$$\le \frac{2p-1}{2\tau^{2-p}}\int_0^\tau \rho^{1-p}R\,d\rho - A_0 + \frac{1}{2c\tau^{2-p}}\int_0^\tau \rho^{1-p}(1-2(1-p)\rho^{2p-1})R\,d\rho \le 0$$

for any $0 < \tau < (1-c)\tau_0$. Hence by Corollary 3.3 and (4.23),

$$\begin{aligned} &Z_p(v,\tau_2) \le Z_p(v,\tau_1) \quad \forall 0 < \tau_1 \le \tau_2, v \in U_p'(\tau_2) \\ \Rightarrow \quad &\int_{U_p'(\tau_2)} Z_p(v,\tau_2)\,dv \le \int_{U_p'(\tau_2)} Z_p(v,\tau_1)\,dv \le \int_{U_p'(\tau_1)} Z_p(v,\tau_1)\,dv \end{aligned}$$

for any $0 < \tau_1 \le \tau_2 < \overline{\tau}_1$. Hence

$$e^{-A_0\tau_2}\widetilde{V}_p(\tau_2) \le e^{-A_0\tau_1}\widetilde{V}_p(\tau_1) \quad \forall 0 < \tau_1 \le \tau_2 < \overline{\tau}_1. \tag{4.25}$$

By (4.25), Lemma 4.1, Lemma 4.2, and the monotone convergence theorem, (4.26)

$$e^{-A_0\tau_2}\widetilde{V}_p(\tau_2) \le \int_{U_p'(\tau_2)} \lim_{\tau\to 0} Z_p(v,\tau)\,dv \le (1-p)^{-n}\int_{\mathbb{R}^n} e^{-|v|^2}\,dv = (\sqrt{\pi}/(1-p))^n$$

holds for any $0 < \tau_2 < \tau_0$. By (4.25) and (4.26) we get (4.16) and the lemma follows. □

5. Monotonicity Property of the rescaled generalized reduced volume

In this section we will assume that $(M,\overline{g})$ is an ancient solution of the Ricci flow in $(-\infty,0)$. We will fix a point $(p_0,t_0) \in M \times (-\infty,0)$ and let g and $\overline{g}$ be related by (0.5). Unless stated otherwise we will also assume that M has nonnegative curvature operator with respect to $g(\tau)$ for any $0 \le \tau < \infty$. We will consider the $\mathcal{L}_p$-length, L_p-distance, $\widetilde{V}_p(\tau)$, etc. all with respect to this point (p_0,t_0). We will derive various scaling properties of these geometric quantities in this section. For any $\overline{\tau} > 0$, let

$$g^{\overline{\tau}}(\tau) = \frac{1}{\overline{\tau}}g(\overline{\tau}\tau) \quad \forall \tau \ge 0.$$

and let $R^{\overline{\tau}}(q,\tau)$ be the scalar curvature of M at q with respect to the metric $g^{\overline{\tau}}(\tau)$. We also let $\mathcal{L}_p^{\overline{\tau}}$, $L_p^{\overline{\tau}}$, $l_p^{\overline{\tau}}$, $V_p^{\overline{\tau}}$ be the corresponding $\mathcal{L}_p$, L_p, l_p, V_p functions with respect to the metric $g^{\overline{\tau}}$. Note that

$$R^{\overline{\tau}}(q,\tau) = \overline{\tau}R(q,\overline{\tau}\tau) \quad \forall q \in M, \tau, \overline{\tau} > 0.$$

For any curve γ in M we let $\gamma^{\overline{\tau}}$ be the curve in M given by $\gamma^{\overline{\tau}}(\rho) = \gamma(\overline{\tau}\rho)$.

Lemma 5.1. *Let $\gamma : (\tau_1, \tau_2) \to M$ be a $\mathcal{L}_p$-geodesic in (τ_1, τ_2). Then for any $\overline{\tau} > 0$, $\gamma^{\overline{\tau}}$ is a $\mathcal{L}_p$-geodesic with respect to $g^{\overline{\tau}}$ in $(\tau_1/\overline{\tau}, \tau_2/\overline{\tau})$. If $\tau_1 = 0$ and γ satisfies* (1.18) *for some $v \in T_{p_0}M$, then $\gamma^{\overline{\tau}}$ satisfies* (1.18) *with v being replaced by $\overline{\tau}^{1-p}v$.*

Proof. Let $X(\tau) = X(\gamma(\tau)) = \gamma'(\tau)$ and let $X^{\overline{\tau}}(\rho) = X^{\overline{\tau}}(\gamma^{\overline{\tau}}(\rho)) = \gamma^{\overline{\tau}\prime}(\rho)$. Then $X^{\overline{\tau}}(\rho) = \overline{\tau}X(\overline{\tau}\rho)$, and $\forall \tau_1 < \rho < \tau_2$, $Y \in T_{\gamma(\overline{\tau}\rho)}M$,

$$(\nabla_X X)(\overline{\tau}\rho) = \frac{D}{d\tau}X(\overline{\tau}\rho) = \frac{1}{\overline{\tau}}\frac{D}{d\rho}X(\overline{\tau}\rho) = \frac{1}{\overline{\tau}^2}(\nabla_{X^{\overline{\tau}}}X^{\overline{\tau}})(\rho)$$

$$\Rightarrow \quad \langle \nabla_X X(\overline{\tau}\rho), Y\rangle_{g(\overline{\tau}\rho)} = \frac{1}{\overline{\tau}}\langle \nabla_{X^{\overline{\tau}}}X^{\overline{\tau}}(\rho), Y\rangle_{g^{\overline{\tau}}(\rho)}, \tag{5.1}$$

$$\begin{cases} \mathrm{Ric}_{g(\overline{\tau}\rho)}(X(\overline{\tau}\rho), Y) = \dfrac{1}{\overline{\tau}}\mathrm{Ric}_{g^{\overline{\tau}}(\rho)}(X^{\overline{\tau}}(\rho), Y) \\ \langle \dfrac{1}{2\overline{\tau}\rho}X(\overline{\tau}\rho), Y\rangle_{g(\overline{\tau}\rho)} = \dfrac{1}{\overline{\tau}}\langle \dfrac{1}{2\rho}X^{\overline{\tau}}(\rho), Y\rangle_{g^{\overline{\tau}}(\rho)} \\ \langle \nabla R(\gamma(\overline{\tau}\rho), \overline{\tau}\rho), Y\rangle_{g(\overline{\tau}\rho)} = \dfrac{1}{\overline{\tau}}\langle \nabla R^{\overline{\tau}}(\gamma(\overline{\tau}\rho), \rho), Y\rangle_{g^{\overline{\tau}}(\rho)}. \end{cases} \tag{5.2}$$

Since γ satisfies (1.11) in (τ_1, τ_2), by (1.11), (5.1), and (5.2), $\forall \tau_1 < \rho < \tau_2, Y \in T_{\gamma(\overline{\tau}\rho)}M$,

$$\left\langle \nabla_{X^{\overline{\tau}}}X^{\overline{\tau}} - \frac{1}{2}\nabla R^{\overline{\tau}} + \frac{p}{\rho}X^{\overline{\tau}} + 2\,\mathrm{Ric}_{g^{\overline{\tau}}(\rho)}(X^{\overline{\tau}}, \cdot), Y \right\rangle_{g^{\overline{\tau}}(\rho)} = 0$$

$$\Rightarrow \quad \nabla_{X^{\overline{\tau}}}X^{\overline{\tau}} - \frac{1}{2}\nabla R^{\overline{\tau}} + \frac{p}{\rho}X^{\overline{\tau}} + 2\,\mathrm{Ric}_{g^{\overline{\tau}}(\rho)}(X^{\overline{\tau}}, \cdot) = 0 \quad \forall \tau_1 < \rho < \tau_2.$$

If $\tau_1 = 0$ and γ satisfies (1.18) for some $v \in T_{p_0}M$, then

$$\lim_{\rho\to 0}\rho^p X^{\overline{\tau}}(\rho) = \lim_{\rho\to 0}\rho^p\overline{\tau}X(\overline{\tau}\rho) = \overline{\tau}^{1-p}\lim_{\rho\to 0}(\overline{\tau}\rho)^p X(\overline{\tau}\rho) = \overline{\tau}^{1-p}v$$

and the lemma follows. □

Lemma 5.2. *For any $q \in M$, $\overline{\tau} > 0$, $\tau > 0$, the following holds.*

(i) $L_p^{\overline{\tau}}(q, \tau) = \dfrac{L_p(q, \overline{\tau}\tau)}{\overline{\tau}^p}$

(ii) $l_p^{\overline{\tau}}(q, \tau) = \overline{\tau}^{1-2p}l_p(q, \overline{\tau}\tau)$.

Proof. Let $\gamma \in \mathcal{F}(q, \overline{\tau}\tau)$. Then $\gamma^{\overline{\tau}} \in \mathcal{F}(q, \tau)$. Let X and $X^{\overline{\tau}}$ be as in the proof of Lemma 5.1. Then

$$\begin{aligned} \mathcal{L}_p^{\overline{\tau}}(q, \gamma^{\overline{\tau}}, \tau) &= \int_0^\tau \rho^p(R^{\overline{\tau}}(\gamma^{\overline{\tau}}(\rho), \rho) + |X^{\overline{\tau}}(\rho)|^2_{g^{\overline{\tau}}(\rho)})\,d\rho \\ &= \int_0^\tau \rho^p(\overline{\tau}R(\gamma(\overline{\tau}\rho), \overline{\tau}\rho) + \overline{\tau}|X(\overline{\tau}\rho)|^2_{g(\overline{\tau}\rho)})\,d\rho \\ &= \frac{1}{\overline{\tau}^p}\int_0^{\overline{\tau}\tau} z^p(R(\gamma(z), z) + |X(z)|^2_{g(z)})\,dz \\ &= \frac{1}{\overline{\tau}^p}\mathcal{L}_p(q, \gamma, \overline{\tau}\tau) \end{aligned} \tag{5.3}$$

Since

$$\gamma \in \mathcal{F}(q, \overline{\tau}\tau) \quad \Leftrightarrow \quad \gamma^{\overline{\tau}} \in \mathcal{F}(q, \tau),$$

by taking infimum in (5.3) over $\gamma \in \mathcal{F}(q, \overline{\tau}\tau)$, (i) follows. By (0.6) and (i), (ii) follows. □

Lemma 5.3. *There exists a constant $C > 0$ such that*

$$|\nabla^{g^{\overline{\tau}}(\tau)} l_p^{\overline{\tau}}|^2(q,\tau) + 4\frac{(1-p)^2}{\tau^{2(1-2p)}} R^{\overline{\tau}}(q,\tau) \le \frac{C}{\tau^{2(1-p)}} l_p^{\overline{\tau}}(q,\tau) \tag{5.4}$$

for any $\overline{\tau} > 0$, $\tau > 0$ and $q \in \Omega_p(\overline{\tau}\tau)$.

Proof. We will use a modification of the proof of this result for the case $p = 1/2$ and $\overline{\tau} = 1$ in [18] to prove the lemma. We will first prove (5.4) for the case $\overline{\tau} = 1$. Let $\tau > 0$, and $q \in \Omega_p(\tau)$. By Theorem 1.2 there exists a unique $\mathcal{L}_p(q,\tau)$-length minimizing $\mathcal{L}_p$-geodesic γ satisfying $\gamma(0) = p_0$, $\gamma(\tau) = q$. Let $X(\rho) = \gamma'(\rho)$ for any $\rho \in [0,\tau]$. Choose $\tau_0 > 2\tau$. Then $\tau < \tau_0/2$ and (4.24) holds with $c = 1/2$. Hence by (4.22) and (4.24),

$$\begin{aligned}(R(q,\tau) + |X(\tau)|^2) &\le \frac{1}{\tau^{p+1}}\left(C_1 L_p(q,\tau) - \int_0^\tau \rho^{p+1} H(X)\,d\rho\right)\\ &\le \frac{1}{\tau^{p+1}}\left(C_1 L_p(q,\tau) + 2\int_0^\tau \rho^p R\,d\rho\right)\\ &\le \frac{C_1+2}{\tau^{p+1}} L_p(q,\tau) \end{aligned}\tag{5.5}$$

where $H(X)$ is given by (3.9) and

$$C_1 = \begin{cases} (1-p) & \text{if } 0 < p < 1/2 \\ p & \text{if } 1/2 \le p < 1. \end{cases}$$

Then by (0.6), (3.1), (5.5),

$$\begin{aligned}|\nabla l_p(q,\tau)|^2 &= \frac{(1-p)^2}{\tau^{2(1-p)}}|\nabla L_p(q,\tau)|^2\\ &= \frac{(1-p)^2}{\tau^{2(1-p)}}[4\tau^{2p}(|X(\tau)|^2 + R(q,\tau)) - 4\tau^{2p}R(q,\tau)]\\ &\le \frac{(1-p)^2}{\tau^{2(1-p)}}\left(\frac{4C_1'}{\tau^{1-p}}L_p(q,\tau) - 4\tau^{2p}R(q,\tau)\right)\\ &\le \frac{4(1-p)C_1'}{\tau^{2(1-p)}} l_p(q,\tau) - \frac{4(1-p)^2}{\tau^{2(1-2p)}}R(q,\tau) \quad \forall \tau > 0, q \in \Omega_p(\tau)\end{aligned}\tag{5.6}$$

where $C_1' = C_1 + 2$. Hence by (5.6) and Lemma 5.2 for any $\overline{\tau} > 0$, $\tau > 0$, $q \in \Omega_p(\overline{\tau}\tau)$,

$$\begin{aligned}|\nabla^{g^{\overline{\tau}}(\tau)} l_p^{\overline{\tau}}(q,\tau)|^2 &= \overline{\tau}\cdot\overline{\tau}^{2(1-2p)}|\nabla l_p(q,\overline{\tau}\tau)|^2\\ &\le \overline{\tau}^{3-4p}\left(\frac{4(1-p)C_1'}{(\overline{\tau}\tau)^{2(1-p)}} l_p(q,\overline{\tau}\tau) - \frac{4(1-p)^2}{(\overline{\tau}\tau)^{2(1-2p)}} R(q,\overline{\tau}\tau)\right)\\ &\le \frac{4(1-p)C_1'}{\tau^{2(1-p)}} l_p^{\overline{\tau}}(q,\tau) - \frac{4(1-p)^2}{\tau^{2(1-2p)}} R^{\overline{\tau}}(q,\tau)\end{aligned}$$

and (5.4) follows. □

Lemma 5.4. *There exists a constant $C > 0$ such that*

$$\left|\frac{\partial l_p^{\overline{\tau}}}{\partial \tau}(q,\tau)\right| \le C\frac{l_p^{\overline{\tau}}(q,\tau)}{\tau} \quad \forall \overline{\tau} > 0, \tau > 0, q \in \Omega_p(\overline{\tau}\tau). \tag{5.7}$$

Proof. We will use a modification of the proof of this result for the case $p = 1/2$ and $\overline{\tau} = 1$ in [18] to prove the lemma. We will first prove (5.7) for the case $\overline{\tau} = 1$. Let $\tau > 0$, and $q \in \Omega_p(\tau)$. Let γ and X be as in the proof of Lemma 5.3. Then by (3.1) and (5.5),

$$\begin{aligned}\frac{\partial L_p}{\partial \tau}(q,\tau) =& \frac{dL_p}{d\tau}(q,\tau) - \nabla L_p\cdot X = \tau^p(R + |X|^2) - 2\tau^p|X|^2\\ =& 2\tau^p R - \tau^p(R + |X|^2) \qquad (5.8)\\ \Rightarrow \quad \left|\frac{\partial L_p}{\partial \tau}(q,\tau)\right| \le& \tau^p(R + |X|^2) \le \frac{C_1'}{\tau} L_p(q,\tau) \qquad (5.9)\end{aligned}$$

where $C_1' > 0$ is as in the proof of Lemma 5.3. By (0.6) and (5.9),

$$\left|\frac{\partial l_p}{\partial \tau}(q,\tau)\right| = \frac{(1-p)}{\tau^{1-p}}\left|\frac{\partial L_p}{\partial \tau}(q,\tau) - \frac{(1-p)}{\tau} L_p(q,\tau)\right| \le C\frac{l_p(q,\tau)}{\tau} \tag{5.10}$$

for any $\tau > 0$ and $q \in \Omega_p(\tau)$. For the general case we let $\overline{\tau} > 0$, $\rho > 0$, and $q \in \Omega_p(\overline{\tau}\rho)$. Then by Lemma 5.2 and (5.10),

$$\begin{aligned}\left|\frac{\partial l_p^{\overline{\tau}}}{\partial \rho}(q,\rho)\right| &= \left|\frac{\partial}{\partial \rho}[\overline{\tau}^{1-2p} l_p(q,\overline{\tau}\rho)]\right| = \overline{\tau}^{2(1-p)}\left|\frac{\partial l_p}{\partial \tau}(q,\overline{\tau}\rho)\right|\\ &\le C\overline{\tau}^{2(1-p)}\frac{l_p(q,\overline{\tau}\rho)}{\overline{\tau}\rho} = C\frac{l_p^{\overline{\tau}}(q,\rho)}{\rho}\end{aligned}$$

and the lemma follows. □

For any $\tau \ge 0$, $q \in M$, we let

$$\widetilde{L}_p(q,\tau) = \tau^{1-p} L_p(q,\tau) \tag{5.11}$$

and

$$\begin{cases} G_p(q,\tau)=\widetilde{L}_p(q,\tau)-\dfrac{n}{2p}\tau \\ G_p(\tau)=\min\limits_{q\in M} G_p(q,\tau). \end{cases} \tag{5.12}$$

Note that by Lemma 1.6 $G_p(\tau)$ is well-defined.

Lemma 5.5. *$G_p(\tau)$ is a decreasing function of $\tau\ge 0$.*

Proof. Let $\tau>0$ and $q\in\Omega_p(\tau)$. Let γ and X be as in the proof of Lemma 5.3. By (4.22) and (5.8),

(5.13)
$$\frac{\partial L_p}{\partial\tau}(q,\tau)=2\tau^pR-\frac{(1-p)}{\tau}L_p(q,\tau)+\frac{1}{\tau}\int_0^\tau\rho^{p+1}H(X)\,d\rho-\frac{(2p-1)}{\tau}\int_0^\tau\rho^pR\,d\rho$$

where the integration is along the curve γ. Putting $b=1/2$ in (3.8),

(5.14)
$$\Delta L_p(q,\tau)\le -2\tau^pR(q,\tau)+\frac{n}{2p\tau^{1-p}}+\frac{2p-1}{\tau}\int_0^\tau\rho^pR\,d\rho-\frac{1}{\tau}\int_0^\tau\rho^{p+1}H(X)\,d\rho$$

By (5.13) and (5.14),

$$\begin{aligned}
&\frac{\partial L_p}{\partial\tau}(q,\tau)+\Delta L_p(q,\tau)\le\frac{n}{2p\tau^{1-p}}-\frac{(1-p)}{\tau}L_p(q,\tau)\quad\forall\tau>0,q\in\Omega_p(\tau)\\
\Rightarrow\ &\frac{\partial\widetilde{L}_p}{\partial\tau}(q,\tau)+\Delta\widetilde{L}_p(q,\tau)\le\tau^{1-p}\left(\frac{\partial L_p}{\partial\tau}(q,\tau)+\Delta L_p(q,\tau)+\frac{(1-p)}{\tau}L_p(q,\tau)\right)\\
&\qquad\qquad\qquad\qquad\quad=\frac{n}{2p}
\end{aligned}$$

(5.15)
$$\Rightarrow\ \frac{\partial G_p}{\partial\tau}(q,\tau)+\Delta G_p(q,\tau)\le 0\quad\forall\tau>0,q\in\Omega_p(\tau).$$

By (5.15), Corollary 3.4, Corollary 3.5, and an argument similar to the proof of Proposition 2.15 of [25],

$$\frac{\partial G_p}{\partial\tau}(q,\tau)+\Delta G_p(q,\tau)\le 0\quad\text{in } M\times(0,\infty) \tag{5.16}$$

in the barrier sense of Perelman [18]. By the same argument as the proof of Lemma 3.1 of [25] but with (5.16) replacing (3.1) in the proof there the lemma follows. □

Corollary 5.1.

$$\min_{q\in M} l_p^{\overline{\tau}}(q,\tau)\le\frac{n(1-p)}{2p\tau^{1-2p}}\quad\forall\overline{\tau}>0,\tau>0. \tag{5.17}$$

Proof. Let $\overline{\tau} > 0$ and $\tau > 0$. By Lemma 5.5,

$$
\begin{aligned}
0 = G_p(0) \ge G_p(\tau) &= \min_{q\in M}\left(\tau^{1-p}L_p(q,\tau) - \frac{n}{2p}\tau\right)\\
&= \min_{q\in M}\left(\frac{\tau^{2(1-p)}}{1-p}l_p(q,\tau) - \frac{n}{2p}\tau\right)
\end{aligned}
$$

$$
\Rightarrow \quad \min_{q\in M} l_p(q,\tau) \le \frac{n(1-p)}{2p\tau^{1-2p}}. \tag{5.18}
$$

By Lemma 5.2 and (5.18), (5.17) follows. □

By Corollary 5.1 for any $\overline{\tau} > 0$ there exists $q(\overline{\tau}) \in M$ such that

$$
l_p^{\overline{\tau}}(q(\overline{\tau}),1) = \min_{q\in M} l_p^{\overline{\tau}}(q,1) \le \frac{n(1-p)}{2p}. \tag{5.19}
$$

Note that (5.17) and (5.19) will be used in the proof of the following lemma.

Lemma 5.6. *For any $r_0 > 0$, $\tau_2 > \tau_1 > 0$, there exists a constant $C_1 = C_1(r_0,\tau_1,\tau_2) > 0$ such that*

$$
R^{\overline{\tau}}(q,\tau) + l_p^{\overline{\tau}}(q,\tau) \le C_1 \tag{5.20}
$$

holds for any $\tau_1 \le \tau \le \tau_2$, $\overline{\tau} > 0$ and $q \in M$ satisfying

$$
d_{g^{\overline{\tau}}(1)}(q(\overline{\tau}),q) \le r_0. \tag{5.21}
$$

Proof. Let $\tau_1 \le \tau \le \tau_2$, $\overline{\tau} > 0$, and $q \in M$ satisfy (5.21). Let $\gamma : [0,d] \to M$ be a minimal normalized geodesic joining q and $q(\overline{\tau})$ with respect to the metric $g^{\overline{\tau}}(1)$ where $d = d_{g^{\overline{\tau}}(1)}(q(\overline{\tau}),q)$. Then by Lemma 5.3,

$$
\begin{aligned}
|l_p^{\overline{\tau}}(q,1)^{\frac12} - l_p^{\overline{\tau}}(q(\overline{\tau}),1)^{\frac12}| &= \left|\int_0^d \frac{\partial}{\partial\rho} l_p^{\overline{\tau}}(\gamma(\rho),1)^{\frac12}\,d\rho\right|\\
&= \left|\int_0^d \langle \nabla^{g^{\overline{\tau}}(1)} l_p^{\overline{\tau}}(\gamma(\rho),1)^{\frac12}, \gamma'(\rho)\rangle_{g^{\overline{\tau}}(1)}\,d\rho\right|\\
&\le \int_0^d |\nabla^{g^{\overline{\tau}}(1)}(l_p^{\overline{\tau}}(\gamma(\rho),1)^{\frac12})|\,d\rho\\
&\le Cd\\
&\le Cr_0
\end{aligned}
$$

$$
\Rightarrow \quad l_p^{\overline{\tau}}(q,1) \le (l_p^{\overline{\tau}}(q(\overline{\tau}),1)^{\frac12} + Cr_0)^2 \quad \forall \tau_1 \le \tau \le \tau_2. \tag{5.22}
$$

By Lemma 5.4 there exists a constant $a > 0$ such that $\forall \overline{\tau} > 0, \tau_1 \le \tau \le \tau_2$,

$$
l_p^{\overline{\tau}}(q,\tau) \le l_p^{\overline{\tau}}(q,1)\left(\tau^a + \frac{1}{\tau^a}\right) \tag{5.23}
$$

holds for any $q \in \Omega_p(\overline{\tau}\tau_2)$. Since $\Omega_p(\overline{\tau}\tau_2)$ is dense in M and $l_p^{\overline{\tau}}(q,\tau)$ is a continuous function, (5.23) holds for any $q \in M$. Hence by (5.19), (5.22), and (5.23),

$$l_p^{\overline{\tau}}(q,\tau) \le (\sqrt{n(1-p)/(2p)} + Cr_0)^2\left(\tau_2^a + \frac{1}{\tau_1^a}\right) \quad \forall \tau_1 \le \tau \le \tau_2. \tag{5.24}$$

By (5.24) and Lemma 5.3, (5.20) follows. □

Since

$$\frac{\partial}{\partial \tau} g_{ij}^{\overline{\tau}} = 2R_{ij}^{\overline{\tau}},$$

by (5.20), Lemma 1.5, Lemma 5.3, Lemma 5.4 and an argument similar to the proof of Lemma 5.6 we have the following lemma.

Lemma 5.7. *For any $r_0 > 0$, $\tau_2 > \tau_1 > 0$, there exists a constant $C > 0$ such that*

$$\begin{cases} |l_p^{\overline{\tau}}(q_1,\tau) - l_p^{\overline{\tau}}(q_2,\tau)| \le C d_{g^{\overline{\tau}}(1)}(q_1,q_2) \\ |l_p^{\overline{\tau}}(q,\rho_1) - l_p^{\overline{\tau}}(q,\rho_2)| \le C|\rho_1 - \rho_2| \end{cases}$$

for any $\overline{\tau} > 0$, $\tau_1 \le \tau \le \tau_2$, $\rho_1,\rho_2 \in [\tau_1,\tau_2]$ and q_1,q_2,q satisfying (5.21).

By Lemma 5.6, Lemma 5.7, and an argument similar to the sketch of proof of Proposition 11.2 of [18] and a diagonalization argument we have

Corollary 5.2. *Suppose $(M,\overline{g})$ has nonnegative curvature operator in $(-\infty,0)$. If $(M,\overline{g})$ is κ-noncollapsing on all scales, then there exist a sequence $\{q_i\}_{i=1}^{\infty} \subset M$ and a sequence $\{\overline{\tau}_i\}_{i=1}^{\infty} \subset \mathbb{R}^+$, $\overline{\tau}_i \to \infty$ as $i \to \infty$, such that $l^{\overline{\tau}_i}(q,\tau)$ converges uniformly on*

$$d_{g^{\overline{\tau}_i}(1)}(q_i,q) \le r_0, \tau_1 \le \tau \le \tau_2$$

as $i \to \infty$ for any $r_0 > 0$ and $\tau_2 > \tau_1 > 0$.

For any $\overline{\tau} > 0$, $\rho > 0$, $0 < p < 1$, let

$$\widetilde{V}_p^{\overline{\tau}}(\rho) = \int_M \rho^{-\frac{(1-p)n}{2}} e^{-l_p^{\overline{\tau}}(q,\rho)}\, dV_{g^{\overline{\tau}}(\rho)}(q).$$

Theorem 5.1. *Suppose $(M,\overline{g})$ is an ancient κ-solution of the Ricci flow. Let g and $\overline{g}$ be related by* (0.5) *for some constant $t_0 < 0$. Let $\overline{\tau}_0 > 0$ for $1/2 < p < 1$ and $\overline{\tau}_0 = 0$ for $p = 1/2$. When $1/2 < p < 1$, suppose also that $(M,g(\tau))$ is compact and satisfies* (1.21) *in $M \times (0,\infty)$ for some constant $c_2 > 0$. Let $A_0 = 0$ for $p = 1/2$ and A_0 be given by* (4.15) *with $c = 1$ for $1/2 < p < 1$. Then for any $1/2 \le p < 1$ there exist constants $A_1 \ge 0$, $A_2 \ge 0$, such that $e^{-W(\overline{\tau},\rho)}\widetilde{V}_p^{\overline{\tau}}(\rho)$ is a monotone decreasing function of $\overline{\tau} > \overline{\tau}_0$ for any ρ satisfying* (0.3) *where*

$$W(\overline{\tau},\rho) = (A_0\rho + A_1\rho^{2p} + A_2\rho^{2p-3}e^{2c_2\overline{\tau}\rho})\overline{\tau} \tag{5.25}$$

with $A_1 = A_2 = 0$ for $p = 1/2$. Moreover (0.4) *holds for any $1/2 \le p < 1$.*

Proof. Let ρ satisfy (0.3), $\overline{\tau}_1 > \overline{\tau}_0$, and $v \in U_p'(\overline{\tau}_1\rho)$. Let $\gamma_v : [0, \overline{\tau}_1\rho] \to M$ be the unique $\mathcal{L}_p(\gamma_v(\overline{\tau}_1\rho), \overline{\tau}_1\rho)$-length minimizing $\mathcal{L}_p$-geodesic given by Theorem 1.2 which satisfies (1.18). By Corollary 3.3 and an argument similar to the proof of Theorem 4.1, $v \in U_p'(\overline{\tau}\rho)$ and $L_p(\gamma(\overline{\tau}\rho), \overline{\tau}\rho) = \mathcal{L}_p(\gamma(\overline{\tau}\rho), \gamma, \overline{\tau}\rho)$ for any $0 < \overline{\tau} \le \overline{\tau}_1$. Let

$$Z_p^{\overline{\tau}}(v, \rho) = \overline{\tau}^{-\frac{n}{2}} \rho^{-\frac{(1-p)n}{2}} e^{-W(\overline{\tau}, \rho)} e^{-\overline{\tau}^{1-2p} l_p(\gamma_v(\overline{\tau}\rho), \overline{\tau}\rho)} J_p(v, \overline{\tau}\rho)$$

where $W(\overline{\tau}, \rho)$ is given by (5.25), $A_1 \ge 0$, $A_2 \ge 0$, are constants to be determined later for $1/2 < p < 1$ and $A_1 = A_2 = 0$ for $p = 1/2$. By Lemma 5.2, Corollary 3.1, (4.18) and (4.22), for any $\overline{\tau}_0 < \overline{\tau} \le \overline{\tau}_1$,

$$\begin{aligned}
&\frac{d}{d\overline{\tau}} \log Z_p^{\overline{\tau}}(v, \rho) \\
&\quad \le -\frac{n}{2\overline{\tau}} - \frac{(1-2p)}{\overline{\tau}^{2p}} l_p(\gamma_v(\overline{\tau}\rho), \overline{\tau}\rho) - \overline{\tau}^{(1-2p)} \rho \frac{d}{d\tau}(l_p(\gamma_v(\overline{\tau}\rho), \overline{\tau}\rho)) + \rho \frac{d}{d\tau} \log J_p(v, \overline{\tau}\rho) \\
&\qquad - (A_0\rho + A_1\rho^{2p}) - 2c_2 A_2 \rho^{2p-2} \overline{\tau} e^{2c_2\overline{\tau}\rho} \\
&\quad \le -\frac{n}{2\overline{\tau}} - \frac{(1-2p)}{\overline{\tau}^{2p}} l_p + \frac{(1-p)}{\overline{\tau}^{1+p}\rho^{1-p}} \left(\int_0^{\overline{\tau}\rho} w^{p+1} H(X)\, dw - (2p-1) \int_0^{\overline{\tau}\rho} w^p R\, dw \right) \\
&\qquad + \frac{(1-p)n}{\overline{\tau}} + \frac{2p-1}{2\overline{\tau}^{2-p}\rho^{1-p}} \int_0^{\overline{\tau}\rho} w^{1-p} R\, dw - \frac{1}{2\overline{\tau}^{2-p}\rho^{1-p}} \int_0^{\overline{\tau}\rho} w^{2-p} H(X)\, dw \\
&\qquad - (A_0\rho + A_1\rho^{2p}) - 2c_2 A_2 \rho^{2p-2} \overline{\tau} e^{2c_2\overline{\tau}\rho} \\
&\quad \le -(2p-1)\frac{n}{2\overline{\tau}} + \frac{(2p-1)}{\overline{\tau}^{2p}} l_p + \frac{(2p-1)}{2\overline{\tau}^{2-p}\rho^{1-p}} \int_0^{\overline{\tau}\rho} w^{1-p} R\, dw - (A_0\rho + A_1\rho^{2p}) \\
&\qquad - \frac{1}{2\overline{\tau}^{2-p}\rho^{1-p}} \int_0^{\overline{\tau}\rho} \left(1 - 2(1-p)\left(\frac{w}{\overline{\tau}}\right)^{2p-1} \right) w^{2-p} H(X)\, dw \\
&\qquad - 2c_2 A_2 \rho^{2p-2} \overline{\tau} e^{2c_2\overline{\tau}\rho}
\end{aligned} \tag{5.26}$$

where the integration is along the curve γ_v. We now divide the proof into two cases.

Case 1:$p = 1/2$.

Then the right hand side of (5.26) is ≤ 0 for any $\overline{\tau}_0 < \overline{\tau} \le \overline{\tau}_1$.

Case 2:$1/2 < p < 1$.

Since M has uniformly bounded Ricci curvature, by Corollary 1.1 we can extend γ_v to a $\mathcal{L}_p$-geodesic on $(0, \infty)$. For any $0 < c < 1$, $\tau > 0$, choose τ_0 such that $\tau < (1-c)\tau_0$. Then by the Hamilton Harnack inequality [8] (4.24) holds. Letting $\tau_0 \to \infty$ and $c \to 1$ in (4.24),

$$H(X(\tau)) \ge -\frac{1}{\tau} R(\gamma(\gamma(\tau), \tau) \quad \forall \tau > 0. \tag{5.27}$$

By (5.27) the right hand side of (5.26) is bounded above by

$$
\begin{aligned}
&\le -(2p-1)\frac{n}{2\overline{\tau}} + \frac{(2p-1)}{\overline{\tau}^{2p}}l_p + \frac{(2p-1)}{2\overline{\tau}^{2-p}\rho^{1-p}}\int_0^{\overline{\tau}\rho} w^{1-p}R\,dw - (A_0\rho + A_1\rho^{2p})\\
&\quad + \frac{1}{2\overline{\tau}^{2-p}\rho^{1-p}}\int_0^{\overline{\tau}\rho}\left(1-2(1-p)\left(\frac{w}{\overline{\tau}}\right)^{2p-1}\right)w^{1-p}R\,dw\\
&\quad - 2c_2A_2\rho^{2p-2}\overline{\tau}e^{2c_2\overline{\tau}\rho}
\end{aligned}
$$

(5.28)
$$
\le -(2p-1)\frac{n}{2\overline{\tau}} + \frac{(2p-1)}{\overline{\tau}^{2p}}l_p - A_1\rho^{2p} - 2c_2A_2\rho^{2p-2}\overline{\tau}e^{2c_2\overline{\tau}\rho}
$$

Since M is compact and satisfies (1.21), by (0.6) and Lemma 1.7 there exists a constant $C_0 > 0$ such that

$$
l_p(\gamma_v(\overline{\tau}\rho), \overline{\tau}\rho) \le C_0\left((\overline{\tau}\rho)^{2p} + \frac{e^{2c_2\overline{\tau}\rho}}{(\overline{\tau}\rho)^{2-2p}}\right). \tag{5.29}
$$

Let $A_1 = (2p-1)C_0$ and $A_2 = (2p-1)C_0/(2c_2\overline{\tau}_0^3)$. Then by (5.28) and (5.29) the right hand side of (5.26) is ≤ 0 for any $\overline{\tau}_0 < \overline{\tau} \le \overline{\tau}_1$.

By case 1 and case 2 and an argument similar to the proof of Theorem 4.1 we get that $e^{-W(\overline{\tau},\rho)}\widetilde{V}_p^{\overline{\tau}}(\rho)$ is a monotone decreasing function of $\overline{\tau} > \overline{\tau}_0$. Hence when $p = 1/2$, $\widetilde{V}_p^{\overline{\tau}}(\rho)$ is a monotone decreasing function of $\overline{\tau} > 0$. We now write

$$
Z_p^{\overline{\tau}}(v,\rho) = e^{-W(\overline{\tau},\rho)}[(\overline{\tau}\rho)^{-(1-p)n}J_p(v,\overline{\tau}\rho)][\overline{\tau}^{-\frac{n}{2}(2p-1)}e^{-\overline{\tau}^{1-2p}l_p(\gamma_v(\overline{\tau}\rho),\overline{\tau}\rho)}]\rho^{\frac{(1-p)n}{2}}.
$$

By Theorem 2.1 and Corollary 3.3 there exist constants $\overline{\tau}_0 > 0$ and $r_1 > 0$ such that $\overline{\mathcal{B}(0,r_1)} \subset U_p'(\overline{\tau}\rho)$ for all $0 < \overline{\tau} \le \overline{\tau}_0$. Since $\overline{\mathcal{B}(0,r_1)}$ is compact and the solution of a $\mathcal{L}_p$-geodesic depends continuously on the initial data, by an argument similar to the proof of Lemma 4.1, for any $\varepsilon > 0$, there exists $\overline{\tau}_1 \in (0,\overline{\tau}_0)$ and a constant $C_1 > 0$ such that $\forall v \in \overline{\mathcal{B}(0,r_1)}$,

$$
(1-p)^{-n} - \varepsilon \le (\overline{\tau}\rho)^{-(1-p)n}e^{-C_1\overline{\tau}\rho}J_p(v,\overline{\tau}\rho) \le (1-p)^{-n} + \varepsilon \quad \forall 0 < \overline{\tau} \le \overline{\tau}_1.
$$

Hence $\forall v \in \overline{\mathcal{B}(0,r_1)}, 0 < \overline{\tau} \le \overline{\tau}_1$,

(5.30)
$$
\begin{cases}
Z_p^{\overline{\tau}}(v,\rho) \le e^{C_1\overline{\tau}\rho - W(\overline{\tau},\rho)}((1-p)^{-n}+\varepsilon)\rho^{\frac{(1-p)n}{2}}[\overline{\tau}^{-\frac{n}{2}(2p-1)}e^{-\overline{\tau}^{1-2p}l_p(\gamma_v(\overline{\tau}\rho),\overline{\tau}\rho)}]\\
Z_p^{\overline{\tau}}(v,\rho) \ge e^{C_1\overline{\tau}\rho - W(\overline{\tau},\rho)}((1-p)^{-n}-\varepsilon)\rho^{\frac{(1-p)n}{2}}[\overline{\tau}^{-\frac{n}{2}(2p-1)}e^{-\overline{\tau}^{1-2p}l_p(\gamma_v(\overline{\tau}\rho),\overline{\tau}\rho)}].
\end{cases}
$$

Let

$$
g(v,\overline{\tau},\rho) = \overline{\tau}^{-\frac{n}{2}(2p-1)}e^{-\overline{\tau}^{1-2p}l_p(\gamma_v(\overline{\tau}\rho),\overline{\tau}\rho)}.
$$

By the proof of Lemma 4.2 there exists constants $C_2 > 0$, $C_3' > 0$, $K_1 > 0$ such that (4.8), (4.9), and (4.13) holds. Let $C_4 = (1-p)K_1/(1+p)$ and

$C_5 = C_3'(1-p)/(2+p)$. Then by (4.8), (4.9), and (4.13),

$$
\begin{aligned}
g(v,\overline{\tau},\rho) &\le \overline{\tau}^{-\frac{n}{2}(2p-1)} e^{-\overline{\tau}^{1-2p}[-C_4(\overline{\tau}\rho)^{2p}+e^{-C_2\overline{\tau}\rho}|v|^2-C_5(\overline{\tau}\rho)^{1+2p}]}\\
&\le \overline{\tau}^{-\frac{n}{2}(2p-1)} e^{-e^{-C_2\overline{\tau}\rho}(|v|/\overline{\tau}^{(2p-1)/2})^2} e^{C_4\overline{\tau}\rho^{2p}+C_5\overline{\tau}^2\rho^{1+2p}}
\end{aligned} \tag{5.31}
$$

for any $v\in\Omega_p(\overline{\tau}\rho)$. Similarly

$$
g(v,\overline{\tau},\rho) \ge \overline{\tau}^{-\frac{n}{2}(2p-1)} e^{-e^{C_2\overline{\tau}\rho}(|v|/\overline{\tau}^{(2p-1)/2})^2} e^{-C_4\overline{\tau}\rho^{2p}-C_5\overline{\tau}^2\rho^{1+2p}e^{C_2\overline{\tau}\rho}} \tag{5.32}
$$

for any $v\in\Omega_p(\overline{\tau}\rho)$. Hence

$$
\begin{aligned}
&\int_{U_p'(\overline{\tau}\rho)} g(v,\overline{\tau},\rho)\,dv\\
&\quad\le e^{C_4\overline{\tau}\rho^{2p}+C_5\overline{\tau}^2\rho^{1+2p}}\overline{\tau}^{-\frac{n}{2}(2p-1)}\int_{T_{p_0}M} e^{-e^{-C_2\overline{\tau}\rho}(|v|/\overline{\tau}^{(2p-1)/2})^2}\,dv\\
&\quad\le e^{C_4\overline{\tau}\rho^{2p}+C_5\overline{\tau}^2\rho^{1+2p}+(nC_2/2)\overline{\tau}\rho}\int_{\mathbb{R}^n} e^{-|v'|^2}\,dv'\\
&\quad\le e^{C_4\overline{\tau}\rho^{2p}+C_5\overline{\tau}^2\rho^{1+2p}+(nC_2/2)\overline{\tau}\rho}\pi^{\frac{n}{2}}.
\end{aligned}
$$

Thus

$$
\limsup_{\overline{\tau}\to 0^+}\int_{U_p'(\overline{\tau}\rho)} g(v,\overline{\tau},\rho)\,dv \le \pi^{\frac{n}{2}}. \tag{5.33}
$$

By (5.30) and (5.33),

$$
\begin{aligned}
&\limsup_{\overline{\tau}\to 0^+}\widetilde{V}_p^{\overline{\tau}}(\rho) \le [(1-p)^{-n}+\varepsilon]\pi^{\frac{n}{2}}\\
\Rightarrow\quad &\limsup_{\overline{\tau}\to 0^+}\widetilde{V}_p^{\overline{\tau}}(\rho) \le (1-p)^{-n}\pi^{\frac{n}{2}} \qquad \text{as } \varepsilon\to 0.
\end{aligned} \tag{5.34}
$$

Similarly by (5.32),

$$
\int_{\mathcal{B}(0,r_1)} g(v,\overline{\tau},\rho)\,dv \ge e^{-C_4\overline{\tau}\rho^{2p}-C_5\overline{\tau}^2\rho^{1+2p}-(nC_2/2)\overline{\tau}\rho}\int_{\mathcal{B}(0,r_2)} e^{-|v'|^2}\,dv'. \tag{5.35}
$$

where $r_2=\overline{\tau}^{-\frac{2p-1}{2}}e^{\frac{C_2\overline{\tau}\rho}{2}}r_1$. Since $r_2\to\infty$ as $\overline{\tau}\to 0$, letting $\overline{\tau}\to 0$ in (5.35),

$$
\liminf_{\overline{\tau}\to 0^+}\int_{\mathcal{B}(0,r_1)} g(v,\overline{\tau},\rho)\,dv \ge \pi^{\frac{n}{2}}. \tag{5.36}
$$

By (5.30) and (5.36),

$$
\begin{aligned}
&\liminf_{\overline{\tau}\to 0^+}\widetilde{V}_p(\rho) \ge [(1-p)^{-n}-\varepsilon]\pi^{\frac{n}{2}}\\
\Rightarrow\quad &\liminf_{\overline{\tau}\to 0^+}\widetilde{V}_p(\rho) \ge (1-p)^{-n}\pi^{\frac{n}{2}} \qquad \text{as } \varepsilon\to 0.
\end{aligned} \tag{5.37}
$$

By (5.34) and (5.37), (0.4) follows. $\square$

6. Smoothness property of the reduced distance

In this section we will prove a result on the reduced distance l and the reduced volume $\widetilde{V}(\tau)$ used by Perelman in [18]. This result was used in the proof of Proposition 11.2 of [18] but no proof was given by Perelman in [18]. We will assume that $(M, \overline{g})$ is an ancient κ-solution with g and $\overline{g}$ being related by (0.5) for some fixed $t_0 < 0$. For any $\tau > 0$, let $\Omega(\tau) = \Omega_{\frac{1}{2}}(\tau)$. We also fix a point $p_0 \in M$ and have $L(q,\tau)$, $\widetilde{V}(\tau)$, etc. all defined with respect to the reference point (p_0, t_0).

By an argument similar to the proof of Theorem 6 of [5] we have

Lemma 6.1. *Let $\tau_2 > \tau_1 > 0$. For any $r_0 > 0$ there exists a unique solution $0 \le f \in C^\infty(\overline{B_0(p_0, r_0)} \times [\tau_1, \tau_2])$ of*

$$(6.1)\qquad \begin{cases} f_\tau = \Delta f - Rf & \text{in } B_0(p_0, r_0) \times (\tau_1, \tau_2) \\ f(q,\tau) = \tau^{-\frac{n}{2}} e^{-l(q,\tau)} & \text{on } \partial B_0(p_0, r_0) \times (\tau_1, \tau_2) \\ f(q,\tau_1) = \tau_1^{-\frac{n}{2}} e^{-l(q,\tau_1)} & \text{in } B_0(p_0, r_0). \end{cases}$$

We now state and prove a result that was used implicitly by Perelman in his proof of Proposition 11.2 but no proof of it was given in [18].

Theorem 6.1. *Suppose $\widetilde{V}(\tau_1) = \widetilde{V}(\tau_2)$ for some $\tau_2 > \tau_1 > 0$. Then $l(q,\tau) \in C^\infty(M \times (\tau_1, \tau_2))$ and satisfies*

$$(6.2)\qquad l_\tau - \Delta l + |\nabla l|^2 - R + \frac{n}{2\tau} = 0$$

in $M \times (\tau_1, \tau_2)$ in the classical sense.

Proof. Suppose $\widetilde{V}(\tau_1) = \widetilde{V}(\tau_1)$ for some $\tau_2 > \tau_1 > 0$. Let $r_0 > 0$ and let f be the solution of (6.1) given by Lemma 6.1. Let

$$Q(\phi) = \int_{\tau_1}^{\tau_2} \int_M \left\{ \nabla l \cdot \nabla \phi + \left(l_\tau + |\nabla l|^2 - R + \frac{n}{2\tau} \right) \phi \right\} dV_{g(\tau)}\, d\tau.$$

By [25],

$$(6.3)\qquad Q(\phi) = 0$$

holds for any Lipschitz function ϕ on $M \times [\tau_1, \tau_2]$ which satisfies

$$\begin{cases} |\phi(q,\tau)| \le Ce^{-l(q,\tau)} & \forall q \in M, \tau_1 \le \tau \le \tau_2 \\ |\nabla \phi(q,\tau)| \le Ce^{-l(q,\tau)} & \forall q \in M, \tau_1 \le \tau \le \tau_2 \end{cases}$$

for some constant $C > 0$. Let $h(q,\tau) = \tau^{-\frac{n}{2}} e^{-l(q,\tau)}$. Then by (6.3) for any $\phi \in C_0^\infty(M \times [\tau_1, \tau_2])$,

$$(6.4)\qquad Q(h\phi) = 0 \quad \Rightarrow \quad \int_{\tau_1}^{\tau_2} \int_M [(h_\tau + Rh)\phi + \nabla h \cdot \nabla \phi]\, dV_{g(\tau)}\, d\tau = 0.$$

Let $0 \le \theta \in C_0^\infty(B_0(p_0, r_0))$. For any $\rho \in (\tau_1, \tau_2]$, let $0 \le \psi \in C_0^\infty(B_0(p_0, r_0) \times [\tau_1, \rho])$ be the solution of

$$\begin{cases} \psi_\tau + \Delta\psi = 0 & \text{in } B_0(p_0, r_0) \times (\tau_1, \rho) \\ \psi(q, \tau) = 0 & \text{on } \partial B_0(p_0, r_0) \times (\tau_1, \rho) \\ \psi(q, \rho) = \theta & \text{in } B_0(p_0, r_0). \end{cases} \tag{6.5}$$

For any $k \in \mathbb{Z}^+$, let $r_k = (2k-1)r_0/(2k)$ and $\eta_k \in C_0^\infty(B_0(p_0, r_0))$, $0 \le \eta_k \le 1$, such that $\eta_k \equiv 1$ on $\overline{B_0(p_0, r_k)}$ and $\eta_k \equiv 0$ on $M \setminus B_0(p_0, r_0)$. Then by (6.1), (6.4), and (6.5),

$$\begin{aligned} &\int_{B_0(p_0,r_0)} (f-h)(q,\rho)\theta(q)\, dV_{g(\rho)} \\ =& \int_{B_0(p_0,r_0)} (f-h)(q,\rho)\psi(q,\rho)\, dV_{g(\rho)} - \int_{B_0(p_0,r_0)} (f-h)(q,\tau_1)\psi(q,\tau_1)\, dV_{g(\tau_1)} \\ =& \int_{\tau_1}^{\rho} \frac{d}{d\tau}\left(\int_{B_0(p_0,r_0)} (f-h)\psi\, dV_{g(\tau)}\right) d\tau \\ =& \int_{\tau_1}^{\rho}\int_{B_0(p_0,r_0)} [(f_\tau - h_\tau)\psi + (f-h)\psi_\tau + R(f-h)\psi]\, dV_{g(\tau)} d\tau \\ =& \int_{\tau_1}^{\rho}\int_{B_0(p_0,r_0)} [\psi\Delta f + (f-h)\psi_\tau]\, dV_{g(\tau)} d\tau \\ &\quad - \int_{\tau_1}^{\rho}\int_{B_0(p_0,r_0)} (h_\tau + Rh)\psi\, dV_{g(\tau)} d\tau. \end{aligned} \tag{6.6}$$

Now by (6.4),

$$\begin{aligned} &\int_{\tau_1}^{\rho}\int_{B_0(p_0,r_0)} (h_\tau + Rh)\psi\eta_k\, dV_{g(\tau)} d\tau \\ =& -\int_{\tau_1}^{\rho}\int_{B_0(p_0,r_0)} \nabla h \cdot \nabla(\psi\eta_k)\, dV_{g(\tau)} d\tau \\ =& -\int_{\tau_1}^{\rho}\int_{B_0(p_0,r_0)} (\eta_k \nabla h \cdot \nabla\psi + \psi\nabla h \cdot \nabla\eta_k)\, dV_{g(\tau)} d\tau \\ =& -\int_{\tau_1}^{\rho}\int_{B_0(p_0,r_0)} (\nabla(h\eta_k) \cdot \nabla\psi + \psi\nabla h \cdot \nabla\eta_k - h\nabla\psi \cdot \nabla\eta_k)\, dV_{g(\tau)} d\tau \\ =& \int_{\tau_1}^{\rho}\int_{B_0(p_0,r_0)} h\eta_k\Delta\psi\, dV_{g(\tau)} d\tau - \int_{\tau_1}^{\rho}\int_{B_0(p_0,r_0)} \psi\nabla h \cdot \nabla\eta_k\, dV_{g(\tau)} d\tau \\ &\quad + \int_{\tau_1}^{\rho}\int_{B_0(p_0,r_0)} h\nabla\psi \cdot \nabla\eta_k\, dV_{g(\tau)} d\tau. \end{aligned} \tag{6.7}$$

Since $\nabla h \in L^\infty(B_0(p_0, r_0) \times [\tau_1, \rho])$ by the proof of Lemma 2.2, letting $k \to \infty$

in (6.7),

$$
\begin{aligned}
&\int_{\tau_1}^{\rho}\int_{B_0(p_0,r_0)}(h_\tau+R)\psi\,dV_{g(\tau)}d\tau\\
(6.8)\qquad &=\int_{\tau_1}^{\rho}\int_{B_0(p_0,r_0)}h\Delta\psi\,dV_{g(\tau)}d\tau-\int_{\tau_1}^{\rho}\int_{\partial B_0(p_0,r_0)}h\frac{\partial\psi}{\partial n}\,d\sigma d\tau.
\end{aligned}
$$

By (6.1), (6.6) and (6.8),

$$
\begin{aligned}
&\int_{B_0(p_0,r_0)}(f-h)(q,\rho)\theta(q)\,dV_{g(\rho)}\\
&=\int_{\tau_1}^{\rho}\int_{B_0(p_0,r_0)}[\psi\Delta f-h\Delta\psi+(f-h)\psi_\tau]\,dV_{g(\tau)}d\tau+\int_{\tau_1}^{\rho}\int_{\partial B_0(p_0,r_0)}h\frac{\partial\psi}{\partial n}\,d\sigma d\tau\\
&=\int_{\tau_1}^{\rho}\int_{B_0(p_0,r_0)}(f-h)(\psi_\tau+\Delta\psi)\,dV_{g(\tau)}d\tau\\
(6.9)\qquad &\le 0.
\end{aligned}
$$

We now choose a sequence of smooth functions $\theta_k\in C_0^\infty(B_0(p_0,r_0))$, $0\le\theta_k\le 1$, such that $\theta_k\to\text{sign}(f-h)_+(q,\rho)$ as $k\to\infty$. Putting $\theta=\theta_k$ in (6.9) and letting $k\to\infty$,

$$
\begin{aligned}
&\int_{B_0(p_0,r_0)}(f-h)_+(q,\rho)\,dV_{g(\rho)}\le 0\quad\forall\tau_1<\rho\le\tau_2\\
(6.10)\qquad &\Rightarrow\quad f\le h\quad\text{ in } B_0(p_0,r_0)\times[\tau_1,\tau_2].
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
&\int_{B_0(p_0,r_0)}(h-f)_+(q,\rho)\,dV_{g(\rho)}\le 0\quad\forall\tau_1<\rho\le\tau_2\\
(6.11)\qquad &\Rightarrow\quad h\le f\quad\text{ in } B_0(p_0,r_0)\times[\tau_1,\tau_2].
\end{aligned}
$$

Since $r_0>0$ is arbitrary, by (6.10) and (6.11),

$$
\begin{aligned}
(6.12)\qquad &\tau^{-\frac{n}{2}}e^{-l(q,\tau)}=h(q,\tau)=f(q,\tau)\quad\forall q\in M,\tau_1\le\rho\le\tau_2\\
&\Rightarrow\quad l(q,\tau)\in C^\infty(M\times[\tau_1,\tau_2]).
\end{aligned}
$$

By (6.1) and (6.12), (6.2) follows. □

By Theorem 6.1 and an argument similar to that of [16] and [25] we have

Theorem 6.2. *Suppose $\widetilde{V}(\tau_1)=\widetilde{V}(\tau_2)$ for some $\tau_2>\tau_1>0$. Then $l(q,\tau)$ satisfies*

$$2\Delta l-|\nabla l|^2+R+\frac{l-n}{\tau}=0\quad\forall q\in M,\tau_1\le\tau\le\tau_2$$

and

$$R_{ij}(q,\tau)-\frac{1}{2\tau}g_{ij}(q,\tau)+\nabla_i\nabla_j l=0$$

in $M\times(\tau_1,\tau_2)$.

Lemma 6.2. *Let* $\overline{\tau}>0$ *and* $q\in M$. *Suppose* γ *is the* $\mathcal{L}(q,\overline{\tau})$*-length minimizing* $\mathcal{L}$*-geodesic given by Theorem* 1.2 *which satisfies* $\gamma(0)=p_0$ *and* $\gamma(\overline{\tau})=q$. *Then for any* $\delta_0\in(0,1)$, *there exists a constant* $C=C(\delta_0)>0$ *such that*

$$\frac{d_{g(\overline{\tau})}(\gamma(\delta\overline{\tau}),q)^2}{\overline{\tau}}\le C(1+l(q,\overline{\tau}))\quad\forall\delta_0\le\delta\le 1. \tag{6.13}$$

Proof. We will use a modification of the proof of Lemma 3.2 of [25] to prove the lemma. Let $\overline{\tau}>0$. Since $\Omega(\overline{\tau})$ is dense in M and $l(q,\overline{\tau})$ is continuous in q, it suffices to prove (6.13) for $q\in\Omega(\overline{\tau})$. Let $\delta_0\le\delta\le 1$. Then

$$\begin{aligned}
&d_{\overline{\tau}}(\gamma(\delta\overline{\tau}),q)\\
&=\int_0^{\overline{\tau}}\frac{d}{d\rho}d_\rho(\gamma(\delta\rho),\gamma(\rho))\,d\rho\\
&=\int_0^{\overline{\tau}}\biggl(\frac{\partial}{\partial\rho}d_\rho(\gamma(\delta\rho),\gamma(\rho))+\delta\nabla_I d_\rho(\gamma(\delta\rho),\gamma(\rho))\cdot\gamma'(\delta\rho)\\
&\qquad\quad+\nabla_{II}d_\rho(\gamma(\delta\rho),\gamma(\rho))\cdot\gamma'(\rho)\biggr)\,d\rho\\
&=I_1+I_2+I_3
\end{aligned}\tag{6.14}$$

where ∇_I and ∇_{II} is the gradient with respect to the first and second argument respectively. Now by (0.6), (3.1), and Lemma 5.3,

$$\begin{aligned}
|\gamma'(\delta\rho)|&=|\nabla l(\gamma(\delta\rho),\delta\rho)|\le C\left(\frac{l(\gamma(\delta\rho),\delta\rho)}{\delta\rho}\right)^{\frac12}\le C(\delta_0\rho)^{-\frac12}\left(\frac{L(\gamma(\delta\rho),\delta\rho)}{2\sqrt{\delta\rho}}\right)^{\frac12}\\
&\le C(\delta_0\rho)^{-\frac12}\left(\frac{L(\gamma(\overline{\tau}),\overline{\tau})}{2\sqrt{\delta\rho}}\right)^{\frac12}\le C(\delta_0\rho)^{-\frac34}\overline{\tau}^{\frac14}\sqrt{l(q,\overline{\tau})}.
\end{aligned}$$

Hence

$$I_2\le C'\overline{\tau}^{\frac12}\sqrt{l(q,\overline{\tau})}. \tag{6.15}$$

Similarly,

$$I_3\le C\overline{\tau}^{\frac12}\sqrt{l(q,\overline{\tau})}. \tag{6.16}$$

For any $0<\rho\le\overline{\tau}$, let $x(\rho)=\gamma(\delta\rho)$, $y(\rho)=\gamma(\rho)$, and

$$r_0(\rho)=(l(q,\overline{\tau})+1)^{-\frac12}\rho^{\frac58}\overline{\tau}^{-\frac18}.$$

Then for any $x \in B_\rho(x(\rho), r_0(\rho))$, by Lemma 5.3,

(6.17)
$$\sqrt{l(x,\rho)} \le \sqrt{l(x(\rho),\rho)} + \frac{C}{\sqrt{\rho}} r_0(\rho) \le \sqrt{l(x(\rho),\rho)} + C(l(q,\overline{\tau})+1)^{-\frac{1}{2}}\rho^{\frac{1}{8}}\overline{\tau}^{-\frac{1}{8}}.$$

By Lemma 5.4 there exists a constant $C > 0$ such that

$$(6.18)\qquad \begin{aligned} & l(x(\rho),\rho) \le \delta^{-C} l(x(\rho),\delta\rho) \quad \forall 0 < \rho \le \overline{\tau} \\ \Rightarrow \quad & l(x(\rho),\rho) \le \delta_0^{-C} \frac{L(q,\overline{\tau})}{2\sqrt{\delta_0 \rho}} \le \delta_0^{-C-\frac{1}{2}} \rho^{-\frac{1}{2}} \overline{\tau}^{\frac{1}{2}} l(q,\overline{\tau}). \end{aligned}$$

By (6.17) and (6.18),

(6.19)
$$\begin{aligned} \sqrt{l(x,\rho)} &\le C\rho^{-\frac{1}{4}}\overline{\tau}^{\frac{1}{4}}\sqrt{l(q,\overline{\tau})} + C(l(q,\overline{\tau})+1)^{-\frac{1}{2}}\rho^{\frac{1}{8}}\overline{\tau}^{-\frac{1}{8}} \quad \forall x \in B_\rho(x(\rho), r_0(\rho)) \\ &\le C(\rho^{-\frac{1}{4}}\overline{\tau}^{\frac{1}{4}} + \rho^{\frac{1}{8}}\overline{\tau}^{-\frac{1}{8}})\sqrt{l(q,\overline{\tau})+1} \quad \forall x \in B_\rho(x(\rho), r_0(\rho)). \end{aligned}$$

By Lemma 5.3 and (6.19),

(6.20)
$$R(x,\rho) \le C\frac{l(x,\rho)}{\rho} \le C\rho^{-1}(\rho^{-\frac{1}{4}}\overline{\tau}^{\frac{1}{4}} + \rho^{\frac{1}{8}}\overline{\tau}^{-\frac{1}{8}})^2(l(q,\overline{\tau})+1) \quad \forall x \in B_\rho(x(\rho), r_0(\rho)).$$

Similarly

$$(6.21)\quad R(x,\rho) \le C\rho^{-1}(\rho^{-\frac{1}{4}}\overline{\tau}^{\frac{1}{4}} + \rho^{\frac{1}{8}}\overline{\tau}^{-\frac{1}{8}})^2(l(q,\overline{\tau})+1) \quad \forall x \in B_\rho(y(\rho), r_0(\rho)).$$

By Lemma 8.3(b) of [18] and (6.20), (6.21),

$$\begin{aligned} \frac{\partial}{\partial\rho} d_\rho(\gamma(\delta\rho),\gamma(\rho)) &\le C[\rho^{-1}(\rho^{-\frac{1}{4}}\overline{\tau}^{\frac{1}{4}} + \rho^{\frac{1}{8}}\overline{\tau}^{-\frac{1}{8}})^2(l(q,\overline{\tau})+1)r_0(\rho) + r_0(\rho)^{-1}] \\ &\le C(\rho^{-\frac{7}{8}}\overline{\tau}^{\frac{3}{8}} + \rho^{-\frac{1}{8}}\overline{\tau}^{-\frac{3}{8}} + \rho^{-\frac{1}{2}} + \rho^{-\frac{5}{8}}\tau^{\frac{1}{8}})\sqrt{l(q,\overline{\tau})+1}. \end{aligned}$$

Hence

$$(6.22)\qquad \begin{aligned} I_1 &\le C\sqrt{l(q,\overline{\tau})+1}\int_0^{\overline{\tau}} (\rho^{-\frac{7}{8}}\overline{\tau}^{\frac{3}{8}} + \rho^{-\frac{1}{8}}\overline{\tau}^{-\frac{3}{8}} + \rho^{-\frac{1}{2}} + \rho^{-\frac{5}{8}}\tau^{\frac{1}{8}})\,d\tau \\ &\le C\overline{\tau}^{\frac{1}{2}}\sqrt{l(q,\overline{\tau})+1}. \end{aligned}$$

By (6.14), (6.15), (6.16), and (6.22), we get (6.13) and the lemma follows. $\square$

We now let $\{\overline{\tau}_i\}_{i=1}^\infty$ be a sequence of positive numbers such that $\overline{\tau}_i \to \infty$ as $i \to \infty$. For any $i \in \mathbb{Z}^+$, $\tau > 0$, let

$$\widetilde{V}_i(\tau) = \int_M \tau^{-\frac{n}{2}} e^{-\bar{l}_i(q,\tau)}\, dV_{g_i(\tau)}$$

where $g_i(\tau) = g(\overline{\tau}_i\tau)/\overline{\tau}_i$ and $\overline{l}_i(q,\tau)$ is the $l = l_{\frac{1}{2}}$ function with respect to $g_i(\tau)$. Since by Lemma 5.2 $\overline{l}_i(q,\tau) = l(q,\overline{\tau}_i\tau)$, $\widetilde{V}_i(\tau) = \widetilde{V}(\overline{\tau}_i\tau)$. As observed by Perelman [18] there exists a sequence $\{q_i\}_{i=1}^{\infty} \subset M$ and a subsequence of $\{\overline{\tau}_i\}_{i=1}^{\infty}$ which we may assume without loss of generality to be the sequence itself such that the sequence of pointed manifold (M, g_i, q_i) will converge in the sense of Hamilton [10] $0 < \tau < \infty$ to some pointed manifold $(\hat{M}, \hat{g}, q_0)$ which also satisfies the backward Ricci flow as $i \to \infty$.

That is there exists a sequence of open sets $\hat{U}_i \subset \hat{M}$ with $q_0 \in \hat{U}_i$ for all $i \in \mathbb{Z}^+$ and a sequence of diffeomorphisms $F_i : \hat{U}_i \to \hat{V}_i$ where $q_i \in \hat{V}_i$ is open in M and $F_i(q_0) = q_i$ such that for any compact set $K \subset M$ there exists $i_0 \in \mathbb{Z}^+$ such that $K \subset \hat{U}_i$ for all $i \ge i_0$. Moreover if $\hat{g}_i = F_i^*(g_i)$ is the pull-back metric of g_i, then the metric $\hat{g}_i$ and all its derivatives will converge to $\hat{g}$ and the corresponding derivatives uniformly on $K \times [a,b]$ as $i \to \infty$ for any $0 < a < b < \infty$.

Moreover $\overline{l}_i(q_i,1) \le n/2$ for all $i \in \mathbb{Z}^+$ and $\overline{l}_i(q,\tau)$ converges uniformly on $\overline{B_{g_i(1)}(q_i,r)} \times [a,b]$ to some function $\hat{l}(q,\tau)$ as $i \to \infty$ for any $r > 0$ and $0 < a < b < \infty$. By Lemma 5.3 and Lemma 5.4 we may assume without loss of generality that $\overline{l}_{i,\tau}(q,\tau)$, $\nabla\overline{l}_i$ converge weakly to $\overline{l}_\tau(q,\tau)$ and $\nabla\overline{l}$ respectively as $i \to \infty$. Then $\overline{l}_\tau$, $|\nabla\overline{l}| \in L^\infty_{loc}(\hat{M} \times (0,\infty))$. Perelman [18] also proved that $\widetilde{V}_i(\tau)$ decreases and converges to some positive constant $\widetilde{V}_0$ which is independent of $\tau \in (0,\infty)$ as $i \to \infty$. Let $\hat{R}_{ij}(q,\tau)$ and $\hat{R}(q,\tau)$ be the Ricci curvature and scalar curvature of $\hat{M}$ with respect to the metric $\hat{g}(q,\tau)$. By an argument similar to the proof of Theorem 6.1 and Theorem 6.2 we have

Theorem 6.3. *$\overline{l}(q,\tau) \in C^\infty(M \times (0,\infty))$ and $\overline{l}(q,\tau)$ satisfies*

$$\overline{l}_\tau - \Delta\overline{l} + |\nabla\overline{l}|^2 - \hat{R} + \frac{n}{2\tau} = 0$$

in $\hat{M} \times (0,\infty)$.

Theorem 6.4. *Let $\hat{q} \in \hat{M}$. Let $\overline{q}_i \in M$ be such that $\hat{q} = \lim_{i\to\infty} F_i^{-1}(\overline{q}_i)$. For each $\rho \ge 1$, $i \in \mathbb{Z}^+$, let $\gamma_i : [0, \overline{\tau}_i\rho] \to M$ be the $\mathcal{L}(\overline{q}_i, \overline{\tau}_i\rho)$-length minimizing $\mathcal{L}$-geodesic given by Theorem 1.2. Let $\gamma_i^{\overline{\tau}_i}(w) = \gamma_i(\overline{\tau}_i w)$, $0 \le w \le \rho$. Then there exists a $\mathcal{L}$-geodesic $\hat{\gamma} : (0,\rho] \to \hat{M}$ with $\hat{\gamma}(\rho) = \hat{q}$ which is a $\mathcal{L}^{\hat{\gamma}(\rho_0)}_{\rho_0,\frac{1}{2}}(\hat{q},\rho)$-length minimizing $\mathcal{L}$-geodesic on $[\rho_0,\rho]$ for any $\rho_0 \in (0,\rho)$ such that for any $\rho_0 \in (0,\rho)$ $\gamma_i^{\overline{\tau}_i}(w)$ will converge uniformly on $\rho_0 \le w \le \rho$ to a $\mathcal{L}$-geodesic of $\hat{M}$ with $\hat{\gamma}(\rho) = \hat{q}$ as $i \to \infty$.*

Proof. Let $\hat{q} \in \hat{M}$ and let $\rho \ge 1$. We choose $\overline{q}_i \in M$ such that

$$\hat{q} = \lim_{i\to\infty} F_i^{-1}(\overline{q}_i).$$

Let $b > a > 0$. Since $d_{g_i(w)}(q_i, \overline{q}_i)$ converges uniformly to $d_{\hat{g}(w)}(q_0, \hat{q})$ on $a \le w \le b$ as $i \to \infty$, there exists a constant $C_1 > 0$ such that

$$d_{g_i(\rho)}(q_i, \overline{q}_i) \le C_1 \quad \forall i \in \mathbb{Z}^+. \tag{6.23}$$

Since $\overline{l}_i$ converges to $\overline{l}$ uniformly on $\overline{B_{g_i(1)}(q_i, r)} \times [a, b]$ as $i \to \infty$ for any $r > 0$, $b > a > 0$, there exists a constant $C_2 > 0$ such that

$$\overline{l}_i(\overline{q}_i, \rho) \le \overline{l}(\hat{q}, \rho) + C_2 \quad \forall i \in \mathbb{Z}^+. \tag{6.24}$$

Let $\delta_0 \in (0, 1)$. By Lemma 5.1 and Lemma 6.2 $\gamma_i^{\overline{\tau}_i} : [0, 1] \to M$ is a minimizing $\mathcal{L}$-geodesic with $\gamma_i^{\overline{\tau}_i}(1) = \overline{q}_i$ and

$$\frac{d_{g(\overline{\tau}_i\rho)}(\gamma_i(\delta\overline{\tau}_i\rho), \overline{q}_i)^2}{\overline{\tau}_i\rho} \le C(1 + l(\overline{q}_i, \overline{\tau}_i\rho)) \quad \forall \delta_0 \le \delta \le 1, i \in \mathbb{Z}^+$$

$$\Rightarrow \quad \frac{d_{g_i(\rho)}(\gamma_i^{\overline{\tau}_i}(\delta\rho), \overline{q}_i)^2}{\rho} \le C(1 + \overline{l}_i(\overline{q}_i, \rho)) \quad \forall \delta_0 \le \delta \le 1, i \in \mathbb{Z}^+. \tag{6.25}$$

By (6.24) and (6.25) there exists a constant $C_3 > 0$ such that

$$\frac{d_{g_i(\rho)}(\gamma_i^{\overline{\tau}_i}(\delta\rho), \overline{q}_i)^2}{\rho} \le C_3 \quad \forall \delta_0 \le \delta \le 1, i \in \mathbb{Z}^+. \tag{6.26}$$

By (6.23) and (6.26),

$$d_{g_i(\rho)}(\gamma_i^{\overline{\tau}_i}(\delta\rho), q_i) \le C_1 + \sqrt{C_3\rho} \quad \forall \delta_0 \le \delta \le 1, i \in \mathbb{Z}^+.$$

Hence by the Hamilton compactness theorem [10] and Lemma 5.1 $\gamma_i^{\overline{\tau}_i}$ will converge uniformly on $\rho_0 \le \tau \le \rho$ to a $\mathcal{L}$-geodesic of $\hat{M}$ with $\hat{\gamma}(\rho) = \hat{q}$ as $i \to \infty$ for any $\rho_0 \in (0, \rho)$. Since δ_0 is arbtiary, $\lim_{i\to\infty} \gamma_i^{\overline{\tau}_i}(\delta\rho)$ exists for any $\delta \in (0, 1)$. For any $0 < \tau \le \rho$, let

$$\hat{\gamma}(\tau) = \lim_{i\to\infty} \gamma_i^{\overline{\tau}_i}(\tau).$$

Then $\hat{\gamma} : (0, \rho) \to \hat{M}$ is a $\mathcal{L}$-geodesic of $\hat{M}$ with $\hat{\gamma}(\rho) = \hat{q}$. Since each $\gamma_i^{\overline{\tau}_i}$ is a $\mathcal{L}(\overline{q}_i, \rho)$-length minimizing $\mathcal{L}$-geodesic, $\hat{\gamma}|_{[\rho_0,\rho]}$ is a $\mathcal{L}_{\rho_0,\frac{1}{2}}^{\hat{\gamma}(\rho_0)}(\hat{q}, \rho)$-length minimizing $\mathcal{L}$-geodesic on $[\rho_0, \rho]$ for any $\rho_0 \in (0, \rho)$ and the theorem follows. □

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