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Some Theorems on Bounded Analytic Functions.

By

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Recently various theorems on bounded analytic functions have been generalized to the case of a multiply connected domain by several authors. For instance, the well-known Schwarz's lemma has been generalized by Ahlfors, Garabedian and Nehari. The theory of complete orthonormal system of analytic functions and of Szegö kernel function play important rôles in such generalizations. The objectives of this paper are to generalize Hardy's theorem on bounded functions (sec. 1–3) and to give a radius of univalence for a certain class of bounded functions (sec. 4–6).

1. Let D be a finite domain in the complex z-plane, bounded by n closed analytic curves Γ_i, $(i=1, 2,\dots, n)$. We define the class Λ of all functions $f(z)$ satisfying the following conditions:

(i) $f(z)$ is single-valued and regular at $z \in D$,

(ii) $f(z)$ has an integral $F(z)=\int_{z_0}^{z} f(t)\,dt$ $(z_0 \in D)$ which may have additive moduli if z describes a closed contour in D but which is continuous in the closed domain $\bar{D}=D+\Gamma$ $(\Gamma=\sum_{i=1}^{n}\Gamma_i)$,

(iii) for every function $f(z)$ there corresponds a summable function $\mu(t)$ on Γ with its square of absolute value $|\mu|^2$, such that for any two points t_1 and t_2 on the same boundary curve Γ_i $(i=1, \dots, n)$

$$F(t_2)-F(t_1)=\int_{t_1}^{t_2}\mu(t)dt$$

holds for the right determinations of the multivalued function $F(z)$. $\mu(t)$ is called the associate function on Γ of the function $f(z)$ in D. It is evident that if $f(z)$ is regular in D and continuous in $\bar{D}$, it belongs to the class Λ and its associate function $\mu(t)$ is given by the boundary value $f(t)$ of $f(z)$.

We consider the space Λ and introduce in Λ the metric

$$\|f\|^2=\oint_\Gamma |\mu(t)|^2 ds=\oint_\Gamma \mu(t)\cdot\overline{\mu(t)}ds\,,$$

where s denotes the arc-length of Γ. By the well-known methods of orthonormalization of regular functions we construct a complete orthonormal system $\{f_i(z)\}$ $(i=0, 1,\ldots)$ by which each function $f(z)\in\Lambda$ may be represented in the form of absolutely and uniformly convergent series in any closed subdomain of D,

$$f(z)=\sum_{i=0}^{\infty}a_i f_i(z),\quad a_i=(f,\bar{f_i}),\quad (f_i,\bar{f_j})=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases},$$

where the scalar product of two functions f and g is defined as $(f,\bar{g})=\oint_\Gamma \mu(t)\cdot\overline{\nu(t)}ds$, $\mu(t)$ and $\nu(t)$ being the respective associate functions of f and g. N. Aronszajn[1] and M. Schiffer[2] proved that *every function of a complete orthonormal system* $\{f_i(z)\}$ *may be chosen to be regular in* $\bar{D}$. In the following section we consider only such a regular system.

2. In this section we shall prove the following

Theorem I. *Every bounded regular function in D belongs to the class* Λ.

Proof. For our purpose it is sufficient only to prove that any bounded regular function $f(z)$ $(|f|\leqq 1)$ satisfies the conditions (ii) and (iii).

Let t_0 be an arbitrary point on the boundary curve Γ_i, z_1 and z_2 be any two points of D in the neighborhood of t_0. Next we map conformally the simply connected domain D_i $(\supset D)$ bounded only by Γ_i onto the unit circle in ζ-plane. Denote the mapping function by $\zeta=\varphi(z)$. Let τ_0, ζ_1 and ζ_2 be the respective images of t_0, z_1 and z_2. Then we obtain

$$(1)\qquad F(z_2)-F(z_1)=\int_{z_1}^{z_2}f(t)dt=\int_{\zeta_1}^{\zeta_2}f\{\varphi^{-1}(\zeta)\}\{\varphi^{-1}(\zeta)\}'d\zeta\,.$$

By using Cauchy's theorem and the boundedness of $f(z)$ in (1),

$$(2)\qquad |F(z_2)-F(z_1)|\leqq M|\zeta_2-\zeta_1|\,,$$

M being a constant such that $|\{\varphi^{-1}(\zeta)\}'|=1/|\varphi'(z)|\leqq M$ in $\bar{D}$. Indeed, there exists the constant M by Schwarz's reflection principle. Denote by D_0 a portion of a neighborhood of t_0 contained in D whose image in ζ-plane is a common part of the unit circle and a neighborhood of τ_0 defined by $|\zeta-\tau_0|<\varepsilon/2M$, ε being an arbitrarily chosen positive number. Hence, corresponding to $\varepsilon>0$ we can, by (2), determine a portion D_0 such that there always holds the inequality

$$(3) \qquad |F(z_2)-F(z_1)|<\varepsilon ,$$

provided that z_1 and z_2 belong to D_0. Accordingly $\lim\limits_{\substack{z\to t_0\\(z\in D)}} F(z)$ must be finite and determinate except additive moduli. Thus we have proved that the condition (ii) is satisfied for $f(z)$.

By the well-known theorem of Fatou every bounded regular function $f(z)$ has a finite boundary value (non-tangential) almost everywhere on Γ. We define the associate function $\mu(t)$ of $f(z)$ by such finite boundary value except a set of null-measure on Γ. On such an exceptional set we define properly, say, $\mu(t)=0$. By the theorem of Lebesgue integral of bounded function, $\mu(t)$ and $|\mu(t)|^2$ are summable on Γ. Next let t_1 and t_2 be arbitrarily fixed points on the same boundary curve Γ_i, $e^{i\theta_1}$ and $e^{i\theta_2}$ be the respective images in ζ-plane, and denote a sequence of arcs in D converging to the arc $\widehat{t_1t_2}$ by $\gamma_i^{(\nu)}(\nu=1, 2,\ldots)$ whose images in ζ-plane are the sequence of concentric circular-arcs $\widehat{r_\nu e^{i\theta_1}, r_\nu e^{i\theta_2}}(r_\nu\to 1, \nu\to\infty)$. Then it follows from (1) that for the right determination of $F(z)$

$$F(z_2^{(\nu)})-F(z_1^{(\nu)})=\int_{\gamma_i^{(\nu)}} f(t)dt$$

$$=\int_{\theta_1}^{\theta_2} f\{\varphi^{-1}(r_\nu e^{i\theta})\}\cdot\{\varphi^{-1}(r_\nu e^{i\theta})\}'\cdot ir_\nu e^{i\theta}d\theta \qquad (\zeta=r_\nu e^{i\theta}),$$

where $z_1^{(\nu)}$ and $z_2^{(\nu)}$ are the endpoints of $\gamma_i^{(\nu)}$ and tend to t_1 and t_2 respectively as $\nu\to\infty$. Letting $\nu\to\infty$ and using the theorem of Lebesgue integral of a sequence of functions, we observe that

$$F(t_2)-F(t_1)=\int_{t_1}^{t_2}\mu(t)dt,$$

because $f\{\varphi^{-1}(r_\nu e^{i\theta})\}\to\mu\{\varphi^{-1}(e^{i\theta})\}$ as $\nu\to\infty$ a. e. on $|\zeta|=1$. Thus the theorem is proved.

3. By means of the above result we shall now generalize the following classical Hardy's theorem to the case of a multiply connected domain.

Hardy's theorem. *Let* $f(z)=\sum_{\nu=0}^{\infty}a_\nu z^\nu$ *be bounded and regular in the unit circle* $|z|<1$, *and define a function* $M(r)=\sum_{\nu=0}^{\infty}|a_\nu|\, r^\nu$, $(z=re^{i\theta})$. *Then it follows that* $M(r)=o(1/\sqrt{1-r})$, $(r\rightarrow 1)$.

For the sake of brevity we shall use the same notation with that of sec. 1 and 2. Let $f(z)$ be again single-valued, regular and bounded in a finite multiply connected domain D and by means of the above result expand $f(z)$ in a series of a regular complete orthonormal system $\{f_\nu(z)\}$ in the form

$$f(z)=\sum_{\nu=0}^{\infty}a_\nu f_\nu(z),\quad a_\nu=(f,\bar{f_\nu}),$$

where the right-hand side is uniformly and absolutely convergent in any closed subdomain of D. Analogously to $M(r)$ in Hardy's theorem, we define a function $M(z)=\sum_{\nu=0}^{\infty}|a_\nu|\cdot|f_\nu(z)|$. Then we obtain the following.

Theorem. II. *For every regular and bounded function* $f(z)$ *in a multiply connected domain D, there holds* $M(z)=o(1/\sqrt{d(z)})$, *when* z *tends to an arbitrary boundary point,* $d(z)$ *being the shortest distance from* z *to the boundary* Γ *of D.*

Proof.

$$\begin{aligned} M(z)&=\sum_{\nu=0}^{\infty}|a_\nu|\cdot|f_\nu(z)|\\ &=\sum_{\nu=0}^{m}|a_\nu|\cdot|f_\nu(z)|+\sum_{\nu=m+1}^{\infty}|a_\nu|\cdot|f_\nu(z)|\\ &\leqq\sum_{\nu=0}^{m}|a_\nu|\cdot M_\nu+\sqrt{\sum_{\nu=m+1}^{\infty}|a_\nu|^2\sum_{\nu=m+1}^{\infty}|f_\nu(z)|^2}\qquad (M_\nu=\operatorname*{Max}_{z\in\bar{D}}|f_\nu(z)|) \qquad (4)\\ &\leqq\sum_{\nu=0}^{m}|a_\nu|\cdot M_\nu+\sqrt{\sum_{\nu=m+1}^{\infty}|a_\nu|^2\sum_{\nu=0}^{\infty}|f_\nu(z)|^2}\\ &=\sum_{\nu=0}^{m}|a_\nu|\cdot M_\nu+\sqrt{\sum_{\nu=m+1}^{\infty}|a_\nu|^2}\ \sqrt{K(z,\bar{z})}, \end{aligned}$$

where $K(z,\zeta)=\sum_{\nu=0}^{\infty}f_\nu(z)\overline{f_\nu(\zeta)}$ $(z,\zeta\in D)$ is called the *Szegö kernel function*[3] of the domain D. It is well-known that at any interior point z of D

$$K(z,\bar{z})\leqq\frac{1}{2\pi d(z)}.^{4)} \qquad (5)$$

By introducing (5) into (4) we obtain

$$M(z) \leq \sum_{\nu=0}^{m} |a_\nu| M_\nu + \frac{1}{\sqrt{2\pi d(z)}} \sqrt{\sum_{\nu=m+1}^{\infty} |a_\nu|^2}.$$

Since $\sum_{\nu=0}^{\infty} |a_\nu|^2 = \|f(z)\|^2 = \oint_{\Gamma} |\mu(t)|^2 ds < \infty$ by Parseval's theorem, for an arbitrarily chosen $\varepsilon > 0$, we can determine a positive integer m such that

$$\sum_{\nu=m+1}^{\infty} |a_\nu|^2 < 2\pi\varepsilon^2.$$

Therefore

$$\sqrt{d(z)} \cdot M(z) < \left(\sum_{\nu=0}^{m} |a_\nu| M_\nu\right) \sqrt{d(z)} + \varepsilon .$$

Hence it follows that for a fixed m

$$\varlimsup_{\substack{z \to t \\ (z \in D)}} \sqrt{d(z)} \cdot M(z) \leq \varepsilon$$

for any boundary point t. The left-hand side is independent of m. Thus we obtain the required result

$$(6) \qquad M(z) = o\left(\frac{1}{\sqrt{d(z)}}\right), \quad (z \to t).$$

Particularly let D be the unit circle $|z| < 1$. Then we can choose as the complete orthonormal system $f_\nu(z) = z^\nu / \sqrt{2\pi}$ $(\nu = 0, 1, \ldots)$ and hence obtain $M(z) = \sum_{\nu=0}^{\infty} \sqrt{2\pi} \cdot |a_\nu| (r^\nu / \sqrt{2\pi})$ $(z = re^{i\theta})$ for $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$. By the above theorem $M(z) = o(1/\sqrt{1-r})$ $(r \to 1)$. Thus Hardy's theorem has been generalized to the case of a multiply connected domain.

4. We shall now retain the same assumptions concerning the domain D and its boundary $\Gamma = \sum_{i=1}^{n} \Gamma_i$ as the preceding sections. Furthermore let a function $\lambda(z)$ be defined to be positive and continuous on each Γ_i. Then, according to Nehari,[5] there exist two functions $K_\lambda(z, \zeta)$ and $L_\lambda(z, \zeta)$ such as follows: for an arbitrarily chosen interior point ζ of D,

(i) $K_\lambda(z, \zeta)$ and $L_\lambda(z, \zeta) - 1/2\pi(z-\zeta)$ are regular functions of z in D,

(ii) $|K_\lambda(z, \zeta)|$ is continuous in $D + \Gamma$, $|L_\lambda(z, \zeta)|$ continuous in $D + \Gamma - C_\varepsilon$, C_ε being any sufficiently small neighborhood of ζ,

(iii) $K_\lambda(z, \zeta)$ and $L_\lambda(z, \zeta)$ are connected by an identity

$$(7) \qquad \lambda(z) \cdot \overline{K_\lambda(z, \zeta)}\, ds_z = \frac{1}{i} L_\lambda(z, \zeta)\, dz, \quad z \in \Gamma,$$

$ds_z = |dz|$, being the linear element of Γ.

It is known that both functions $K_\lambda(z, \zeta)$ and $L_\lambda(z, \zeta)$ are uniquely determined by the above three conditions (i), (ii) and (iii). Moreover, for every regular function $f(z)$ in D such that $\oint_\Gamma \lambda(z)|f(z)|^2 ds_z < \infty$ (in the Lebesgue sense), there arises the relation

$$\oint_\Gamma \lambda(z) f(z) \overline{K_\lambda(z, \zeta)}\, ds_z = f(\zeta), \tag{8}$$

which is called *the reproducing property of the function* $K_\lambda(z, \zeta)$.

5. By means of two domain functions $K_\lambda(z, \zeta)$ and $L_\lambda(z, \zeta)$ we shall now construct a theorem on the radius of univalence of the function $f(z)$ belonging to a certain class. Without loss of generality we assume that the domain D contains the origin. Consider a class S_λ which consists of every function $f(z)$ satisfying the following conditions;

(a) $\log(f(z)/z)$ is single-valued and regular in D,

(b) $\overline{\lim_{z\to t}} \log(f(z)/z) \leqq \lambda(t)$ for every boundary point t on Γ.

Instead of $L_\lambda(z, \zeta)$ connected with the function $\lambda(z)$ we construct another domain function $L_\mu(z, \zeta)$ connected with the function $\mu(z) = 1/\lambda(z) (>0)$ which can be expanded by (i) in the form

$$L_\mu(z, \zeta) = \frac{1}{2\pi(z-\zeta)} + a_\mu(\zeta) + O(|z-\zeta|). \tag{9}$$

Moreover we introduce a new function $Q(z, \zeta)$ by using L_μ, a_μ, and K_λ

$$Q(z, \zeta) = 4\pi^2 \left(L_\mu(z, \zeta) - a_\mu(\zeta) \frac{K_\lambda(z, \zeta)}{K_\lambda(\zeta, \zeta)} \right)^2 \tag{10}$$

$$(K_\lambda(\zeta, \zeta) > 0 \text{ c.f. } 5))$$

which is expanded in the neighborhood of $z = \zeta$ in the form

$$\frac{1}{(z-\zeta)^2} + \text{regular part.}$$

By the residue theorem we obtain for every function $f(z) \in S_\lambda$

$$\frac{f'(\zeta)}{f(\zeta)} - \frac{1}{\zeta} = \frac{1}{2\pi i} \oint_\Gamma \log \frac{f(z)}{z} Q(z, \zeta)\, dz, \tag{11}$$

and by the definition of the class S_λ

$$\left| \frac{f'(\zeta)}{f(\zeta)} - \frac{1}{\zeta} \right| \leqq \frac{1}{2\pi} \oint_\Gamma \lambda(z) \cdot |Q(z, \zeta)|\, ds_z \tag{12}$$

$$=2\pi\oint_{\Gamma}\lambda(z)\cdot\Big(|L_{\mu}(z,\zeta)|^2-\overline{\alpha_{\mu}(\zeta)}\,\frac{\overline{K_{\lambda}(z,\zeta)}L_{\mu}(z,\zeta)}{K_{\lambda}(\zeta,\zeta)}$$

$$-\alpha_{\mu}(\zeta)\frac{K_{\lambda}(z,\zeta)\overline{L_{\mu}(z,\zeta)}}{K_{\lambda}(\zeta,\zeta)}+|\alpha_{\mu}(\zeta)|^2\frac{|K_{\lambda}(z,\zeta)|^2}{K_{\lambda}(\zeta,\zeta)^2}\Big)ds_z\,.$$

Here we obtain from the identity (7) for $\mu(z)$ instead of $\lambda(z)$

$$L_{\mu}(z,\zeta)=i\mu(z)\overline{K_{\mu}(z,\zeta)}\Big(\frac{ds_z}{dz}\Big)$$

$$=\frac{i}{\lambda(z)}\overline{K_{\mu}(z,\zeta)}\Big(\overline{\frac{dz}{ds_z}}\Big)\qquad\Big(\because\ \Big|\frac{ds_z}{dz}\Big|=1 \text{ on } \Gamma\Big)$$

$$(7')\qquad\therefore\quad \frac{1}{i}\lambda(z)L_{\mu}(z,\zeta)ds_z=\overline{K_{\mu}(z,\zeta)dz}\,.$$

Therefore we observe that by (7′)

$$(13)\qquad \oint_{\Gamma}\lambda(z)|L_{\mu}(z,\zeta)|^2ds_z=\overline{\frac{1}{i}\oint_{\Gamma}K_{\mu}(z,\zeta)L_{\mu}(z,\zeta)dz}$$

$$=K_{\mu}(\zeta,\zeta)\quad\text{(by residue theorem)},$$

$$(14)\qquad \oint_{\Gamma}\overline{\alpha_{\mu}(\zeta)}\cdot\lambda(z)\frac{\overline{K_{\lambda}(z,\zeta)}L_{\mu}(z,\zeta)}{K_{\lambda}(\zeta,\zeta)}ds_z$$

$$=\frac{i\overline{\alpha_{\mu}(\zeta)}}{K_{\lambda}(\zeta,\zeta)}\overline{\oint_{\Gamma}K_{\lambda}(z,\zeta)K_{\mu}(z,\zeta)dz}$$

$$=0\quad\text{(by Cauchy's theorem)},$$

similarly

$$(15)\qquad \oint_{\Gamma}\alpha_{\mu}(\zeta)\cdot\lambda(z)\frac{K_{\lambda}(z,\zeta)\overline{L_{\mu}(z,\zeta)}}{K_{\lambda}(\zeta,\zeta)}ds_z=0\,,$$

$$(16)\qquad \oint_{\Gamma}|\alpha_{\mu}(\zeta)|^2\lambda(z)\frac{|K_{\lambda}(z,\zeta)|^2}{K_{\lambda}(\zeta,\zeta)^2}ds_z$$

$$=\frac{|\alpha_{\mu}(\zeta)|^2}{K_{\lambda}(\zeta,\zeta)^2}\oint_{\Gamma}\lambda(z)K_{\lambda}(z,\zeta)\overline{K_{\lambda}(z,\zeta)}ds_z$$

$$=\frac{|\alpha_{\mu}(\zeta)|^2}{K_{\lambda}(\zeta,\zeta)}\quad\text{(by reproducing property (8))}.$$

Introducing (13), (14), (15) and (16) into (12), we obtain

$$\Big|\frac{f'(\zeta)}{f(\zeta)}-\frac{1}{\zeta}\Big|\leqq 2\pi\Big[K_{\mu}(\zeta,\zeta)+\frac{|\alpha_{\mu}(\zeta)|^2}{K_{\lambda}(\zeta,\zeta)}\Big],$$

$$(17)\qquad\therefore\quad \Big|\zeta\frac{f'(\zeta)}{f(\zeta)}-1\Big|\leqq 2\pi|\zeta|\Big[K_{\mu}(\zeta,\zeta)+\frac{|\alpha_{\mu}(\zeta)|^2}{K_{\lambda}(\zeta,\zeta)}\Big].$$

Accordingly there holds

$$Re\left\{\zeta \frac{f'(\zeta)}{f(\zeta)}\right\} \geqq 0 \tag{18}$$

within the largest circle in D about the origin in which the inequality

$$2\pi|\zeta|\left[K_\mu(\zeta, \zeta) + \frac{|a_\mu(\zeta)|^2}{K_\lambda(\zeta, \zeta)}\right] \leqq 1$$

is satisfied. Hence, by Kobori's theorem,[6] we obtain the following

Theorem III. *Let $f(z)$ be in S_λ, then $f(z)$ is univalent (and yet star-like) in the largest circle about the origin all of whose points satisfy the inequality*

$$2\pi|\zeta|\left[K_\mu(\zeta, \zeta) + \frac{|a_\mu(\zeta)|^2}{K_\lambda(\zeta, \zeta)}\right] \leqq 1, \quad (\lambda(z)\mu(z) \equiv 1). \tag{19}$$

Thus, by our theorem, the radius of univalence for every function $\in S_\lambda$ can be estimated by domain functions K_λ, K_μ, and a_μ.

Particularly consider the case where $\mu(z) = 1/M$ (a positive const.) everywhere on Γ. In this case it follows that

$K_{1/M}(z, \zeta) = MK(z, \zeta)$

($K(z, \zeta)$ denotes Szegö kernel function described in sec. 3)

and

$$L_{1/M}(z, \zeta) = \frac{1}{2\pi(z-\zeta)} + O(|z-\zeta|)$$

i. e. $a_{1/M}(\zeta) \equiv 0$[7] ($\zeta \in D$). Therefore our theorem reduces to the following

Nehari's theorem.[8] *Let S be a class of regular functions $f(z)$ which satisfy the conditions that* (i) $\log(f(z)/z)$ *is single-valued and regular in D and* (ii) $\overline{\lim_{z \to t}} |\log(f(z)/z)| \leqq M$ *for every boundary point t. Then every function $f(z) \in S$ is univalent in the largest circle about the origin all of whose points satisfy.*

$$2\pi|z| K(z, z) \leqq \frac{1}{M}. \tag{20}$$

In his proof it is shown that this value of the radius of univalence is the best possible. We cannot, however, assert that our theorem in the more general case also gives a best possible value of the radius for every function $f(z) \in S_\lambda$.

6. Instead of using the function $K_\lambda(z, \zeta)/K_\lambda(\zeta, \zeta)$ in (10) of the previous section, we may use an arbitrary function $g(z)$ which is regular in D, $g(\zeta)=1$ and $\oint_\Gamma \lambda(z)|g(z)|^2 ds<\infty$, and construct a function similar to $Q(z,\zeta)$ by K_μ, α_μ and g. By the same method of estimation as in the sec. 5 it follows that $f(z)\in S_\lambda$ is univalent in the largest circle about the origin, all of whose points satisfy the inequality

$$2\pi|\zeta|\cdot[K_\mu(\zeta,\zeta)+|\alpha_\mu(\zeta)|^2\cdot\|g(z)\|^2]\leqq 1\,, \tag{21}$$

where $\|g(z)\|^2=\oint_\Gamma \lambda(z)|g(z)|^2 ds$. On the other hand, Nehari[9] proved that

$$\|g(z)\|^2\geqq\frac{1}{K_\lambda(\zeta,\zeta)}\,, \tag{22}$$

and yet the equality in (22) is possible only for $g(z)\equiv\dfrac{K_\lambda(z,\zeta)}{K_\lambda(\zeta,\zeta)}$. By considering Nehari's theorem and comparing the inequality (19) with (21), we observe that the radius of univalence determined by (19) is not smaller than that determined by (21). Therefore, in direction of our approach, it is preferable to use the function $K_\lambda(z,\zeta)/K_\lambda(\zeta,\zeta)$ in the estimation of the radius of univalence.

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