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Biharmonic Green's functions and harmonic degeneracy*

By

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The harmonic and biharmonic classification theories of Riemannian manifolds have developed in somewhat opposite directions. In harmonic classification theory, the existence of the Green's function was first explored, and then its relations to various harmonic null classes established. In biharmonic classification theory, a rather complete array of relations between quasiharmonic and biharmonic null classes has been developed, without any reference to biharmonic Green's functions. The reason is that no explicit tests for the existence of these functions have been known. Such tests have recently been found for biharmonic Green's functions γ which, roughly speaking, satisfy the conditions $\gamma=\Delta\gamma=0$ on the ideal boundary of the Riemannian manifold [5], [6]. The road is now open to finding relations between the class O_{Γ}^{N} of Riemannian N-manifolds which do not carry γ, and other null classes considered in classification theory.

The present study is devoted to harmonic null classes. The first question here is: is there any relation between O_{Γ}^{N} and the class O_{G}^{N} of Riemannian N-manifolds that do not carry harmonic Green's functions? We shall show:

The strict inclusion

$$O_{G}^{N}<O_{\Gamma}^{N}$$

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holds for every dimension $N \geq 2$.

It is known that $O_G^N < O_{HP}^N < O_{HB}^N < O_{HD}^N = O_{HC}^N$, where the O_{HX}^N, $X = P, B, D, C$, are the classes of Riemannian N-manifolds not admitting nonconstant harmonic functions whice are positive, bounded, Dirichlet finite, or bounded Dirichlet finite, respectively. Where does O_Γ^N fit into this scheme? We shall prove that its behavior is quite different from that of the O_{HX}^N classes:

O_Γ^N *neither contains nor is contained in any of the classes* O_{HX}^N *for any* $X = P, B, D, C$, *or any* $N \geq 2$.

Let $\tilde{O}^N$ be the complement of a null class O^N with respect to the totality of Riemannian N-manifolds. The nonvoidness of the classes $\tilde{O}_\Gamma^N \cap \tilde{O}_{HX}^N$, $O_\Gamma^N \cap O_{HX}^N$, $O_\Gamma^N \cap \tilde{O}_{HX}^N$ offers no difficulty, but since $O_G^N < O_\Gamma^N$ and $O_G^N < O_{HP}^N$, any counterexample to prove $\tilde{O}_\Gamma^N \cap O_{HP}^N \neq \phi$ must lie in the "narrow" space $\tilde{O}_G^N \cap O_{HP}^N$. We have succeeded in finding counterexamples only in the cases $N = 2$ and $N > 4$. Fortunately the cases $N = 3, 4$ can be settled by means of a counterexample for $\tilde{O}_{QX}^N \cap O_{HP}^N \neq \phi$ recently constructed by L. Chung [1]; here Q is the family of quasiharmonic functions q, defined by $\Delta q = 1$.

We also take up the class $O_{HL^p}^N$ of Riemannian N-manifolds which admit no harmonic functions of finite L^p norm, $p \geq 1$. We show that this class shares the property of the above O_{HX}^N:

The classes

$$\tilde{O}_\Gamma^N \cap \tilde{O}_{HL^p}^N, \quad O_\Gamma^N \cap O_{HL^p}^N, \quad O_\Gamma^N \cap \tilde{O}_{HL^p}^N, \quad \tilde{O}_\Gamma^N \cap O_{HL^p}^N$$

are all nonvoid for every $p \geq 1$ *and every* $N \geq 2$.

1. We start with a test for O_Γ^N. Given a Riemannian N-manifold $R, N \geq 2$, let $\Delta = d\delta + \delta d$ be the Laplace-Beltrami operator. On a regular subregion Ω of R, let $\gamma_\Omega(x, y)$ be the Green's function for the biharmonic equation $\Delta^2 u = \Delta\Delta u = 0$, with the biharmonic fundamental singularity y, and boundary data $\gamma_\Omega = \Delta\gamma_\Omega = 0$ on $\beta_\Omega = \partial\Omega$. In terms of the harmonic Green's function $g_\Omega(x, y)$ on $\bar{\Omega}$, we have

$$\gamma_\Omega(x, y) = \int_\Omega g_\Omega(x, z) g_\Omega(z, y) dz ,$$

with dz the volume element at z. We know that $\Delta_x \gamma_\Omega(x, y) = g_\Omega(x, y)$. As Ω exhausts R, we obtain the directed limit

$$\gamma(x, y) = \lim_{\Omega \to R} \gamma_\Omega(x, y) = \int_R g_R(x, z) g_R(z, y) dz ,$$

which, if it exists, is the biharmonic Green's function γ on R, with $\Delta_x \gamma(x, y) = g_R(x, y)$.

Take a fixed regular subregion R_0 of R, set $S_0 = R - \bar{R}_0$, $\alpha_0 = \partial R_0$, and choose a regular subregion Ω with $\bar{R}_0 \subset \Omega$. Let H be the family of harmonic functions. On $\bar{\Omega} \cap \bar{S}_0$, take

$$\omega_\Omega \in H(\Omega \cap S_0) \cap C(\bar{\Omega} \cap \bar{S}_0),\ \omega_\Omega | \alpha_0 = 1,\ \omega_\Omega | \beta_\Omega = 0 .$$

According as the limit $\omega = \lim_{\Omega \to R} \omega_\Omega$ is $\equiv 1$ or < 1 on S_0, R is, by definition, parabolic or hyperbolic. We denote by O_G^N the class of parabolic N-manifolds; it is known that these are precisely those on which $\lim_{\Omega \to R} g_\Omega(x, y) \equiv \infty$.

Let O_Γ^N be the class of Riemannian N-manifolds on which $\gamma \equiv \infty$. The fact that the finiteness of γ and the degeneracy of ω are independent of x, y, and S_0 will be a consequence of the following simple test:

$$R \in \tilde{O}_\Gamma^N \Leftrightarrow \omega \in L^2(S_0) . \tag{1}$$

The test was established in [5]; here we give an alternate proof which is slightly more "direct" in that it does not make use of Harnack's inequality.

First suppose $\gamma(x, y)$ exists for some $x, y \in R$,

$$\gamma(x, y) = \int_R g_R(x, z) g_R(z, y) dz .$$

Since the existence of γ entails that of g_R, we may assume henceforth that $R \in \tilde{O}_G^N$. We shall show that $\omega \in L^2(S_0)$, $S_0 = R - \bar{R}_0$, for any regular subregion R_0 of R. Take regular subregions R_1 and Ω of R with $\bar{R}_0 \cup x \cup y \subset R_1 \subset \bar{R}_1 \subset \Omega$ and set $\alpha_1 = \partial R_1$, $S_1 = R - \bar{R}_1$,

$$m_{1\Omega} = \min_{z \in \alpha_1} g_\Omega(z, x),\ M_{1\Omega} = \max_{z \in \alpha_1} g_\Omega(z, x),$$

$$m_{2\Omega} = \min_{z \in \alpha_1} g_\Omega(z, y),\ M_{2\Omega} = \max_{z \in \alpha_1} g_\Omega(z, y),$$

$$m_{3\Omega} = \min_{\alpha_1} \omega_\Omega ,\ M_{3\Omega} = \max_{\alpha_1} \omega_\Omega ,$$

$$k_{1\Omega} = \frac{M_{3\Omega}^2}{m_{1\Omega} m_{2\Omega}},\ k_{2\Omega} = \frac{M_{1\Omega} M_{2\Omega}}{m_{3\Omega}^2} .$$

Denote by m_1, M_1, m_2, M_2, m_3, M_3, k_1, k_2, the corresponding limits as $\Omega\to R$. Then

$$\omega_\Omega^2(z)\leq k_{1\Omega}g_\Omega(z,x)g_\Omega(z,y)$$

on $\alpha_1\cup\beta_\Omega$, hence on $\bar{\Omega}\cap\bar{S}_1$. A fortiori,

$$\omega^2(z)\leq k_1 g_R(z,x)g_R(z,y) \text{ on } \bar{S}_1 .$$

By the symmetry of g_R,

$$\begin{aligned}\|\omega\|_2^2 &= \int_{S_0}\omega^2(z)dz=C+\int_{S_1}\omega^2(z)dz\\ &\leq C+k_1\int_{S_1}g_R(x,z)g_R(z,y)dz\\ &< C_1+k_1\int_R g_R(x,z)g_R(z,y)dz\\ &= C_1+k_1\gamma(x,y)<\infty .\end{aligned}$$

Therefore, $\gamma(x,y)<\infty$ for some (x,y) implies $\omega\in L^2(S_0)$ for any $S_0=R-\bar{R}_0$.

Conversely, suppose $\omega\in L^2(S_0)$ for some $S_0=R-\bar{R}_0$. Take any x, $y\in R$ and choose R_1, Ω as above. Then

$$g_\Omega(z,x)g_\Omega(z,y)\leq k_{2\Omega}\omega_\Omega^2(z)$$

on $\alpha_1\cup\beta_\Omega$, hence on $\bar{\Omega}\cap\bar{S}_1$, and therefore

$$g_R(z,x)g_R(z,y)\leq k_2\omega^2(z) \text{ on } \bar{S}_1 .$$

It follows that

$$\begin{aligned}\gamma(x,y) &= C+\int_{S_1}g_R(x,z)g_R(z,y)dz\\ &\leq C+k_2\int_{S_1}\omega^2(z)dz<\infty .\end{aligned}$$

We conclude that $\omega\in L^2(S_0)$ for some $S_0=R-\bar{R}_0$ implies $\gamma(x,y)<\infty$ for any $x,y\in R$. This proves our criterion (1). As a consequence, the finiteness of $\gamma(x,y)$ and the nondegeneracy of ω are independent of x, y, and R_0.

2. By means of (1), we now tackle our first problem, that of determining the relation between O_Γ^N and O_G^N.

Theorem 1. *$O_G^N < O_\Gamma^N$ for $N \geq 2$.*

Proof. We already observed that $O_G^N \subset O_\Gamma^N$, and we only have to prove the strictness of the inclusion.

Let $(x, y_1, \cdots, y_{N-1})$ be Cartesian coordinates, and take the N-cylinder

$$T = \{(x, y_1, \cdots, y_{N-1}) \mid x > 0, \ |y_i| \leq 1, i = 1, \cdots, N-1\},$$

where for each i the faces $y_i = 1$ and $y_i = -1$ are identified by a parallel translation perpendicular to the x-axis. Endow T with the metric

$$ds^2 = x^\alpha dx^2 + x^{\alpha/(N-1)} \sum_{i=1}^{N-1} dy_i^2,$$

α a constant. For $h(x) \in H$,

$$\Delta h(x) = -x^{-\alpha}(x^\alpha x^{-\alpha} h'(x))' = 0,$$

hence $h(x) = ax + b$, with a, b arbitrary constants. For $R_0 = \{1 < x < 2\}$, $S_0 = \{0 < x < 1\} \cup \{x > 2\}$, the harmonic measure on $\{0 < x < 1\}$ is $\omega(x) = x$, hence $T \in \tilde{O}_G^N$. On the other hand, for $\alpha \leq -3$,

$$\|\omega\|_2^2 \geq c \int_0^1 x^2 x^\alpha dx = \infty$$

and therefore $T \in O_\Gamma^N$. The theorem follows.

3. We proceed to relations between O_Γ^N and the harmonic null classes O_{HX}^N, with $X = P, B, D, C$. We recall that $O_G^N < O_{HP}^N < O_{HB}^N < O_{HD}^N = O_{HC}^N$ (e.g. [7]). To begin with, the Euclidean N-ball gives trivially

$$\tilde{O}_\Gamma^N \cap \tilde{O}_{HX}^N, \ X = P, B, D, C, \ N \geq 2.$$

To see that

$$O_\Gamma^N \cap O_{HX}^N \neq \phi, \ X = P, B, D, C, \ N \geq 2,$$

consider the N-cylinder

$$T = \{|x| < \infty, |y_i| \leq 1, i = 1, \cdots, N-1\},$$

with the Euclidean metric. Every $h(x) \in H$ has the form $h(x) = ax + b$, which is unbounded for $a\neq 0$. Therefore, $\omega(x)\equiv 1$, and $T\in O_G^N<O_{HX}^N$. Moreover, $\|\omega\|_2^2>c\int^\infty 1\cdot dx=\infty$, hence $T\in O_\Gamma^N$.

4. Next we give an N-manifold which carries HX functions but no γ.

Theorem 2. $O_\Gamma^N\cap\tilde{O}_{HX}^N\neq\phi$, $X=P,B,D,C$, $N\geq 2$.

Proof. Take the "short" N-cylinder

$$T=\{|x|<1,\ |y_i|\leq 1,\ i=1,\cdots,N-1\}$$

with the metric

$$ds^2=\lambda^2(x)dx^2+\lambda(x)^{2/(N-1)}\sum_{i=1}^{N-1}dy_i^2\,,$$

where $\lambda\in C^2((-1,1))$. For $h(x)\in H$,

$$\Delta h=-\lambda^{-2}(\lambda^2\lambda^{-2}h')'=0\,,$$

$h=ax+b$, and the Dirichlet integral is

$$D(h)=\int_T h'^2\lambda^{-2}dV=c\int_{-1}^1\lambda^{-2}\lambda^2dx<\infty,$$

hence $x\in HD$ and $T\in\tilde{O}_{HD}^N\supset\tilde{O}_{HX}^N$. On the other hand,

$$\|\omega\|_2^2>c\int^1(ax+b)^2\lambda^2dx\,.$$

For $\lambda=(1-x^2)^\alpha$, $\alpha\leq-\frac{1}{2}$, this gives $\|\omega\|_2=\infty$ and $T\in O_\Gamma^N$.

5. Our next problem is to find an N-manifold which carries γ but no HX functions.

Theorem 3. $\tilde{O}_\Gamma^N\cap O_{HX}^N\neq\phi$, $X=P,B,D,C$, $N\geq 2$.

The proof will be given in Nos. 5-9.

The case $N>4$ offers no difficulty. In fact, on the Euclidean N-space E^N we have for every $h\in HP(E^N)$, $x\in E^N$, $r=|x|$, $r<R<\infty$, the Harnack inequality

$$\left(\frac{R}{R+r}\right)^{N-2}\frac{R-r}{R+r}h(0)\leq h(x)\leq\left(\frac{R}{R-r}\right)^{N-2}\frac{R+r}{R-r}h(0)$$

which for $R\to\infty$ gives $h=\text{const}$, $E^N\in O_{HP}^N\subset O_{HX}^N$. The harmonic measure on $\{r>1\}$ is $\omega(x)=r^{-N+2}$ and

$$\|\omega\|_2^2=c\int_1^\infty r^{-2N+4}r^{N-1}dr<\infty\,,$$

hence $E^N\in\tilde{O}_\Gamma^N$.

6. The above argument fails if $N\leq 4$, as then $E^N\in O_\Gamma^N$. However, L. Chung [1] has communicated to the author the construction of an intricate N-manifold belonging to $\tilde{O}_{QC}^N\cap O_{HP}^N$, $N>2$. Since $O_\Gamma^N\subset O_{QC}^N$, as can readily be seen, this manifold also gives $\tilde{O}_\Gamma^N\cap O_{HX}^N\neq\phi$ for $N>2$.

7. Chung's example does not apply to the case $N=2$. For this dimension we make use of an example originally constructed to show the strictness of $O_G<O_{HP}$ for Riemann surfaces [4]. For our present purpose the surface has to be endowed with a metric with the property $\omega\in L^2$.

The surface is constructed as follows. From the unit disk $|z|<1$ remove the radial slits

$$R_{mn}^\nu=\{r_{2\mu}\leq r<r_{2\mu+1},\quad \varphi=r\cdot 2\pi/2^{m+\lambda}\},$$

where $\mu=q_m\cdot 2^n$; $m,n=1,2,\cdots$; $\{q_m\}$ the sequence of odd primes; $r_i=1-2^{-i}$; $\nu=1,\cdots,2^{m+\lambda}$; the $\lambda=\lambda(\mu)$ positive integers to be specified later. In each sector

$$S_{mk}=\{(k-1)\cdot 2\pi/2^{m-1}\leq\varphi\leq k\cdot 2\pi/2^{m-1}\},$$

$k=1,\cdots,2^{m-1}$, identify by pairs those edges of the R_{mn}^ν that are symmetrically located with respect to the bisecting half-ray d_{mk} of S_{mk}, the edges facing d_{mk} being identified, and similarly the edges away from d_{mk}. The edges of a slit on d_{mk} are thus identified, and the left edge of a slit on $\varphi=(k-1)\cdot 2\pi/2^{m-1}$ is identified with the right edge of the slit on $\varphi=k\cdot\pi/2^{m-1}$. Denote by p and $p_m=p_n(p)$ the points so identified on the slits. If p is an end point of a slit on the boundary of S_{mk}, there are 2^{m-1} identified point $p_m^i(p)$, $i=1,\cdots,2^{m-1}$. After these identifications for each $m=1,2,\cdots$, we have a surface R.

8. To make R into a Riemann surface we endow it with a conformal structure by means of a covering of R by open sets V and their homeomorphic mappings $t=h(p)$ onto parametric disks $|t|<\rho$ in the following manner. For a point p not on a slit, let V be a disk about p disjoint from all slits and let $T=h(p)=z(p)$ be the projection of $p\in R$ into $|z|<1$. If p lies on the edge of a slit but is not an end point of it, V is to consist of two half-disks, about p and $p_m(p)$, with diameters on the corresponding slits. The half-disks are transferred, by rotations $\tilde{p}(p)$ about $z=0$, to form a full disk, and the mapping h is taken as $t=z(\tilde{p}(p))$.

If p is an end point of slit R_{mn}^{ν} not on the boundary of S_{mk}, the neighborhood V of p is to consist of two identical slit disks, about p and $p_m(p)$, transferred by rotations about $z=0$ to form a 2-sheeted Riemannian disk. The mapping h is taken as $t=[z(\tilde{p}(p))]^{1/2}$. If p is an end point of a slit on the boundary of S_{mn}, V is similarly chosen to consist of 2^{m-1} slit disks about the points $p_m^i(p)$, $i=1,\cdots,2^{m-1}$, rotated about $z=0$ to form a 2^{m-1}-sheeted disk. Now the mapping h is $t=[z(\tilde{p}(p))]^{2^{-m+1}}$. By means of the covering $\{V\}$ and the mappings h so chosen, R has become a Riemann surface.

9. The rather intricate proof that every $h\in HP(R)$ reduces to a constant will not be reproduced in the present study where the main interest is with γ. We merely recall that the proof is based on first showing that every $h\in HP$ has an axis of symmetry and then proving inductively that there are infinitely many axes of symmetry. Therefore h must reduce to a constant.

It remains to turn R into a Riemannian manifold which continues excluding all nonconstant HP functions, but which nevertheless admits the biharmonic Green's function γ.

The function

$$\omega(z(p))=-\log|z(p)|/\log 2$$

is the harmonic measure on that part of R which lies above $\{\frac{1}{2}<|z|<1\}$, its value and harmonicity being unaffected by the rotations about $z=0$ of the partial regions combining into the single or multiple disk V. In view of $N=2$, endowing R with a conformal structure does not alter the harmonicity of ω. Any conformal metric $ds_0(p)=\lambda(t(p))|dt|$ on R can be reduced to another conformal metric $ds(p)=\mu(p)ds_0(p)$

where, as p tends to the ideal boundary of R, $\mu(p)\to 0$ so rapidly that the volume of R is finite. Since $\omega\in B$, we have $\|\omega\|_2<\infty$, and therefore $R\in\tilde{O}_{\Gamma}^{2}$. The introduction of the conformal metric has no bearing on the class HP, so that we continue having $R\in O_{HP}^{2}\subset O_{HX}^{2}$.

The proof of Theorem 3 is herewith complete.

10. We now take up the class $O_{HL^p}^{N}$ of Riemannian manifolds which do not carry harmonic functions with a finite L^p norm, $p\geq 1$. The case $p=\infty$ gives the class HB already discussed and is therefore excluded in the sequel. In view of the Euclidean N-ball we have

$$\tilde{O}_{\Gamma}^{N}\cap\tilde{O}_{HL^p}^{N}\neq\phi,\ p\geq 1,\ N\geq 2.$$

To prove the nonvoidness of the classes $O_{\Gamma}^{N}\cap O_{HL^p}^{N}$, $O_{\Gamma}^{N}\cap\tilde{O}_{HL^p}^{N}$, and $\tilde{O}_{\Gamma}^{N}\cap O_{HL^p}^{N}$, it turns out that we can make use of the examples exhibited in Sario-Wang [8], Chung-Sario-Wang [3], and Chung-Sario [2] to establish the nonvoidness of the classes $O_{HX}^{N}\cap O_{HL^p}^{N}$, $O_{HX}^{N}\cap\tilde{O}_{H^mC}^{N}$, and $\tilde{O}_{QX}^{N}\cap O_{HL^p}^{N}$, respectively. For the convenience of the reader we reproduce the quite short proofs, slightly modified for the present purpose.

11. First we exclude both γ and HL^p functions.

Theorem 4. $O_{\Gamma}^{N}\cap O_{HL^p}^{N}\neq\phi$, $p\geq 1$, $N\geq 2$.

Proof. We know from No. 3 that the N-cylinder

$$T=\{|x|<\infty,\ |y_i|\leq\pi,\ i=1,\cdots,N-1\}$$

with the Euclidean metric belongs to O_G^N, hence to O_{Γ}^{N}. Every $h\in H$ can be expanded into a series $h=\sum f_nG_n$, where $f_nG_n=f_n(x)G_n(y)\in H$, $y=(y_1,\cdots,y_{N-1})$, and G_n ranges over all products of the form

$$G_n(y)=\prod_{i=1}^{N-1}{}^{\cos}_{\sin}\,n_iy_i$$

with $n=(n_1,\cdots,n_{N-1})$, the n_i integers ≥ 0, and $f_0(x)=h_0(x)\in H$. Set $\eta^2=\sum_{i=1}^{N-1}n_i^2$. In view of $\Delta(f_nG_n)=-(f_n''G_n-\eta^2f_nG_n)=0$, we have

$$h=h_0(x)+\sum{}'(a_ne^{\eta x}+b_ne^{-\eta x})G_n\,,$$

where the sum $\sum'$ is extended over all $n\neq(0,\cdots,0)$.

Assume there exists an $h\in HL^p$ and choose a continuous function $\rho_0(x)\geq 0$ with supp $\rho_0\subset(0,1)$, $\int_0^1\rho_0 dx=1$. Suppose $a_n\neq 0$ for some $n\neq(0,\cdots,0)$. Then for $\rho_t(x)=\rho_0(x-t)$, $\varphi_t=\rho_t G_n$, with $t>0$ a real number,

$$(h,\varphi_t)\sim ce^{\eta t}\int_t^{t+1}\rho_t\,dx=ce^{\eta t},$$

hence $|(h,\varphi_t)|\to\infty$ as $t\to\infty$. But for $p^{-1}+q^{-1}=1$,

$$\|\varphi_t\|_q=c\left(\int_t^{t+1}\rho_t^q dx\right)^{1/q}=\text{const}<\infty$$

for $p\geq 1$, and $|(h,\varphi_t)|\leq\|h\|_p\|\varphi_t\|_q=\text{const}<\infty$ for all $t>0$. It follows that $a_n=0$ for all $n\neq(0,\cdots,0)$. An analogous argument with $t<0$, $t\to-\infty$ shows that $b_n=0$ for all $n\neq(0,\cdots,0)$. Therefore $h=h_0(x)=ax+b$. But $\|h_0\|_p=\infty$ unless $a=b=0$, and we have proved that $T\in O^N_{HL^p}$.

12. Next we exhibit an N-manifold without γ but with HL^p functions.

Theorem 5. $O^N_\Gamma\cap\tilde{O}^N_{HL^p}\neq\phi$, $p\geq 1$, $N\geq 2$.

Proof. On the N-cylinder

$$T=\{|x|<\infty,|y_i|\leq 1,\ i=1,\cdots,N-1\},$$

choose the metric

$$ds^2=e^{-x^2}dx^2+e^{-x^2/(N-1)}\sum_{i=1}^{N-1}dy_i^2.$$

For $h(x)\in H$ we have $h=ax+b$, and therefore $T\in O^N_G<O^N_\Gamma$. On the other hand, $x\in HL^p$, since

$$\|x\|_p^p=c\int_{-\infty}^{\infty}|x|^p e^{-x^2}dx<\infty.$$

13. The remaining case is an N-manifold carrying γ but no HL^p functions.

Theorem 6. $\tilde{O}^N_\Gamma\cap O^N_{HL^p}\neq\phi$, $p\geq 1$, $N\geq 2$.

Proof. Consider the N-space M with the metric

$$ds^2=\varphi(r)dr^2+\psi(r)^{1/(N-1)}\sum_{i=1}^{N-1}\lambda_i(\theta)d\theta_i^2,$$

where $\varphi, \psi \in C^2[0, \infty)$,

$$\varphi(r)=\begin{cases}1 & \text{for} \quad r<\frac{1}{2}, \\ e^{-r} & \text{for} \quad r>1,\end{cases} \qquad \psi(r)=\begin{cases}r^{2(N-1)} & \text{for} \quad r<\frac{1}{2}, \\ e^{r} & \text{for} \quad r>1,\end{cases}$$

and the λ_i are trigonometric functions of $\theta=(\theta_1, \cdots, \theta_{N-1})$ such that the metric is Euclidean on $\{r<\frac{1}{2}\}$. For $h(r) \in H(\{r>1\})$, we have $\Delta h(r)=-(e^r h'(r))'=0$. Therefore, $\omega(r)=e^{1-r}$ on $\{r \geq 1\}$, and

$$\|\omega\|_2^2=c \int_1^\infty e^{2-2r} dr<\infty ,$$

hence $M \in \tilde{O}_T^N$.

To see that $M \in O_{HL^p}^N$, expand $h \in HL^p(M)$ into a series $h=\sum f_n(r) S_n(\theta)$, where $f_n S_n \in H$ and the S_n are spherical harmonics. If $f_{n_0} \not\equiv 0$ for some $n_0 \geq 0$, the maximum principle applied to $f_{n_0} S_{n_0}$ gives $|f_{n_0}|>c_0>0$ on $[1, \infty)$.

In the case $p=1$, take $g(r) \in C^\infty[0, \infty), 0<g<1$, with $g(r)=(2r)^{-1}$ for $r>1$. Then

$$\|h\|_1 \geq c\left|\int_M h g S_{n_0} dV\right| \geq \left|c_1+c_2 \int_1^\infty f_{n_0} g dr\right|$$

$$\geq \left|c_1+c_0 c_2 \int_1^\infty g dr\right|=\infty ,$$

a contradiction. If $p>1$, take q with $p^{-1}+q^{-1}=1$. Then $g S_{n_0} \in L^q$, and $(\cdot, g S_{n_0})$ is a linear functional on L^p. Since

$$|(h, g S_{n_0})|=\left|c \int_0^\infty f_{n_0} g dr\right|=\infty ,$$

we have a contradiction with $h \in L^p$, and conclude that $M \in O_{HL^p}^N$ for all $p \geq 1$.

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