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Classical flows with discrete spectra

Dedicated to Professor Atuo Komatsu for his 60th birthday

By

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Introduction

In his paper [12] von Neumann showed that unitary equivalence of flows implies their metrical equivalence in the case of ergodic flows with discrete spectra. More precisely, if two one-parameter groups of unitary operators induced by two ergodic flows have discrete spectra and are unitarily equivalent, then these flows are metrically equivalent. Moreover they can be realized "canonically" as rotations on compact abelian groups.

Up to the present time many results are obtained with regard to the set of eigenvalues, which forms an additive subgroup of real numbers, and eigenfunctions. However this does not finish the investigation of the eigenvalues, eigenfunctions, discrete spectrum, and etc., if we consider the flows (φ_t) on the manifolds M generated by the differential equations on them. For instance, we do not know even whether the ranks of the additive groups of eigenvalues of (φ_t) are finite or not. [1], [2].

In this paper we consider the case when flow (φ_t) is ergodic and the manifold M is compact, then we can consider M as the total space of a locally trivial smooth fibre space, whose base space is a torus and fibres are submanifolds of M, and moreover φ_t is fibre-preserving (§2). Especially, if (φ_t) has discrete spectrum, then M is diffeomorphic to a torus and (φ_t) can be regarded as the well known quasi-periodic motion on it (§3). These results show the relationship between the spectral types of flows and the structures of manifolds on which the flows are defined. In §4 we discuss some generalizations of §3 and some characterizations of quasi-periodic motion. In §5 we generalize the notion of eigenfunctions by introducing the concept of oscillatory functions.

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§1. Preliminaries from ergodic theory

In this § we briefly enumerate necessary definitions and theorems from the ergodic theory. For details refer to [5], [8], [9] and [11].

Definition A. A *flow* is the triple (M, μ, φ_t) of a probability space M with measure μ and a one-parameter group of transformations φ_t of M which preserve the measure μ. We assume the measurability of (φ_t) with respect to "time" t.

A flow (φ_t) induces naturally a one-parameter group of unitary operators $\{U_t\}$ on the Hilbert space $H=L^2(M, \mu)$ of complex valued square summable functions defined on M:

$$(U_t f)(x)=f(\varphi_t x), \quad \text{for} \quad f\in H.$$

By the decomposition theorem of Stone, these U_t have the following spectral resolution:

$$U_t=\int e^{2\pi i t\lambda} dE(\lambda),$$

where $\{E(\lambda)\}$ is a resolution of identity of H.

Definition B. Let $H^{(\lambda)}=(E(\lambda)-E(\lambda-0))H$. We call λ an *eigenvalue* of the flow (φ_t) when $\dim H^{(\lambda)}>0$ and an element of $H^{(\lambda)}$ an *eigenfunction* of the flow (φ_t). The fact that λ is an eigenvalue is equivalent with the existence of such $f_\lambda \in H$, $f_\lambda \neq 0$ that

$$U_t f_\lambda = e^{2\pi i t\lambda} f_\lambda \quad \text{for all } t.$$

Definition C. A flow (φ_t) is called *ergodic*, when the condition

$$\mu\{\varphi_t A \ominus A\} = 0^{1)} \quad \text{for all } t$$

implies $\mu(A)=0$ or $\mu(A)=1$.

Theorem A. *A flow (φ_t) is ergodic iff $\lambda=0$ is a simple eigenvalue.*

Theorem B. *Let (φ_t) be an ergodic flow on a probability space (M, μ), Λ be the set of eigenvalues of the flow (φ_t)* and $H^{(\lambda)}$ *be the set of eigenfunctions which belong to the eigenvalue λ. Then*

(i) *Λ is an additive subgroup of the real number group R.*

(ii) *If $f_\lambda \in H^{(\lambda)}$ then $|f_\lambda| = constant$ a.e..*

(iii) *For all $\lambda \in \Lambda$ there exist $\varphi_\lambda \in H^{(\lambda)}$ such that $\varphi_\lambda/\varphi_\mu = \varphi_{\lambda-\mu}$, $|\varphi_\lambda|=1$, and $\sum_{i=1}^{n} k_i\lambda_i = 0$ $(k_i \in Z)$ implies $\Pi_{i=1}^{n} \varphi_{\lambda_i}^{k_i} = 1$.*

(iv) *Let $H_d = \sum_{\lambda\in\Lambda} \oplus H^{(\lambda)}$ and $\mathcal{B}_d = \mathcal{B}(H_d)$ be the smallest Borel algebra which makes the elements of H_d measurable, then $H_d = L^2(\mathcal{B}_d)$.*

§2. Fibre structure

We introduce a fibre structure on manifold from the given flow. First we shall give definitions of some fundamental notions.

Definition 1. *Classical flow* or *classical dynamical system*

1) We denote the symmetric difference of two sets A and B with $A \ominus B$:

$$A \ominus B = A \cup B - A \cap B$$

means the triple (M, μ, φ_t) formed by a C^∞-manifold M, a finite measure μ on M defined by a positive continuous density (we assume $\mu(M)=1$) and a one-parameter group (φ_t) of diffeomorphisms of M which preserve the measure μ.

Definition 2. Let (M, μ, φ_t) and (N, ν, ϕ_t) be classical flows. (M, μ, φ_t) is *C^ρ-isomorphic* to (N, ν, ϕ_t) as classical flows, when there exists C^ρ-diffeomorphism ψ: $M \to N$ such that $\psi\circ\varphi_t=\phi_t\circ\psi$ for all t, and $\psi(\mu)=\nu$. We denote it by:

$$(M, \mu, \varphi_t) \underset{C^\rho}{\simeq} (N, \nu, \phi_t)$$

Now let us give the definition of quasi-periodic motion which will be necessary for our theorems.

Definition 3. Let $T^n=R^n/Z^n=\{(x_1, \cdots, x_n);\ x_i\in R \bmod 1\ i=1, 2, \cdots, n\}$ be the n-dimensional torus with the usual Lebesgue measure, $dm=dx_1\cdots dx_n$. *Jacobi flow* with frequencies $\omega_1, \cdots, \omega_n$ is a classical flow (T^n, m, ψ_t) where (ψ_t) is the one-parameter group of transformations defined by,

$$\psi_t x_i = x_i+\omega_i t, \bmod 1,\ i=1, \cdots, n.$$

Lemma. *An orbit of Jacobi flow with frequencies $\omega_1, \cdots, \omega_n$ is everywhere dense on T^n, if and only if $\omega_1, \cdots, \omega_n$ are linearly independent over Z, i.e. $k_1\omega_1+\cdots+k_n\omega_n=0$ for $k_i\in Z$ implies $k_1=\cdots=k_n=0$.*

In this case we call this Jacobi flow a *quasi-periodic motion.*

We can prove easily this lemma with the help of theorem A. It is also easy to prove it directly.

Let (M, μ, φ_t) be a classical flow and Λ^ρ*) be the set of eigenvalues of (φ_t) whose eigenfunctions are C^ρ-differentiable.

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*) Λ^ρ forms an additive subgroup of Λ. cf theorem B, in 1

Theorem 1. *Let (M, μ, φ_t) be a classical ergodic flow and M be compact. If $\lambda_1, \cdots, \lambda_r \in \Lambda^\rho (\rho \geqq 1)$ are linearly independent over Z, then we can consider M as the total space of a locally trivial $C^{\rho-1}$ smooth fibre space, whose base space is an r-dimensional torus T^r and whose fibres are C^ρ-submanifolds of M of codimension r and numbers of their connected components are finite. The flow (φ_t) is fibre-preserving and the flow which is naturally induced on the base space T^r is quasi-periodic motion with frequencies $\lambda_1, \cdots, \lambda_r$. In addition, the fibres can be assumed to be connected, in this case, however, the frequencies of the induced flow on the base T^r are different from $\lambda_1, \cdots, \lambda_r$.*

Remark. In theorem 1, the assumption of the ergodicity of the flow (φ_t) can be weakened. In fact the existence of only one orbit which is everywhere dense in M is sufficient. We note that if (φ_t) is ergodic, then almost all orbits are everywhere dense in M.

In proving theorem 1, it is essential to prove the following fundamental lemma.

Fundamental lemma. *Under the assumptions of theorem 1, let $f_1, \cdots, f_r$ be differentiable eigenfunctions of class $C^\rho (\rho \geqq 1)$ which belong to the eigenvalues $\lambda_1, \cdots, \lambda_r \in \Lambda^\rho$ respectively. Then $df_1, \cdots, df_r$*$) \in T^*(M)$ are linearly independent everywhere.*

Proof of fundamental lemma: the proof will be devided into three parts:

Step 1. As $f_j(x) (j=1, \cdots, r)$ are continuous and eigenfunctions of the ergodic flow (φ_t),

$$|f_j(x)| = \text{constant} \quad (\text{say}, =1),$$

$$(*) \qquad f_j(x) = e^{2\pi i t \lambda_j} f_j(x) \qquad \text{for all } x \in M \text{ and } t.$$

*) Though f_j are complex valued, however, as will be seen in step 1 of the proof, we can put $f_j(x) = e^{2\pi i \theta_j(x)}$, $\theta_j(x) \in R/Z$, so we write df_j in the place of $d\theta_j$.

Now let us define the mappings P_k and the sets $M_k(\alpha)$, $1\leqq k \leqq r$, as following:

$$P_k\colon M\to \boldsymbol{T}^k\cong T^k\colon x \rightsquigarrow (f_1(x),\cdots,f_k(x))$$

where $\qquad \boldsymbol{T}=\{Z\in C;\ |z|=1\}$

$$M_k(\alpha)=P_k^{-1}(\alpha)\subset M,\ \alpha=(\alpha_1,\cdots,\alpha_k)\in \boldsymbol{T}^k.$$

We define the flow (ψ_t^k) on $\boldsymbol{T}^k$ as follows:

$$\psi_t^k;\ \boldsymbol{T}^k\to\boldsymbol{T}^k\colon (\alpha_1,\cdots,\alpha_K)\rightsquigarrow(e^{2\pi i t\lambda_1}\alpha_1,\cdots,e^{2\pi i t\lambda_k}\alpha_k).$$

That is, flow (ψ_t^k) is a quasi-periodic motion with frequencies $\lambda_1,\cdots,\lambda_k$ on $\boldsymbol{T}^k$. The following diagramm is obviously commutative:

$$\begin{array}{ccc} M & \xrightarrow{\varphi_t} & M \\ \Big\downarrow P_k & & \Big\downarrow P_k \\ \boldsymbol{T}^k & \xrightarrow{\psi_t^k} & \boldsymbol{T}^k \end{array}$$

As M is compact and the orbit of ψ_t^k is everywhere dense in $\boldsymbol{T}^k$ from lemma, P_k is onto mapping.

Step 2. Let us assume that $df_1,\cdots,df_k$ are linearly independent everywhere, i.e. the mapping P_k is full rank everywhere on M.

Then as is well known, the sets $M_k(\alpha)$ become C^ρ-submanifolds of codimension k. Moreover, for any $\alpha_0\in\boldsymbol{T}^k$, there exists some neighbourhood $U_0\subset\boldsymbol{T}^k$ of α_0 and we can define a structure of direct product in $M_k(U_0)=P_k^{-1}(U_0)\subset M$, i.e. there exists a diffeomorphism ι_0:

$$\iota_0\colon U_0\times M_k(\alpha_0)\longrightarrow M_k(U_0)$$

such that

$$\iota_0(\{\alpha\}\times M_k(\alpha_0))=M_k(\alpha)\quad\text{for any }\alpha\in U_0.$$

This shows the local triviality. It is known, but for the completeness we will show it.

For instance, this can be shown as follows: We define a Riemannian metric in M. Let $N(M_k)$ be normal bundle of M_k

$=M_k(\alpha_0)$. This bundle is a trivial bundle because of the linear independence of $df_1, \cdots, df_k$. As is known in the theory of Riemannian geometry, there exists some open neighbourhood V of M_k in $N(M_k)$ and C^{p-1}-embedding,

$$\mathrm{Exp}_{M_k}\colon V \to M,$$

such that, for any vector $L \in V$, $\mathrm{Exp}_{M_k}(L)$ is the terminal point of the geodesic arc l of M, whose initial vector is L and $\|l\| = \|L\|$. (Exp_{M_k} is called exponential mapping at M_k and $\mathrm{Exp}_{M_k}(V)$ tubular neighbourhood of M_k.)

Let $N_x = \mathrm{Exp}_{M_k}(N(M_k)_x \cap V)$, $x \in M_k$, where $N(M_k)_x$ is fibre of $N(M_k)$ on x. If we take the neighborhood U_0 of α_0 in T^k sufficiently small, then the following mapping ι_0 is one to one:

$$\iota_0\colon U_0 \times M_k(\alpha_0) \longrightarrow M_k(U_0)\colon (\alpha, x) \rightsquigarrow M_k(\alpha) \cap N_x.$$

ι_0^{-1} is obviously C^{p-1}-differentiable, and rank of ι_0^{-1} is equal to $\dim M$ on $M_k(\alpha_0)$. Therefore, by taking U_0 less if necessary, ι_0 is a C^{p-1}-diffeomorphism. That is to show.

we have now verified that (M, P_k, T^k, M_k) is a local trivial C^{p-1}-smooth fibre space over T^k, whose fibres are C^p-submanifolds of M. See for details R. L. Bishop & R. J. Crittenden [3], S. T. Hu [6] and N. Steenrod [10], Par I.

Next we show that the number of connected components of fibre M_k is finite. Let $\widetilde{M}$ be the space which is obtained by regarding each connected component of fibres to be one point. Naturally $\widetilde{M}$ becomes a manifold. ($f_1, \cdots, f_k$ are the local coordinates of it!) Moreover this space is obviously a covering space of the torus T^k. So there exist certain integers $z_1, \cdots, z_m$, $z_1 \neq 0, \cdots, z_m \neq$, $m \leqq k$, and

$$\widetilde{M} \cong R/_{z_1 Z} \times \cdots \times R/_{z_m Z} \times R^{k-m}.$$

As the flow (φ_t) preserves the connected components of fibres, it induces naturally a flow $(\tilde{\varphi}_t)$ on $\widetilde{M}$:

$$\tilde{\varphi}_t\colon \widetilde{M} \longrightarrow \widetilde{M}\colon (x_1, \cdots, x_k) \rightsquigarrow (x_1 + \lambda_1 t, \cdots, x_k + \lambda_k t).$$

If $m<K$, then this will contradict to the ergodicity of the flow (φ_t). Therefore $m=k$. This shows that the number of connected components of M_k is finite:

$$\#(M_k)=Z_1\times\cdots\times Z_m.$$

Step 3. Now by induction we will prove that $df_1,\cdots,df_r$ are linearly independent everywhere. It is clear that df_1 is linearly independent everywhere, so it is sufficient to prove the linear independence of $df_1,\cdots,df_{k+1}$ under the assumption of linear independence of $df_1,\cdots,df_k$.

Let us assume that df_{k+1} is linearly dependent to $df_1,\cdots,df_k$ at point $x_0\in M$. Then as φ_t are diffeomorphisms of M and the eigenfunctions satisfy the property (*) in step 1, so df_{k+1} is linearly dependent to $df_1,\cdots,df_k$ at every point $\varphi_t\ x_0\in M$. Namely df_{k+1} is linearly dependent to $df_1,\cdots,df_k$ on the closure of orbit which pass through the point x_0, C_{x_0}:

$$C_{x_0}=\overline{\bigcup_{-\infty<t<\infty}\varphi_t x_0}\subset M.$$

This shows that when we consider the function f_{k+1} as a function on $M_k(\alpha)$, every point of $C_{x_0}\cap M_k(\alpha)$ is a critical point of f_{k+1}. Therefore by the well-known theorem of Sard, the measure of the set $f_{k+1}(C_{x_0}\cap M_k(\alpha))$ in $\boldsymbol{T}^1$ is zero. By the lemma and compactness of C_{x_0}, $P_{k+1}(C_{x_0})=\boldsymbol{T}^{k+1}$, so $f_{k+1}(C_{x_0}\cap M_k(\alpha))=\boldsymbol{T}^1$. But it is obviously a contradiction. This was to be proven. **q.e.d.**

The proof of theorem 1 is contained in the proof of the fundamental lemma.

Corollary. *Under the assumptions of theorem* 1, Λ^p *is finitely generated and rank of* $\Lambda^p\leqq$ *dimension of* M.

§3 The case of discrete spectrum

In this § , we consider the special case when the flow has a discrete spectrum (pure point spectrum).

Theorem 2. *Let (M, μ, φ_t) be classical ergodic flow and M be compact. If the flow (φ_t) has discrete spectrum and all the eigenfunctions are C^ρ-differentiable $(\rho \geqq 1)$ i.e. $\sum_{\lambda \in \Lambda^\rho} \oplus H^{(\lambda)} = L^2(M, \mu)$, then (M, μ, φ_t) is C^ρ-isomorphic to a quasi-periodic motion (T^n, m, ψ_t) as classical flows;*

$$(M, \mu, \varphi_t) \underset{C^\rho}{\simeq} (T^n, m, \psi_t),$$

where $n = \dim M$.

Proof: As is seen in the proof of fundamental lemma, it is sufficient to prove that, rank of $\Lambda^\rho = \dim M$.

Let r be the rank of Λ^ρ, and $\lambda_1, \cdots, \lambda_r \in \Lambda^\rho$ linearly independent over Z. Let $f_1(x), \cdots, f_r(x)$ be C^ρ-differentiable eigenfunctions belonging to $\lambda_1, \cdots, \lambda_r$ respectively. Now let λ be any eigenvalue $\in \Lambda^\rho$ and f_λ be a differentiable eigenfunction which belongs to λ.

As r is the rank of Λ^ρ, so $\lambda, \lambda_1, \cdots, \lambda_r$ are linearly dependent over Z, i.e.

$$\exists k \neq 0,\ k_1, \cdots, k_r \in Z,$$

such that

$$k = k_1\lambda_1 + \cdots + k_r\lambda_r.$$

By theorem B, (iii) in §1, we can assume

$$f_\lambda^k(x) = f_1^{k_1}(x) \cdots f_r^{k_r}(x).$$

From the continuity of $f_\lambda(x)$, it must be constant on every connected component of fibre $M_r(\alpha)$. By the assumption, $\{f_\lambda;\ \lambda \in \Lambda^\rho\}$ forms a C.O.N.S. of $L^2(M, \mu)$. Therefore, if $\dim M_r(\alpha) = \dim M - r > 0$, it obviously contradicts to the theorem B, (iv). It is also easy to prove it directly. **q.e.d.**

§4. Some remarks and conjecture

Can we replace the assumption of differentiability of eigenfunctions by the one of continuity? The author has not succeeded in verifiing it, but it is very likely that it is correct. We will show the partial solution for it, though it is far from the complete solution.

In this connection, we remark that if this conjecture is verified, then we will have the following interesting result as a corollary of it and a theorem of von Neumann and Halmos. (von Neumann and Halmos [13] p. 349 Theorem 6.) As a matter of fact, this result can be easily proven, if we use a theorem of Lie groups: "a compact group is a Lie group" and the above mentioned theorem. But we will give the proof for the completeness.

Theorem 3. *(von Neumann-Halmos) Let (M, μ, φ_t) be a classical ergodic flow. If we can define the metric $d(x, y)$, compatible with the original topology of M, such that, for which M is complete and the flow (φ_t)* is equi-continuous with respect to t, *i.e.*

$$\forall\varepsilon>0,\ \exists\delta>0,\ d(x, y)<\delta \textit{ implies}$$

$$d(\varphi_t x, \varphi_t y)<\varepsilon \textit{ for all } t,$$

then (M, μ, φ_t) is C^0-isomorphic to a quasi-periodic motion (T^n, m, ψ_t) as classical flows;

$$(M, \mu, \varphi_t) \underset{C^0}{\simeq} (T^n, m, \psi_t),$$

where $$n = \dim M.$$

Proof: First we prove that we can define a multiplication on M so that it becomes (with the original topology of M) a compact abelian group and (φ_t) is a rotaion, i.e. $\exists\{x_t\}$: one-paremeter group of M such that

$$\varphi_t x = x_t \cdot x \quad \text{for } \forall x \in M.$$

Now we show that we can assume that (φ_t) is isometric with respect to d. For, if we define the new metric $d'(x, y)$ by

$$d'(x, y) = \sup_{-\infty<t<\infty} \{\min(1, d(\varphi_t x, \varphi_t y))\},$$

then (φ_t) is isometric with respect of d':

$$d'(x, y)=d'(\varphi_t x, \varphi_t y) \qquad \text{for all } t.$$

(It is clear d and d' define the same topology on M.)

As the flow (φ_t) is isometric, ergodic and preserving the measure μ, it is easy to prove that M is totally bounded. By the way, M is complete so M is compact.

Next we show M is abelian group. Let take a point x_0 such that $N=\bigcup_t \varphi_t x_0$ is dense in M. For $x=x_t=\varphi_t x_0$, $y=x_s\in N$, we define $p(x, y)=x_{t+s}\in N$ and $r(x)=x_{-t}\in N$. Then,

$$\begin{aligned} d(p(x, y),\ p(x', y'))&=d(x_{t+s}, x_{t'+s'}) \\ &\leqq d(x_{t+s}, x_{t+s'})+d(x_{t+s'}, x_{t'+s'}) \\ &=d(x_s, x_{s'})+d(x_t, x_{t'}) \\ &=d(y, y')+d(x, x'), \end{aligned}$$

$$\begin{aligned} d(r(x), r(y))&=d(x_{-t}, x_{-s}) \\ &=d(x_{-t+t+s}, x_{-s+t+s}) \\ &=d(y, x), \end{aligned}$$

where $\qquad x'=x_{t'},\ y'=x_{s'}\in N.$

This shows $p(x,y)$ and $r(x)$ are uniformly continuous on $N\times N$ and N respectively. But $N\times N$ and N are everywhere dense in $M\times M$ and M respectively, therefore $p(x,y)$ and $r(x)$ each has a unique continuous extension, to $M\times M$ and M respectively. We define for every $x, y\in M$, $x\cdot y=p(x, y)$ and $x^{-1}=r(x)$; it is clear that with these definitions M becomes an abelian topological group. If we define $p'(x, y)=\varphi_t y$ for any $x=x_t\in N$ and arbitrary y, then $p'(x,y)$ is a continuous extension of the original $p(x,y)$ and therefore $\varphi_t y=x_t\cdot y$.

We have now proven that M is a compact abelian group and (φ_t) is a rotation. As is known in the theory of Lie groups, there exists an isomorphism h: $M\to T^n$. If we define (ψ_t) by $\psi_t=h\circ\varphi_t\circ h^{-1}$ and m' by $m'=h(\mu)$, then from the ergodicity of (ψ_t) which follows from the ergodicity of (φ_t) it is easy to prove that (T^n, m', ψ_t) is a quasi-periodic motion. **q.e.d.**

Using this theorem 3, we prove the next.

Theorem 4. *Let (M, μ, φ_t) be classical ergodic flow and M be compact. If the flow (φ_t) has discrete spectrum and all the eigenfunctions are continuous, in addition, if they separate M i.e. for any $x, y \in M$, $x \neq y$, there exists some continuous eigenfunction $f_\lambda(x)$ such that $f_\lambda(x) \neq f_\lambda(y)$; then (M, μ, φ_t) is C^0-isomorphic to a quasi-periodic motion (T^n, m, ψ_t) as classical flows:*

$$(M, \mu, \varphi_t) \underset{C^0}{\simeq} (T^n, m, \psi_t),$$

where $$n = \dim M.$$

Proof: From the assumptions there exist continuous eigenfunctions $f_1, f_2, \cdots, f_n, \cdots$ which belong to eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots$ respectively and form C.O.N.S. of $L^2(M, \mu)$.

Let define a metric d by

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} |f_n(x) - f_n(y)|, \quad x, y \in M.$$

Now the topology defined by this metric d is Hausdorff by the assumption. As M is compact, so this topology is equivalent with the original one. Therefore, if we show that the flow (φ_t) is isometric with respect to this metric d, then we get theorem 4 from theorem 3.

$$\begin{aligned} d(\varphi_t x, \varphi_t y) &= \sum_{n=1}^{\infty} \frac{1}{2^n} |f_n(\varphi_t x) - f_n(\varphi_t y)| \\ &= \sum \frac{1}{2^n} |e^{2\pi i \lambda_n t}(f_n(x) - f_n(y))| \\ &= d(x, y). \end{aligned}$$

This is to be proven. **q.e.d.**

§5. Generalization of the concept of eigenfunctions

In this §, we will give a generalization of the concept of (differentiable) eigenfunctions.

It is clear that if $f(x)$ is an eigenfunction, then $f(x)=f(y)$ implies $f(\varphi_t x)=f(\varphi_t y)$ for all t.

Definition 4. Let (M, μ, φ_t) be a classical flow, and $f\colon M\to C$ be a complex valued function. We call $f(x)$ *oscillatory function* of the flow (φ_t), if $f(x)=f(y)$ implies $f(\varphi_t x)=f(\varphi_t y)$ for all t. In this case the flow (φ_t) is said to be *oscillated* with respect to $f(x)$.

Definition 5. Let (M, μ, φ_t) be a classical flow and $f\colon M\to C$ be a complex valued function. We call $f(x)$ an *essentially eigen-function*, if there exists a homeomorphism h of C such that $h\circ f$ is an eigen-function of the flow (φ_t). Then we can show

Theorem 5. *A non-constant differentiable oscillatory function of a classical ergodic flow (φ_t) is an essentially eigen-function, hence the flow (φ_t) has a non-constant eigen-function.*

It is an almost direct corollary of the well-known Poincaré-Bendixson theorem, (see for details, Coddington and Levinson [4]) if we refer to the following well-known theorem of the ergodic theory. So we omit the proof.

Theorem C. (*Hopf* [7] p. 29) *Let (M, μ, φ_t) be a classical ergodic flow, then almost all orbits of the flow (φ_t) are everywhere dense on M.*

Concluding this paper, we want to emphasize the importance of the concept of the flow homorphism.

Definition 6. Let (M, φ_t) and (N, ψ_t) be classical flows. We call the differentiable mapping f of M into N *flow homeomorphism*, if $\psi_t \circ f = f \circ \varphi_t$ for all t.

In theorem 1 and 2, the quasi-periodic motion on a torus plays the role of (N, ψ_t), and in theorem **5**, N is $C \cong R^2$.

We note that theorem 1, 2, 3 and 4 for the case when the (φ_t) is a discrete flow, i.e. (φ^n) generated by one diffeomorphism φ which preserves the measure μ can be verified analogously.

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