J. Math. Kyoto Univ. (JMKYAZ)
12-1 (1972) 129-140

Polyharmonic classification of Riemannian manifolds

By

Cecilia WANG and Leo SARIO

(Communicated by Professor Kusunoki, July 1, 1971)

A *polyharmonic function* is a C^{2n} solution, $n \geq 2$, of the equation

$$\Delta^n u = 0. \tag{1}$$

We sometimes also use the term *n-harmonic* to specify the degree. The object of the present study is a polyharmonic classification of Riemannian manifolds, i.e. the problem of existence of polyharmonic functions with various boundedness properties. We shall show that much of the biharmonic classification theory developed in Nakai-Sario [4], [5], Sario-Wang-Range [9], and Kwon-Sario-Walsh [2], can be generalized to the polyharmonic case. The higher degree brings forth fascinating new versality, as various boundedness conditions can be separately imposed on the functions and the iterates of the Laplacian.

In §1 we introduce the quasipolyharmonic classification of Riemannian manifolds based on the equation $\Delta^n u = 1$, and characterize the corresponding null classes in terms of the harmonic Green's function. Polyharmonic projection and decomposition are the topics of §2. As an application we find a necessary and sufficient condition for the existence of a solution of the polyharmonic Dirichlet problem. We also briefly discuss the classification theory associated with the class of q-polyharmonic functions.

The work was sponsored by the U.S. Army Research Office-Durham, Grant DA-ARO-D-31-124-71-G20, University of California, Los Angeles.

§1. Quasipolyharmonic classification

1. On a smooth noncompact Riemannian manifold R of dimension $m \geq 2$ with a smooth metric tensor (g_{ij}), the Laplace-Beltrami operator is

$$\Delta \cdot = -\frac{1}{\sqrt{g}} \sum_{i=1}^{m} \frac{\partial}{\partial x^i} \sum_{j=1}^{m} \sqrt{g}\, g^{ij} \frac{\partial \cdot}{\partial x^j}, \tag{2}$$

where $x=(x^1, \cdots, x^m)$ is a local coordinate system, $g = \det(g_{ij})$, and $(g^{ij})=(g_{ij})^{-1}$. We call a C^{2n} function *quasipolyharmonic* or *n-quasi-harmonic* if it satisfies

$$\Delta^n u = c$$

with some constant c. For the purpose of the classification of manifolds, we normalize by setting

$$Q_n = \{u \in C^{2n} \mid \Delta^n u = 1\}. \tag{3}$$

For a given class X of functions we denote by O_X the class of Riemannian manifolds on which there exist no nonconstant functions in X, and by X_T the class of functions which is mapped into X by a given operator T. We shall characterize the manifolds $R \notin O_{Q_n X_0 X_{1\Delta} \cdots X_{(n-1)\Delta^{n-1}}}$ in terms of the function $G_n 1$ determined by the harmonic Green's function $g(x, y)$ on R by

$$Gf = \int_R g(\cdot, y) f(y) dy \qquad \text{for} \quad n=1, \tag{4}$$

and by

$$G_n f = G \cdots G f \qquad \text{for} \quad n > 1, \tag{5}$$

where the iteration $G \cdots G$ is taken n times.

2. Positive $\Delta^{n-1} f$. Let P, B, and D be the classes of nonnegative functions, bounded functions, and functions with finite Dirichlet integrals, respectively. We shall write $G_n f \in X$ to mean $G_n|f|<\infty$ for $X=P$; $|\sup_R G_n f|<\infty$ for $X=B$; $D(G_n f)<\infty$ for $X=D$; and $|\sup_R G_n f|<\infty$, $D(G_n f)<\infty$ for $X=C=B\cap D$. It is known [5] that $R\notin O_{QX}$ if and only if $G1\in X$, and, whenever the integrals are well defined, $D(G_n 1)=G(G_{n-1}1, G_{n-1}1)$ with

(6) $$G(f_1, f_2)=\int_{R\times R} g(x, y)f_1(x)f_2(y)dx\,dy.$$

Theorem 1. *A Riemannian manifold R belongs to $O_{Q_n XP_\Delta \cdots P_{\Delta^{n-1}}}$ if and only if $G_n 1\notin X$, where $X=P$, B, D, or C.*

Proof. By the equality $\Delta G_n 1=G_{n-1}1$, $G_n 1\in X$ implies $G_n 1\in Q_n XP_\Delta\cdots P_{\Delta^{n-1}}$, and therefore $R\notin O_{Q_n XP_\Delta\cdots P_{\Delta^{n-1}}}$.

To prove the converse, consider first the case $X=P$. Since $R\notin O_{Q_n PP_\Delta\cdots P_{\Delta^{n-1}}}$, there exists a function $f\in Q_n PP_\Delta\cdots P_{\Delta^{n-1}}$. Clearly f is nonnegative superharmonic and thus harmonizable. For any regular subregion $\Omega\subset R$,

(7) $$f=h_f^\Omega+G_\Omega h_{\Delta f}^\Omega+\cdots+G_{n\Omega}1.$$

Here h_f^Ω is harmonic on Ω, continuous on $\bar{\Omega}$, equal to f on $R-\Omega$, and $G_\Omega f=\int_R g_\Omega(\cdot, y)f(y)dy$ with $g_\Omega(\cdot,\cdot)$ the harmonic Green's function on Ω, $g_\Omega|R-\Omega=0$. By the harmonizability of f, the limit of $G_\Omega h_{\Delta f}^\Omega+\cdots+G_{n\Omega}1$ as $\Omega\to R$ exists. Since all terms in $G_\Omega h_{\Delta f}^\Omega+\cdots+G_{n\Omega}1$ are nonnegative, they converge separately as $\Omega\to R$. By the monotone convergence theorem, $G_n 1=\lim_{\Omega\to R}G_{n\Omega}1$.

For $X=B$, the proof of the existence of $G_n 1$ is similar to that in the case $X=P$. The boundedness of $G_n 1$ follows from the boundedness of $f-h_f^\Omega$.

In the case $X=D$, let $f\in Q_n DP_\Delta\cdots P_{\Delta^{n-1}}$. For every regular subregion $\Omega\subset R$,

(8) $$D(G_{\Omega}h^{\Omega}_{\Delta f}+\cdots+G_{n\Omega}1)\leq D(f)<\infty.$$

Since

(9) $$D(G_{\Omega}h^{\Omega}_{\Delta f}+\cdots+G_{(n-1)\Omega}h^{\Omega}_{\Delta^{n-1}f},\ G_{n\Omega}1)$$
$$=G_{\Omega}(h^{\Omega}_{\Delta f}+\cdots+G_{(n-2)\Omega}h^{\Omega}_{\Delta^{n-1}f},\ G_{(n-1)\Omega}1)\geq 0,$$

we have

(10) $$D(G_{n\Omega}1)\leq D(G_{\Omega}h^{\Omega}_{\Delta f}+\cdots+G_{n\Omega}1)<D(f),$$

or equivalently,

(11) $$G_{\Omega}(G_{n\Omega}1,\ G_{n\Omega}1)<D(f).$$

The monotone convergence theorem yields

(12) $$D(G_n 1)=G(G_{n-1}1,\ G_{n-1}1)<\infty.$$

The theorem for $X=C$ is a consequence of $X=B$ and $X=D$.

Corollary. *The following inclusion relations are valid*:

$$O_{QP}\subset O_{Q_nPP_{\Delta}\dots P_{\Delta^{n-1}}}\begin{matrix}\subset O_{Q_nBP_{\Delta}\dots P_{\Delta^{n-1}}}\subset\\ \subset O_{Q_nDP_{\Delta}\dots P_{\Delta^{n-1}}}\subset\end{matrix}O_{Q_nCP_{\Delta}\dots P_{\Delta^{n-1}}}.$$

3. Denote by $h_f^{n\Omega}$ the n-harmonic function on Ω with $\Delta^i h_f^{n\Omega}=\Delta^i f$ on $R-\Omega$ for $i=0,\dots,n-1$. Consider the class $H^n=H^n(R)$ of n-harmonic functions on R, and the class

$$H^{n*}=\{f\mid \lim_{\Omega\to R}h_f^{n\Omega}<\infty\}.$$

Set $h_f^n=\lim_{\Omega\to R}h_f^{n\Omega}$.

Proposition. *A Riemannian manifold R belongs to $O_{Q_nH^{n*}}$ if and only if $G_n 1=\infty$.*

Proof. If $G_n 1 < \infty$, then $G_n 1 \in Q_n$. On a regular region Ω, we have the decomposition

$$(13) \qquad G_n 1 = h^{n\Omega}_{G_n 1} + G_{n\Omega} 1.$$

Since $G_n 1 = \lim_{\Omega \to R} G_{n\Omega} 1$, $\lim_{\Omega \to R} h^{n\Omega}_{G_n 1} = 0$. Therefore $G_n 1 \in H^{n*}$ and $R \notin O_{Q_n H^{n*}}$.

Conversely, let $f \in Q_n H^{n*}$. By (7) and $f \in H^{n*}$, the limit $G_{n\Omega} 1$ exists, and the proposition follows from the monotone convergence theorem.

Corollary. $O_{Q_n H^{n*}} = O_{Q_n P \dots P_{\Delta^{n-1}}}$.

4. Bounded $\Delta^{n-1} f$. Consider a function $f \in Q_n X_{\Delta^{n-1}}$. Clearly $\Delta^{n-1} f \in QX$. Thus we have

$$(14) \qquad O_{QX} \subset O_{Q_n X_{\Delta^{n-1}}}.$$

We shall show that the converse is also true for $X = P$, C, D, or C.

Lemma. *For* $n \geq 1$,

$$(15) \qquad O_{Q_n X_{\Delta^{n-1}}} = O_{QX}$$

with $X = P$, B, D, *or* C.

Proof. Let $R \notin O_{QX}$. It is known that $G1 < \infty$. Since $G1 \in C^\infty$, there exists a function $f_2 \in C^\infty$ with $\Delta f_2 = G1$ (cf. [3]). Clearly $f_2 \in Q_2 X_\Delta$. By repeating the above process, we can find $f_3, \cdots, f_n \in C^\infty$ such that $f_i \in Q_i X_{\Delta^{i-1}}$ for $i \geq 3$. In particular, $f_n \in Q_n X_{\Delta^{n-1}}$, and Lemma 1 follows.

Theorem 2. *A Riemannian manifold R belongs to* $O_{Q_n X_0 \cdots X_{(n-2)\Delta^{n-2}} B_{\Delta^{n-1}}}$ *if and only if* $G1 \notin B$, *where* $X_i = P$, B, *or* $P \cap B$, *and* $i = 0, \cdots, n-2$.

Proof. By Lemma 1, $O_{QB}=O_{Q_nB_{\Delta^{n-1}}}\subset O_{Q_nX_0\cdots X_{(n-2)\Delta^{n-2}}B_{\Delta^{n-1}}}$. Consequently $R\notin O_{Q_nX_0\cdots X_{(n-2)\Delta^{n-2}}B_{\Delta^{n-1}}}$ implies $G1\in B$.

Conversely, if $G1\in B$ then $G_n1\in B$. Theorem 2 follows from $G_n1\in Q_nX_0\cdots X_{(n-2)\Delta^{n-2}}B_{\Delta^{n-1}}$.

Corollary. *For $X_i=P$, B, or $P\cap B$, and $Y_i=X_i$, D, or C,*

$$O_{Q_nBP_\Delta\cdots P_{\Delta^{n-1}}}\subset O_{QB}=O_{Q_nX_0\cdots X_{(n-2)\Delta^{n-2}}B_{\Delta^{n-1}}}\subset O_{Q_nY_0\cdots Y_{(n-2)\Delta^{n-2}}B_{\Delta^{n-1}}}.$$

The last inclusion is a consequence of $O_{QB}=O_{Q_nB_{\Delta^{n-1}}}\subset O_{Q_nY_0\cdots Y_{(n-2)\Delta^{n-2}}B_{\Delta^{n-1}}}$.

5. Bounded Dirichlet finite $\Delta^{n-1}f$. *We assert:*

Theorem 3. *A Riemannian manifold R belongs to $O_{Q_nZ_0\cdots Z_{(n-2)\Delta^{n-2}}C_{\Delta^{n-1}}}$ if and only if $G1\notin C$, where $Z_i=P$, B, D or C.*

Proof. Since $O_{QC}=O_{Q_nC_{\Delta^{n-1}}}\subset O_{Q_nZ_0\cdots Z_{(n-2)\Delta^{n-2}}C_{\Delta^{n-1}}}$, $R\notin O_{Q_nZ_0\cdots Z_{(n-2)\Delta^{n-2}}C_{\Delta^{n-1}}}$ implies $G1\in C$.

Conversely. if $G1\in C$, then $G_k1\in B$ and

$$D(G_k1)=G(G_{k-1}1,\ G_{k-1}1)\leq(\sup_R G_{k-1}1)^2G(1,\ 1)<\infty \tag{16}$$

for $k\leq n$. Thus $G_n1\in Q_nZ_0\cdots Z_{(n-2)\Delta^{n-2}}C_{\Delta^{n-1}}$, and Theorem 3 follows.

Corollary. *For $Z_i=P$, B, D or C,*

$$O_{QC}=O_{Q_nZ_0\cdots Z_{(n-2)\Delta^{n-2}}C_{\Delta^{n-1}}}. \tag{17}$$

§2. Polyharmonic projection and decomposition

6. Denote by $M_1=M_1(R)$ the class of bounded continuous harmonizable functions on a Riemannian manifold R. We shall show that every function $f\in M_1^n=M_1M_{1\Delta}\cdots M_{1\Delta^{n-1}}$ can be written uniquely as $u+g$ with u an n-harmonic function and $g\in N_1^n=N_1N_{1\Delta}\cdots N_{\Delta^{n-1}}$. Here N_1 is the subclass of potentials in M_1, that is, functions with null harmonic parts.

Theorem 4. *On every Riemannian manifold R which carries QB-functions,*

$$M_1^n = H^n M_1^n \oplus N_1^n. \tag{18}$$

Proof. We may write

$$f = h_f^\Omega + G_\Omega h_{\Delta f}^\Omega + \cdots + G_{n-1\Omega} h_{\Delta^{n-1}f}^\Omega + G_{n\Omega}\Delta^n f \tag{19}$$

for every regular subregion $\Omega \subset R$. Since $R \notin O_{QB}$ is equivalent with $\sup_R G1 < \infty$, we have $\sup_R G_k 1 < \infty$ and $\sup_R G_k h_{\Delta^k f} < \infty$ for $n \geq h \geq 1$. By the Lebesgue dominated convergence theorem,

$$G_n \Delta^n f = f - (h_f + Gh_{\Delta f} + \cdots + G_{n-1} h_{\Delta^{n-1}f}). \tag{20}$$

Since $\Delta G_i g = G_{i-1} g$, and $G_i g \in N_1$ for $g \in C^\infty$ and all i (cf. [9]), we see that $(f - G_n \Delta^n f) \in H^n M_1^n$, and $G_n \Delta^n f \in N_1^n$. Therefore $f = (f - G_n \Delta^n f) + G_n \Delta^n f$ is the desired decomposition provided we can prove the uniqueness. Let $u \in H^n \cap N_1^n$. Since $\Delta^{n-1} u \in H \cap N_1$, $\Delta^{n-1} u \equiv 0$ on R. Thus we have $u \in H^{n-1} N_1^{n-1}$. On repeating the above reasoning we conclude that $u \equiv 0$ on R, and the proof is complete.

We shall call the n-harmonic function u in Theorem 4 the *polyharmonic projection* of $f \in M_1^n$. On Wiener's harmonic boundary α, it is clear that $u|\alpha = f|\alpha$, $\Delta u|\alpha = \Delta f|\alpha$, $\cdots$, and $\Delta^{n-1} u|\alpha = \Delta^{n-1} f|\alpha$.

7. If we restrict the class of polyharmonic functions to M_1^n, then we have the following direct sum decomposition:

Theorem 5. *On a Riemannian manifold $R \notin O_{QB}$ every function $f \in H^n M_1^n$ can be written uniquely as*

$$f = u + G_i v \tag{21}$$

with $u \in H^i B$ and $v \in H^{n-i-1} B$, $n > i \geq 0$. Equivalently,

$$f = h_0 + Gh_1 + \cdots + G_{n-1} h_{n-1} \tag{22}$$

with $h_i \in HB$.

8. Denote by $M_2(R)=M_2$ the Royden algebra, consisting of bounded Dirichlet finite Tonelli functions. Set

$$M_2^n = M_2 M_{2\Delta} \cdots M_{2\Delta^{n-1}}. \tag{23}$$

Let $N_2(R)=N_2$ be the potential subalgebra of M_2, and define N_2^n in analogy with M_2^n.

Theorem 6. *On a Riemannian manifold R which carries QC-functions,*

$$M_2^n = H^n M_2^n \oplus N_2^n. \tag{24}$$

Proof. As in the proof of Theorem 5,

$$f - G_n \Delta^n f = h_f + G h_{\Delta f} + \cdots + G_{n-1} h_{\Delta^{n-1} f}.$$

Since

$$\begin{aligned} D(G_i h_{\Delta^i f}) &= G(G_{i-1} h_{\Delta^i f},\ G_{i-1} h_{\Delta^i f}) \\ &\leq (\sup_R G_{i-1} h_{\Delta^i f})^2 G(1,\ 1) < \infty, \end{aligned}$$

we obtain $f - G_n \Delta^n f \in H^n M_2^n$. In the same manner as in Theorem 5, we can show that $(f - G_n \Delta^n f) + G_n \Delta^n f$ is the desired unique decomposition of f.

Theorem 7. *Let R be a Riemannian manifold which carries QC-functions. Every* $f \in H^n M_2^n$ *has the unique decomposition*

$$f = u + G_i v, \tag{25}$$

with $u \in H^i C$, *and* $v \in H^{n-i-1} C$ *for* $n > i \geq 0$. *Equivalently,*

$$f = h_0 + G h_1 + \cdots + G_{n-1} h_{n-1} \tag{26}$$

with $h_i \in HC$.

9. The polyharmonic projection and decomposition theorem have thus been proved for certain subclasses of M_1 and M_2. It is natural to ask whether the theorems remain true if we suppress the boundedness condition in the definition of M_1 and M_2. Denote by M_3 the class of continuous harmonizable functions and by M_4 the class of Tonelli functions with finite Dirichlet integrals. Consider the family

$$M^n_{iX_i} = \prod_{k=0}^{n-1} M_{i\Delta^k}(F^k_{X_i})_{\pi\Delta^k},\ i=1,2,3,4, \tag{27}$$

where π is the harmonic projection and

$$F^k_{X_i} = \{f \mid G_k f \in X_i\} \tag{28}$$

with $X_1 = B$, $X_2 = C$, $X_3 = P$, and $X_4 = D$. Define $N^n_{iX_i}$ analogously.

Theorem 8. *On an arbitrary hyperbolic Riemannian manifold* R,

$$M^n_{iX_i} = H^n M^n_{iX_i} \oplus N^n_{iX_i} \tag{29}$$

for $i=1, 2, 3, 4$.

We note that $M^n_1 \equiv M^n_{1B}$ if $R \notin O_{QB}$, and $M^n_2 \equiv M^n_{2C}$ if $R \notin O_{QC}$. Therefore Theorem 8 is weaker than Theorems 4 and 6. Its proof is analogous. We also have the following decomposition:

Theorem 9. *Let* R *be an arbitrary hyperbolic Riemannian manifold. Then every* $f \in H^n M^n_{iX_i}$ *can be written uniquely as*

$$f = u + G_i v, \tag{30}$$

with $u \in H^i$ *and* $v \in H^{n-i-1}$ *for* $n > i \geq 0$. *Equivalently*

$$f = h_0 + G h_1 + \cdots + G_{n-1} h_{n-1} \tag{31}$$

with $h_i \in H$.

10. Polyharmonic Dirichlet problem. Given bounded continuous functions $f_0, f_1, \cdots, f_{n-1}$ on the Royden harmonic boundary β, the polyharmonic Dirichlet problem is to find a function u on R with

$$(32)\qquad \begin{cases} u \in H^n CC_\Delta \cdots C_{\Delta^{n-1}} \\ u \mid \beta = f_0,\ \Delta u \mid \beta = f_1,\ \Delta^{n-1} u \mid \beta = f_{n-1}. \end{cases}$$

We shall assume that $f_0, f_1, \cdots, f_{n-1}$ can be extended continuously to functions in Royden's algebra. Unconditional solvability of the above problem is not expected, since there are Riemannian manifolds on which the only n-harmonic functions are constants. Theorem 6 enables us to show:

Theorem 10. *$R \notin O_{QC}$ is a necessary and sufficient condition for problem* (32) *to have a solution. The solution is unique.*

Proof. Let $g_0, g_1, \cdots, g_{n-1} \in M_2$ be the extended functions of f_0, $f_1, \cdots, f_{n-1}$ respectively. Theorem 6 implies that $h_{g_0} + G h_{g_1} + \cdots + G_{n-1} h_{g_{n-1}}$ is the unique solution of (32).

Conversely, consider $f_0 = 0, f_1 = 1$. There exists a function $u \in H^2 CC_\Delta$ with $u \mid \beta = 0$ and $\Delta u \mid \beta = 1$. By the maximum principle, $\Delta u \equiv 1$ on R. Thus $u \in QC$, and the proof is complete.

Making use of Wiener's harmonic boundary α we obtain similarly (cf. Tanaka [10]):

Theorem 11. *Given bounded continuous functions $f_0, f_1, \cdots, f_{n-1}$ on the Wiener harmonic boundary α, $R \notin O_{QB}$ is necessary and sufficient for the existence of a function $u \in H^n BB_\Delta \cdots B_{\Delta^{n-1}}$ on R with $\Delta^k u \mid \alpha = f_k$, $k = 0, 1, \cdots, n-1$.*

11. Let q be a nonnegative C^2 function on R. We call a C^{2n} function u *q-polyharmonic* if it satisfies the equation

$$(33)\qquad (\Delta + q)^n u = 0,$$

and *q-quasipolyharmonic* if

$$(34)\qquad (\varDelta+q)^n u = c e_q,$$

where c is a constant and e_q the q-elliptic measure of R (cf. [11]). Set

$$(35)\qquad Q_{qn}=\{u\in C^{2n}\mid(\varDelta+q)^n u=e^q\}.$$

We shall show that the nondegeneracy of a manifold with respect to Q_{qn}-functions is determined by an operator on the q-harmonic Green's function $g_q(x, y)$, and the q-elliptic measure e_q on R. Consider the operator G_{qn} defined by

$$(36)\qquad \begin{cases} G_q e_q=\int_R g_q(x, y)e_q(y)dy & \text{for } n=1,\\ G_{qn}e_q=G_q\cdots G_q e_q & \text{for } n>1,\end{cases}$$

where $G_q\cdots G_q$ means iteration n times. Denote by E the class of functions with finite energy integrals, and set $K=E\cap B$. In analogy with Theorems 1–3 we have:

Theorem 12. *Let R be a q-hyperbolic Riemannian manifold.*

(i) $R\notin O_{Q_{qn}XP_\varDelta\ldots P_{\varDelta^{n-1}}}$ *if and only if* $G_{qn}e_q\in X$ *for* $X=P, B, E,$ *or* K.

(ii) $R\notin O_{Q_{qn}BB_\varDelta\ldots B_{\varDelta^{n-1}}}$ *if and only if* $G_q e_q\in B$.

(iii) $R\notin O_{Q_{qn}KK_\varDelta\ldots K_{\varDelta^{n-1}}}$ *if and only if* $G_q e_q\in K$.

Theorems 4–11, with obvious modifications, also remain valid for q-polyharmonic functions.

References

[1] C. Constantinescu-A. Cornea, *Ideale Ränder Riemannscher Flächen*, Springer, 1963, 244 pp.

[2] Y.K. Kwon-L. Sario-B. Walsh, *Behavior of biharmonic functions on Wiener's and Royden's compactifications*, Ann. Inst. Fourier (Grenoble), (to appear).

[3] B. Malgrange, *Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution*, Ann. Inst. Fourier (Grenoble) **6** (1955/56), 271-355.

[4] M. Nakai-L. Sario, *Biharmonic classification of Riemannian manifolds*, Bull. Amer. Math. Soc. **77** (1971), 432-436.

[5] ——————, *Quasiharmonic classification of Riemannian manifolds*, Proc. Amer. Math. Soc. (to appear).

[6] ——————, *A property of biharmonic functions with Dirichlet finite Laplacians*, Math· Scand. (to appear).

[7] ——————, *Dirichlet finite biharmonic functions with Dirichlet finite Laplacians*, Math. Z. 122 (1971), 203-216.

[8] L. Sario-M. Nakai, *Classification of Riemann surfaces*, Springer, 1970, 446 pp.

[9] L. Sario-C. Wang-M. Range, *Biharmonic projection and decomposition*, Ann. Acad. Sci. Fenn. Ser. A.I. No. 494 (1971), 14 pp.

[10] H. Tanaka, *On Wiener functions of order* m, (to appear).

[11] C. Wang-L. Sario, *The class of* (p, q)-*biharmonic functions*, Pacific J. (to appear).