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Asymptotic behavior of multitype Galton-Watson processes

By

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0. Introduction

The asymptotic behavior of the distributions of multitype Galton-Watson processes has been studied by many mathematicians. According to the author's knowledge, Jirina [8] for subcritical processes is the first paper on this subject, and Chistyakov [4] and Mullikin [10] for critical processes followed. But they assumed that (i) *the second moments* (in the subcritical case) *or the third moments* (in the critical case) *are finite* and (ii) *the mean matrix is positively regular*. Joffe and Spitzer [9] obtained the results for discrete time processes without the hypothesis (i), and Sevastyanov [14] extended them for cotinuous time processes. Their results are final for the processes satisfying the condition (ii). However, when the condition (ii) fails, somewhat different phenomena occur. Chistyakov [3] illustrated it for the continuous time subcritical processes with the hypothesis (i). For the continuous time critical processes, the results of Savin and Chistyakov [12] for the processes with three particle types and the hypothesis (i) are very suggestive.

In this paper, we shall give the whole asymptotic behavior of discrete and continuous time multitype Galton-Watson processes without the hypotheses (i) and (ii) (but with some weaker hypotheses). The processes are decomposed into elementary subprocesses. When the elementary subprocesses have positively regular mean matrices, the results naturally coincide with those of [8], [9], [10] and [14]. But when they are reducible, the rate that the generating functions tend to the extinction probabilities are different from that of the positively regular cases. Furthermore for the processes with discrete time we must take some care of the periodicity.

We shall give the definitions and notations in section 1. In section 2 we shall deal with the discrete time noncritical processes having aperiodic mean matrices, while we shall deal with those having periodic mean matrices in section 3. Sections 4 and 5 are devoted to the study of the discrete time critical processes. The results for the continuous time processes are summarized in section 6, and some examples are given in section 7.

1. Definitions and notations

We designate the set of all integers between m and n by $\langle m, n\rangle$ and put $Z_+ = \langle 0, \infty\rangle$, $S = Z_+^N$ ($N \in \langle 1, \infty\rangle$). If two vectors $s_1 = (s_1^1, \cdots, s_1^N)$ and $s_2 = (s_2^1, \cdots, s_2^N)$ satisfy $s_1^i > s_2^i$ [$s_1^i \geqq s_2^i$] for all $i \in \langle 1, N\rangle$, we say that s_1 is *larger* [resp. *not less*] than s_2 and write as $s_1 > s_2$ [resp. $s_1 \geqq s_2$]. Thus we can naturally define the *maximum*, *minimum*, *monotony*, *etc.* of a sequence of vectors. Further, these notions and notations are extended for matrices in the natural way. For example a matrix A is called *nonnegative* if all its components are nonnegative, and in this case we write as $A \geqq 0$. Let A be a nonnegative square matrix of order k. We call A *positively regular* if $A^n > 0$ for some $n \in \langle 1, \infty\rangle$, where A^n means the n-fold product of the matrix A. Also the matrix A is called *irreducible* if for each $i, j \in \langle 1, k\rangle$, $i \neq j$, there is an $n \in \langle 1, \infty\rangle$ such that $A_j^i(n) > 0$, where $A_j^i(n)$ is the (i, j)-component of the matrix A^n. Hence each nonnegative matrix of order 1 is always irreducible. We also call a square matrix a with nonnegative off-diagonal elements to be *irreducible* if the matrix $a + lI$ ($\geqq 0$) is irreducible for some $l > 0$ in the above sense, where I is the identity matrix. For two vectors s_1 and s_2, we define new vectors $s_1 s_2$ and s_1/s_2 (for $s_2 > 0$) by

$$s_1 s_2 = (s_1^1 s_2^1, \cdots, s_1^N s_2^N), \quad s_1/s_2 = (s_1^1/s_2^1, \cdots, s_1^N/s_2^N).$$

For each $s \in R^N$ and $x \in S$ we set

$$s^x = (s^1)^{x^1} \cdots (s^N)^{x^N}, \text{ where } s = (s^1, \cdots, s^N), \ x = (x^1, \cdots, x^N).$$

Finally we denote the i-th canonical unit basis by e_i, i.e. $e_i^j = \delta_j^i$

where δ_j^i is the Kronecker's delta.

Now we shall call a Markov chain $X=(Z(n), P_x)$ on S a *discrete time N-type Galton-Watson process* (*DGWP* for brevity), if its probability generating functions

$$F^x(n;s) \equiv \sum_{y\in S} P_x\{Z(n)=y\}s^y, \quad x\in S, \quad n\in\langle 0, \infty\rangle, \quad 0\leqq s\leqq 1,$$

are given by

$$F^x(n;s)=F(n;s)^x, \tag{1.1}$$

for some vector functions $F(n;s)=(F^1(n;s), \cdots, F^N(n;s))$. Then it is clear that $F(n;s)$ is given by the n-fold iteration of the vector probability generating function $F(s)\equiv F(1;s)$:

$$\begin{aligned} &F(n+1;s)=F(F(n;s)), \quad n\in\langle 0, \infty\rangle, \\ &F(0;s)=s, \quad 0\leqq s\leqq 1, \end{aligned} \tag{1.2}$$

where

$$F^i(s)=\sum_{y\in S} P^i(y)s^y, \quad i\in\langle 1, N\rangle, \tag{1.3}$$

with $P^i(y)\geqq 0$ and $\sum_{y\in S} P^i(y)\leqq 1$. Since the family of generating functions $\{F(n;s)\}$ uniquely determines a DGWP, we sometimes call $\{F(n;s)\}$ itself a DGWP.

Similarly a Markov process $X=(Z(t), P_x)$ on S is called a *continuos time N-type Galton-Watson Process* (*CGWP*), if its probility generating functions $F^x(t;s)$ are given by

$$F^x(t;s)=F(t;s)^x, \quad x\in S, \quad t\in[0, \infty), \quad 0\leqq s\leqq 1, \tag{1.4}$$

where $F(t;s)=(F^1(t;s), \cdots, F^N(t;s))$ is the unique solution of

$$\begin{aligned} &\frac{dF(t;s)}{dt}=f(F(t;s)), \quad t>0, \\ &F(0;s)=s, \quad 0\leqq s<1, \end{aligned} \tag{1.5}$$

where

$$f^i(s)=\sum_{y\in S} p^i(y)s^y, \quad i\in\langle 1, N\rangle, \tag{1.6}$$

with $p^i(y)\geqq 0$, $y\neq e_i$, and $\sum_{y\in S} p^i(y)\leqq 0$. Also, we sometimes call the family of generating functions $\{F(t;s)\}$ itself a CGWP.

It is shown by Sevastyanov ([13], [14]) that for a DGWP [CGWP] there exists the least nonnegative fixed point q of $F(s)$ [resp. zero point q of $f(s)$] in the cube $0\leqq s\leqq 1$, and it is stable in the sense of

$$\lim_{n\to\infty} F(n;s)=q \;\; [\text{resp. } \lim_{t\to\infty} F(t;s)=q], \quad 0\leqq s\leqq q. \tag{1.7}$$

Especially it holds

$$P_{e_i}\{T<\infty\}=\lim_{n\to\infty} F^i(n;0)=q^i$$

$$[\text{resp. } P_{e_i}\{T<\infty\}=\lim_{t\to\infty} F^i(t;0)=q^i],$$

where T is the first hitting time for trap state $0\in S$, namely the extinction time. Hence we shall call q the *extinction probability* of the DGWP [resp. CGWP]. Let $R(s)=q-F(s)$ and $R(n;s)=q-F(n;s)$. An objective of the present paper is to obtain an exact estimate of $R(n;s)$ tending to 0.

For a DGWP, we shall assume

$$q>0 \quad \text{and} \quad F_j{}^i(q)<\infty, \quad i,j\in\langle 1,N\rangle, \tag{D}$$

where $F_j{}^i(s)=\partial F^i/\partial s^j$ if it exists and $F_j{}^i(s)=\lim_{\xi\uparrow s} F_j{}^i(\xi)$ otherwise. Note that when the DGWP is subcritical, or critical with no final classes, $q=1>0$ holds. We call the matrix

$$A\equiv[A_j{}^i]_{i,j=1}^N=[F_j{}^i(q)]_{i,j=1}^N$$

the *q-mean matrix* of DGWP. Since $A\geqq 0$, there exists a nonnegative characteristic root $\rho(A)$ of A which is not smaller in absolute value than any other characteristic roots (cf. Gantmacher [6]). We call it the *Perron-Frobenius root* (*P-F root* for brevity) of the matrix A. From the definition of q, the inequality $\rho(A)\leqq 1$ easily follows. It is known that by a change of suffixes the nonnegative matrix A is represented as

$$A=\begin{pmatrix} \tilde{A}_1 & 0 & \cdots & 0 \\ & \tilde{A}_2 & 0\cdots & 0 \\ & & \cdots\cdots & \\ * & & & \cdots \\ & & & \tilde{A}_g \end{pmatrix}, \tag{1.8}$$

where each $\tilde{A}_\alpha$ is an irreducible square matrix of order $m_\alpha \in \langle 1, N\rangle$ ($\sum_{\alpha=1}^{g} m_\alpha = N$). We set

$$\Gamma^i = \{j \in \langle 1, N\rangle ; A_j{}^i(n) > 0 \quad \text{for some} \quad n \in \langle 1, \infty\rangle\} \cup \{i\},$$

$$\Delta_\alpha = \langle \sum_{\beta=1}^{\alpha-1} m_\beta + 1, \sum_{\beta=1}^{\alpha} m_\beta \rangle \quad (\Delta_1 = \langle 1, m_1\rangle).$$

Since every $\tilde{A}_\alpha$ is irreducible, $\Delta_\beta \subset \Gamma_i$ if $\Gamma^i \cap \Delta_\beta \neq \phi$, and $\Gamma^i = \Gamma^{i'}$ if $i, i' \in \Delta_\alpha$. Hence Γ^i is a disjoint union of some Δ_β's and it is same for all $i \in \Delta_\alpha$, which we denote by Γ_α. We also set $\bar{\Gamma}_\alpha = \Gamma_\alpha - \Delta_\alpha$. The Γ_α-part $(s^i)_{i \in \Gamma_\alpha}$ [$\bar{\Gamma}_\alpha$-part $(s^i)_{i \in \bar{\Gamma}_\alpha}$, Δ_α-part $(s^i)_{i \in \Delta_\alpha}$] of a vector $s = (s^1, \cdots, s^N)$ is denoted by s_α[resp. $\bar{s}_\alpha$, $\tilde{s}_\alpha$]. From (1. 3) and (1. 8) it follows that the generating function $F^i(s)$ for $i \in \Gamma_\alpha [i \in \bar{\Gamma}_\alpha]$ only depends on s_α[resp. $\bar{s}_\alpha$]. Hence we can write as $F(s)_\alpha = F(s_\alpha)_\alpha$[resp. $\bar{F}(s)_\alpha = \bar{F}(\bar{s}_\alpha)_\alpha$]. Similarly, since $F^i(n; s)$ for $i \in \Gamma_\alpha [i \in \bar{\Gamma}_\alpha]$ only depends on s_α[resp. $\bar{s}_\alpha$] by (1. 2), we can write as

$$(1.9) \qquad F(n; s)_\alpha = F(n; s_\alpha)_\alpha, \quad 0 \leqq s_\alpha \leqq 1$$
$$[\text{resp. } \bar{F}(n; s)_\alpha = \bar{F}(n; \bar{s}_\alpha)_\alpha, \quad 0 \leqq \bar{s}_\alpha \leqq 1].$$

We set $S_\alpha = \{x_\alpha = (x^i)_{i \in \Gamma_\alpha} ; x^i \in \langle 0, \infty\rangle\}$. Then the family of generating functions $\{F(n; s_\alpha)_\alpha ; n \in \langle 0, \infty\rangle\}$ forms a DGWP on S_α, which we denote by $X_\alpha = (Z_\alpha(n), P^\alpha_{x_\alpha})$. Note that the extinction probability of the DGWP X_α is equal to the Γ_α-part q_α of the extinction probability q of the original DGWP X by (1. 7), and hence the submatrix $A_\alpha \equiv [A_j{}^i]_{i, j \in \Gamma_\alpha}$ coincides with the q-mean matrix of X_α. Further it follows

$$F_j{}^i(n; q) = F_j{}^i(n; q_\alpha) = (A_\alpha(n))_j{}^i = A_j{}^i(n), \quad i, j \in \Gamma_\alpha.$$

Since $\rho(A) \leqq 1$, $\rho_\alpha \equiv \rho(A_\alpha) \leqq 1$ holds. We call the DGWP X_α *critical* if $\rho_\alpha = 1$ and *noncritical* if $\rho_\alpha < 1$.

For a CGWP, we assume

$$(\text{C}) \qquad q > 0 \quad \text{and} \quad f_j{}^i(q) < \infty, \quad i, j \in \langle 1, N\rangle.$$

We call the matrix

$$a \equiv [a_j{}^i]_{i, j=1}^{N} = [f_j{}^i(q)]_{i, j=1}^{N}$$

the *infinitesimal q-mean matrix* of the CGWP X. Since (1. 6) implies $a+lI\geqq 0$ for some $l>0$, there is a real characteristic root $\rho(a)$ of a which is not smaller in real part than any other characteristic roots of a. In this case $\rho(a)\leqq 0$ holds (cf. Ogura [11]). By a change of suffixes the matrix a is represented as

$$(1.10)\qquad a=\begin{bmatrix} \tilde{a}_1 & 0 & \cdots\cdots & 0 \\ & \tilde{a}_2 & 0 \cdots & 0 \\ & & & \cdots\cdots \\ * & & & \cdots \\ & & & \tilde{a}_g \end{bmatrix},$$

where each $\tilde{a}_\alpha$ is an irreducible square matrix of order $m_\alpha(\sum_{\alpha=1}^{g} m_\alpha = N)$. We define the sets $\varDelta_\alpha$, $\varGamma_\alpha$ and $\overline{\varGamma}_\alpha$ as in the discrete time case but from the matrix $a+lI$ $(\geqq 0)$ instead of A. By (1.6) and (1.10) the function $f^i(s)$ for $i\in\varGamma_\alpha[i\in\overline{\varGamma}_\alpha]$ only depends on s_α[resp. $\bar{s}_\alpha$], and we write as

$$(1.11)\quad f(s)_\alpha=f(s_\alpha)_\alpha,\ 0\leqq s_\alpha\leqq 1,\ [\text{resp. } \bar{f}(s)_\alpha=\bar{f}(\bar{s}_\alpha)_\alpha,\ 0\leqq \bar{s}_\alpha\leqq 1].$$

Hence $F^i(t;s)$ for $i\in\varGamma_\alpha[i\in\overline{\varGamma}_\alpha]$ only depends on s_α[resp. $\bar{s}_\alpha$] by (1.5), so that we can write as

$$(1.12)\qquad F(t;s)_\alpha=F(t;s_\alpha)_\alpha,\ 0\leqq s_\alpha\leqq 1,$$

$$[\text{resp. } \bar{F}(t;s)_\alpha=\bar{F}(t;\bar{s})_\alpha,\ 0\leqq \bar{s}_\alpha\leqq 1].$$

We designate the CGWP $\{F(t;s_\alpha)_\alpha;t\in[0,\infty)\}$ by $X_\alpha=(Z_\alpha(t),P^\alpha_{x_\alpha})$. The extinction probability of the CGWP X_α is equal to the $\varGamma_\alpha$-part q_α of that q of the CGWP X, and the submatrix $a_\alpha\equiv[a_j{}^i]_{i,j\in\varGamma_\alpha}$ coincides with the infinitesimal q-mean matrix of X_α. Moreover, setting

$$A(t)\equiv[A_j{}^i(t)]_{i,j=1}^N=\exp(ta),$$

$$A_\alpha(t)\equiv[A_{\alpha j}^i(t)]_{i,j\in\varGamma_\alpha}=\exp(ta_\alpha),$$

we have

$$(1.13)\qquad F_j{}^i(t;q)=F_j{}^i(t;q_\alpha)=A_{\alpha j}^i(t)=A_j{}^i(t),\quad i,j\in\varGamma_\alpha.$$

Since $\rho(a)\leqq 0$, $\sigma_\alpha\equiv\rho(a_\alpha)\leqq 0$ holds. We call the CGWP X_α *critical* if $\sigma_\alpha=0$, and *noncritical* if $\sigma_\alpha<0$.

2. Noncritical aperiodic DGWP

In this section we shall deal with noncritical DGWP's with the assumption

$$\text{(DN)} \qquad \sum_{y \in S} P^i(y) y^j q^y \log y^j < \infty, \quad i, j \in \langle 1, N \rangle.$$

We shall also assume that all the matrices in this section are aperiodic, i.e.

$$\text{G.C.D. } \{n \in \langle 1, \infty \rangle ; A_j{}^i(n) > 0\} = 1, \quad i, j \in \Delta_\alpha.$$

Since $\tilde{A}_\alpha$ is irreducible, it is positively regular if it is not equal to the zero matrix of order 1. Hence there correspond positive right and left eigenvectors $\tilde{u}_\alpha = (\tilde{u}_\alpha{}^i)_{i \in \Delta_\alpha}$ and $\tilde{v}_\alpha = (\tilde{v}_{\alpha i})_{i \in \Delta_\alpha}$ to the *P-F* root $\tilde{\rho}_\alpha \equiv \rho(\tilde{A}_\alpha)$;

$$\tilde{A}_\alpha \tilde{u}_\alpha = \tilde{\rho}_\alpha \tilde{u}_\alpha, \quad \tilde{v}_\alpha A_\alpha = \tilde{\rho}_\alpha \tilde{v}_\alpha,$$

with the normalizations

$$\sum_{i \in \Delta_\alpha} \tilde{v}_{\alpha i} \tilde{u}_\alpha{}^i = 1, \quad \sum_{i \in \Delta_\alpha} \tilde{u}_\alpha{}^i = 1$$

(Gantmacher [6]). It is also known that as $n \to \infty$

$$\text{(2.1)} \qquad \tilde{A}_\alpha{}^n = \tilde{\rho}_\alpha{}^n (\tilde{A}_\alpha{}^* + o(1)),$$

where $\tilde{A}_\alpha{}^* = [\tilde{A}_{\alpha j}^{*i}] \equiv [\tilde{u}_\alpha{}^i \tilde{v}_{\alpha j}]_{i, j \in \Delta_\alpha}$. Of course it holds

$$\text{(2.2)} \qquad \tilde{A}_\alpha \tilde{A}_\alpha{}^* = \tilde{A}_\alpha{}^* \tilde{A}_\alpha = \tilde{\rho}_\alpha \tilde{A}_\alpha{}^*, \quad \tilde{A}_\alpha{}^* \tilde{A}_\alpha{}^* = \tilde{A}_\alpha{}^*.$$

In order to define the 'rank ν_α of α', we shall introduce the semi-order '$<$' in the space of indices $\langle 1, g \rangle$ by

$$\beta < \alpha \quad \text{if} \quad \Delta_\beta \subset \Gamma_\alpha.$$

Next we define the rank $\nu_\beta(r)$ of β w.r.t. r by

$$\text{(2.3)} \qquad \nu_\beta(r) = \begin{cases} \max\{\nu_\gamma(r) ; \gamma \lneqq \beta\}, & \text{if } \tilde{\rho}_\beta \neq r, \\ \max\{\nu_\gamma(r) ; \gamma \lneqq \beta\} + 1, & \text{if } \tilde{\rho}_\beta = r, \end{cases}$$

inductively, where we agree on $\max \phi = -1$. Then the *rank* ν_α *of* α is given by

$$\text{(2.4)} \qquad \nu_\alpha = \nu_\alpha(\rho_\alpha).$$

Note that $\nu_\alpha \in \langle 0, g-1 \rangle$ since $\tilde{\rho}_\beta = \rho_\alpha$ for some $\beta < \alpha$.

To state the theorem we shall define one more set: $I_+(x)$

$=\{\alpha\in\langle 1,g\rangle;\,x_\alpha\neq 0\}$, $x\in S$.

Theorem 2.1. *Let a DGWP $X=(Z(n),P_x)$ satisfy Conditions (D) and (DN) for each $\alpha\in\langle 1,g\rangle$ with $\rho_\alpha<1$, and all the matrices $\tilde{A}_\alpha$ be aperiodic. Then,* 1) *for each $\alpha\in\langle 1,g\rangle$ there correspond monotone nonincreasing functions $R^{*i}(s_\alpha)$ in $0\leqq s_\alpha\leqq q_\alpha$, $i\in\Delta_\alpha$, such that as $n\to\infty$*

$$R^i(n;s)=n^{\nu_\alpha}\rho_\alpha{}^n(R^{*i}(s_\alpha)+o(1)),\quad i\in\Delta_\alpha, \tag{2.5}$$

where $o(1)$ is uniform in s on $0\leqq s_\alpha\leqq q_\alpha$. The $R^{*i}(s_\alpha)$ are determined inductively w.r.t. the semiorder '$<$' by Lemmas 2.1 and 2.4 below. Further, if $\rho_\alpha>0$, no $R^{*i}(s_\alpha)$, $i\in\Delta_\alpha$, are identically zero. 2) *For each $x\in S$ such that $\rho_\alpha<1$ holds for all $\alpha\in I_+(x)$, and $\rho_\alpha>0$ for some $\alpha\in I_+(x)$, there corresponds a probability distribution $\{P_x{}^*(y)\}$ on $S-\{0\}$ satisfying*

$$\lim_{n\to\infty}P_x\{Z(n)=y\,|\,n<T<\infty\}=P_x{}^*(y). \tag{2.6}$$

We shall prove this theorem by the induction w.r.t. the semiorder '$<$'. When α is minimal, $\Gamma_\alpha=\Delta_\alpha$ and $A_\alpha=\tilde{A}_\alpha$. Hence the q-mean matrix A_α is positively regular, if $\rho_\alpha>0$, i.e. $A_\alpha\neq[0]$.[1] In this case there are the following excellent results given by Joffe and Spitzer [9]:[2]

Lemma 2.1. (Joffe and Spitzer.) *Let the q-mean matrix A_α of the DGWP X_α is positively regular and $\rho_\alpha<1$. Then there exist a monotone nonincreasing function $K_\alpha{}^*(s)$ in $0\leqq s_\alpha\leqq q_\alpha$ and a distribution $\{P^{\alpha*}(y_\alpha)\}$ on S_α, such that*

$$\lim_{n\to\infty}\frac{q_\alpha-F(n;s)_\alpha}{\rho_\alpha{}^n}=K_\alpha{}^*(s_\alpha)\tilde{u}_\alpha,\quad 0\leqq s_\alpha\leqq q_\alpha, \tag{2.7}$$

$$\lim_{n\to\infty}P^\alpha_{x_\alpha}\{Z_\alpha(n)=y_\alpha\,|\,n<T<\infty]=P^{\alpha*}(y_\alpha),\quad x_\alpha,y_\alpha\in S_\alpha-\{0\}. \tag{2.8}$$

Further $K_\alpha{}^*(s_\alpha)\not\equiv 0$ if and only if (DN) holds.

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1) If $A_\alpha=[0]$, (2·5) is always satisfied.

2) It seems there are two errors in their paper; the second inequality in (4·10) and that in (4·42). But their assertions are valid.

When α is not minimal, $\Gamma_\alpha \neq \phi$ and the q-mean matrix A_α is represented as

$$(2.9)\qquad A_\alpha = \begin{bmatrix} \bar{A}_\alpha & 0 \\ A_\alpha' & \tilde{A}_\alpha \end{bmatrix},$$

where

$$\bar{A}_\alpha = [A_j{}^i]_{i, j \in \bar{\Gamma}_\alpha}, \quad A_\alpha' = [A_j{}^i]_{i \in \Delta_\alpha, j \in \bar{\Gamma}_\alpha} \neq 0.$$

We put $\bar{\rho}_\alpha = \rho(\bar{A}_\alpha)$. Then ρ_α is equal to the maximum $\tilde{\rho}_\alpha \vee \bar{\rho}_\alpha$ of $\tilde{\rho}_\alpha$ and $\bar{\rho}_\alpha$.

Let $R(s) = q - F(s)$ and $R(n; s) = q - F(n; s)$. Then it is given by Joffe and Spitzer [8] ((4.6)) that

$$(2.10)\qquad R(s) = (A - E(s))(q - s), \quad 0 \leqq s \leqq q,$$

$$(2.11)\qquad E_j{}^i(s) = \sum_{y \in S} P^i(y) y^j \left\{ q^{y - e_j} - \int_0^1 (q - (q - s)\xi)^{y - e_j} d\xi \right\},$$

where we agree on $s^y = 0$ for $y \notin S$. (2.11) implies

$$(2.12)\qquad \begin{aligned} &0 \leqq E(s_2) \leqq E(s_1) \leqq A, \quad 0 \leqq s_1 \leqq s_2 \leqq q, \\ &E(s) \to 0, \quad \text{as} \quad s \to q \quad \text{in} \quad 0 \leqq s \leqq q. \end{aligned}$$

We set $E(n; s) = E(F(n; s))$ and $C(n; s) = A - E(n; s)$. We define matrices $E(n; s)_\alpha, C(n; s)_\alpha, \bar{E}(n; s)_\alpha$, etc. in the natural way. From (1.3), (1.8), (1.9) and (2.11) it follows

$$(2.13)\qquad E(n; s)_\alpha = E(n; s_\alpha)_\alpha, \; C(n_\alpha; s)_\alpha = C(n; s_\alpha)_\alpha, \; 0 \leqq s_\alpha \leqq q_\alpha.$$

Hence (2.10) implies

$$R(n+1; s)_\alpha = R(n+1; s_\alpha)_\alpha = C(n; s_\alpha)_\alpha R(n; s_\alpha)_\alpha, \quad 0 \leqq s_\alpha \leqq q_\alpha,$$

and with the aid of (2.9) and (2.13)

$$(2.14)\qquad \tilde{R}(n+1; s_\alpha)_\alpha = \tilde{C}(n; s_\alpha)_\alpha \tilde{R}(n; s_\alpha)_\alpha + C(n; s_\alpha)_\alpha' \bar{R}(n; \bar{s}_\alpha)_\alpha.$$

Using (2.14) inductively, we obtain

$$(2.15)\qquad \begin{aligned} \tilde{R}(n+1; s_\alpha)_\alpha = {} & \tilde{D}_\alpha(n, -1)(\tilde{q}_\alpha - \tilde{s}_\alpha) \\ & + \sum_{l=0}^{n} \tilde{D}_\alpha(n, l) C(l; s_\alpha)_\alpha' \bar{R}(l; \bar{s}_\alpha)_\alpha, \end{aligned}$$

where

(2.16) $$\tilde{D}_\alpha(n, l) = \tilde{D}_\alpha(n, l; s_\alpha) = \begin{cases} \tilde{C}(n; s_\alpha)_\alpha \tilde{C}(n-1; s_\alpha)_\alpha \cdots \tilde{C}(l+1; s_\alpha)_\alpha, & l \in \langle -1, n-1 \rangle, \\ I, & l = n. \end{cases}$$

Lemma 2.2. *If Condition (DN) and the inequality $\rho_\alpha < 1$ are satisfied, then it holds*

(2.17) $$\sum_{n=1}^{\infty} E(n; 0)_\alpha < \infty.$$

Proof. From the convexity of the function $F^i(n; s+(q-s)\xi)$ in $0 \leqq \xi \leqq 1$, it follows $q_\alpha - F(n; 0)_\alpha \leqq A_\alpha{}^n q_\alpha$. Applying[3] the same arguments as in the proof of Lemma 2.5 below to the matrices A^n, we obtain

$$A_\alpha{}^n q_\alpha \leqq n^{\nu_\alpha} \rho_\alpha^n K q_\alpha \leqq \theta r^n q_\alpha,$$

where K is a positive square matrix with the indices in Γ_α, and r and θ are contants with $\rho_\alpha < r < 1$ and $\theta > 0$. Hence it follows $F(n; 0)_\alpha \geqq (1 - \theta r^n) q_\alpha$, and we obtain the conclusion by the same arguments as in Joffe and Spitzer [9] (pp. 424-425) with the aid of (2.11).

Lemma 2.3. *The relations $\tilde{\rho}_\alpha > 0$ and (2·17) imply the existence of the limit*

(2.18) $$\lim_{n \to \infty} \tilde{D}_\alpha(n, l; s_\alpha) \tilde{\rho}_\alpha{}^{-n+l} = \tilde{D}_\alpha{}^*(l; s_\alpha)$$

uniformly in $0 \leqq s_\alpha \leqq q_\alpha$. Further it holds

(2.19) $$0 < \tilde{D}_\alpha{}^*(l; s_\alpha) \leqq \tilde{A}_\alpha{}^*, \quad 0 \leqq s_\alpha \leqq q_\alpha, \quad l \in \langle -1, \infty \rangle.$$

Proof.[4] Let

(2.20) $$\varepsilon_n = \max\{E_j{}^i(n; 0)/A_j{}^i; i, j \in \Delta, A_j{}^i > 0\}.$$

Then, since $A_j{}^i = 0$ implies $E_j{}^i(n; 0) = 0$ from (2.12), it is clear that

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[3] Or equivalently, we may use the Jordan's normal form of A reminding the asymptotic forms of its products.

[4] In the proofs of the following theorems and lemmas we shall often abbreviate the suffix α and the variable s where there are no confusions.

(2.21) $$0 \leqq \tilde{E}(n; s) \leqq \tilde{E}(n; 0) \leqq \varepsilon_n \tilde{A},$$

(2.22) $$\sum_{n=0}^{\infty} \varepsilon_n \leqq \sum_{n=0}^{\infty} \sum_{i,j \in \Delta} \tilde{E}_j{}^i(n; 0)/\tilde{A}_j{}^i < \infty,$$

by (2.17). On the other hand, there is a sequence $\alpha_n \to 0$, $1 \geqq \alpha_n \geqq 0$, by (2.1) satisfying

$$(1-\alpha_n)\tilde{A}^* \leqq \tilde{A}^n \tilde{\rho}^{-n} \leqq (1+\alpha_n)\tilde{A}^*.$$

Hence it follows

(2.23) $$\tilde{\rho}^{-n+l}\tilde{D}(n, l) \leqq \tilde{\rho}^{-n+l}\tilde{A}^{n-l} \leqq (1+\alpha_{n-l})\tilde{A}^*,$$

and with the aid of (2.21)

$$\tilde{\rho}^{-n+l}\tilde{D}(n, l) \geqq \tilde{\rho}^{-n+l}\tilde{A}^{n-l} \prod_{k=l+1}^{n} (1-\varepsilon_k)$$
$$= (1-\alpha_{n-l}) \prod_{k=l+1}^{n} (1-\varepsilon_k)\tilde{A}^* \geqq (1-\alpha_{n-l} - \sum_{k=l+1}^{n} \varepsilon_k)\tilde{A}^*,$$

for all large l with $\varepsilon_k \leqq 1$, $k \in \langle l, \infty \rangle$. Therefore we obtain

(2.24) $$-(\alpha_{n-l} + \sum_{k=l+1}^{n} \varepsilon_k)\tilde{A}^* \leqq \tilde{\rho}^{-n+l}\tilde{D}(n, l) - \tilde{A}^* \leqq \alpha_{n-l}\tilde{A}^*.$$

Now take any $\varepsilon > 0$. Then by (2.22) we can choose an n_0 such that $\sum_{k=n_0+1}^{\infty} \varepsilon_k \leqq \varepsilon$. Further, it holds

$$\tilde{\rho}^{-n_1+l}\tilde{D}(n_1, l) - \tilde{\rho}^{-n_2+l}\tilde{D}(n_2, l) = (\tilde{\rho}^{-n_1+n_0}\tilde{D}(n_1, n_0)$$
$$-\tilde{\rho}^{-n_2+n_0}\tilde{D}(n_2, n_0))\tilde{\rho}^{-n_0+l}\tilde{D}(n_0, l), \quad n_1, n_2 \geqq n_0,$$

and $\tilde{\rho}^{-n_0+l}\tilde{D}(n_0, l)$ is bounded in n_0 because of (2.23). Hence it follows that the sequence $\tilde{\rho}^{-n+l}\tilde{D}(n, l)$, $n \in \langle l+1, \infty \rangle$, is a Cauchy sequence uniformly in $0 \leqq s_\alpha \leqq q_\alpha$. So we obtain (2.18). Now we shall show (2.19). Letting $n \to \infty$ in (2.24), we have $\tilde{D}^*(n_0) > 0$ for all sufficiently large n_0. Since $\tilde{D}(n, l) = \tilde{D}(n, n_0)\tilde{D}(n_0, l)$, it holds

(2.25) $$\tilde{D}^*(l) = \tilde{\rho}^{-n_0+l}\tilde{D}^*(n_0)\tilde{D}(n_0, l).$$

On the other hand it follows from (2.11) that $A_j{}^i > 0$ implies $A_j{}^i - E_j{}^i > 0$, so that

$$C_j{}^i(k) > 0, \quad \text{if} \quad A_j{}^i > 0.$$

Since the matrix $\tilde{A}$ is positively regular $\tilde{A}^{n_0-l} > 0$ for a large n_0. Combining these facts with (2.25) we have $\tilde{D}^*(l)>0$. The relation $\tilde{D}^*(l)\leqq A^*$ is clear, if we let $n\to\infty$ in (2.23).

Corollary 2.1. *Suppose that Condition (DN) holds and α is minimal w.r.t. the semiorder '$<$' with $\rho_\alpha<1$. Then the limit of (2.7) is uniform in $0\leqq s_\alpha\leqq q_\alpha$ and $K_\alpha{}^*(0)>0$.*

The proof is clear from Lemma 2.2 and 2.3, since,

$$\tilde{R}(n; s_\alpha)_\alpha=\tilde{D}_\alpha(n, -1; s_\alpha)(\tilde{q}_\alpha-\tilde{s}_\alpha)$$

in this case.

Now we assume that for all $\beta\lneqq\alpha$

$$\tilde{R}(n; s_\beta)_\beta=n^{\nu_\beta}\rho_\beta{}^n(\tilde{R}_\beta{}^*(s_\beta)+o(1)),\quad 0\leqq s_\beta\leqq q_\beta, \tag{2.26}$$

as $n\to\infty$, where $o(1)$ is uniform in $0\leqq s_\alpha\leqq q_\alpha$. Then it follows as $n\to\infty$

$$\bar{R}(n; s_\alpha)_\alpha=n^{\bar{\nu}_\alpha}\bar{\rho}_\alpha{}^n(\bar{R}_\alpha{}^*(\bar{s}_\alpha)+o(1)),\quad 0\leqq\bar{s}_\alpha\leqq\bar{\rho}_\alpha, \tag{2.27}$$

for some vector valued function $\bar{R}_\alpha{}^*(\bar{s}_\alpha)$, where $o(1)$ is uniform in $0\leqq\bar{s}_\alpha\leqq\bar{\rho}_\alpha$ and

$$\bar{\nu}_\alpha=\max\{\nu_\beta(\bar{\rho}_\alpha);\beta\lneqq\alpha\}.$$

Hence, it is enough for (2.5) to prove following

Lemma 2.4. *Let (2.17), (2.27) and $\rho_\alpha<1$ hold. Then it follows*

$$\tilde{R}(n; s_\alpha)_\alpha=n^{\nu_\alpha}\rho_\alpha{}^n(\tilde{R}_\alpha{}^*(s_\alpha)+o(1)),\quad 0\leqq s_\alpha\leqq q_\alpha, \tag{2.28}$$

where $o(1)$ is uniform in $0\leqq s_\alpha\leqq q_\alpha$, ν_α and $\tilde{R}_\alpha{}^*(s_\alpha)$ are given separately in the following three cases: (i) *if $\rho_\alpha=\tilde{\rho}_\alpha>\bar{\rho}_\alpha$, then $\nu_\alpha=0$ and*

$$\tilde{R}_\alpha{}^*(s_\alpha)=\tilde{D}^*(-1; s_\alpha)+\sum_{l=0}^{\infty}\tilde{D}_\alpha{}^*(l; s_\alpha)_\alpha\bar{R}(l;\bar{s}_\alpha)_\alpha\rho_\alpha{}^{-l-1}, \tag{2.29}$$

(ii) *if $\rho_\alpha=\bar{\rho}_\alpha>\tilde{\rho}_\alpha$, then $\nu_\alpha=\bar{\nu}_\alpha$ and*

$$\tilde{R}_\alpha{}^*(s_\alpha)=(\rho_\alpha I-\tilde{A}_\alpha)^{-1}A_\alpha{}'\bar{R}_\alpha{}^*(\bar{s}_\alpha), \tag{2.30}$$

and (iii) *if $\rho_\alpha=\tilde{\rho}_\alpha=\bar{\rho}_\alpha>0$, $\nu_\alpha=\bar{\nu}_\alpha+1$ and*

$$\tilde{R}_\alpha{}^*(s_\alpha) = \tilde{A}_\alpha{}^* A_\alpha{}' \bar{R}_\alpha{}^*(\bar{s}_\alpha)/\rho_\alpha \nu_\alpha. \tag{2.31}$$

Proof.[4] (i) When $\rho=\tilde{\rho}>\bar{\rho}$, we divide the sum in (2.15) into $\sum_{l=0}^{n_0}$ and $\sum_{l=n_0+1}^{n}$. For each $\rho>r>\bar{\rho}$ we have from (2.23) and (2.27) that

$$\tilde{\rho}^{-n-1}\tilde{D}(n, l)C(l)'\bar{R}(l) \leqq (r\rho^{-1})^l c,$$

where c is a positive vector with the indices in Δ. Hence it follows

$$\rho^{-n-1}\sum_{l=n_0+1}^{n}\tilde{D}(n, l)C(l)'\bar{R}(l) \leqq \frac{(r\rho^{-1})^{n_0}}{1-r\rho^{-1}}c<\varepsilon, \quad n\in\langle n_0+1, \infty\rangle,$$

for all sufficiently large n_0. Similarly, for all large n_0, it holds

$$\sum_{l=n_0+1}^{n}\tilde{D}^*(l)C(l)'\bar{R}(l)\rho^{-l}<\varepsilon, \quad n\in\langle n_0+1, \infty\rangle,$$

uniformly in $0\leqq s\leqq q$. But for a fixed n_0 (2.18) implies

$$\rho^{-n-1}\{\tilde{D}(n, -1)(\tilde{q}-\tilde{s})+\sum_{l=0}^{n_0}\tilde{D}(n, l)C(l)'\bar{R}(l)\}$$

$$\longrightarrow \tilde{D}^*(-1)(\tilde{q}-\tilde{s})+\sum_{l=0}^{n_0}\tilde{D}^*(l)C(l)'\bar{R}(l)\rho^{-l-1}$$

as $n\to\infty$, uniformly in $0\leqq s\leqq q$. Hence we have (2.28) with $\nu=0$ and R^* given by (2.29).

(ii) When $\rho=\bar{\rho}>\tilde{\rho}$, we shall exploit (2.15) in the form of

$$\tilde{R}(n+1)=\tilde{D}(n, -1)(\tilde{q}-\tilde{s})+\sum_{l=0}^{n}\tilde{D}(n, n-l)C(n-l)'\bar{R}(n-l),$$

dividing the sum into $\sum_0^{n_0}$ and $\sum_{n_0+1}^{n}$. From (2.23) and (2.27) it follows

$$(n+1)^{-\nu}\rho^{-n-1}\tilde{D}(n, n-l)C(n-l)'\bar{R}(n-l)\leqq(\tilde{\rho}\rho^{-1})^l c,$$

so that

$$(n+1)^{-\nu}\rho^{-n-1}\sum_{l=n_0+1}^{n}\tilde{D}(n, n-l)C(n-l)'\bar{R}(n-l)\leqq\frac{(\tilde{\rho}\rho^{-1})^{n_0}}{1-\tilde{\rho}\rho^{-1}}c<\varepsilon,$$

$$n\in\langle n_0+1, \infty\rangle,$$

for all sufficiently large n_0. Similarly, it holds for all large n_0 that

$$\sum_{l=n_0+1}^{n}\rho^{-l-1}\tilde{A}^l A'\bar{R}^*<\varepsilon, \quad \text{uniformly } 0\leqq s\leqq q,$$

by means of $\bar{R}^*(s)\leqq\bar{R}^*(0)<\infty$. Since

$$A\geqq C(n)\geqq A-E(n;0)\to A \tag{2.32}$$

as $n\to\infty$, we have for a fixed $l\in\langle 0, n_0\rangle$ that

$$\lim_{n\to\infty}\tilde{D}(n, n-l)=\tilde{A}^l, \quad \text{uniform in} \quad 0\leqq s\leqq q.$$

Hence it follows from (2.27) that

$$\lim_{n\to\infty}(n+1)^{-\nu}\rho^{-n-1}\sum_{l=0}^{n_0}\tilde{D}(n, n-l)C(n-l)'\bar{R}(n-l)=\sum_{l=0}^{n_0}\rho^{-l-1}\tilde{A}^l A'\bar{R}^*,$$

uniform in $0\leqq s\leqq q$. Finally (2.23) and the inequality $\rho>\tilde{\rho}$ imply

$$\lim_{n\to\infty}(n+1)^{-\nu}\rho^{-n-1}\tilde{D}(n, -1)(\tilde{q}-\tilde{s})=0, \quad \text{uniformly in} \quad 0\leqq s\leqq q.$$

Combining the above facts we obtain the conclusion.

(iii) Suppose that $\rho=\tilde{\rho}=\bar{\rho}>0$. From (2.24), (2.22), (2.32) (2.27) we can find n_0 and $n_1\in\langle 1, \infty\rangle$ satisfying

$$-\tilde{c}l^\nu\varepsilon\leqq\rho^{-n}\tilde{D}(n, l)C(l)'\bar{R}(l)-\tilde{A}^*A'\bar{R}^*l^\nu\leqq\tilde{c}l^\nu\varepsilon,\ l\in\langle n_0, n-n_1\rangle, \tag{2.33}$$

for some vector $\tilde{c}>0$. Now we divide the sum in (2.15) like as

$$\sum_0^n=\sum_0^{n_0}+\sum_{n_0+1}^{n-n_1}+\sum_{n-n_1+1}^{n}\equiv\mathrm{I}+\mathrm{II}+\mathrm{III}.$$

Since the functions

$$\rho^{-n-1}(n+1)^{-\nu}\tilde{D}(n, l)C(l)'\bar{R}(l), \quad l\in\langle 0, n\rangle, \quad n\in\langle 0, \infty\rangle,$$

are bounded in l, n and s on $0\leqq s\leqq q$, it holds

$$\lim_{n\to\infty}\rho^{-n-1}(n+1)^{-\nu-1}(\mathrm{I}+\mathrm{III})=0, \quad \text{uniformly in} \quad s.$$

Further it follows from (2.33) that

$$-\tilde{c}\varepsilon\rho^{-1}\leqq\rho^{-n-1}(n+1)^{-\nu-1}\mathrm{II}-\tilde{A}^*A'\bar{R}^*\rho^{-1}(n+1)^{-\nu-1}\sum_{l=n_0+1}^{n-n_1}l^\nu\leqq\tilde{c}\varepsilon\rho^{-1}.$$

Hence by the fact that

$$\lim_{n\to\infty}(n+1)^{-\nu-1}\sum_{l=n_0+1}^{n-n_1}l^\nu=1/(\nu+1),$$

and the boundedness of $\bar{R}^*$ in s, we have

$$\lim_{n\to\infty}\rho^{-n-1}(n+1)^{-\nu-1}\sum_{l=0}^{n}\tilde{D}(n, l)C(l)'\bar{R}(l)=\tilde{A}^*A'\bar{R}^*/\rho(\nu+1),$$

uniformly in $0 \leqq s \leqq q$. But since (2.23) implies

$$\lim_{n\to\infty}(n+1)^{-\nu-1}\rho^{-n-1}\tilde{D}(n, -1)(\tilde{q}-\tilde{s})=0, \quad \text{uniformly in} \quad 0\leqq s\leqq q,$$

we obtain the conclusion.

Note that the procedure to determine ν_α from $\bar{\nu}_\alpha$ by Lemma 2.4 coincides with that of (2.3)-(2.4). Further, we have

Lemma 2.5. *Under Condition (DN), the function* $R_\alpha{}^{*i}(s_\alpha)$ *determined by Lemmas 2.1 and 2.4 for each* $i\in\Delta_\alpha$, $\alpha\in\langle 1, g\rangle$ *with* $0<\rho_\alpha<1$, *is not identically zero.*

Proof. If α is minimal w.r.t. the semiorder '$<$', the assertion is clear by Lemma 2.1. If $\rho_\alpha=\tilde{\rho}_\alpha>\bar{\rho}_\alpha$, it is also clear from (2.19) and Lemmas 2.4 and 2.2. To deal with other cases, we assume that $R_\beta{}^{*i}(s_\beta)\not\equiv 0$ for all $i\in\Delta_\beta$ with $\beta\lneqq\alpha$ sarisfying $\rho_\beta>0$. We choose a maximal element β_0 in the set $\{\beta\lneqq\alpha; \nu_\beta(\bar{\rho}_\alpha)=\bar{\nu}_\alpha\}$. This β_0 is also maximal in the set $\{\beta\lneqq\alpha\}$, since in general $\beta<\alpha$ implies $\rho_\beta\leqq\rho_\alpha$, and $\beta<\alpha$, $\rho_\beta=\rho_\alpha$ imply $\nu_\beta\leqq\nu_\alpha$. Indeed, if it is not maximal in $\{\beta\lneqq\alpha\}$, there is a β such that $\beta_0\lneqq\beta\lneqq\alpha$. Then it follows $\rho_{\beta_0}=\rho_\beta=\bar{\rho}_\alpha$, and so $\nu_{\beta_0}=\nu_\beta=\bar{\nu}_\alpha$, which implies $\bar{\nu}_\alpha=\nu_\beta(\bar{\rho}_\alpha)$ and leads a contradiction. Now, since $\bar{\nu}_\alpha=\nu_{\beta_0}(\bar{\rho}_\alpha)$, it follows

$$\bar{R}^{*i}(\bar{s}_\alpha)=\tilde{R}^{*i}_{\beta_0}(s_{\beta_0})\not\equiv 0, \quad i\in\Delta_{\beta_0},$$

by (2.26) and (2.27), and since β_0 maximal in the set $\{\beta\lneqq\alpha\}$ it holds

$$A_j{}^i>0, \quad \text{for some} \quad i\in\Delta_\alpha \quad \text{and} \quad j\in\Delta_{\beta_0}.$$

Hence the conclusion is clear from (2.30)-(2.31) since $\tilde{A}_\alpha{}^*>0$ and, when $\rho_\alpha>\tilde{\rho}_\alpha$, $(\rho_\alpha I-\tilde{A}_\alpha)^{-1}>0$.

Proof of Theorem 2.1. Since 1) is clear from the previous arguments, we have only to show 2). Combining the equality

$$P_x\{T<\infty\}=\lim_{n\to\infty}F(n;0)^x=q^x$$

with the Markov property, we obtain

$$\sum_{y\in S}P_x\{Z(n)=y, T<\infty\}s^y=\sum_{y\in S}P_x\{Z(n)=y\}q^y s^y=F(n;qs)^x.$$

Hence it follows

$$(2.34)\qquad \sum_{y\in S} P_x\{Z(n)=y|n<T<\infty\}s^y = 1-\frac{q^x-F(n;qs)^x}{q^x-F(n;0)^x}.$$

Further by mean of (2. 5) and (1. 7) it holds as $n\to\infty$

$$(2.35)\quad q^x-F(n;qs)^x = \sum_{\alpha\in I_+(x)}\sum_{i\in\Delta_\alpha} x^i q^{x-e_i} n^{\nu_\alpha}\rho_\alpha{}^n (R^{*i}(q_\alpha s_\alpha)+o(1)),$$

where $o(1)$ is uniform in $0\leqq s\leqq 1$. Hence there exists the limit

$$F_x{}^*(s) = \lim_{n\to\infty}\sum_{y\in S} P_x\{Z(n)=y|n<T<\infty\}s^y,$$

uniformly in $0\leqq s\leqq 1$. Since $R^{*i}(q_\alpha)=0$, $i\in\Delta_\alpha$, it is easily seen that $F_x{}^*(1)=1$. Thus $F_x{}^*(s)$ is a generating function of a probability distribution and we obtain the conclusions.

Remark 2. 1. We can calculate the support of the limit distribution $\{P_x{}^*(y)\}$ more precisely. Let $\rho_x=\max\{\rho_\alpha;\alpha\in I_+(x)\}$, $\nu_x=\max\{\nu_\alpha;\alpha\in I_+(x),\rho_\alpha=\rho_x\}$ and $I^*(x)=\{\alpha\in I_+(x);\rho_\alpha=\rho_x,\nu_\alpha=\nu_x\}$. Then it is clear from (2. 5), (2. 6), (2. 34) and (2. 35) that the support of the limit distribution $\{P_x{}^*(y)\}$ is contained in the set

$$\{x=(x^1,\cdots,x^N)\in S; x^i=0,\quad i\notin \bigcup_{\alpha\in I^*(x)}\Gamma_\alpha\}-\{0\}.$$

Remark 2. 2. It can also be calculated how the limit distributions $\{P_x{}^*(y)\}$ depend on $x\in S-\{0\}$. Indeed, it follows from (2. 34) and (2. 35) that

$$\sum_{y\in S} P_x{}^*(y)s^y = F_x{}^*(s) = 1-\frac{\sum_{\alpha\in I^*(x)}\sum_{i\in\Delta_\alpha} x^i q^{x-e_i}R^{*i}(q_\alpha s_\alpha)}{\sum_{\alpha\in I^*(x)}\sum_{i\in\Delta_\alpha} x^i q^{x-e_i}R^*{}_i(0)}.$$

Further, if $\tilde\rho_\alpha\geqq\bar\rho_\alpha$ or α is minimal w.r.t. the semiorder '$<$', it hold

$$\tilde R_\alpha{}^*(s_\alpha) = K_\alpha{}^*(s_\alpha)\tilde u_\alpha,$$

for some monotone nonincreasing function $K_\alpha{}^*(s_\alpha)$, since (2. 7) holds, and (2. 29) and (2. 31) imply $\tilde A_\alpha \tilde R_\alpha{}^*(s_\alpha)=\tilde\rho_\alpha \tilde R_\alpha{}^*(s_\alpha)$. In the case of $\tilde\rho_\alpha<\bar\rho_\alpha$, (2. 30) will give us the sufficient information for the purpose.

Remark 2. 3. From (2. 5) it easily follows that

(2. 36) $$R^{*i}(F(n;s)_\alpha)=\rho_\alpha{}^n R^{*i}(s_\alpha), \quad i\in\Delta_\alpha,$$

if $0<\rho_\alpha<1$. Hence the coefficients of the power series $(\log R^*(s)/R^*(0))/\log\rho_\alpha$ give a stationary measure of the DGWP X_α on $S_\alpha-\{0\}$.

3. Noncritical periodic DGWP

In this section we shall deal with the noncritical DGWP's with the periodic matrices $\tilde{A}_\alpha$. It is known that, by a change of suffixes, an irreducible nonnegative matrix $M\neq[0]$ is represented as

(3. 1) $$M=\begin{bmatrix} 0 & M_1 & 0\cdots\cdots 0 \\ 0 & 0 & M_2\ 0\cdots 0 \\ \cdots & \cdots & \cdots \\ 0\cdots\cdots\cdots 0 & & M_{d-1} \\ M_d & 0\cdots\cdots\cdots 0 & \end{bmatrix},$$

where every 0 matrix on the diagonal is a square matrix and each $Q_\alpha\equiv M_\alpha\cdots M_d M_1\cdots M_{\alpha-1}$ is positively regular (Doob [5] pp. 177-178). We shall call the positive integer d the *period of the matrix M.* Of course the d-fold product M^d of M is given

(3. 2) $$M^d=\begin{bmatrix} Q_1 & 0\cdots\cdots\cdots & 0 \\ 0 & Q_2\ 0\cdots\cdots & 0 \\ \cdots & \cdots & \cdots \\ 0\cdots\cdots\cdots & 0 & Q_d \end{bmatrix}.$$

Lemma 3. 2. *The P-F root of the matrix Q_α is equal to $\rho(M)^d$.*

Proof. The set of all characteristic roots of M^d is the union of the sets of characteristic roots of Q_α, $\alpha\in\langle 1,d\rangle$, by means of (3. 2). On the other hand it holds $\rho(M^d)=\rho(M)^d$ by the Frobenius' theorem on the characteristic roots of a polynomial in a matrix. Hence we have

(3. 3) $$\rho(M)^d=\max\{\rho(Q_\alpha);\alpha\in\langle 1,d\rangle\}.$$

Suppose that $\rho(M)^d=\rho(Q_{\alpha_0})$. Then, because of the positive regularity of Q_{α_0}, there corresponds a positive eigenvector u_{α_0} of Q_{α_0} to $\rho(M)^d$;

$$Q_{\alpha_0}u_{\alpha_0}=M_{\alpha_0}\cdots M_d M_1\cdots M_{\alpha_0-1}u_{\alpha_0}=\rho(M)^d u_{\alpha_0}.$$

Operating the matrix $M_\alpha\cdots M_{\alpha_0-1}$ if $\alpha\in\langle 1,\alpha_0-1\rangle$ (and the matrix $M_\alpha\cdots M_d M_1\cdots M_{\alpha_0-1}$ if $\alpha\in\langle\alpha_0+1,d\rangle$) from the left, we have $Q_\alpha u_\alpha=\rho(M)^d u_\alpha$, where $u_\alpha=M_\alpha\cdots M_{\alpha_0-1}u_{\alpha_0}$ if $\alpha\in\langle 1,\alpha_0-1\rangle$ (and $u_\alpha=M_\alpha\cdots M_d M_1\cdots M_{\alpha_0-1}u_{\alpha_0}$ if $\alpha\in\langle\alpha_0+1,d\rangle$). But $u_\alpha\neq 0$ since every Q_α is positively regular, and hence $\rho(M)^d$ is a characteristic root of Q_α. Therefore $\rho(Q_\alpha)\geqq\rho(M)^d$ and so $\rho(Q_\alpha)=\rho(M)^d$ by means of (3.3).

For each $d\in\langle 1,\infty\rangle$, the family of generating functions $\{F(nd;s_\alpha)_\alpha;\ n\in\langle 0,\infty\rangle\}$ forms a DGWP on S_α, which we denote by $X_\alpha^{(d)}$.

Lemma 3.3. *The least nonnegative fixed point of $F(d;s_\alpha)_\alpha$ is equal to the Γ_α-part q_α of the extinction probability q of the DGWP X. Hence the q-mean matrix of the DGWP $X_\alpha^{(d)}$ coincides with the d-fold product A_α^d of A_α, and if Conditions (D) and (DN) are satisfied for the DGWP X_α then they are also satisfied for the DGWP $X_\alpha^{(d)}$.*

Proof. Let r_α be the least nonnegative fixed point of $F(d;s_\alpha)_\alpha$. Then it holds $r_\alpha\leqq q_\alpha$ since q_α is a nonnegative fixed point of $F(d;s_\alpha)_\alpha$. Hence it follows

$$r_\alpha=\lim_{n\to\infty}F(nd;r_\alpha)_\alpha=q_\alpha$$

from (1.7). The remaining assertions except for that on (DN) are clear. But the assertion on (DN) can be easily seen if we make use of the same arguments as in Athreya [1] or Sevastyanov [14] Chapter III, § 3.

Now let $\tilde d_\alpha\in\langle 1,m_\alpha\rangle$ be the period of the irreducible matrix $\tilde A_\alpha$ in (1.8), and

$$d_\alpha=\text{L.C.M.}\{\tilde d_\beta;\ \Delta_\beta\subset\Gamma_\alpha\}$$

(we set $\tilde d_\alpha=1$ if $\tilde\rho_\alpha=0$). Then by a change of the suffixes, we have

$$\tilde A_\beta^{d_\alpha}=\begin{bmatrix}\tilde A_{\beta 1}^{(\alpha)} & 0 & \cdots\cdots & 0\\ 0 & \tilde A_{\beta 2}^{(\alpha)} & 0\cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots\cdots & 0 & \tilde A_{\beta\tilde d_\beta}^{(\alpha)}\end{bmatrix},\quad \Delta_\beta\subset\Gamma_\alpha, \tag{3.4}$$

where $\tilde{A}_{\beta\gamma}^{(\alpha)}$ is an irreducible aperiodic nonnegative square matrix of order $m_{\beta\gamma}\in\langle 1, m_{\beta}\rangle$ $(\sum_{\gamma=1}^{\tilde{d}_{\beta}} m_{\beta\gamma}=m_{\beta})$. We define from (3.4) the sets $\Delta_{\beta\gamma}$, $\Gamma_{\beta\gamma}^{(\alpha)}$ and $S_{\beta\gamma}^{(\alpha)}$, the vectors $s_{\beta\gamma}^{(\alpha)}$ and $\tilde{s}_{\beta\gamma}$, and the matrices $A_{\beta\gamma}^{(\alpha)}$ as we defined Δ_{β}, Γ_{α}, etc., in section 1; for example

$$\Delta_{\beta\gamma}=\langle \sum_{p=1}^{\beta-1} m_p+\sum_{q=1}^{\gamma-1} m_{\beta q}+1,\ \sum_{p=1}^{\beta-1} m_p+\sum_{q=1}^{\gamma} m_{\beta q}\rangle.$$

Note that $m_{\beta\gamma}$ (and hence $\Delta_{\beta\gamma}$) is independent of d_{α} which satisfies $\tilde{d}_{\alpha}|d_{\alpha}$. We also define the DGWP $X_{\beta\gamma}^{(\alpha)}$ by the family of vector generating functions $\{F(nd_{\alpha};s_{\beta\gamma}^{(\alpha)})_{\beta\gamma}^{(\alpha)};n\in\langle 0,\infty\rangle\}$. By Lemma 3.2 and the representation (3.4), our DGWP $X_{\beta\gamma}^{(\alpha)}$ satisfies the assumptions of Theorem 2.1. As in section 2, we shall introduce the semiorder '$<_{\alpha}$' in the space of the suffixes $\{(\beta,p)\}$ by

$$(\delta,q)<_{\alpha}(\beta,p) \quad \text{if} \quad \Delta_{\delta q}\subset\Gamma_{\beta p}^{(\alpha)}.$$

Then the *rank* $\nu_{\alpha\gamma}$ *of* (α,γ) is defined by

$$(3.5)\qquad \nu_{\beta p}^{(\alpha)}(r)=\begin{cases}\max\{\nu_{\delta q}(r);(\delta,q)\lneqq_{\alpha}(\beta,p)\}, & \text{if}\quad \tilde{\rho}_{\beta}\neq r,\\ \max\{\nu_{\delta q}(r);(\delta,q)\lneqq_{\alpha}(\beta,p)\}+1, & \text{if}\quad \tilde{\rho}_{\beta}=r,\end{cases}$$

$(\max\phi=-1)$, and $\nu_{\alpha\gamma}=\nu_{\alpha\gamma}^{(\alpha)}(\rho_{\alpha})$.

Lemma 3.3. *Let Conditions (D) and (DN) be satisfied for all* $\alpha\in\langle 1,g\rangle$ *with* $\rho_{\alpha}<1$. *Then for each* $\alpha\in\langle 1,g\rangle$ *and* $\gamma\in\langle 1,\tilde{d}_{\alpha}\rangle$ *with* $\rho_{\alpha}<1$, *there correspond monotone nonincresing functions* $R^{*i}(s_{\alpha\gamma}^{(\alpha)})$ *in* $0\leqq s_{\alpha\gamma}^{(\alpha)}\leqq q_{\alpha\gamma}^{(\alpha)}$, $i\in\Delta_{\alpha\gamma}$, *such that it holds as* $n\to\infty$

$$(3.6)\qquad R^i(nd_{\alpha};s)=n^{\nu_{\alpha\gamma}}\rho_{\alpha}^{nd_{\alpha}}(R^{*i}(s_{\alpha\gamma}^{(\alpha)})+o(1)),\quad i\in\Delta_{\alpha\gamma},$$

where $o(1)$ *is uniform in* s *on* $0\leqq s_{\alpha\gamma}^{(\alpha)}\leqq q_{\alpha\gamma}^{(\alpha)}$. *Further, if* $\rho_{\alpha}>0$, *any* $R^{*i}(s_{\alpha\gamma}^{(\alpha)})$, $i\in\Delta_{\alpha}$, *is not identically zero.*

For each $x\in S$, we set

$$d_x=\text{L.C.M.}\{d_{\alpha};\alpha\in I_+(x)\}.$$

Theorem 3.1. *Let a DGWP* $X=(Z(n),P_x)$ *satisfy Conditions (D) and (DN) for each* $\alpha\in\langle 1,g\rangle$ *with* $\rho_{\alpha}<1$. *Then* 1) *for each* $\alpha\in\langle 1,g\rangle$ *with* $\rho_{\alpha}<1$ *and* $\gamma\in\langle 1,\tilde{d}_{\alpha}\rangle$, *it holds as* $n\to\infty$

(3. 7) $$R^i(nd_\alpha+l;s)=n^{\nu_{\alpha\gamma}}\rho_\alpha{}^{nd_\alpha}(R^{*i}(F(l;s_\alpha)^{(\alpha)}_{\alpha\gamma})+o(1)),$$

$$l\in\langle 0,d_\alpha-1\rangle,\quad i\in\Delta_{\alpha\gamma},\quad 0\leqq s_\alpha\leqq q_\alpha,$$

where $o(1)$ *is uniform in* s *on* $0\leqq s_\alpha\leqq q_\alpha$. *Further, if* $\rho_\alpha>0$, *then any* $R^{*i}(F(l;s_\alpha)^{(\alpha)}_{\alpha\gamma})$, $i\in\Delta_\alpha$, *is not identically zero.* 2) *For each* $x\in S$ *such that* $\rho_\alpha<1$ *for all* $\alpha\in I_+(x)$, *and* $\rho_\alpha>0$ *for some* $\alpha\in I_+(x)$, *there correspond a probability distributions* $\{P^*_{xl}(y)\}$ *on* $S-\{0\}$ *satisfying*

(3. 8) $$\lim_{n\to\infty}P_x\{Z(nd_x+l)\,|\,nd_x+l<T<\infty\}=P^*_{xl}(y),\,l\in\langle 0,d_x-1\rangle.$$

Proof. Repeating the arguments in the proof of Theorem 2. 1, we have only to show the nontriviality of the functions $R^{*i}(F(l;s_\alpha)^{(\alpha)}_{\alpha\gamma})$, $i\in\Delta_\alpha$, for $\rho_\alpha>0$. It follows from (3. 6) that

(3. 9) $$R^{*i}(F(md_\alpha;s)^{(\alpha)}_{\alpha\gamma})=\rho_\alpha{}^{md_\alpha}R^{*i}(s^{(\alpha)}_{\alpha\gamma}),\quad i\in\Delta_{\alpha\gamma}.$$

Since $F(l;0)^{(\alpha)}_{\alpha\gamma}\leqq F(md_\alpha;0)^{(\alpha)}_{\alpha\gamma}$, $l\leqq md_\alpha$, it is clear that $\rho_\alpha>0$ implies

$$R^{*i}(F(l;0)^{(\alpha)}_{\alpha\gamma})\geqq R^{*i}(F(md_\alpha;0)^{(\alpha)}_{\alpha\gamma})=\rho_\alpha{}^{md_\alpha}R^{*i}(0)>0,\ i\in\Delta_{\alpha\gamma},\ l\geqq md_\alpha,$$

and we obtain the conclusion.

Remark 3. 1. With the aid of Lemmas 2. 1 and 2. 4, we can determine the functions $R^{*i}(s^{(\alpha)}_{\alpha\gamma})$ inductively w.r.t. the semiorder '$<_\alpha$' in the space of the suffixes $\{(\beta,p)\,;\,\Delta_{\beta p}\subset\Gamma^{(\alpha)}_{\alpha\gamma}\}$.

4. Asymptotic behavior of critical DGWP

Since we have studied the noncritical DGWP's in the previous sections we shall study the critical ones in this and the next sections. We assume Condition (D) and

(DC) $$F^i_{jk}(q)<\infty,\quad i,j,k\in\Gamma_\alpha,$$

where $F^i_{jk}(s)=\partial^2F^i(s)/\partial s^j\partial s^k$ if it exists and $F^i_{jk}(s)=\lim_{\xi\uparrow s}F^i_{jk}(\xi)$ otherwise. We set

(4. 1) $$\mu_\alpha=1/2^{\nu_\alpha(1)},\quad \mu_{\alpha\gamma}=1/2^{\nu^{(\alpha)}_{\alpha\gamma}(1)},$$

where $\nu_\alpha(1)$ and $\nu^{(\alpha)}_{\alpha\gamma}(1)$ are those defined by (2. 3) and (3. 5). The object of this section is to prove the next two theorems:

Theorem 4.1. *Let a DGWP* $X=(Z(n), P_x)$ *satisfy Conditions (D) and (DC) for each* $\alpha\in\langle 1, g\rangle$ *with* $\rho_\alpha=1$, *and every matrix* $\tilde{A}_\alpha$ *be aperiodic. Then, for each* $\alpha\in\{1, g\rangle$ *with* $\rho_\alpha=1$, *there correspond constants* $R^{*i}>0$, $i\in\Delta_\alpha$, *such that*

$$\lim_{n\to\infty} n^{\mu_\alpha} R^i(n; s)=R^{*i}, \quad i\in\Delta_\alpha, \tag{4.2}$$

for each s satisfying $0\leqq s_\alpha\leqq q_\alpha$ *and*

$$\tilde{s}_\beta<\tilde{q}_\beta, \quad \text{if} \quad \beta<\alpha, \quad \tilde{\rho}_\beta>0. \tag{4.3}$$

The constants R^{*i} *are determined inductively w.r.t. the semiorder* '$<$' *from Lemmas 4.2 and 4.7 below.*

Theorem 4.2. *Let a DGWP* $X=(Z(n), P_x)$ *satisfy Conditions (D) and (DC) for each* $a\in\langle 1, g\rangle$ *with* $\rho_\alpha=1$. *Then, for each* $\alpha\in\langle 1, g\rangle$ *with* $\rho_\alpha=1$, *and* $\gamma\in\langle 1, \tilde{d}_\alpha\rangle$, *there correspond constants* $R^{*i}>0$, $i\in\Delta_{\alpha\gamma}$, *such that*

$$\lim_{n\to\infty} n^{\mu_{\alpha\gamma}} R^i(n;s)=R^{*i}, \quad i\in\Delta_{\alpha\gamma}, \tag{4.4}$$

for each s satisfying $0\leqq s_\alpha\leqq q_\alpha$ *and (4.3).*

Proof of Theorem 4.2 assuming Theorem 4.1. By the same arguments as in the proof of Lemma 3.3, we have from Theorem 4.1 that

$$\lim_{n\to\infty}(nd_\alpha)^{\mu_{\alpha\gamma}} R^i(nd_\alpha;s)=R^{*i}, \quad i\in\Delta_{\alpha\gamma},$$

for each s satisfying $0\leqq s_\alpha\leqq q_\alpha$ and (4.3). But since $F(l; s)$ also satisfies $0\leqq F(l; s)_\alpha\leqq q_\alpha$ and (4.3) for such an s, if follow

$$\lim_{n\to\infty}(nd_\alpha+l)^{\mu_{\alpha\gamma}} R^i(nd_\alpha+l; s)=\lim_{n\to\infty}(nd_\alpha)^{\mu_{\alpha\gamma}} R^i(nd_\alpha; F(l; s))=R^{*i},$$

$$i\in\Delta_{\alpha\gamma}, \quad l\in\langle 0, d_\alpha-1\rangle.$$

Remark 4.1. Combining Theorems 3.1 and 4.2, we of course obtain the whole asymptotic behavior of a DGWP satisfying conditions (D) and (DC) for all $\alpha\in\langle 1, g\rangle$.

Now we shall prove Theorem 4.1 without haste. In the following in this section, we assume that the hypotheses of Theorem 4.1 are satisfied, unless otherwise is stated.

Lemma 4.1. *If* $\tilde{\rho}_\alpha=1$, *then*

$$B_\alpha\equiv\frac{1}{2}\sum_{i,j,k\in\Delta_\alpha}\tilde{v}_{\alpha i}F^i_{jk}(q)\tilde{u}_\alpha{}^j\tilde{u}_\alpha{}^k>0.$$

Proof.[4] Suppose first that $\bar{\Gamma}=\phi$ and $F(s)=F(0)+As$. Then it follows

$$q=F(n;q)=F(n;0)+A^n q.$$

Letting $n\to\infty$ we have $\lim_{n\to\infty}A^n q=0$ by (1.7) which implies $\rho<1$. Next we shall assume that $\bar{\Gamma}\neq\phi$ and $\tilde{F}(s)=\tilde{F}_0(\bar{s})+\tilde{H}(\bar{s})\tilde{s}$ with $\tilde{F}_0(\bar{s})\not\equiv 0$. Then it follows that $\tilde{H}(\bar{q})=\tilde{A}$ and

$$\tilde{F}(n;s)=\tilde{F}_0(\bar{F}(n-1;\bar{s}))+\sum_{l=1}^{n-1}\tilde{H}(\bar{F}(n-1;\bar{s}))\cdots\tilde{H}(\bar{F}(l;\bar{s}))$$

$$\times\tilde{F}_0(\bar{F}(l-1;\bar{s}))+\tilde{H}(\bar{F}(n-1;\bar{s}))\cdots\tilde{H}(\bar{F}(0;\bar{s}))\tilde{s}.$$

Hence it follows

$$\tilde{q}=\tilde{F}(n;q)=\sum_{l=1}^{n}\tilde{A}^{n-l}\tilde{F}_0(\bar{q})+\tilde{A}^n\tilde{q}.$$

Since $\sum_{l=1}^n\tilde{A}^{n-l}\tilde{F}_0(\bar{q})>0$ for a large n, it hold $\tilde{q}>\tilde{A}^n\tilde{q}$. Hence we have $\rho(\tilde{A})^n=\rho(\tilde{A}^n)<1$ by the mini-max principle (cf. Gantmacher [6] II. p. 65).

For an $\alpha\in\langle 1,g\rangle$ which is minimal w.r.t. the semiorder '$\prec$', we exploit the following

Lemma 4.2. (Joffe and Spitzer [9]). *If the q-mean matrix* A_α *is positively regular with* $\rho_\alpha=1$, *it holds*

$$R^i(n;s)=\frac{\tilde{u}_\alpha{}^i\tilde{v}_\alpha\cdot(1_\alpha-s_\alpha)}{1+nB_\alpha\tilde{v}_\alpha\cdot(1_\alpha-s_\alpha)}(1+o(1)),\quad i\in\Delta_\alpha, \tag{4.6}$$

as $n\to\infty$, *where* $o(1)$ *is uniform in* $0\leqq s_\alpha\leqq 1_\alpha$, $s_\alpha\neq 1_\alpha$.

Note that q_α is equal to the Γ_α-part 1_α of the vector $1=(1,\cdots,1)$ in this case.

To study the case when α is not minimal, we prepare some lammas.

Lemma 4.3. *Let* $\tilde{\rho}_\alpha=1$ *and* $0\leqq s_\alpha\leqq q_\alpha$, $s_\alpha\neq q_\alpha$. *Then the relation*

$$(4.7)\qquad \lim_{n\to\infty}\frac{\bar{R}(n+k;\bar{s}_\alpha)_\alpha}{\tilde{v}_\alpha\cdot\tilde{R}(n;s_\alpha)_\alpha}=0,\quad k\in\langle 0,\infty\rangle,$$

implies

$$(4.8)\qquad \lim_{n\to\infty}\frac{\tilde{R}(n;s_\alpha)_\alpha}{\tilde{v}_\alpha\cdot\tilde{R}(n;s_\alpha)_\alpha}=\tilde{u}_\alpha.$$

Proof.[4] First of all we note that

$$(4.9)\qquad \tilde{v}\cdot\tilde{R}(n;s)>0,\quad n\in\langle n_0,\infty\rangle,\quad 0\leqq s\leqq q,\quad s\neq q,$$

for some $n_0\in\langle 1,\infty\rangle$. Indeed, for each $i\in\Delta$ and $j\in\Gamma$ there corresponds an $n_j{}^i\in\langle 1,\infty\rangle$ such that $A_j{}^i(n_j{}^i)>0$. Hence the positive regularity of $\tilde{A}$ implies

$$A_j{}^i(n)\geqq A_i{}^i(n-n_j{}^i)A_j{}^i(n_j{}^i)>0$$

for all sufficiently large n. So such $F^i(n;s)$ depends on every variable s^j with $j\in\Gamma$, and we obtain (4.9). Now using (2.14) inductively, we obtain

$$(4.10)\quad \tilde{R}(n+1)=\tilde{D}(n,n-m-1)\tilde{R}(n-m)+\sum_{l=n-m}^{n}\tilde{D}(n,l)C(l)'\bar{R}(l).$$

We take the sequences ε_n and α_n in the proof of Lemma 2.3. In our case the sequence ε_n may not satisfy (2.22), but it tends to zero as $n\to\infty$ and satisfies (2.24) with $\rho=1$. Combining (2.24) and (4.10) we have

$$(1-\alpha_{m+1}-\sum_{k=n-m}^{n}\varepsilon_k)\tilde{A}^*\tilde{R}(n-m)+\sum_{l=n-m}^{n}(1-\alpha_{n-l}-\sum_{k=l+1}^{n}\varepsilon_k)\tilde{A}^*C(l)'\bar{R}(l)$$

$$\leqq\tilde{R}(n+1)\leqq(1+\alpha_{m+1})\tilde{A}^*\tilde{R}(n-m)+\sum_{l=n-m}^{n}(1+\alpha_{n-l})\tilde{A}^*C(l)'\bar{R}(l).$$

Hence it follows, for each m and n with $n-m\in\langle n_0,\infty\rangle$,

$$(4.11)\qquad \frac{(1-\alpha_{m+1}-\sum_{k=n-m}^{n}\varepsilon_k)\tilde{P}(n,m)+\sum_{l=n-m}^{n}(1-\alpha_{n-l}-\sum_{k=l+1}^{n}\varepsilon_k)\tilde{Q}(n,l)}{(1+\alpha_{m+1})+\sum_{l=n-m}^{n}(1+\alpha_{n-l})\tilde{v}\cdot\tilde{Q}(n,l)}$$

$$\leqq \frac{\tilde{R}(n+1)}{\tilde{v}\cdot\tilde{R}(n+1)}$$

$$\leqq \frac{(1+\alpha_{m+1})\tilde{P}(n,m)+\sum_{l=n-m}^{n}(1+\alpha_{n-l})\tilde{Q}(n,l)}{(1-\alpha_{m+1}-\sum_{k=n-m}^{n}\varepsilon_k)+\sum_{l=n-m}^{n}(1-\alpha_{n-l}-\sum_{k=l+1}^{n}\varepsilon_k)\tilde{v}\cdot\tilde{Q}(n,l)},$$

where

$$\tilde{P}(n,m)=\frac{\tilde{A}^*\tilde{R}(n-m)}{\tilde{v}\cdot\tilde{R}(n-m)},\quad \tilde{Q}(n,l)=\frac{\tilde{A}^*C(l)'\bar{R}(l)}{\tilde{v}\cdot\tilde{R}(n-m)}.$$

But $\tilde{P}(n,m)=\tilde{u}$ by the definition of $\tilde{A}^*$, and $\tilde{Q}(n,l)\to 0$ as $n\to\infty$ by (4.7) and (2.32). Hence, letting $n\to\infty$ in (4.11), we have

$$\frac{(1-\alpha_{m+1})\tilde{u}^i}{1+\alpha_{m+1}}\leqq \varliminf_{n\to\infty}\frac{R^i(n+1)}{\tilde{v}\cdot\tilde{R}(n+1)}\leqq \varlimsup_{n\to\infty}\frac{R^i(n+1)}{\tilde{v}\cdot\tilde{R}(n+1)}$$

$$\leqq \frac{(1+\alpha_{m+1})\tilde{u}^i}{1-\alpha_{m+1}},\quad i\in\Delta,\quad m\in\langle 1,\infty\rangle.$$

Now we obtain (4.8) by letting $m\to\infty$.

Lemma 4.4. *There are functions $B^i_{jk}(s_\alpha)$ and $G_j{}^i(s_\alpha)$ in $0\leqq s_\alpha\leqq q_\alpha$ such that*

$$R^i(s_\alpha)=\sum_{j\in\Delta_\alpha}A_j{}^i(q^j-s^j)-\sum_{j,k\in\Delta_\alpha}B^i_{jk}(s_\alpha)(q^j-s^j)(q^k-s^k)$$
$$+\sum_{j\in\Gamma_\alpha}(A_j{}^i-G_j{}^j(s_\alpha))(q^j-s^j),\quad i\in\Delta_\alpha,\quad 0\leqq s_\alpha\leqq q_\alpha, \tag{4.12}$$

where

$$0\leqq B^i_{jk}(s_\alpha{}^{(1)})\leqq B^i_{jk}(s_\alpha{}^{(2)})\leqq \tfrac{1}{2}F^i_{jk}(q),\quad 0\leqq s_\alpha{}^{(1)}\leqq s_\alpha{}^{(2)}\leqq q_\alpha,$$
$$B^i_{jk}(s_\alpha)\to\tfrac{1}{2}F^i_{jk}(q),\quad \text{as}\quad s_\alpha\to q_\alpha\quad \text{in}\quad 0\leqq s_\alpha\leqq q_\alpha,\quad i,j,k\in\Delta_\alpha, \tag{4.13}$$

$$0\leqq G_j{}^i(s_\alpha)\leqq 2E_j{}^i(s_\alpha),\quad i\in\Delta_\alpha,\quad j\in\Gamma_\alpha. \tag{4.14}$$

Proof. Integrating by parts the integral in (2.11), we have

$$E_j{}^i(s)=\sum_{k\in\Gamma}B^i_{jk}(s)(q^k-s^k),\quad i\in\Delta,\quad 0\leqq s\leqq q,$$
$$B^i_{jk}(s)=\sum_{y\in S}P^i(y)(y^jy^k-y^j\delta_k{}^j)\int_0^1(q-(q-s)\xi)^{y-e_j-e_k}(1-\xi)d\xi. \tag{4.15}$$

Combining this with (2.10) we have

$$R^i(s) = \sum_{j\in\Delta} A_j{}^i (q^j - s^j) - \sum_{j,k\in\Delta} B^i_{jk}(s)(q^j - s^j)(q^k - s^k)$$

$$+ \sum_{j\in\Gamma} (A_j{}^i - E_j{}^i(s))(q^j - s^j) - \sum_{j\in\Delta\, k\in\Gamma} B^i_{jk}(s)(q^j - s^j)(q^k - s^k),$$

$$0 \leqq s \leqq q.$$

Since $B^i_{jk}(s) = B^i_{kj}(s)$ by (4.15), the last term is equal to

$$-\sum_{j\in\Gamma}\sum_{k\in\Delta} B^i_{jk}(s)(q^k - s^k)(q^j - s^j),$$

and we obtain (4.12) with

$$G_j{}^i(s) = E_j{}^i(s) + \sum_{k\in\Delta} B^i_{kj}(s)(q^k - s^k). \tag{4.16}$$

Further (4.13) follows from (4.15), and (4.14) follows from (4.15)-(4.16).

Note that, if we replace s_α in (4.12) by $F(n; s_\alpha)_\alpha$, we obtain

$$R^i(n+1; s_\alpha) = \sum_{j\in\Delta_\alpha} A_j{}^i R^j(n; s_\alpha) - \sum_{j,k\in\Delta_\alpha} B^i_{jk}(n; s_\alpha) R^j(n; s_\alpha) \tag{4.17}$$

$$\times R^k(n; s_\alpha) + \sum_{j\in\Gamma_\alpha} (A_j{}^i - G_j{}^i(n; s_\alpha)) R^j(n; s_\alpha),$$

$$i \in \Delta_\alpha, \quad 0 \leqq s_\alpha \leqq q_\alpha,$$

where

$$B^i_{jk}(n; s_\alpha) = B^i_{jk}(F(n; s_\alpha)_\alpha), \quad G_j{}^i(n; s_\alpha) = G_j{}^i(F(n; s_\alpha)_\alpha). \tag{4.18}$$

Hence it follows, when $\tilde{\rho}_\alpha = 1$,

$$a_{n+1} - a_n = -b_n a_n{}^2 + c_n, \tag{4.19}$$

where

$$\begin{cases} a_n = a_{\alpha n}(s_\alpha) = \tilde{v}_\alpha \cdot \tilde{R}(n; s_\alpha)_\alpha \\ b_n = b_{\alpha n}(s_\alpha) = \dfrac{\sum_{i,j,k\in\Delta_\alpha} \tilde{v}_{\alpha i} B^i_{jk}(n; s_\alpha) R^j(n; s_\alpha) R^k(n; s_\alpha)}{a_{\alpha n}(s_\alpha)^2} \\ c_n = c_{\alpha n}(s_\alpha) = \sum_{i\in\Delta_\alpha, j\in\Gamma_\alpha} \tilde{v}_{\alpha i}(A_j{}^i - G_j{}^i(n; s_\alpha)) R^j(n; \bar{s}_\alpha). \end{cases} \tag{4.20}$$

Note that (4.9) is rewritten as

$$a_n > 0, \quad n \in \langle n_0, \infty\rangle, \quad 0 \leqq s_\alpha \leqq q_\alpha, \quad s_\alpha \neq q_\alpha, \tag{4.21}$$

for some $n_0 \in \langle 1, \infty \rangle$. Further

(4. 22) $$\lim_{n\to\infty} a_n = 0$$

by (1. 7), and

(4. 23) $$0 \leqq \underline{b}^* \equiv \varliminf_{n\to\infty} b_n \leqq \varlimsup_{n\to\infty} b_n \equiv \bar{b}^* < \infty$$

by (4. 13) and the inequality $\tilde{v}_\alpha > 0$. Finally it holds

(4. 24) $$c_n \geqq 0, \qquad n \in \langle n_1, \infty \rangle,$$

for some $n_1 \in \langle 0, \infty \rangle$ by means of (4. 14), (1. 7) and the fact that $A_j{}^i = 0$ implies $E_j{}^i(s_\alpha) = 0$.

Now we assume that

(4. 25) $$\lim_{n\to\infty} n^{\mu_\beta} R^i(n; s) = R^{*i}, \qquad i \in \Delta_\beta,$$

for each $\beta \lneqq \alpha$ with $\rho_\beta = 1$ and s satisfying $0 \leqq s_\alpha \leqq q_\alpha$ and (4. 3), where R^{*i} are constants with $R^{*i} > 0$. Then we have

Lemma 4. 5. 1) *If* $\bar{\rho}_\alpha < 1$, *it holds*

(4. 26) $$c_n = o(1/n^2), \quad as \quad n \to \infty.$$

2) *If* $\bar{\rho}_\alpha = 1$,

(4. 27) $$\lim_{n\to\infty} n^{\bar{\mu}_\alpha} \bar{R}(n; s)_\alpha = \bar{R}_\alpha{}^*,$$

for each s *with* $0 \leqq s_\alpha \leqq q_\alpha$ *and* (4. 3), *where*

(4. 28) $$\bar{\mu}_\alpha = \min\{\mu_\beta; \beta \lneqq \alpha, \ \rho_\beta = 1\}.$$

Further, it holds

(4. 29) $$\lim_{n\to\infty} n^{\bar{\mu}_\alpha} c_n = \tilde{v}_\alpha A_\alpha{}' \bar{R}_\alpha{}^* \equiv c^* > 0.$$

Proof. (4. 26) is clear from (4. 20) and Theorem 2. 1. (4. 27) is also clear by (4. 25). Hence (4. 29) except for the relation $c^* > 0$ follows with the aid of (4. 14) and (1. 7). But $c^* > 0$ is easily seen if we repeat the same arguments as in the proof of Lemma 2. 5.

The next lemma plays an important role in the following.

Lemma 4.6. *Let sequences* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *satisfy* (4.19) *and* (4.21)-(4.24). *Then,* 1) (4.26) *implies*

$$1/\bar{b}^* \leqq \varliminf_{n\to\infty} na_n \leqq \varlimsup_{n\to\infty} na_n \leqq 1/\underline{b}^*. \tag{4.30}$$

2) *If*

$$0 < \underline{b}^* \leqq \bar{b}^* < \infty, \tag{4.31}$$

$$\lim_{n\to\infty} n^{\mu} c_n = c^*, \tag{4.32}$$

for some $0 < \mu \leqq 1$, *then it holds*

$$\sqrt{\frac{c^*}{\bar{b}^*}} \leqq \varliminf_{n\to\infty} n^{\mu/2} a_n \leqq \varlimsup_{n\to\infty} n^{\mu/2} a_n \leqq \sqrt{\frac{c^*}{\underline{b}^*}}\ . \tag{4.33}$$

Proof. 1) By (4.22) and (4.23), it holds

$$\frac{b_n}{1-a_n b_n} \leqq M, \quad n \in \langle n_2, \infty \rangle$$

for some $M>0$ and $n_2 \in \langle 1, \infty \rangle$. Hence it follows from (4.19), (4.21) and (4.24) that

$$\frac{1}{a_{n+1}} - \frac{1}{a_n} \leqq \frac{b_n a_n}{a_{n+1}} = \frac{b_n}{(1-a_n b_n) + c_n/a_n} \leqq M, \quad n \in \langle n_3, \infty \rangle,$$

where $n_3 = n_0 \vee n_1 \vee n_2$. Summing up these inequalities from n_3 to n we have

$$\frac{1}{a_n} \leqq (n-n_3) M + \frac{1}{a_{n_3}}, \quad n \in \langle n_3, \infty \rangle,$$

so that, by means of (4.26),

$$\lim_{n\to\infty} c_n/a_n = \lim_{n\to\infty} c_n/a_n^2 = 0.$$

Hence we obtain (4.30) since (4.19) implies

$$\frac{1}{n}\left\{\frac{1}{a_n} - \frac{1}{a_{n_3}}\right\} = \frac{1}{n} \sum_{l=n_3}^{n-1} \frac{b_l - c_l/a_l^2}{1 - b_l a_l + c_l/a_l}\ .$$

2) Setting $\xi_n = n^{\mu/2} a_n$, we have from (4.19) that

$$b_n \xi_n^2 - n^{\mu} c_n + n^{\mu/2} (\xi_{n+1} - \xi_n) = a_{n+1} O(n^{\mu-1}),$$

as $n \to \infty$. Since $0 < \mu \leqq 1$, this with (4.22) implies the basic equality

(4.34) $$\lim_{n\to\infty}\{b_n\xi_n^2 - n^\mu c_n + n^{\mu/2}(\xi_{n+1}-\xi_n)\} = 0.$$

Now we shall show that the sequence $\{\xi_n\}$ is bounded. Suppose that $\{\xi_n\}$ is unbounded, and let

$$n_1 = 1, \quad n_k = \min\{n; \xi_n > \xi_{n_{k-1}} \vee k\}, \quad k \in \langle 2, \infty\rangle.$$

Then it follows

(4.35) $$\xi_{n_k} > \xi_{n_{k-1}} \vee k \geqq \xi_{n_k-1}, \quad k \in \langle 2, \infty\rangle,$$

(4.36) $$\lim_{k\to\infty} \xi_{n_k} = \infty.$$

By (4.35) we have $\xi_{n_k} > \xi_{n_k-1}$, and hence by (4.34)

$$\overline{\lim_{k\to\infty}}\{b_{n_k-1}\xi^2_{n_k-1} - (n_k-1)^\mu c_{n_k-1}\} \leqq 0.$$

Hence with the aid of (4.32) and (4.31) we have

(4.37) $$\overline{\lim_{k\to\infty}}\, \xi^2_{n_k-1} \leqq c^*/\underline{b}^* < \infty,$$

and from (4.34)

(4.38) $$\lim_{k\to\infty}(\xi_{n_k} - \xi_{n_k-1}) = -\lim_{k\to\infty}\frac{b_{n_k-1}\xi^2_{n_k-1} - (n_k-1)^\mu c_{n_k-1}}{(n_k-1)^{\mu/2}} = 0.$$

(4.37) and (4.38) imply the boundedness of the sequence $\{\xi_{n_k}\}$, which is a contradiction. We note that, by means of the boundedness of the sequence $\{\xi_n\}$, (4.38) is valid for any subsequence $\{n_k\}$. To prove (4.33), we set

$$\underline{\xi}^* = \underline{\lim}_{n\to\infty}\, \xi_n, \quad \bar{\xi}^* = \overline{\lim_{n\to\infty}}\, \xi_n.$$

First we shall show that $\underline{\xi}^* = \bar{\xi}^* \equiv \xi^*$ implies

$$\sqrt{c^*/\bar{b}^*} \leqq \xi^* \leqq \sqrt{c^*/\underline{b}^*}.$$

Indeed if $(0\leqq)\,\xi^* < \sqrt{c^*/\bar{b}^*}$ for example, it holds by (4.34) and (4.32) that

$$n^{\mu/2}(\xi_{n+1}-\xi_n) \geqq n^\mu c_n - b_n\xi_n^2 - \varepsilon \geqq c^* - \bar{b}^*(\xi^*)^2 - 2\varepsilon > 0, \quad n \in \langle N_0, \infty\rangle$$

for some $N_0 \in \langle 1, \infty\rangle$. Hence it follows

$$\xi_n - \xi_{N_0} \geqq (c^* - \bar{b}^*(\xi^*)^2 - 2\varepsilon)\sum_{k=N_0}^{n-1}\frac{1}{k^{\mu/2}},$$

which contradicts the boundedness of $\{\xi_n\}$. Next we shall show that (4.33) holds even when $\underline{\xi}^* < \bar{\xi}^*$. Since the situations do not differ, we suppose $\bar{\xi}^* > \sqrt{c^*/\underline{b}^*}$ and lead a contaradiction. Take a constant ξ in $\bar{\xi}^* > \xi > \underline{\xi}^* \vee \sqrt{c^*/\underline{b}_*}$, and let

$$n_0 = \min\{n; \xi_n > \xi\},$$

$$m_k = \min\{n \in \langle n_{k-1}+1, \infty\rangle; \xi_n < \xi\},$$

$$n_k = \min\{n \in \langle m_k+1, \infty\rangle; \xi_n > \xi\}, \quad k \in \langle 1, \infty\rangle.$$

Then it holds

$$\xi_{n_k} > \xi_{n_k-1} \vee \xi, \quad k \in \langle 1, \infty\rangle. \tag{4.39}$$

Indeed, the inequality $\xi_{n_k} > \xi$ is clear from the definitions, and $\xi_{n_k} > \xi_{n_k-1}$ is also clear since $\xi_{n_k-1} \leqq \xi$ if $n_k - 1 \in \langle m_k+1, \infty\rangle$, and $\xi_{n_k-1} = \xi_{m_k} < \xi$ if $n_k - 1 = m_k$. Now it follows from (4.34) and (4.39) that for any $\varepsilon > 0$ there is a k_1 satisfying

$$\xi_{n_k-1}^2 < \frac{(n_k-1)^{\mu} c_{n_k-1} + \varepsilon}{b_{n_k-1}}, \quad k \in \langle k_1, \infty\rangle .$$

Combining this inequality with (4.39), (4.38) and (4.32), we obtain

$$\xi^2 \leqq \varliminf_{k\to\infty} \xi_{n_k}^2 = \varliminf_{k\to\infty} \xi_{n_k-1}^2 \leqq \frac{c^* + \varepsilon}{\underline{b}^*},$$

which contradicts the inequality $\xi > \sqrt{c^*/\underline{b}^*}$.

Corollary 4.1. *(4.33) is still valid even if we replace the assumption (4.19) by (4.34) where* $\xi_n = n^{\mu/2} a_n$.

Now we are ready to prove the next lemma which completes the proof of Theorem 4.1:

Lemma 4.7. *Let* $\rho_\alpha = 1$, *and (4.25) hold. Then it follows*

$$\lim_{n\to\infty} n^{\mu_\alpha} \tilde{R}(n; s)_\alpha = \tilde{R}_\alpha^*, \tag{4.40}$$

for all s satisfying $0 \leqq s_\alpha \leqq q_\alpha$ *and (4.3), where* μ_α *and* $\tilde{R}_\alpha^*$ *are given separately in the following three cases*; (i) *if* $1 = \tilde{\rho}_\alpha > \bar{\rho}_\alpha$, *then* $\mu_\alpha = 1$ *and*

(4. 41) $$\tilde{R}_\alpha{}^* = \tilde{u}_\alpha / B_\alpha,$$

(ii) *if* $1=\bar{\rho}_\alpha > \tilde{\rho}_\alpha$, *then* $\mu_\alpha = \bar{\mu}_\alpha$ *and*

(4. 42) $$\tilde{R}_\alpha{}^* = (I-\tilde{A}_\alpha)^{-1} A_\alpha{}' \bar{R}_\alpha{}^*,$$

and (iii) *if* $1=\tilde{\rho}_\alpha = \bar{\rho}_\alpha$, *then* $\mu_\alpha = \bar{\mu}_\alpha/2$ *and*

(4. 43) $$\tilde{R}_\alpha{}^* = \left(\frac{\tilde{v}_\alpha A_\alpha{}' \bar{R}_\alpha{}^*}{B_\alpha}\right)^{1/2} \tilde{u}_\alpha.$$

Proof. (i) When $1=\tilde{\rho}>\bar{\rho}$, it holds (4. 26) by Lemma 4. 5. Hence it follows (4. 30) by Lemma 4. 6, and we have (4. 7) by Theorem 2. 1. Therefore (4. 8) holds by Lemma 4. 3, and so

(4. 44) $$\lim_{n\to\infty} b_n = B$$

by (4. 20), (4. 18), (1. 7) and (4. 13). Now appealing to Lemma 4. 6 1) again, we have $\lim_{n\to\infty} n\, a_n = 1/B$ to obtain (4. 40) with $\mu=1$ and $\tilde{R}^*$ given by (4. 41) from (4. 8).

(ii) When $1=\bar{\rho}>\tilde{\rho}$, it holds

(4. 45) $$n^\mu \tilde{R}(n;s) \leqq \tilde{c}, \quad n \in \langle 0, \infty \rangle,$$

for $\mu=\bar{\mu}$. Indeed, combing (2. 15) with (2. 23) and (4. 27) we have have

$$(n+1)^\mu \tilde{R}(n+1) \leqq (n+1)^\mu \tilde{A}^{n+1}\tilde{q} + (n+1)^\mu \sum_{l=0}^{n} \tilde{A}^{n-l} A' \bar{R}(l)$$

$$\leqq (n+1)^\mu \theta_1 \tilde{\rho}^{n+1} \tilde{A}^* \tilde{q} + (n+1)^\mu \theta_2 \tilde{\rho}^n \sum_{l=1}^{n} \tilde{\rho}^{-l} l^{-\mu} \tilde{A}^* A' (\bar{R}^* + K),$$

where θ_1, θ_2 and K are positive constants. But since

$$\sum_{l=1}^{n} \tilde{\rho}^{-l} l^{-\mu} \sim n^{-\mu} \tilde{\rho}^{-n} / (-\log \tilde{\rho}), \quad \text{as} \quad n\to\infty,$$

(4. 45) follows.

Now by means of (4. 10) it holds

$$\sum_{l=0}^{m} \tilde{D}(n, n-l) C(n-l)' (n+1)^\mu \bar{R}(n-l) \leqq (n+1)^\mu \tilde{R}(n+1)$$

$$\leqq \left(\frac{n+1}{n-m}\right)^\mu \tilde{A}^{m+1} (n-m)^\mu \tilde{R}(n-m) + \sum_{l=0}^{m} \tilde{A}^l A' (n+1)^\mu \bar{R}(n-l).$$

Hence letting $n\to\infty$ we have from (4.45) that

$$\sum_{l=0}^{m} \tilde{A}^l A' \bar{R}^* \leqq \varliminf_{n\to\infty} n^{\mu}\tilde{R}(n) \leqq \varlimsup_{n\to\infty} n^{\mu}\tilde{R}(n) \leqq \tilde{A}^{m+1}\tilde{c} + \sum_{l=0}^{m} \tilde{A}^l A' \bar{R}^*.$$

But $\tilde{A}^{m+1}\to 0$ as $m\to\infty$ since $\tilde{\rho}<1$, and we obtain the conclusion.

(iii) In the case of $1=\tilde{\rho}=\bar{\rho}$, we shall first prove (4.44). Since the sequence $a_n(0)$ is monotone nonincreasing in n, it follows from (4.19) and (4.20) that

$$0\leqq \frac{c_n(0)}{a_n(0)} \leqq b_n(0)\, a_n(0) \to 0, \quad \text{as} \quad n\to\infty.$$

Hence it holds from (4.29) that

$$\lim_{n\to\infty} 1/n a_n(0) = 0.$$

Further, for each $0\leqq s\leqq q$ satisfying (4.3) we can find an $l\in\langle 0,\infty\rangle$ by (1.7) such that $s\leqq F(l;0)\leqq q$, whence it follows $R(n;s)\geqq R(n+l;0)$ and

$$\lim_{n\to\infty} 1/n^{\beta} a_n(s) = 0.$$

Hence we have (4.7) by (4.27), so that (4.8) and (4.44) by Lemma 4.3 and (4.20). Now since $B>0$ by Lemma 4.1, it follows from (4.44) and Lemma 4.6 2) that

$$\lim_{n\to\infty} n^{\beta/2} a_n = \sqrt{c^*/B}.$$

Hence we have the conclusion with the aid of (4.8).

Remark 4.2. The vectors $\tilde{R}_\alpha{}^*$ given above are positive. The proof is similar to that of Lemma 2.5.

Remark 4.3. It is clear from the proof that (4.40) holds for all s with $0\leqq s_\alpha\leqq q_\alpha$, $s_\alpha\neq q_\alpha$ in case of (i), and for all s satisfying $0\leqq s_\alpha\leqq q_\alpha$, $s_\alpha\neq q_\alpha$ and (4.27) in case of (ii). Further, it can be seen that if we assume Condition (DE) in the next section (4.40) (and hence (4.2)) holds for all s with $0\leqq s_\alpha\leqq q_\alpha$, $s_\alpha\neq q_\alpha$ in all cases.

5. Asymptotic behavior of $Z(n)/n$ of critical DGWP

In this section we shall give the asymptotic behavior of the distributions

$$Q_x(n;u)=P_x\left\{\frac{Z(n)}{n}\leqq u \mid n<T<\infty\right\},\quad u\in R_+^N,$$

of critical DGWP's. We shall assume for each $\alpha\in\langle 1,g\rangle$ with $\tilde{\rho}_\alpha=1$ that

$$\text{(DE)}\qquad \sum_{i,j,k\in\Delta_{\alpha\gamma}}\tilde{v}_{\alpha\gamma i}F^i_{jk}(\tilde{d}_\alpha;q)\xi^j\xi^k\geqq c_{\alpha\gamma}\Big(\sum_{i\in\Delta_{\alpha\gamma}}\tilde{v}_{\alpha\gamma i}\xi^i\Big)^2,$$

$$\tilde{\xi}_{\alpha\gamma}=(\xi^i)_{i\in\Delta_{\alpha\gamma}}>0,\quad \gamma\in\langle 1,\tilde{d}_\alpha\rangle,$$

where $c_{\alpha\gamma}$ is a positive constant and $\tilde{v}_{\alpha\gamma}$ is the positive left eigenvector of $\tilde{A}^{(\alpha)}_{\alpha\gamma}$ corresponding to the P-F root 1. When the matrix $\tilde{A}_\alpha$ is aperiodic, it is clear that $\tilde{d}_\alpha=1$, and Condition (DE) is reduced to

$$\text{(5. 1)}\qquad \sum_{i,j,k\in\Delta_\alpha}\tilde{v}_{\alpha i}F^i_{jk}(q)\xi^j\xi^k\geqq c_\alpha\Big(\sum_{i\in\Delta_\alpha}\tilde{v}_{\alpha i}\xi^i\Big)^2,\quad \tilde{\xi}_\alpha=(\xi^i)_{i\in\Delta_\alpha}>0,$$

for some $c_\alpha>0$. We set

$$s^{(n)}=s^{(n,\lambda)}=(q^1\exp(-\lambda^1/n),\cdots,q^N\exp(-\lambda^N/n)),$$

for each $\lambda=(\lambda^1,\cdots,\lambda^N)\geqq 0$. Our object in this section is to prove the following

Theorem 5. 1. *Let a DGWP $X=(Z(n),P_x)$ satisfy Conditions (D), (DC) for each $\alpha\in\langle 1,g\rangle$ with $\rho_\alpha=1$ and (DE) for each $\alpha\in\langle 1,g\rangle$ with $\tilde{\rho}_\alpha=1$, and the matrices $\tilde{A}_\alpha$ be aperiodic. Then,* 1) *for each $\alpha\in\langle 1,g\rangle$ with $\rho_\alpha=1$, there correspond nontrivial nonnegative functions $\psi^i(\lambda_\alpha)$, $i\in\Delta_\alpha$, such that*

$$\text{(5. 2)}\qquad \lim_{n\to\infty}n^{\mu_\alpha}R^i(n;s^{(n,\lambda)})=\psi^i(\lambda_\alpha),\quad i\in\Delta_\alpha,$$

for each $\lambda\geqq 0$ satisfying

$$\text{(5. 3)}\qquad \tilde{\lambda}_\beta>0,\quad \text{if}\quad \beta<\alpha,\quad \tilde{\rho}_\beta>0.$$

The functions $\psi^j(\lambda_\alpha)$, $i\in\Delta_\alpha$, are determined inductively w.r.t. the semiorder '$<$' by Lemmas 5.1 and 5.3 below. 2) *For each $x\in S$ with $\rho_\alpha=1$ for some $\alpha\in I_+(x)$, the distributions $Q_x(n;u)$, $n\in\langle 1,\infty\rangle$, converge as $n\to\infty$ to a probability distribution $Q_x^*(u)$ on R_+^N given by*

$$\int_{R_+^N} e^{-\lambda\cdot u} \mathrm{d}Q_x^*(u) = 1 - \frac{\sum\limits_{\substack{\alpha\in I_+(x)\\ \mu_\alpha=\mu_x}} \sum\limits_{i\in\Delta_\alpha} x^i q^{x-e_i} \psi^i(\lambda_\alpha)}{\sum\limits_{\substack{\alpha\in I_+(x)\\ \mu_\alpha=\mu_x}} \sum\limits_{i\in\Delta_\alpha} x^i q^{x-e_i} R^{*i}} . \tag{5.4}$$

where $\mu_x = \min\{\mu_\alpha ; \alpha\in I_+(x)\}$.

Theorem 5.2. *Let a DGWP* $X=(Z(n), P_x)$ *satisfy Conditions (D), (DC) for each* $\alpha\in\langle 1, g\rangle$ *with* $\rho_\alpha=1$ *and (DE) for each* $\alpha\in\langle 1, g\rangle$ *with* $\tilde{\rho}_\alpha=1$. *Then,* 1) *for each* $\alpha\in\langle 1, g\rangle$ *with* $\rho_\alpha=1$ *and* $\gamma\in\langle 1, \tilde{d}_\alpha\rangle$, *there correspond nonnegative functions* $\psi^i(\lambda_{\alpha\gamma}^{(\alpha)})$, $i\in\Delta_{\alpha\gamma}$, *such that*

$$\lim_{n\to\infty} (nd_\alpha + l)^{\mu_{\alpha\gamma}} R^i(nd_\alpha + l; s^{(nd_\alpha+l,\lambda)}) = \psi^i(\omega_l(\lambda)_{\alpha\gamma}^{(\alpha)}), \tag{5.5}$$

$$i\in\Delta_{\alpha\gamma},\quad l\in\langle 0, d_\alpha - 1\rangle,$$

for each $\lambda\geqq 0$ *with (5.3), where* $\omega_l(\lambda) = A^l\{q\lambda\}/q$. 2) *For each* $x\in S$ *with* $\rho_\alpha=1$ *for some* $\alpha\in I_+(x)$, *the distributions* $Q_x(nd_x + l; u)$, $u\in R_+^N$, *converge as* $n\to\infty$ *to a probability distribution* $Q_{xl}^*(u)$ *on* R_+^N.

Throughout in the following in this section we always assume the hypotheses of Theorem 5.2. Further, we shall assume for the moment that every $\tilde{A}_\alpha$ is aperiodic. Then, for an $\alpha\in\langle 1, g\rangle$ which is minimal w.r.t. the semiorder '$\prec$', there is the following excellent

Lemma 5.1. (Joffe and Spitzer [9]). *If the q-mean matrix* A_α *is positively regular with* $\rho_\alpha=1$, *it holds (5.2) with* $\mu_\alpha=1$ *and*

$$\psi^i(\lambda_\alpha) = \frac{\tilde{u}^i \tilde{v}\cdot(q_\alpha\lambda_\alpha)}{1 + B_\alpha \tilde{v}_\alpha\cdot(q_\alpha\lambda_\alpha)} . \tag{5.6}$$

To deal with the case when α is not minimal, we prepare a lemma.

Lemma 5.2. *Suppose that* $\tilde{\rho}_\alpha=1$ *and* $\lambda\geqq 0$ *satisfies (5.3). Then the relation*

$$\lim_{n\to\infty} \frac{\bar{R}(n-m+l; s^{(n,\lambda)})_\alpha}{\tilde{v}_\alpha\cdot \tilde{R}(n-m; s_\alpha^{(n,\lambda)})_\alpha} = 0,\quad l\in\langle 0, m\rangle,\quad m\in\langle 0, \infty\rangle, \tag{5.7}$$

implies

$$\lim_{n\to\infty}\frac{\tilde{R}(n;s_\alpha^{(n,\lambda)})_\alpha}{\tilde{v}_\alpha\cdot\tilde{R}(n;s_\alpha^{(n,\lambda)})_\alpha}=\tilde{u}_\alpha. \tag{5.8}$$

Further the relation

$$\lim_{k\to\infty}\sup_{n\geq k}\frac{\bar{R}(k-m+l;\bar{s}_\alpha^{(n,\lambda)})_\alpha}{\tilde{v}_\alpha\cdot\tilde{R}(k-m;s_\alpha^{(n,\lambda)})_\alpha}=0,\ l\in\langle 0,m\rangle,\ m\in\langle 0,\infty\rangle, \tag{5.9}$$

implies

$$\lim_{k\to\infty}\sup_{n\geq k}\max_{i\in\Delta_\alpha}\left|\frac{R^i(k;s_\alpha^{(n,\lambda)})}{\tilde{v}_\alpha\cdot\tilde{R}(k;s_\alpha^{(n,\lambda)})_\alpha}-\tilde{u}_\alpha^{\,i}\right|=0. \tag{5.10}$$

The proof is similar to that of Lemma 4.3 and will be omitted.

Here we assume

$$\lim_{n\to\infty} n^{\mu_\beta}R^i(n;s^{(n,\lambda)})=\psi^i(\lambda_\beta),\quad i\in\Delta_\beta,$$

for all $\beta\lneqq\alpha$ with $\rho_\beta=1$. Then it follows, if $\bar{\rho}_\alpha=1$, that

$$\lim_{n\to\infty} n^{\bar{\mu}_\alpha}\bar{R}(n;s^{(n,\lambda)})_\alpha=\bar{\psi}_\alpha(\bar{\lambda}_\alpha), \tag{5.11}$$

for some $\bar{\psi}_\alpha(\bar{\lambda}_\alpha)=(\bar{\psi}^i(\bar{\lambda}_\alpha))_{i\in\bar{\Gamma}_\alpha}$.

Lemma 5.3. *Let* $\rho_\alpha=1$, *and* (5.11) *hold if* $\bar{\rho}_\alpha=1$. *Then it follows*

$$\lim_{n\to\infty} n^{\mu_\alpha}\tilde{R}(n;s^{(n,\lambda)})=\tilde{\psi}_\alpha(\lambda_\alpha), \tag{5.12}$$

for all $\lambda\geqq 0$ *with* (5.3), *where* μ_α *are those in section 4 and* $\tilde{\psi}_\alpha(\lambda_\alpha)$ *are given separately in the following three cases*: (i) *if* $1=\tilde{\rho}_\alpha>\bar{\rho}_\alpha$, *then*

$$\tilde{\psi}_\alpha(\lambda_\alpha)=\frac{\tilde{v}_\alpha\cdot(\tilde{q}_\alpha\tilde{\lambda}_\alpha)\tilde{u}_\alpha}{1+\tilde{v}_\alpha\cdot(\tilde{q}_\alpha\tilde{\lambda}_\alpha)\{B_\alpha-\chi_\alpha(\lambda_\alpha)\}}, \tag{5.13}$$

where

$$\chi_\alpha(\lambda_\alpha)=\sum_{k=0}^{\infty}\frac{\tilde{v}_\alpha A_\alpha{}'\bar{A}_\alpha^{\,k}\{\bar{q}_\alpha\bar{\lambda}_\alpha\}}{\{\tilde{v}_\alpha\cdot(A_\alpha^{\,k}\{q_\alpha\lambda_\alpha\})_{\tilde{\alpha}}\}\{\tilde{v}_\alpha\cdot(A_\alpha^{\,k+1}\{q_\alpha\lambda_\alpha\})_{\tilde{\alpha}}\}}, \tag{5.14}$$

(ii) *if* $1=\bar{\rho}_\alpha>\tilde{\rho}_\alpha$, *then*

(5. 15) $$\tilde{\psi}_\alpha(\lambda_\alpha) = (I-\tilde{A}_\alpha)^{-1} A_\alpha' \bar{\psi}_\alpha(\bar{\lambda}_\alpha),$$

and (iii) *if* $1=\tilde{\rho}_\alpha=\bar{\rho}_\alpha$, *then*

(5. 16) $$\tilde{\psi}_\alpha(\lambda_\alpha) = \left(\frac{\tilde{v}_\alpha A_\alpha' \bar{\psi}_\alpha(\bar{\lambda}_\alpha)}{B_\alpha}\right)^{1/2} \tilde{u}_\alpha.$$

Proof. (i) With the notations in (4. 20), $a_n>0$ holds for all $n\in\langle 0, \infty\rangle$ since we have assumed (5. 3) and $\tilde{\rho}_\alpha>0$. Hence it follows from (4. 19)

(5. 17) $$\frac{1}{n}\left\{\frac{1}{a_n(s^{(n)})} - \frac{1}{a_0(s^{(n)})}\right\}$$
$$= \frac{1}{n}\sum_{k=0}^{n-1} \frac{b_k(s^{(n)})}{1-b_k(s^{(n)})a_k(s^{(n)})+c_k(s^{(n)})/a_k(s^{(n)})}$$
$$- \frac{1}{n}\sum_{k=0}^{n-1} \frac{c_k(s^{(n)})}{a_k(s^{(n)})\, a_{k+1}(s^{(n)})}\ .$$

By the same arguments as in the proof of Lemma 2. 2, it holds

(5. 18) $$\bar{R}(k-m+l;\bar{s}^{(n)}) \leqq \bar{A}^{k-m+l}(\bar{q}-\bar{s}^{(n)}) \leqq \frac{\theta_1 r^{k-m+l}}{n}\bar{q},$$
$$k\in\langle m-l, \infty\rangle,$$

for some $\theta_1=\theta_1(\lambda)>0$ and $\rho<r<1$. Similarly, by the convexity of the function $F^i(n;s+(q-s)\xi)$ in $0\leqq\xi\leqq 1$, we have

(5. 19) $$\tilde{R}(k-m;s^{(n)}) \geqq \tilde{A}(k-m;s^{(n)})\,(\tilde{q}-\tilde{s}^{(n)}), \quad k\in\langle m, \infty\rangle,$$

where $\tilde{A}(k;s)=[F_j{}^i(k;s)]_{i,j\in\Delta}$. Further it can be seen that for each $r<\tilde{r}<1$ there is a vector $0<\eta\leqq q$ satisfying (4. 3) such that

(5. 20) $$F(\eta)\geqq\eta \quad \text{and} \quad \rho(\tilde{A}(1;\eta))>\tilde{r}.$$

Indeed, since $F^i(n;0)\uparrow q^i$ as $n\uparrow\infty$, it is enough to take an $F(n;0)$ with a sufficiently large n as the vector η. Since the matrix $\tilde{A}(1;\eta)$ is also positively regular, it follow from (5. 20) that

(5. 21) $$\tilde{A}(k;\eta) \geqq \tilde{A}(1;\eta)^k \geqq \tilde{r}^k(1-\delta_k)\tilde{A}^*(\eta),$$

where $\tilde{A}^*(\eta)$ is a positive matrix and $\{\delta_k\}$ is a sequence with $\delta_k\to 0$ as $k\to\infty$ and $0\leqq\delta_k\leqq 1$. But since there is a $k_0\in\langle 1, \infty\rangle$ with

$$\eta\leqq s^{(k)}\leqq s^{(n)}\leqq q, \quad n\in\langle k, \infty\rangle, \quad k\in\langle k_0, \infty\rangle,$$

we have from (5. 19) that

$$(5.22)\qquad \tilde{R}(k-m;s^{(n)})\geqq\frac{\theta_2\tilde{r}^{k-m}(1-\delta_{k-m})\tilde{A}^*(\eta)\tilde{q}}{n},\quad n\geqq k\geqq m\vee k_0,$$

for some $\theta_2=\theta_2(\lambda)>0$. Combining (5. 18) and (5. 22) we obtain (5. 9), and hence (5. 10) by Lemma 5. 2. Since $B^i_{jk}(k;s)\to F^i_{jk}(q)/2$ as $k\to\infty$ uniformly in $0\leqq s\leqq q$, it follows from (5. 10) and (4. 20) that

$$(5.23)\qquad \lim_{k\to\infty}\sup_{n\geqq k}|b_k(s^{(n)})-B|=0.$$

Hence it also follows from (4. 22) that

$$(5.24)\qquad \lim_{k\to\infty}\sup_{n\geqq k}b_k(s^{(n)})a_k(s^{(n)})=0.$$

Letting $m=l=0$ in (5. 18) and (5. 22), we have

$$\frac{c_k(s^{(n)})}{a_k(s^{(n)})}\leqq\frac{\theta_1 r^k\tilde{v}A'\bar{q}}{\theta_2\tilde{r}^k(1-\delta_k)\tilde{v}\tilde{A}^*(\eta)\tilde{q}},\quad n\geqq k\geqq k_0,$$

so that

$$(5.25)\qquad \lim_{k\to\infty}\sup_{n\geqq k}c_k(s^{(n)})/a_k(s^{(n)})=0.$$

To estimate the sequence $c_k(s^{(n)})/a_k(s^{(n)})a_{k+1}(s^{(n)})$, we shall exploit (5. 22) for an $\tilde{r}$ with

$$\sqrt{r}<\tilde{r}<1.$$

Then it is clear from (5. 18) and (5. 22) that

$$\frac{1}{n}\,\frac{c_k(s^{(n)})}{a_k(s^{(n)})a_{k+1}(s^{(n)})}\leqq\theta_3\frac{r_k}{\tilde{r}^{2k}},\quad n\geqq k\geqq k_0,$$

for some $\theta_3>0$. As for $k\in\langle 0,k_0\rangle$, it is not difficult to see that

$$\frac{1}{n}\,\frac{c_k(s^{(n)})}{a_k(s^{(n)})a_{k+1}(s^{(n)})}\leqq M_k,\ n\in\langle k,\infty\rangle.$$

Since

$$\sum_{k=0}^{k_0}M_k+\sum_{k=k_0+1}^{\infty}\theta_3\frac{r^k}{\tilde{r}^{2k}}<\infty,$$

we can apply the Lebesgue's convergence theorem, obtaining

$$\lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{1}{n}\frac{c_k(s^{(n)})}{a_k(s^{(n)})a_{k+1}(s^{(n)})}=\chi(\lambda), \tag{5.26}$$

with the help of

$$\lim_{n\to\infty} nR(k;s^{(n,\lambda)})=A^k\{q\lambda\}. \tag{5.27}$$

Combining (5.23)-(5.26) with (5.17), we have

$$\lim_{n\to\infty} n\,a_n(s^{(n)})=\frac{\tilde{v}\cdot(\tilde{q}\tilde{\lambda})}{1+\tilde{v}\cdot(\tilde{q}\tilde{\lambda})\{B-\chi(\lambda)\}}.$$

Hence we have (5.12) with $\psi^i(\lambda)$ given by (5.13) because of (5.8).

(ii) By the convexity of the function $F^i(l;s^{(l)}+(s^{(n+1)}-s^{(l)})\xi)$ in $0\leqq\xi\leqq 1$, we have

$$\begin{aligned} R^i(l;s^{(l)})-R^i(l;s^{(n+1)}) &= F^i(l;s^{(n+1)})-F^i(l;s^{(l)}) \\ &\leqq \sum_{j\in\Gamma} F_j{}^i(l;s^{(n+1)})\,(s^{(n+1)}-s^{(l)})^j, \end{aligned} \tag{5.28}$$

for each $i\in\Gamma$ and $n+1\geqq l$. Similarly it holds

$$\begin{aligned} R^i(l;s^{(n+1)}) &= F^i(l;q)-F^i(l;s^{(n+1)}) \\ &\geqq \sum_{j\in\Gamma} F_j{}^i(l;s^{(n+1)})\,(q-s^{(n+1)})^j. \end{aligned} \tag{5.29}$$

Since

$$(s^{(n+1)}-s^{(l)})^j\leqq\theta'\frac{1}{n+1}\frac{n+1-l}{l}\leqq\theta(q-s^{(n+1)})^j\frac{n+1-l}{l}, \tag{5.30}$$

$$n+1\geqq l\vee n_0,$$

for some $\theta>0$ and $n_0\in\langle 1,\infty\rangle$, it follows from (5.28), (5.29) and (4.27) that

$$0\leqq\bar{R}(l;s^{(l)})-\bar{R}(l;s^{(n+1)})\leqq\frac{(n+1-l)\,\theta}{l}\bar{R}(l;0)\leqq\frac{(n+1-l)\,\bar{c}}{l^{1+\mu}}, \tag{5.31}$$

$$n+1\geqq l\vee n_0,$$

for some vector $\bar{c}$. Hence, substituting $l=n-l$, we have for any fixed m

$$\lim_{n\to\infty}(n+1)^{\mu}\sum_{l=1}^{m}\tilde{D}(n,n-l;s^{(n+1)})C(n-l;s^{(n+1)})'\bar{R}(n-l;s^{(n+1)})$$

$$=\lim_{n\to\infty}\sum_{l=0}^{m}\tilde{D}(n, n-l; s^{(n+1)})C(n-l; s^{(n+1)})'(n-l)^{\mu}\bar{R}(n-l; s^{(n-l)})$$

$$=\sum_{l=0}^{m}\tilde{A}^{l}A'\bar{\Psi}(\bar{\lambda}).$$

Now we can obtain (5. 12) with (5. 15) by the same arguments as in the proof of Lemma 4. 7 (ii).

(iii) By Lemma 4. 7 (iii), the sequence $n^{\beta/2}\tilde{R}(n; s^{(n+1)})$ is bounded in $n\in\langle 1, \infty\rangle$ so that we have by the same way as to (5. 31) that

$$0\leqq\tilde{R}(n+1; s^{(n)})-\tilde{R}(n+1; s^{(n+1)})\leqq\frac{\tilde{c}}{n^{1+\beta/2}},\quad n\geqq n_0, \tag{5. 32}$$

for some vector $\tilde{c}$ and $n_0\in\langle 1, \infty\rangle$. Let

$$\alpha_n=\alpha_n(\lambda)=n^{\beta/2}a_n(s^{(n)}),\ \beta_n=\beta_n(\lambda)=b_n(s^{(n)}),\ \gamma_n=\gamma_n(\lambda)=n^{\beta}c_n(s^{(n)}).$$

Then (4. 19) and (5. 32) imply

$$\alpha_{n+1}-\alpha_n=n^{-\beta/2}(-\beta_n\alpha_n^2+\gamma_n)+O\left(\frac{1}{n}\right),$$

as $n\to\infty$, so that

$$\lim_{n\to\infty}\{n^{\beta/2}(\alpha_{n+1}-\alpha_n)+(\beta_n\alpha_n^2-\gamma_n)\}=0. \tag{5. 33}$$

Further, by means of (4. 20) and Assumptions (DC) and (DE), it holds

$$\infty>\bar{\beta}=\overline{\lim_{n\to\infty}}\,\beta_n(\lambda)\geqq\underline{\lim_{n\to\infty}}\,\beta_n(\lambda)=\underline{\beta}>0,$$

for some $\bar{\beta}=\bar{\beta}(\lambda)$ and $\underline{\beta}=\underline{\beta}(\lambda)$. Hence, appealing to Corollary 4. 1, we obtain from (5. 33) that

$$\sqrt{\gamma^*/\bar{\beta}}\leqq\underline{\lim_{n\to\infty}}\,\alpha_n\leqq\overline{\lim_{n\to\infty}}\,\alpha_n\leqq\sqrt{\gamma^*/\underline{\beta}}, \tag{5. 34}$$

where

$$\gamma^*=\lim_{n\to\infty}\gamma_n=\tilde{v}A'\bar{\Psi}(\bar{\lambda}).$$

Combining (5. 34), (5. 32) and (4. 29), we obtain (5. 7). Hence (5. 8) follows by Lemma 5. 2, and also

$$\lim_{n\to\infty} \beta_n(\lambda) = B.$$

Hence, again using Corollary 4. 1, we obtain from (5. 33)

$$\lim_{n\to\infty} \alpha_n(\lambda) = \sqrt{\gamma^*/B}.$$

Now (5. 12) with (5. 16) is proved, since (5. 8) is valid.

Proof of Theorem 5. 1. Since 1) is clear from Lemmas 5. 1 and 5. 3, we shall show 2). By the similar arguments as for (2. 34), it is easily seen that

$$\int_{R_+^N} e^{-\lambda\cdot u} dQ_x(n; u) = 1 - \frac{q^x - F(n; s^{(n,\lambda)})^x}{q^x - F(n; 0)^x}.$$

Further, it follows from (5. 2), (4. 2) and (1. 7) that

$$q^x - F(n; s^{(n)})^x = n^{-\mu_x} \sum_{\substack{\alpha\in I_+(x)\\ \mu_\alpha=\mu_x}} \sum_{i\in\Delta_\alpha} x^i q^{x-e_i} \psi^i(\lambda_\alpha) + o(n^{-\mu_x}),$$

$$q^x - F(n; 0)^x = n^{-\mu_x} \sum_{\substack{\alpha\in I_+(x)\\ \mu_\alpha=\mu_x}} \sum_{i\in\Delta_\alpha} x^i q^{x-e_i} R^{*i} + o(n^{-\mu_x}),$$

as $n\to\infty$. Hence it follows

$$\lim_{n\to\infty} \int_{R_+^N} e^{-\lambda\cdot u} dQ_x(n; u) = \psi_x(\lambda),$$

where $\psi_x(\lambda)$ is given by the right side of (5. 4). Further $\psi_x(\lambda)$ is a Laplace transform of a nonnegative measure $dQ_x^*(u)$ on R_+^N. Since $\lim_{\lambda\downarrow 0} \psi^i(\lambda_\alpha) = 0$ by (5. 6) and (5. 13)-(5. 16)[5], it hold $\lim_{\lambda\downarrow 0} \psi_x(\lambda) = 1$. Hence the nonnegative measure $dQ_x^*(u)$ is a probability measure and we obtain the conclusion.

We note that the parallel assertions to those of Remarks 2. 1 and 2. 2 are also valid in this case. Further, we have

Remark 5. 1. *It holds*

(5. 35) $$\psi^i(\omega_l(\lambda)_\alpha) = \psi^i(\lambda_\alpha),$$

[5] More precisely, one may take λ_α with the form of $\tilde{\lambda}_\alpha = \theta\tilde{q}$, $\bar{\lambda}_\alpha = \theta^2\bar{q}$ where $\theta > 0$, $\theta \downarrow 0$, in the case of $1 = \tilde{\rho}_\alpha > \bar{\rho}_\alpha$.

where $\omega_l(\lambda)=A^l\{q\lambda\}/q$.

Proof. From (5. 6) and (5. 13)-(5. 16), it is enough to show (5. 35) in the case of (5. 13). But this is not difficult since

$$\tilde{\phi}(\omega_1(\lambda))=\frac{\tilde{v}\cdot(A\{q\lambda\})^{\sim}\tilde{u}}{1+\tilde{v}\cdot(A\{q\lambda\})^{\sim}B-\tilde{v}\cdot(A\{q\lambda\})^{\sim}\chi(\omega_1(\lambda))}$$
$$=\frac{\tilde{v}\cdot(A\{q\lambda\})^{\sim}\tilde{u}}{1+\tilde{v}\cdot(A\{q\lambda\})^{\sim}(B-\chi(\lambda))+\tilde{v}A'\{\bar{q}\bar{\lambda}\}/\tilde{v}\cdot(\tilde{q}\tilde{\lambda})}$$
$$=\tilde{\phi}(\lambda).$$

As to Theorem 5. 2, we have the next lemma from Theorem 5. 1 by the same arguments as those to lead Lemma 3. 3 from Theorem 2. 1.

Lemma 5. 4. *There exist nontrivial limits*

(5. 36) $$\lim_{n\to\infty}(nd_\alpha)^{\mu_{\alpha\gamma}}R^i(nd_\alpha;s^{(nd_\alpha;\lambda)})=\psi^i(\lambda_{\alpha\gamma}^{(\alpha)}),\quad i\in\Delta_{\alpha\gamma},$$

for each $\lambda\geqq0$ *with* (5. 3), $\alpha\in\langle1,g\rangle$ *with* $\rho_\alpha=1$ *and* $\gamma\in\langle1,\tilde{d}_\alpha\rangle$.

Proof of Theorem 5. 2. First we set

$$F(l)=F(l;s^{(nd+l,\lambda)}),\quad s(\omega)=s^{(nd,\omega_l(\lambda))},$$
$$F\vee s=(F^1(l)\vee s^1(\omega),\cdots,F^N(l)\vee s^N(\omega)).$$

Then it is clear that

(5. 37) $$R^i(nd+l;s^{(nd+l,\lambda)})=R^i(nd;F(l)).$$

Further by the differentiability of the function $F^i(nd;F(l)+(s(\omega)-F(l))\xi)$ it holds

(5. 38) $$|R^i(nd;F(l))-R^i(nd;s(\omega))|\leqq\sum_{j\in\Gamma}F_j{}^i(nd;c)|F^j(l)-s^j(\omega)|\leqq\sum_{j\in\Gamma}F_j{}^i(nd;F\vee s)|F^j(l)-s^j(\omega)|,$$

where c is a vector with $c\leqq F\vee s$. Similarly

(5. 39) $$R^i(nd;F\vee s)\geqq\sum_{j\in\Gamma}F_j{}^i(nd;F\vee s)(q^j-F^j(l)\vee s^j(\omega)).$$

On the other hand, since

$$F^j(l)=q^j-\sum A_k{}^j(l)q^k\lambda^k/nd+O\left(\frac{1}{n^2}\right)$$

$$=q^j(1-\omega_l^j(\lambda)/nd)+O\Big(\frac{1}{n^2}\Big),$$

$$s^j(\omega)=q^j(1-\omega_l^j(\lambda)/nd)+O\Big(\frac{1}{n^2}\Big),$$

as $n\to\infty$, it follows

$$(5.40)\qquad \begin{aligned}&|F^j(l)-s^j(\omega)|\leqq k_1/n^2,\\ &q^j-F^j(l)\vee s^j(\omega)\geqq k_2/n, \quad n\in\langle n_0,\infty\rangle, \quad j\in\Gamma,\end{aligned}$$

for some k_1, $k_2>0$ and $n_0\in\langle 1,\infty\rangle$. Combining (5.37)-(5.40), we have

$$(5.41)\qquad |R^i(nd+l;s^{(nd+l)})-R^i(nd;s(\omega))|\leqq\frac{k_1}{nk_2}R^i(nd;F\vee s)$$

$$\leqq\frac{k_1}{nk_2}R^i(nd;0), \quad n\in\langle n_0,\infty\rangle.$$

Hence it follows from (5.36) and (5.37) that

$$\lim_{n\to\infty}(nd+l)^{\mu}R^i(nd+l;s^{(nd+l)})=\lim_{n\to\infty}(nd)^{\mu}R^i(nd;s(\omega))$$

$$=\psi^i(\omega_l(\lambda)), \quad i\in\Delta, \quad l\in\langle 0,d-1\rangle.$$

The assertion of 2) is easily seen from (4.4) and (5.5) by the same arguments as in the proof of Theorem 5.1.

6. Asymptotic behavior of CGWP

In this section we shall deal with CGWP's $X=(Z(t),P_x)$ satisfying Condition (C). Since the matrix

$$\tilde{A}_\alpha(t)=[A_j{}^i(t)]_{i,j\in\Delta_\alpha}=\exp(t\tilde{a}_\alpha), \quad t>0,$$

is always positive by the irreducibility of $\tilde{a}_\alpha$, the periodicity does not appear. There also correspond positive right and left eigenvectors $\tilde{u}_\alpha=(\tilde{u}_\alpha{}^i)_{i\in\Delta_\alpha}$ and $\tilde{v}_\alpha=(\tilde{v}_{\alpha i})_{i\in\Delta_\alpha}$ of the matrix $\tilde{a}_\alpha$ to the *P-F* root $\tilde{\sigma}_\alpha\equiv\rho(\tilde{a}_\alpha)$;

$$\tilde{a}_\alpha\tilde{u}_\alpha=\tilde{\sigma}_\alpha\tilde{u}_\alpha, \quad \tilde{v}_\alpha\tilde{a}_\alpha=\tilde{\sigma}_\alpha\tilde{v}_\alpha,$$

with the normalizations

$$\sum_{i\in\Delta_\alpha}\tilde{v}_{\alpha i}\tilde{u}_\alpha{}^i=1,\quad \sum_{i\in\Delta_\alpha}\tilde{u}_\alpha{}^i=1.$$

We set $\delta_p=1/2^p$, $p\in\langle 0,\infty\rangle$. Then the family of the generating functions $\{F(n\delta_p;s)\,;n\in\langle 0,\infty\rangle\}$ forms a DGWP on S, which we shall denote by $X^{(\delta_p)}$. The extinction probability of $X^{(\delta_p)}$ is equal to that of the original CGWP X, and the q-mean matrix $A^{(\delta_p)}$ of $X^{(\delta_p)}$ is equal to $\exp(\delta_p a)$. Similarly, the family of the generating functions $\{F(n\delta_p;s_\alpha)_\alpha;n\in\langle 0,\infty\rangle\}$ forms a DGWP $X_\alpha{}^{(\delta_p)}$ with the q-mean matrix $A_\alpha{}^{(\delta_p)}\equiv\exp(\delta_p a_\alpha)$. Here we set the condition

(CN) $$\sum_{y\in S}p^i(y)\,y^j q^y\log y^j<\infty,\quad i,j\in\Gamma_\alpha,$$

where $p^i(y)$ are those in (1.6)

Lemma 6.1. *It is necessary and sufficient for Condition (CN) to hold that*

(6.1) $$E_{e_i}\{Z^j(t)\,q^{Z(t)}\log Z^j(t)\}<\infty,\quad i,j\in\Gamma_\alpha,\quad t>0.$$

Proof. For a $j\in\langle 1,N\rangle$ with $q^j<1$, both (CN) and (6.1) are automatically satisfied since the function

$$y^jq^y\log y^j=\{y^j(q^j)^{y^j}\log y^j\}\prod_{i\neq j}(q^i)^{y^i}$$

is bounded in $y\in S$. But for a $j\in\langle 1,N\rangle$ with $q^j=1$, it is not difficult to show the necessity by the similar arguments as in the proof of Sevastyanov [13] Theom 2.4.7, and the sufficiency from the arguments as in Athreya [1] (pp. 49-50).

Now as in (2.3)-(2.4), we shall define $\nu_\beta(r)$ by

$$\nu_\beta(r)=\begin{cases}\max\{\nu_\gamma(r)\,;\gamma\gneqq\beta\}, & \text{if}\quad \tilde{\sigma}_\beta\neq r,\\ \max\{\nu_\gamma(r)\,;\gamma\gneqq\beta\}+1, & \text{if}\quad \tilde{\sigma}_\beta=r,\end{cases}$$

inductively ($\max\phi=-1$), and ν_α by $\nu_\alpha=\nu_\alpha(\sigma_\alpha)$. Then setting $R(t;s)=q-F(t;s)$, we have the following

Theorem 6.1. *Let a CGWP $X=(Z(t),P_x)$ satisfy Conditions (C) and (CN) for each $\alpha\in\langle 1,g\rangle$ with $\sigma_\alpha<0$. Then,* 1) *for each $\alpha\in\langle 1,g\rangle$ with $\sigma_\alpha<0$ there correspond monotone nonincreasing func-*

tions $R^{*i}(s_\alpha)$ *in* $0 \leqq s_\alpha \leqq q_\alpha$, $i \in \Delta_\alpha$, *such that as* $t \to \infty$

$$R^i(t; s) = t^{\nu_\alpha} e^{t\sigma_\alpha}(R^{*i}(s_\alpha) + o(1)), \quad i \in \Delta_\alpha, \tag{6.2}$$

where $o(1)$ *is uniform in* s *on* $0 \leqq s_\alpha \leqq q_\alpha$. *Further every* $R^{*i}(s_\alpha)$ *is not identically zero.* 2) *For each* $x \in S$ *such that* $\sigma_\alpha < 0$ *for all* $\alpha \in I_+(x)$, *there corresponds a probability distribution* $\{P_x^*(y)\}$ *on* $S - \{0\}$ *satisfying*

$$\lim_{t \to \infty} P_x\{Z(t) = y | t < T < \infty\} = P_x^*(y). \tag{6.3}$$

Proof. By means of Theorem 2.1 and (6.1), there are monotone nonincreasing functions $R^{*i}(s)$, $i \in \Delta$, which are independent of the choice of $p \in \langle 0, \infty \rangle$, such that

$$R^i(n\delta_p; s) = (n\delta_p)^{\nu} e^{n\delta_p \sigma}\{R^{*i}(s) + o(1)\}, \quad i \in \Delta, \tag{6.4}$$

as $n \to \infty$, where $o(1)$ is uniform in $0 \leqq s \leqq q$. Hence it holds by (2.36) that

$$R^{*i}(F(t; s)) = e^{t\sigma} R^{*i}(s), \tag{6.5}$$

for each $t \geqq 0$ with the form of $n/2^p$ first, and then for all $t \geqq 0$ by means of the continuity of $R^{*i}(s)$ in $0 \leqq s \leqq q$ and of $F(t; s)$ it t. Now (6.4) and (6.5) imply

$$\lim_{n \to \infty} \left(\frac{(R^i(n; F(\tau; s))}{(n+\tau)^{\nu} e^{(n+\tau)\sigma}} - R^{*i}(s) \right) = 0 \tag{6.6}$$

uniformly in $0 \leqq s \leqq q$ and $0 \leqq \tau \leqq 1$. Since each $t \geqq 0$ is represented as $t = n + \tau$, $0 \leqq \tau < 1$, where $n \to \infty$ as $t \to \infty$, we obtain (6.2) from (6.6). The assertion 2) is clear from (6.2) if we repeat the arguments in the proof of Theorem 2.1.

Remark 6.1. The procedure to determine the $R^{*i}(s_\alpha)$, $i \in \Delta_\alpha$, is not complicated. Indeed we have only to repeat the analogous way along Lemmas 2.1 and 2.4 in the case of DGWP. Of course the parallel assertions to those of Remarks 2.1–2.3 are also valid in this case.

To deal with the critical CGWP, we shall assume

$$f^i_{jk}(q) < \infty, \quad i, j, k \in \langle 1, N \rangle, \tag{CC}$$

(CE) $$\sum_{i,j,k\in\Delta_\alpha}\tilde{v}_{\alpha i}f^i_{jk}(q)\xi^j\xi^k\geqq c_\alpha(\sum_{i\in\Delta_\alpha}\tilde{v}_{\alpha i}\xi^i)^2,\quad \tilde{\xi}_\alpha=(\xi^i)_{i\in\Delta_\alpha}\geqq 0,$$

for some $c_\alpha>0$.

Lemma 6.2. *Condition* (*CC*) *implies*

(6.7) $$F^i_{jk}(t;q)<\infty,\quad i,j,k\in\langle 1,N\rangle,\quad t>0.$$

Further, (*CE*) *and* $\tilde{\sigma}_\alpha=0$ *imply*

(6.8) $$\sum_{i,j,k\in\Delta_\alpha}\tilde{v}_{\alpha i}F^i_{jk}(t;q)\xi^j\xi^k\geqq c_\alpha(t)(\sum_{i\in\Delta_\alpha}\tilde{v}_{\alpha i}\xi^i)^2,\quad \tilde{\xi}_\alpha=(\xi^i)_{i\in\Delta_\alpha}>0,$$

for some $c_\alpha(t)>0$.

Proof. The first assertion is well known (eg. Sevastyanov [12] Theorem 4.7.3). To show the second assertion, we shall use the relations

$$F^i_{jk}(t;q)=\sum_{l,m,n\in\Gamma}\int_0^t A_l{}^i(t-\tau)f^l_{mn}(q)A_j{}^m(\tau)A_k{}^n(\tau)\,d\tau$$

$$\geqq\sum_{l\in\Delta}\int_0^t A_l{}^i(t-\tau)f^l_{jk}(q)A_j{}^j(\tau)A_k{}^k(\tau)\,d\tau$$

(ibid. (4.7.16)). Then it follows

$$\sum_{i,j,k\in\Delta}\tilde{v}_iF^i_{jk}(t;q)\xi^j\xi^k\geqq\sum_{i,j,k\in\Delta}\int_0^t\tilde{v}_if^i_{jk}(q)A_j{}^j(\tau)A_k{}^k(\tau)\xi^j\xi^k d\tau,$$

which implies (6.8), since $A_j{}^j(\tau)\to 1$ as $\tau\downarrow 0$.

Setting $\mu_\alpha=1/2^{\nu_\alpha(0)}$, we have the following

Theorem 6.2. *Let a CGWP* $X=(Z(t),P_x)$ *satisfy Conditions* (*C*) *and* (*CC*). *Then for each* $\alpha\in\langle 1,g\rangle$ *with* $\sigma_\alpha=0$, *there correspond constants* $R^{*i}>0$, $i\in\Delta_\alpha$, *such that*

(6.8) $$\lim_{t\to\infty}t^{\mu_\alpha}R^i(t;s)=R^{*i},\quad i\in\Delta_\alpha,\quad 0\leqq s<q.$$

The proof is clear from Theorem 4.1 and (6.7), and will be omitted.

Theorem 6.3. *Let a CGWP* $X=(Z(t),P_x)$ *satisfy Conditions*

(C), (CC) and (CE) for each $\alpha\in\langle 1, g\rangle$ *with* $\tilde{\sigma}_\alpha=0$. *Then,* 1) *for each* $\alpha\in\langle 1, g\rangle$ *with* $\sigma_\alpha=0$, *there correspond nonnegative functions* $\psi^i(\lambda_\alpha)$, $i\in\Delta_\alpha$, *such that*

$$\lim_{t\to\infty} t^{\mu_\alpha}R^i(t; s^{(t,\lambda)}) = \psi^i(\lambda_\alpha), \quad i\in\Delta_\alpha, \quad \lambda_\alpha>0. \tag{6.9}$$

2) *For each* $x\in S$ *with* $\sigma_\alpha=0$ *for some* $\alpha\in I_+(x)$, *the distributions*

$$Q_x(t, u) = P_x\left\{\frac{Z(t)}{t}\leqq u \mid t<T<\infty\right\}, \quad u\in R_+{}^N,$$

converge as $t\to\infty$ *to a probability distribusion* $Q_x{}^*(u)$ *on* $R_+{}^N$.

Proof. By means of Theorem 5.1 and (6.8), there are nonnegative functions $\psi^i(\lambda)$, $i\in\Delta$, which are independent of the choice of $p\in\langle 0, \infty\rangle$, such that

$$\lim_{n\to\infty} (n\delta_p)^{\mu}R^i(n\delta_p; s^{(n\delta_p,\lambda)}) = \psi^i(\lambda), \quad i\in\Delta, \quad \lambda>0. \tag{6.10}$$

Further, (5.35) implies

$$\psi^i(\omega_t(\lambda)) = \psi^i(\lambda), \tag{6.11}$$

for each $t\geqq 0$ with the form of $n/2^p$, where $\omega_t(\lambda) = A(t)(q\lambda)/q$. Since the function $1-\psi^i(\lambda)/R^{*i}$ is a Laplace transform of a probability distribution, it is continuous in $\lambda>0$. Hence the function $\psi^i(\omega_t(\lambda))$ is continuous in t, and so (6.11) holds for all $t\geqq 0$. Now representing each $t\geqq 0$ as $t=n+\tau$, $0\leqq\tau<1$, we have

$$R^i(t; s^{(t,\lambda)}) = R^i(n; F(\tau; s^{(t,\lambda)})). \tag{6.12}$$

But by the same reason as of (5.41) it holds

$$|R^i(n; F(\tau; s^{(t,\lambda)}) - R^i(n; s^{(n,\omega_\tau(\lambda))})| \leqq \frac{K}{n} R^i(n; 0), \quad n\in\langle n_0, \infty\rangle.$$

Hence it follows from (6.8) and (6.10)-(6.12) that

$$\lim_{t\to\infty} t^{\mu}R^i(t; s^{(t,\lambda)}) = \lim_{n\to\infty} n^{\mu}R^i(n; s^{(n,\omega_\tau(\lambda))}) = \psi^i(\omega_\tau(\lambda)) = \psi^i(\lambda).$$

The assertion of 2) is clear from (6.9) and (6.8).

7. Examples

In this section we shall give four examples. The first two are those proposed by Jirina [8] and Sevastyanov [14] as examples which, because of the failure of the positive regularity, do not satisfy their theorems. But these are contained in our scheme, and the direct calculations show that the asymptotic forms coincide with those given by our theorems. Example 3 is of reducible cases, where the asymptotic behaviors are also calculated directly and coincide with those given by our theorems. However, all the marginal distributions of $Q_x{}^*(u)$ in Examples 1–3 are of exponential type. In Example 4 we shall show with aid of our theorems that there really exists a case when a certain marginal distribution of $Q_x{}^*(u)$ is not of exponential type. Naturally the distribution is the same type of that in Savin and Chistyakov [12].

Example 1. Let $\Phi(\xi)=\sum_{j=0}^{\infty}p_j\xi^j$ be a one-dimensional probability generating function with $p_0>0$, $\Phi''(1)<\infty$ if $\Phi'(1)=1$, and consider the two-type DGWP X with the generating functions

$$F^1(s^1, s^2)=\Phi(s^2),\quad F^2(s^1, s^2)=\Phi(s^1). \tag{7.1}$$

Let q_0 be the least nonnegative fixed point of $\Phi(\xi)$ and set $\rho=\Phi'(q_0)$. Then it is well known that $\Phi'(1)\neq 1$ implies $\rho<1$, and $\Phi'(1)=1$ implies $\rho=1$. The extinction probability q of X is equal to (q_0, q_0), and the q-mean matrix A is given by $\begin{bmatrix}0 & \rho\\ \rho & 0\end{bmatrix}$. Hence it follows that $\Delta_1=\Gamma_1=\{1, 2\}$ and $\rho_1=\tilde{\rho}_1=\rho$. We can calculate the n-step generating functions $F(n;s)$ precisely:

$$F^i(n;s)=\begin{cases}\Phi(n;s^i), & i=1,2, \quad \text{if } n \text{ is even},\\ \Phi(n;s^{i+1}), & i=1,2, \quad \text{if } n \text{ is odd},\end{cases} \tag{7.2}$$

where $\Phi(n;\xi)$ is the n-step iteration of $\Phi(\xi)$ and $i+1$ is identified with 1 if $i=2$. Here we shall divide it into three cases.

(i) *When* $\rho=0$, it follows $F(n;s)\equiv 1$, $n\in\langle 1,\infty\rangle$, and all the situations are trivial.

(ii) *When* $0<\rho<1$, the one-dimensional (or positively regular case) arguments assure the existence of a nonincreasing function $K^*(\xi)$ and of a distribution $\{P^*(j)\}$ on $\langle 1,\infty\rangle$ such that

$$\lim_{n\to\infty}\{q_0-\Phi(n;\xi)\}/\rho^n=K^*(\xi),\quad 0\leqq\xi\leqq q_0, \tag{7.3}$$

$$(7.3)\qquad 1-\lim_{n\to\infty}\frac{q_0-\Phi(n;q_0\xi)}{q_0-\Phi(n;0)}=\sum_{j=1}^{\infty}P^*(j)\xi^j,\quad 0\leqq\xi\leqq 1.$$

Combining (7. 2) and (7. 3) we obtain

$$(7.4)\qquad \begin{aligned}&\lim_{n\to\infty}R^i(2n;s)/\rho^{2n}=K^*(s^i),\quad 0\leqq s\leqq q,\quad i=1,2,\\&\lim_{n\to\infty}R^i(2n+1;s)/\rho^{2n}=\rho K^*(s^{i+1}),\quad 0\leqq s\leqq q,\quad i=1,2,\end{aligned}$$

$$(7.5)\qquad \begin{aligned}&\lim_{n\to\infty}P_x\{Z(2n)=y|2n<T<\infty\}=\frac{x^1P^*(y^1)+x^2P^*(y^2)}{x^1+x^2}\\&\lim_{n\to\infty}P_x\{Z(2n+1)=y|2n+1<T<\infty\}=\frac{x^1P^*(y^2)+x^2P^*(y^1)}{x^1+x^2},\\&\qquad x=(x^1,x^2)\neq 0.\end{aligned}$$

(iii) *Let* $\rho=1$. Also in this case the one-dimensional arguments tell us

$$(7.6)\qquad \begin{aligned}&\lim_{n\to\infty}n\{1-\Phi(n;\xi)\}=2/\Phi''(1),\quad 0\leqq\xi<1,\\&\lim_{n\to\infty}n\{1-\Phi(n;\exp(-\eta/n))\}=\frac{\eta}{1+\Phi''(1)\eta/2},\quad \eta\geqq 0.\end{aligned}$$

Hence by means of (7. 2) it follows

$$(7.7)\qquad \lim_{n\to\infty}nR^i(n;s)=2/\Phi''(1),\quad 0\leqq s<1,$$

$$(7.8)\qquad \begin{aligned}&\lim_{n\to\infty}E_x\{\exp(-\lambda\cdot Z(2n)/2n)|2n<T\}\\&\quad=\frac{1}{x^1+x^2}\left\{\frac{x^1}{1+\Phi''(1)\lambda^1/2}+\frac{x^2}{1+\Phi''(1)\lambda^2/2}\right\},\\&\lim_{n\to\infty}E_x\{\exp(-\lambda\cdot Z(2n+1)/(n+1))|2n+1<T\}\\&\quad=\frac{1}{x^1+x^2}\left\{\frac{x^1}{1+\Phi''(1)\lambda^2/2}+\frac{x^2}{1+\Phi''(1)\lambda^1/2}\right\},\end{aligned}$$

for each $x=(x^1,x^2)\neq 0$ and $\lambda=(\lambda^1,\lambda^2)>0$. From (7. 8) it follows

$$(7.9)\qquad Q^*_{x0}(u)=\frac{1}{x^1+x^2}\{x^1(1-e^{-2u^1/\Phi''(1)})+x^2(1-e^{-2u^2/\Phi''(1)})\},$$ [6]

[6] This means, in terms of measures,

$$Q^*_{x0}(E^1\times E^2)=\frac{1}{x^1+x^2}\left\{\frac{2x^1}{\Phi''(1)}\int_{E^1}e^{-2u^1/\Phi''(1)}du^1I_{E^2}(0)+\frac{2x^2}{\Phi''(1)}\int_{E^2}e^{-2u^2/\Phi''(1)}du^2I_{E^1}(0)\right\},$$

where $I_E(\cdot)$ is the indicator function.

(7.9) $$Q^*_{x1}(u)=\frac{1}{x^1+x^2}\{x^1(1-e^{-2u^2/\Phi''(1)})+x^2(1-e^{-2u^1/\Phi''(1)})\},$$

for each $x=(x^1,x^2)\neq 0$ and $u=(u^1,u^2)\in R_+{}^2$.

Example 2. Let $\Phi(\xi)$, q_0 and ρ be those given in Example 1. We consider the two-type DGWP X with the generating functions

(7.10) $$F^1(s^1,s^2)=\Phi(s^2),\quad F^2(s^1,s^2)=s^1.$$

The extinction probability is equal to (q_0,q_0) and the q-mean matrix is $A=\begin{bmatrix}0&\rho\\1&0\end{bmatrix}$. Hence $\Delta_1=\Gamma_1=\{1,2\}$ and $\rho_1=\tilde{\rho}_1=\sqrt{\rho}$. The n-step generating functions $F(n;s)$ is given by

(7.11) $$F^i(n;s)=\begin{cases}\Phi(n/2;s^i), & i=1,2,\quad \text{if } n \text{ is even},\\ \Phi(\{n-(-1)^i\}/2;s^{i+1}), & i=1,2,\quad \text{if } n \text{ is odd}.\end{cases}$$

(i) *When* $\rho=0$, $F(n;s)\equiv 1$ for all $n\in\langle 2,\infty\rangle$.

(ii) *When* $0<\rho<1$, it holds

(7.12) $$\lim_{n\to\infty}R^i(2n;s)/\rho^n=K^*(s^i),\quad 0\leqq s\leqq q,\quad i=1,2,$$
$$\lim_{n\to\infty}R^i(2n+1;s)/\rho^n=\rho^{\{1-(-1)^i\}/2}K^*(s^{i+1}),\quad 0\leqq s\leqq q,\quad i=1,2,$$

where $K^*(\xi)$ is that of (7.3). Here we assume

(7.13) $$\sum_{j=0}^{\infty}p_j j\log j<\infty,\quad \text{if}\quad \Phi'(1)<1.$$

Then $K^*(\xi)\not\equiv 0$ and we have

(7.14) $$\lim_{n\to\infty}P_x\{Z(2n)=y|2n<T<\infty\}=\frac{x^1P^*(y^1)+x^2P^*(y^2)}{x^1+x^2},$$
$$\lim_{n\to\infty}P_x\{Z(2n+1)=y|2n+1<T<\infty\}=\frac{x^1P^*(y^2)+x^2P^*(y^1)}{x^1+x^2},$$
$$x=(x^1,x^2)\neq 0.$$

(iii) *When* $\rho=1$, we also have (7.7)-(7.9) but with $\Phi''(1)$ replaced by $\Phi''(1)/2$.

Example 3. Let $\phi(\xi)$ be a one-dimensional infinitesimal generating function with $\phi''(1)<\infty$ and $\phi(0)>0$. We consider the two-type CGWP with the infinitesimal generating functions

$$(7.15)\qquad f^1(s^1,s^2)=\phi(s^1),\quad f^2(s^1,s^2)=b(s^1-1)+c(1-s^2),$$

where b and c are constants with $0<b\leqq c$. Let q_1 be the least non-negative zero point of $\phi(\xi)$ and put $\sigma=\phi'(q_1)$. Then $\phi'(1)\neq 0$ implies $\sigma<0$, and $\phi'(1)=0$ implies $\sigma=0$. The extinction probability is given by $q=(q^1,q^2)$ where $q^1=q_1$ and $q^2=1-b(1-q_1)/c$, and the infinitesimal q-mean matrix is $a=\begin{bmatrix}\sigma & 0\\ b & -c\end{bmatrix}$. Hence it follows $\Delta_1=\{1\}$, $\Delta_2=\{2\}$, $\Gamma_1=\{1\}$ and $\Gamma_2=\{1,2\}$. Now we can define the one-type CGWP $\{\Phi(t;\xi)\}$ with the infinitesimal generating function $\phi(\xi)$:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t}(t;\xi)=\phi(\Phi(t;\xi)),\quad \Phi(0;\xi)=\xi,\quad 0\leqq\xi\leqq 1.$$

Then our CGWP $\{F(t;s)\}$ is given by

$$(7.16)\qquad \begin{aligned} F^1(t;s)&=\Phi(t;s^1),\\ F^2(t;s)&=e^{-ct}\int_0^t e^{c\tau}(b\Phi(\tau;s^1)+c-b)\,\mathrm{d}\tau+s^2\\ &=q^2+e^{-ct}\left\{b\int_0^t e^{c\tau}(\Phi(\tau;s^1)-q^1)\,\mathrm{d}\tau+s^2-q^2\right\}.\end{aligned}$$

The CGWP $X_1\equiv\{F^1(t;s)\}$ is divided into two cases.

(i) *When* $\sigma<0$, the one-dimensional arguments assure the existence of a monotone nonincreasing function $K^*(\xi)$ and a distribution $\{P^*(j)\}$ on $\langle 1,\infty\rangle$ satisfying

$$(7.17)\qquad \begin{aligned} &\lim_{t\to\infty}\{q_1-\Phi(t;\xi)\}/e^{\sigma t}=K^*(\xi),\quad 0\leqq\xi\leqq q,\\ &1-\lim_{t\to\infty}\frac{q_1-\Phi(t;q_1\xi)}{q_1-\Phi(t;0)}=\sum_{j=1}^{\infty}P^*(j)\xi^j,\quad 0\leqq\xi\leqq 1.\end{aligned}$$

Hence it follows

$$(7.18)\qquad \begin{aligned} &\lim_{t\to\infty}R^1(t;s)/e^{\sigma t}=K^*(s^1),\quad 0\leqq s^1\leqq q^1.\\ &\lim_{t\to\infty}P_{(x^1,0)}\{Z(t)=(y^1,y^2)\,|\,t<T<\infty\}\\ &\qquad=\begin{cases}P^*(y^1), & y^2=0,\\ 0, & \text{otherwise},\quad \text{for each } x^1\in\langle 1,\infty\rangle.\end{cases}\end{aligned}$$

(ii) *In case of* $\sigma=0$, the one-dimensional arguments also tell us

(7. 19)
$$\lim_{t\to\infty} t\{1-\Phi(t;\xi)\}=2/\phi''(1),\quad 0\leqq\xi\leqq 1,$$
$$\lim_{t\to\infty} t\{1-\phi(t;\exp(-\eta/t))\}=\frac{\eta}{1+\phi''(1)\eta/2},\quad \eta\geqq 0.$$

Hence it follows that

(7. 20)
$$\lim_{t\to\infty} tR^1(t;s)=2/\phi''(1),\quad 0\leqq s^1\leqq 1,$$
$$\lim_{t\to\infty} P_{(x^1,0)}\left\{\frac{Z(t)}{t}\leqq(u^1,u^2)\,|\,t<T\right\}=1-e^{-2u^1/\phi''(1)},$$

for each $x^1\in\langle 1,\infty\rangle$ and $u\in R_+^2$.

The CGWP $X_2=X=\{F(t;s)\}$ is divided into four cases.

(i) *When* $-c<\sigma<0$, the *P-F* root $\sigma_2=\rho(a)$ is equal to σ. It follows from (7. 16) and (7. 17) that

(7. 21)
$$\lim_{t\to\infty} R^2(t;s)/e^{\sigma t}=\frac{b}{c+\sigma}K^*(s^1),\quad 0\leqq s\leqq q,$$
$$\lim_{t\to\infty} P_x\{Z(t)=(y^1,y^2)\,|\,t<T<\infty\}$$
$$=\begin{cases}P^*(y^1), & y^2=0,\\ 0, & \text{otherwise},\quad \text{for each } x\neq 0.\end{cases}$$

(ii) *When* $\sigma<-c<0$, it holds $\sigma_2=-c$, and

(7. 22)
$$\lim_{t\to\infty} R^2(t;s)/e^{-ct}=b\int_0^\infty e^{c\tau}(q_1-\Phi(\tau;s^1))\,d\tau+q^2-s^2,\quad 0\leqq s\leqq q,$$
$$\lim_{t\to\infty} P_{(x^1,x^2)}\{Z(t)=y\,|\,t<T<\infty\}=P^*(y),\quad x^2\neq 0,$$

where the distribution $\{P^*(y)\}$ is given by

$$\sum_{y\neq 0} P^*(y)s^y=1-\frac{b\int_0^\infty e^{c\tau}(q_1-\Phi(\tau;q^1s^1))d\tau+q^2(1-s^2)}{b\int_0^\infty e^{c\tau}(q_1-\Phi(\tau;0))\,d\tau+q^2,},\quad 0\leqq s\leqq 1.$$

(iii) *In case of* $\sigma=-c<0$, it holds $\sigma_2=\sigma=-c$, and

(7. 23)
$$\lim_{t\to\infty} R^2(t;s)/te^{\sigma t}=bK^*(s^1),\quad 0\leqq s\leqq q,$$
$$\lim_{t\to\infty} P_x\{Z(t)=(y^1,y^2)\,|\,t<T<\infty\}$$

$$= \begin{cases} P^*(y^1), & y^2=0, \\ 0, & \text{otherwise, for each } x \neq 0. \end{cases}$$

(iv) *When* $-c<\sigma=0$, it follows $\sigma_2=0$ and the CGWP X is critical with $q=1$. By means of (7.16) and (7.19) it holds

$$(7.24)\quad \begin{aligned} &\lim_{t\to\infty} tR^2(t;s) = \frac{2b}{c\phi''(1)}, \quad 0\leqq s<1, \\ &\lim tR^2(t;(e^{-\lambda^1/t}, e^{-\lambda^2/t})) = \frac{b}{c}\,\frac{\lambda^2}{1+\phi''(1)\lambda^1/2}, \quad (\lambda^1,\lambda^2)>0. \end{aligned}$$

Hence with the aid of (7.19) and (7.20) it follows

$$(7.25)\qquad Q_x^*(u^1,u^2) = 1-e^{-2u^1/\phi''(1)}, \quad x\neq 0, \quad u\in R_+^2.$$

Example 4. Let $\Phi(\xi)$ be a one-dimensional probability generating function with $\Phi'(1)=1$ and $0<\Phi''(1)\equiv 2B_1<\infty$. We consider two-type DGWP X given by the generating functions $F^1(s^1,s^2)=\Phi(s^1)$ and $F^2(s^1,s^2)$ with $F_1^2(1)\equiv A'>0$, $F_2^2(1)=1$ and $0<F_{22}^2(1)\equiv 2B_2<\infty$. Then the extinction probability is equal to $1=(1,1)$ and the q-mean matrix is $A=\begin{bmatrix}1 & 0\\ A' & 1\end{bmatrix}$. Hence $\Delta_1=\{1\}$, $\Delta_2=\{2\}$, $\Gamma_1=\{1\}$, $\Gamma_2=\{1,2\}$ acd $\tilde{\rho}_1=\tilde{\rho}_2=\rho_1=\rho_2=1$. From (7.6), we have

$$(7.26)\quad \begin{aligned} &\lim_{t\to\infty} nR^1(n;s) = 1/B_1, \quad 0\leqq s^1<1, \\ &\lim_{n\to\infty} nR^1(n;s^{(n,\lambda)}) = \frac{\lambda^1}{1+B_1\lambda^1}, \quad \lambda^1>0. \end{aligned}$$

Now by Lemmas 4.7 and (5.3)

$$(7.27)\quad \begin{aligned} &\lim_{n\to\infty} n^{1/2}R^2(n;s) = \sqrt{A'/B_1B_2}, \quad 0\leqq s<1, \\ &\lim_{n\to\infty} n^{1/2}R^2(n;s^{(n,\lambda)}) = \sqrt{\frac{A'\lambda^1}{B_2(1+B_1\lambda^1)}}, \quad \lambda>0. \end{aligned}$$

Hence by Theorem 5.2 2), it follows

$$(7.28)\qquad \lim_{n\to\infty} E_{(x^1,x^2)}\{\exp(-\lambda\cdot Z(n)/n)\,|\,n<T\}$$

$$= \begin{cases} \dfrac{1}{1+B_1\lambda^1}, & x_2=0, \\ 1-\left(1-\dfrac{1}{1+B_1\lambda^1}\right)^{1/2}, & x_2\neq 0 \end{cases}$$

that is

$$(7.29)\qquad Q^{*}_{(x^1,x^2)}(u)=\begin{cases}1-e^{-u_1/B_1}, & x_2=0,\\ \dfrac{1}{2B}\displaystyle\int_0^{u^1} e^{-\xi/B_1}{}_1F_1\Big(-\frac{1}{2};\,-2;\,\frac{\xi}{B}\Big)d\xi, & x_2\neq 0,\end{cases}$$

where ${}_1F_1$ is the Barnes' generalized hypergeometric function:

$$\begin{aligned}{}_1F_1(-1/2;\,-2;\,\xi)&=\sum_{k=0}^{\infty}\frac{(-1/2)_k}{(-2)_k}\frac{\xi^k}{k!}\\ &=\sum_{k=0}^{\infty}\frac{(k-1/2)(k-1-1/2)\cdots(1-1/2)}{(k+1)!\,k!}\xi^k.\end{aligned}$$

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