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# On amenable transformation semigroups II

By

Kôkichi SAKAI

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## § 0. Introduction

This paper is the continuation of my preceding paper [20] which is referred as Part I. So we shall employ the same notations and definitions as in Part I. Let $S$ be a semigroup, $\boldsymbol{X}=(S, X)$ a transformation semigroup (denoted by $\tau$-semigroup briefly) and $\mathfrak{A}$ any $\boldsymbol{X}$-linear space. In Part I we have studied several necessary and sufficient conditions for $\mathfrak{A}$ to be amenable, extremely amenable or quasi-extremely amenable, which are stated in terms of some intrinsic properties of $\mathfrak{A}$, $\mathfrak{A}^*$ or $\boldsymbol{X}$. While the purpose in this paper is to show that the amenability or extreme amenability of $\mathfrak{A}$ can be characterized by means of certain properties concerning actions of $\boldsymbol{X}$ on various objects, and to apply these characterizations to the case of left amenability of any $\boldsymbol{S}_l$-linear spaces. The above characterizations are certain generalizations of the results on left amenability of $B(S)$ or of some $\boldsymbol{S}_l$-linear spaces investigated by a number of authors, e.g., T. Mitchell [14], B. E. Johnson [9], E. E. Granirer [8] and so on.

All the discussions in this paper do not depend on the topological structure of underlying $\tau$-semigroup and do not require any additional conditions on $\mathfrak{A}$ except that $\mathfrak{A}$ is an $\boldsymbol{X}$-linear space or $\boldsymbol{X}$-algebra. The amenability associated with topological $\tau$-semigroups will be studied in the author's paper [21], in which we shall apply the present results to the cases of function spaces of special kinds.

In §1 we shall define various actions of $\boldsymbol{X}$, say compact or compact convex actions of $\boldsymbol{X}$, $\boldsymbol{X}$-modules and representations of $\boldsymbol{X}$, and introduce the concept of $\mathfrak{A}$-admissibility for them.

In §2 it is shown that for any amenable $\boldsymbol{X}$-linear space and extremely amenable $\boldsymbol{X}$-algebra there correspond the so-called "fixed point property" on compact convex actions of $\boldsymbol{X}$ and on compact actions of $\boldsymbol{X}$ respectively. The fixed point properties corresponding to the left amenability of groups or semigroups are studied by many authors, e.g., S. Kakutani [10], M. M. Day [3], N. W. Rickert [16], T. Mitchell [12–14], A. T. Lau [11], L. N. Argabright [1] and so on. All these results are derived systematically from our Theorems 2.5–2.7.

B. E. Johnson proved in his article "Cohomology in Banach algebras" that a locally compact group $G$ is amenable if and only if for any Banach $L^1(G)$-module $\mathfrak{X}$ the cohomology group $H^1(L^1(G), \mathfrak{X}^*)$ is trivial and also introduced the concept of amenability for Banach algebras and of strong amenability for $C^*$-algebras. In §3 we shall extend the Johnson's result to the cases of not only any amenable $\boldsymbol{X}$-linear spaces but also extremely amenable $\boldsymbol{X}$-algebras.

In §4 we shall show that the amenability of any $\boldsymbol{X}$-linear space $\mathfrak{A}$ can be characterized by means of certain properties, say "invariant extension property", "Reiter-Glicksberg inequality" or "existence of invariant map", of $\mathfrak{A}$-admissible representations of $\boldsymbol{X}$. These properties are considered by many authors in the cases of amenable or extremely amenable semigroups, say M. M. Day [2], E. E. Granirer [8] and so on.

The discussion in §5 is the analogy of §4 for the case of extreme amenability of any $\boldsymbol{X}$-algebra.

**Notation.** Throughout this paper, unless otherwise noted, $X$ is a nonvoid space, $S$ is a semigroup, $\boldsymbol{X}=(S, X)$ or $\boldsymbol{X}$ is a $\tau$-semigroup, $\boldsymbol{S}_l=(S, S)$ or $\boldsymbol{S}_l$ is the $\tau$-semigroup defined by $S$, where $S$ acts on itself as left translations, and $\mathfrak{A}$ is an $\boldsymbol{X}$-linear space or $\boldsymbol{X}$-algebra. Here by an $\boldsymbol{X}$-linear space [resp. $\boldsymbol{X}$-algebra] $\mathfrak{A}$ we mean a closed linear subspace [resp. subalgebra] of $B(X)$ (=the commutative $C^*$-algebra of all bounded complex valued functions on $X$) with the pro-

perties as follows:

(i) $\mathfrak{A}$ contains the characteristic function $I_X$ of $X$,

(ii) $f \in \mathfrak{A}$ implies $\bar{f} \in \mathfrak{A}$ where $\bar{f}(x)$ is the complex conjugate of $f(x)$,

(iii) $f \in \mathfrak{A}$ implies ${}_s f \in \mathfrak{A}$ for all $s \in S$ where ${}_s f(x) = f(sx)$.

The following notations are used freely:

$\delta(x)$ = the element in $\mathfrak{A}^*$ defined by $\delta(x)f = f(x)$ $(f \in \mathfrak{A})$ for $x \in X$,

$\delta$ = the map $X \ni x \mapsto \delta(x) \in \mathfrak{A}^*$,

$M_p(\mathfrak{A}) = \{\delta(x);\ x \in X\}$, $M_f(\mathfrak{A})$ = the convex hull of $M_p(\mathfrak{A})$,

$M(\mathfrak{A})$ [resp. $M(\mathfrak{A})$] = the $\sigma(\mathfrak{A}^*, \mathfrak{A})$-closure of $M_f(\mathfrak{A})$ [resp. $M_p(\mathfrak{A})$],

$IM(\mathfrak{A}) = \{\varphi \in M(\mathfrak{A});\ \varphi({}_s f) = \varphi(f)$ for all $s \in S$ and $f \in \mathfrak{A}\}$,

$\mathfrak{IM}(\mathfrak{A}) = IM(\mathfrak{A}) \cap \mathfrak{M}(\mathfrak{A})$.

Then $M(\mathfrak{A})$ is the space of all means on $\mathfrak{A}$ and $\mathfrak{M}(\mathfrak{A})$ is the space of all multiplicative means on $\mathfrak{A}$ if $\mathfrak{A}$ is an $\boldsymbol{X}$-algebra. We recall now that an $\boldsymbol{X}$-linear space $\mathfrak{A}$ is said to be *amenable* if $IM(\mathfrak{A})$ is nonempty, and that an $\boldsymbol{X}$-algebra $\mathfrak{A}$ is said to be *extremely* [resp. *quasi-extremely*] *amenable* if $\mathfrak{IM}(\mathfrak{A})$ [resp. $IM(\mathfrak{A}) \cap$ (the convex hull of $\mathfrak{M}(\mathfrak{A})$)] is nonempty.

For any locally convex topological linear (l.c.t.l.) spaces $E$ and $F$ over the same scalar field $\boldsymbol{R}$ (the real field) or $\boldsymbol{C}$ (the complex field) we write:

$l(E, F)$ = the space of all continuous linear maps of $E$ to $F$,

$E^*$ = the topological dual space of $E$ with the $\sigma(E^*, E)$-topology,

$T^*$ = the adjoint map of $T \in l(E, F)$,

$\langle f, \varphi \rangle$ = the value of $\varphi \in E^*$ at $f \in E$.

For any commutative Banach [resp. $C^*$]-algebras $A$ and $B$ with unit over $\boldsymbol{C}$ we write:

$\mathrm{Hom}(A, B)$ = the space of all continuous homomorphisms from $A$ to $B$ as Banach [resp. $C^*$-] algebras,

$\mathfrak{M}(A)$ = the space of all maximal ideals of $A$ with the Gelfand topology.

By $C(Z)$ we denote the commutative $C^*$-algebra of all bounded continuous complex valued functions on a topological space $Z$. We assume always all the topological spaces treated in this paper to be Hausdorff.

## § 1. Representations of $\tau$-semigroups

In this section we shall define various actions of $\boldsymbol{X}$ and introduce the concept of $\mathfrak{A}$-admissibility for them. At first we define $\tau(c)$-semigroups, compact $\tau(c)$-semigroups, compact convex $\tau(c)$-semigroups and $S$-modules. A $\tau$-semigroup $(S, Y)$ is called a *$\tau(c)$-semigroup* if $Y$ is a topological space and each $s \in S$ acts on $Y$ as a continuous map. A $\tau$-semigroup $(S, Z)$ is a *compact $\tau(c)$-semigroup* or *compact action of $S$* if it is a $\tau(c)$-semigroup where $Z$ is a compact space. A $\tau$-semigroup $(S, Q_E)$ is a *compact convex $\tau(c)$-semigroup* or *compact convex action of $S$* if $Q_E$ is a compact convex subset of a l.c.t.l. space $E$ and each $s \in S$ acts on $Q_E$ as a continuous affine map. Further a $\tau$-semigroup $(S, E)$ is called an *$S$-module* if $E$ is a l.c.t.l. space and each $s \in S$ acts on $E$ as a continuous linear map.

**Definition 1.1.** *Let $E$ be a l.c.t.l. space and $\mathfrak{A}$ be an $\boldsymbol{X}$-linear space.*

(1) *A map $\rho$ of $X$ into $E$ is said to be $\mathfrak{A}$-admissible if the function $x \mapsto <\rho(x), z>$ on $X$ belongs to $\mathfrak{A}$ for any $z \in E^*$.*

(2) *A system $\{S, E, \rho\}$* [resp. $\{S, Q_E, \rho\}$] *is called an $\boldsymbol{X}$-module* [*resp. compact convex action of $\boldsymbol{X}$*] *if $(S, E)$* [*resp. $(S, Q_E)$*] *is an $S$-module* [*resp. compact convex action of $S$*] *and $\rho$ is a homomorphism from $\boldsymbol{X}$ to $(S, E)$* [*resp. $(S, Q_E)$*], *that is, $\rho$ is a map from $X$ to $E$* [*resp. $Q_E$*] *satisfying $\rho(sx) = s\rho(x)$ for all $(s, x) \in S \times X$. Moreover they are said to be $\mathfrak{A}$-admissible if so is $\rho$ in the sense of* (1).

(3) *A map $\rho$ of $X$ to a compact space $Z$ is said to be $\mathfrak{A}$-c-admissible if the function $x \mapsto f(\rho(x))$ on $X$ belongs to $\mathfrak{A}$ for any $f \in C(Z)$.*

(4) *A system $\{S, Z, \rho\}$ is called a compact action of $\boldsymbol{X}$ if $(S, Z)$ is a compact action of $S$ and $\rho$ is a homomorphism from $\boldsymbol{X}$ to $(S, Z)$. Further it is said to be $\mathfrak{A}$-c-admissible if so is $\rho$ in the sense of* (3).

Now let us define the action of $S$ on $\mathfrak{A}^*$ by $s\varphi(f) = \varphi({}_sf)$ $(f \in \mathfrak{A})$ for any $(s, \varphi) \in S \times \mathfrak{A}^*$. Then we get an $S$-module $(S, \mathfrak{A}^*)$, a compact

convex action $(S, M(\mathfrak{A}))$ and a compact action $(S, \mathfrak{M}(\mathfrak{A}))$ of $S$. Since the map $\delta$ of $X$ to $\mathfrak{A}^*$ satisfies $s\delta(x)=\delta(sx)$ for any $(s, x)\in S\times X$, we have

**Lemma 1.2.** (1) *For any* $\boldsymbol{X}$*-linear space* $\mathfrak{A}$, $\{S, \mathfrak{A}^*, \delta\}$ [*resp.* $\{S, M(\mathfrak{A}), \delta\}$] *is an* $\mathfrak{A}$*-admissible* $\boldsymbol{X}$*-module* [*resp. compact convex action of* $\boldsymbol{X}$].

(2) *For any* $\boldsymbol{X}$*-algebra* $\mathfrak{A}$, $\{S, \mathfrak{M}(\mathfrak{A}), \delta\}$ *is an* $\mathfrak{A}$*-c-admissible compact action of* $\boldsymbol{X}$.

*Proof.* To see (1) it suffices to show the $\mathfrak{A}$-admissibility of $\delta$. Let $F$ be any $\sigma(\mathfrak{A}^*, \mathfrak{A})$-continuous linear functional on $\mathfrak{A}^*$. Then there exists an $f\in\mathfrak{A}$ such that $<\varphi, F>=<f, \varphi>$ for any $\varphi\in\mathfrak{A}^*$. So the function $x\mapsto<\delta(x), F>=f(x)$ on $X$ is in $\mathfrak{A}$. This show that $\delta$ is $\mathfrak{A}$-admissible. To see (2) let $\mathfrak{A}$ be an $\boldsymbol{X}$-algebra and $F\in C(\mathfrak{M}(\mathfrak{A}))$. Then since $C(\mathfrak{M}(\mathfrak{A}))$ is isomorphic to $\mathfrak{A}$ as a $C^*$-algebra by the Gelfand transform, there exists an $f\in\mathfrak{A}$ such that $F(\varphi)=<f, \varphi>$ for any $\varphi$ $\mathfrak{M}(\mathfrak{A})$. Hence the function $x\mapsto F(\delta(x))=f(x)$ on $X$ is in $\mathfrak{A}$. This means that the map $\delta$ of $X$ to $\mathfrak{M}(\mathfrak{A})$ is $\mathfrak{A}$-$c$-admissible if $\mathfrak{A}$ is an $\boldsymbol{X}$-algebra.

q.e.d.

**Remark 1.3.** Let $(S, Y)$ be a $\tau$-semigroup. For any fixed $y\in Y$ define the map $\hat{y}$ of $S$ into $Y$ by $S\ni s\mapsto\hat{y}(s)=sy\in Y$, then $\hat{y}$ gives a homomorphism from $\boldsymbol{S}_l=(S, S)$ to $(S, Y)$. For an $\boldsymbol{S}_l$-linear space $\mathfrak{A}$, an $S$-module $(S, E)$ is said to be *A*-$\mathfrak{A}$ [*resp. E*-$\mathfrak{A}$]*-admissible* if the $\boldsymbol{S}_l$-module $\{S, E, \hat{v}\}$ is $\mathfrak{A}$-admissible (see Definition 1.1) for all [resp. some] $v\in E$. The *A*-$\mathfrak{A}$ [resp. *E*-$\mathfrak{A}$]-admissibility of any compact convex action of $S$ is defined similarly. For an $\boldsymbol{S}_l$-algebra $\mathfrak{A}$, a compact action $(S, Z)$ of $S$ is said to be *A*-$\mathfrak{A}$ [*resp. E*-$\mathfrak{A}$]*-c-admissible* if the compact action $\{S, Z, \hat{z}\}$ of $S_l$ is $\mathfrak{A}$-$c$-admissible for all [resp. some] $z\in Z$. We say now that an $\boldsymbol{S}_l$-linear space $\mathfrak{A}$ is *left* [*resp. E-left*]*-introverted* if the function $s\mapsto<_s f, \varphi>$ on $S$ belongs to $\mathfrak{A}$ for all $f\in\mathfrak{A}$ and all [resp. some] $\varphi\in\mathfrak{A}^*$, and an $\boldsymbol{S}_l$-algebra $\mathfrak{A}$ is *left* [*resp. E-left*]*-m-introverted* if the function $s\mapsto<_s f, \varphi>$ on $S$ belongs to $\mathfrak{A}$ for all $f\in\mathfrak{A}$ and all [resp. some] $\varphi\in\mathfrak{M}(\mathfrak{A})$. Under these

definitions we have

**Lemma 1.4.** (1) *For any $\boldsymbol{S}_l$-linear space $\mathfrak{A}$ the $S$-module $(S, \mathfrak{A}^*)$ or compact convex action $(S, M(\mathfrak{A}))$ of $S$ are $A$-$\mathfrak{A}$ [resp. $E$-$\mathfrak{A}$]-admissible if and only if $\mathfrak{A}$ is left [resp. $E$-left]-introverted.*

(2) *For any $\boldsymbol{S}_l$-algebra $\mathfrak{A}$ the compact action $(S, \mathfrak{M}(\mathfrak{A}))$ of $S$ is $A$-$\mathfrak{A}$ [resp. $E$-$\mathfrak{A}$]-c-admissible if and only if $\mathfrak{A}$ is left [resp. $E$-left]-m-admissible.*

Let $E$ and $F$ be l.c.t.l. spaces over the same field $\boldsymbol{R}$ or $\boldsymbol{C}$ and $\mathfrak{A}$ an $\boldsymbol{X}$-linear space. Then we define a representation of $\boldsymbol{X}=(S, X)$ as follows.

**Definition 1.5.** (1) *A map $\rho$ of $X$ to $l(E, F)$ is said to be $\mathfrak{A}$-admissible if the function $x \mapsto <\rho(x)v, z>$ on $X$ belongs to $\mathfrak{A}$ for any $v \in E$ and $z \in F^*$.*

(2) *A quadruplet $\boldsymbol{D}=\{T, E, \rho, F\}$ is called a representation of $\boldsymbol{X}$ if:*

(i) *$\{T, E\}$ is an antirepresentation of $S$ on $E$, that is, $T$ is a map of $S$ to $l(E, E)$ such that $T_s T_t = T_{ts}$ for any $s, t \in S$,*

(ii) *$\rho$ is a map of $X$ to $l(E, F)$ such that $\rho(sx)=\rho(x)T_s$ for any $s \in S$* and *$x \in X$.*

*Moreover $\boldsymbol{D}$ is said to be $\mathfrak{A}$-admissible if so is $\rho$ in the sense of* (1).

**Definition 1.6.** (1) *A quadruplet $\{T, A, \rho, B\}$ is called a $B$ [resp. $C^*$]-representation of $\boldsymbol{X}$ if:*

(i) *$A$ and $B$ are commutative Banach [resp. $C^*$-] algebras over $\boldsymbol{C}$ with unit,*

(ii) *$\{T, A\}$ is a $B$ [resp. $C^*$]-antirepresentation of $S$ on $A$, that is, $T$ is an antihomomorphism of $S$ to $Hom(A, A)$,*

(iii) *$\rho$ is a map of $X$ to $Hom(A, B)$ such that $\rho(sx)=\rho(x)T_s$ for any $s \in S$ and $x \in X$.*

(2) *For an $\boldsymbol{X}$-algebra $\mathfrak{A}$, a $B$-representation $\{T, A, \rho, B\}$ of $\boldsymbol{X}$ is said to be $\mathfrak{A}$-m-admissible if the function $x \mapsto <\rho(x)v, \varphi>$ on*

*$X$ belongs to $\mathfrak{A}$ for any $v \in A$ and $\varphi \in \mathfrak{M}(B)$.*

Now we define the antirepresentation $\{L, \mathfrak{A}\}$ of $S$ on an $\boldsymbol{X}$-linear space $\mathfrak{A}$ by $L_s f = {}_s f$ $(f \in \mathfrak{A})$ for any $s \in S$. Then the quadruplet $\{L, \mathfrak{A}, \delta, \boldsymbol{C}\}$ is a representation of $\boldsymbol{X}$, which is denoted by $\boldsymbol{L}(\mathfrak{A})$ for brevity. The next lemma is obvious.

**Lemma 1.7.** (1) *For an $\boldsymbol{X}$-linear space $\mathfrak{A}$, $\boldsymbol{L}(\mathfrak{A})$ is an $\mathfrak{A}$-admissible representation of $\boldsymbol{X}$.*

(2) *For an $\boldsymbol{X}$-algebra $\mathfrak{A}$, $\boldsymbol{L}(\mathfrak{A})$ is an $\mathfrak{A}$-m-admissible $C^*$-representation of $\boldsymbol{X}$.*

**Remark 1.8** Let $\{T, E\}$ be an antirepresentation of $S$ on a l.c.t.l. space $E$. Then $\{T, E, T, E\}$ is regarded as a representation of $\boldsymbol{S}_l$. So we say that for an $\boldsymbol{S}_l$-linear space $\mathfrak{A}$ $\{T, E\}$ is *$\mathfrak{A}$-admissible* if $\{T, E, T, E\}$ is $\mathfrak{A}$-admissible as a representation of $\boldsymbol{S}_l$. Similarly the $\mathfrak{A}$-$m$-admissibility of any $B$-antirepresentation of $S$ is defined for an $\boldsymbol{S}_l$-algebra $\mathfrak{A}$. Then we get

**Lemma 1.9.** (1) *For an $\boldsymbol{S}_l$-linear space $\mathfrak{A}$ the antirepresentation $\{L, \mathfrak{A}\}$ of $S$ on $\mathfrak{A}$ is $\mathfrak{A}$-admissible if and only if $\mathfrak{A}$ is left-introverted.*

(2) *For an $\boldsymbol{S}_l$-algebra $\mathfrak{A}$ the $C^*$-antirepresentation $\{L, \mathfrak{A}\}$ of $S$ on $\mathfrak{A}$ is $\mathfrak{A}$-m-admissible if and only if $\mathfrak{A}$ is left-m-introverted.*

## § 2. Fixed point property corresponding to amenability

We say now that a $\tau$-semigroup $(S, Y)$ has a *fixed point* [resp. *finite stable set*] if there exists a point $y_0$ [resp. nonempty finite subset $Y_0$] in $Y$ such that $sy_0 = y_0$ [resp. $sY_0 = Y_0$] for all $s \in S$. By M. M. Day it is proved in [3] that a semigroup $S$ is left amenable if and only if any compact convex action of $S$ has a fixed point. Further it is proved by T. Mitchell [12] [resp. A. T. Lau [11]] that $S$ is extremely [resp. quasi-extremely] left amenable if and only if any compact action of $S$ has a fixed point [resp. finite stable set]. We can prove the Day's result by the simple two facts as follows: (i) *$S$ is left amenable if and only if the compact convex action $(S, M(B(S)))$ of $S$ has a*

*fixed point and* (ii) *for any compact convex action* $(S, Q)$ *of* $S$ *there exists a homomorphism from* $(S, M(B(S)))$ *to* $(S, Q)$. The Mitchell's and Lau's results can be also by the same method as the Day's one.

We show that for any amenable $\boldsymbol{X}$-linear space and extremely amenable $\boldsymbol{X}$-algebra there correspond the fixed point property on compact convex actions of $\boldsymbol{X}$ and on compact actions of $\boldsymbol{X}$ respectively. To this aim we prepare some lemmas which are the most essential in our discussions not only in this section but also in later each section. Let $Q$ be a compact convex subset of a l.c.t.l. space $E$, $\mathfrak{A}$ an $\boldsymbol{X}$-linear space and $\rho$ an $\mathfrak{A}$-admissible map of $X$ into $Q$. By $O(\rho)$ and $CO(\rho)$ we denote the closure and the convex closure of $\{\rho(x); x \in X\}$ respectively. Under these notations we prove

**Lemma 2.1.** (1) *There corresponds uniquely a map* $\hat{\rho}: M(\mathfrak{A}) \ni \varphi \mapsto \hat{\rho}(\varphi) \in CO(\rho)$ *such that*

$$(\#) \qquad <\hat{\rho}(\varphi), z> = \varphi(<\rho(x), z>) \qquad \textit{for any} \quad z \in E^*.$$

*By this* $\hat{\rho}$, $\mathfrak{M}(\mathfrak{A})$ *is mapped into* $O(\rho)$.

(2) *Let* $T$ *be a continuous affine map of* $Q$ *into itself such that the map* $T\rho: X \ni x \mapsto T\rho(x) = T(\rho(x)) \in Q$ *is also* $\mathfrak{A}$*-admissible. Then the map* $\hat{\rho}$ *and* $(T\rho)^\wedge$ *corresponding to* $\rho$ *and* $T\rho$ *as in* (1) *respectively satisfy*

$$(\#\#) \qquad (T\rho)^\wedge(\varphi) = T(\hat{\rho}(\varphi)) \qquad \textit{for any} \quad \varphi \in M(\mathfrak{A}).$$

*Proof.* (1) The uniqueness of such $\hat{\rho}$ follows from the fact that the $\sigma(E, E^*)$-topology in $E$ is separated. For any $\varphi = \sum_{i=1}^n \lambda_i \delta(x_i) \in M_f(\mathfrak{A})$, we put $\hat{\rho}(\varphi) = \sum_{i=1}^n \lambda_i \rho(x_i) \in CO(\rho)$. Then $\hat{\rho}(\varphi)$ satisfies (#) for any $\varphi \in M_f(\mathfrak{A})$. Let $\varphi$ be any element in $M(\mathfrak{A})$. Then we can take a net $\{\varphi_\alpha\}$ in $M_f(\mathfrak{A})$ such that $\varphi = w^*\text{-}\lim_\alpha \varphi_\alpha$. Since $CO(\rho)$ is compact, we choose a subnet $\{\varphi_\beta\}$ of $\{\varphi_\alpha\}$ such that the net $\{\hat{\rho}(\varphi_\beta)\}$ in $CO(\rho)$ converges to some point $q \in Q$. Recalling that the function $x \mapsto <\rho(x), z>$ is in $\mathfrak{A}$ for any $z \in E^*$, we have

$$<q, z> = \lim_\beta <\hat{\rho}(\varphi_\beta), z> = \lim_\beta \varphi_\beta(<\rho(x), z>) = \varphi(<\rho(x), z>)$$

for any $z \in E^*$. Hence putting $\hat{\rho}(\varphi)=q$, we get a map $\hat{\rho}: M(\mathfrak{A}) \to CO(\rho)$ with (♯). Further if $\varphi \in \mathfrak{M}(\mathfrak{A})$ then the above net $\{\varphi_\alpha\}$ can be taken from $M_p(\mathfrak{A})$. So $\hat{\rho}(\varphi) \in O(\rho)$ for any $\varphi \in \mathfrak{M}(\mathfrak{A})$. Thus (1) is shown.

(2) Since $T$ is affine, (♯♯) is obvious for any $\varphi \in M_f(\mathfrak{A})$. Let $\varphi$ be any element in $M(\mathfrak{A})$. Then, as noted in (1), there exists a net $\{\varphi_\beta\}$ in $M_f(\mathfrak{A})$ such that $\varphi = w^*\text{-}\lim_\beta \varphi_\beta$ and $\hat{\rho}(\varphi) = \lim_\beta \hat{\rho}(\varphi_\beta)$. So by the continuity of $T$ we have

$$< T(\hat{\rho}(\varphi)), z > = \lim_\beta < T(\hat{\rho}(\varphi_\beta)), z > = \lim_\beta \varphi_\beta(< T\rho(x), z >)$$

$$= \varphi(< T\rho(x), z >) = <(T\rho)^\wedge(\varphi), z>$$

for any $z \in E^*$. Hence (♯♯) holds for any $\varphi \in M(\mathfrak{A})$. q.e.d.

The next is proved by the same consideration as in the above lemma.

**Lemma 2.2.** *Let $\mathfrak{A}$ be an $\boldsymbol{X}$-algebra and $\rho$ an $\mathfrak{A}$-c-admissible map of $X$ to a compact space $Z$.*

(1) *There corresponds uniquely a map $\hat{\rho}: \mathfrak{M}(\mathfrak{A}) \ni \varphi \to \hat{\rho}(\varphi) \in Z$ such that $\varphi(f(\rho(x))) = f(\hat{\rho}(\varphi))$ for any $f \in C(Z)$.*

(2) *Let $T$ be a continuous map of $Z$ into itself such that the map $T\rho: X \ni x \mapsto T\rho(x) = T(\rho(x)) \in Z$ is also $\mathfrak{A}$-c-admissible. Then the maps $\hat{\rho}$ and $(T\rho)^\wedge$ satisfy $(T\rho)^\wedge(\varphi) = T(\hat{\rho}(\varphi))$ for any $\varphi \in \mathfrak{M}(\mathfrak{A})$.*

**Lemma 2.3.** *Let $\mathfrak{A}$ be an $\boldsymbol{X}$-linear space and $\{S, Q_E, \rho\}$ an $\mathfrak{A}$-admissible compact convex action of $\boldsymbol{X}$. Then there corresponds a homomorphism $\hat{\rho}$ from $(S, M(\mathfrak{A}))$ to $(S, Q_E)$.*

*Proof.* The map $\hat{\rho}$ defined by $\rho$ as in Lemma 2.1 is our desired one. To see this it suffices to show that $\hat{\rho}$ is a homomorphism from $(S, M(\mathfrak{A}))$ to $(S, Q_E)$. Since $s\rho(x) = \rho(sx)$ for any $(s, x) \in S \times X$, we have

$$< s\hat{\rho}(\varphi), z > = \varphi(< s\rho(x), z >) = \varphi(< \rho(sx), z >)$$

$$= s\varphi(< \rho(x), z >) = < \hat{\rho}(s\varphi), z > \qquad (z \in E^*),$$

and hence $s\hat{\rho}(\varphi)=\hat{\rho}(s\varphi)$ for any $(s, \varphi)\in S\times M(\mathfrak{A})$. q.e.d.

Similarly the next follows from Lemma 2.2.

**Lemma 2.4.** *Let $\mathfrak{A}$ be an $\boldsymbol{X}$-algebra and $\{S, Z, \rho\}$ an $\mathfrak{A}$-c--admissible compact action of $\boldsymbol{X}$. Then there exists a homomorphism of $(S, \mathfrak{M}(\mathfrak{A}))$ to $(S, Z)$.*

By virtue of Lemmas 2.3 and 2.4 we get easily the fixed point properties corresponding to the amenability and extreme amenability. we say now that any compact or compact convex action $\{S, Y, \rho\}$ of $\boldsymbol{X}$ has *a fixed point* and *a finite stable set* if so has $(S, Y)$ respectively.

**Theorem 2.5.** *An $\boldsymbol{X}$-linear space $\mathfrak{A}$ is amenable if and only if any $\mathfrak{A}$-admissible compact convex action of $\boldsymbol{X}$ has a fixed point.*

*Proof.* If $\mathfrak{A}$ is amenable then $(S, M(\mathfrak{A}))$ has a fixed point. So by Lemma 2.3 any $\mathfrak{A}$-admissible compact convex action of $\boldsymbol{X}$ has a fixed point. If $\mathfrak{A}$ is not amenable then the $\mathfrak{A}$-admissible compact convex action $\{S, M(\mathfrak{A}), \delta\}$ of $\boldsymbol{X}$ (see Lemma 1.2) does not have any fixed point. q.e.d.

As noted in Proposition 2.1 in Part I, an $\boldsymbol{X}$-algebra $\mathfrak{A}$ is extremely amenable if and only if the compact action $(S, \mathfrak{M}(\mathfrak{A}))$ of $S$ has a fixed point, and $\mathfrak{A}$ is quasi-extremely amenable if and only if $(S, \mathfrak{M}(\mathfrak{A}))$ has a finite stable set. Hence the next two theorems are proved by the same as in Theorem 2.5 using Lemma 2.4 instead of Lemma 2.3.

**Theorem 2.6.** *An $\boldsymbol{X}$-algebra $\mathfrak{A}$ is extremely amenable if and only if any $\mathfrak{A}$-c-admissible compact action of $\boldsymbol{X}$ has a fixed point.*

**Theorem 2.7.** *An $\boldsymbol{X}$-algebra $\mathfrak{A}$ is quasi-extremely amenable if and only if any $\mathfrak{A}$-c-admissible compact action of $\boldsymbol{X}$ has a finite stable set.*

Especially we have

**Theorem 2.8.** *Let* $\boldsymbol{X}=(S, X)$ *be a* $\tau$*-semigroup. Then*

(1) $B(X)$ *is amenable if and only if any compact convex action of* $\boldsymbol{X}$ *has a fixed point.*

(2) $B(X)$ *is [quasi]-extremely amenable if and only if any compact action of* $\boldsymbol{X}$ *has a fixed point [resp. finite stable set].*

*Proof.* These assertions follow from Theorems 2.5~2.7 and the fact that any compact convex or compact actions of $\boldsymbol{X}$ are always $B(X)$- or $B(X)$-$c$-admissible respectively. q.e.d.

Finally let us apply Theorems 2.5 and 2.6 to the cases of any $\boldsymbol{S}_l$-linear space and $\boldsymbol{S}_l$-algebra respectively. Considering only actions of $S$ not of $\boldsymbol{S}_l=(S, S)$, we obtain

**Theorem 2.9.** *Let* $\mathfrak{A}$ *be an* $\boldsymbol{S}_l$*-linear space. For the following conditions the implication* (1)⇨(2)⇨(3) *holds. If* $\mathfrak{A}$ *is E-left-introverted then* (2)⇨(1). *Moreover if* $\mathfrak{A}$ *is left-introverted then all the conditions are mutually equivalent.*

(1) $\mathfrak{A}$ *is left amenable.*

(2) *Any* $E$-$\mathfrak{A}$*-admissible compact convex action of* $S$ *has a fixed point.*

(3) *Any* $A$-$\mathfrak{A}$*-admissible compact convex action of* $S$ *has a fixed point.*

*Proof.* Let $(S, Q)$ be any $E$-$\mathfrak{A}$-admissible compact convex action of $S$. Then, as noted in Remark 1.3, we can get at least one compact convex action $\{S, Q, \rho\}$ of $\boldsymbol{S}_l$ which is $\mathfrak{A}$-admissible. Hence (1)⇨(2)⇨(3) hold by Theorem 2.5. Moreover if $\mathfrak{A}$ is left [resp. $E$-left]-introverted then $(S, M(\mathfrak{A}))$ is $A$-$\mathfrak{A}$ [resp. $E$-$\mathfrak{A}$]-admissible (see Lemma 1.4 (1)). So If $\mathfrak{A}$ is left [resp. $E$-left]-introverted and (3) [resp. (2)] holds, then $(S, M(\mathfrak{A}))$ has a fixed point and hence $\mathfrak{A}$ is left amenable. q.e.d.

Analogously combining Theorem 2.6 and Lemma 1.4 (2) we have

**Theorem 2.10.** *Let $\mathfrak{A}$ be an $\boldsymbol{S}_l$-algebra. For the following conditions the implication* (1)⇨(2)⇨(3) *hold. If $\mathfrak{A}$ is E-left-m-introverted then* (1) *follows from* (2). *Moreover if $\mathfrak{A}$ is left-m-introverted then* (1) *follows from* (3).

(1) *$\mathfrak{A}$ is extremely left amenable.*

(2) *Any E-$\mathfrak{A}$-c-admissible compact action of S has a fixed point.*

(3) *Any A-$\mathfrak{A}$-c-admissible compact action of S has a fixed point.*

Theorems 2.9 and 2.10 are the same as in L. N. Argabright [1] and T. Mitchell [14] respectively. In Theorem 2.10 replacing the terms "extremely" and "fixed point" by "quasi-extremely" and "finite stable set" respectively, we can get a theorem concerning the quasi-extreme left amenability of any $\boldsymbol{S}_l$-algebras.

## § 3. Cohomological characterization of amenability

B. E. Johnson proved in his article [9] that a locally compact group $G$ is amenable if and only if for any Banach $L^1(G)$ module $\mathfrak{X}$ the cohomology group $H^1(L^1(G), \mathfrak{X}^*)$ is trivial, and also introduced the concepts of amenable Banach algebras and of strongly amenable $C^*$-algebras. In this section we shall extend the Johnson's results to the cases of any amenable $\boldsymbol{X}$-linear spaces and extremely amenable $\boldsymbol{X}$-algebras respectively. To this aim we begin with some notations and definitions, say 1-cocycles and coboundary maps. Let $E$ be a l.c.t.l. space, $Y$ a space and $f$ a map from $Y$ to $E$. By $O(f)$ and $CO(f)$ we denote the closure and the convex closure of $\{f(y); y \in Y\}$ in $E$ respectively. Moreover we denote by $A(Y, E)$ the space of all the maps $f$ from $Y$ to $E$ for which $CO(f)$ is compact.

**Definition 3.1.** *Let $(S, E)$ be an S-module and $\boldsymbol{M}=\{S, E, \rho\}$ an $\boldsymbol{X}$-module.*

(1) *$(S, E)$ is said to be almost periodic (a.p.) if the map $S \ni s \mapsto sv \in E$ belongs to $A(S, E)$ for any $v \in E$. Similarly $\boldsymbol{M}$ is said to be almost periodic (a.p.) if $(S, E)$ is a.p. and $\rho \in A(X, E)$.*

(2) *A function $f \in A(S, E)$ is called a 1-cocycle on $S$ if $f(st)$*

$=sf(t)+f(s)$ *for any* $s, t \in S$. *By* $Z(S, E)$ *we denote the space of all such* 1-*cocycles on* $S$.

(3) *A pair* $(f, h) \in A(S, E) \times A(X, E)$ *is called a* 1-*cocycle on* $\boldsymbol{X}$ *if* $h(sx)=sh(x)+f(s)$ *for any* $s \in S$ *and* $x \in X$. *By* $Z(\boldsymbol{X}, \boldsymbol{M})$ *we denote the space of all such* 1-*cocycles on* $\boldsymbol{X}$ *and put* $Z(S, \boldsymbol{M})=\{f \in A(S, E);$ $(f, h) \in Z(\boldsymbol{X}, \boldsymbol{M})$ *for some* $h \in A(X, E)\}$.

It is easily seen that $Z(S, \boldsymbol{M}) \subseteq Z(S, E)$. Let $\boldsymbol{M}=\{S, E, \rho\}$ be an a.p. $\boldsymbol{X}$-module. For any $v \in E$ define $\partial v \in A(S, E)$ and $\partial' v \in A(X, E)$ by $\partial v(s)=sv-v$ $(s \in S)$ and $\partial' v(x)=\rho(x)-v$ $(x \in X)$ respectively. Then for any $s \in S$ and $x \in X$,

$$\partial' v(sx)=\rho(sx)-v=s(\rho(x)-v)+sv-v=s\partial' v(x)+\partial v(s)$$

Hence $(\partial v, \partial' v) \in Z(\boldsymbol{X}, \boldsymbol{M})$ and $\partial v \in Z(S, \boldsymbol{M})$.

**Definition 3.2.** *Let* $\boldsymbol{M}=\{S, E, \rho\}$ *be an a.p.* $\boldsymbol{X}$-*module.*

(1) *The map* $\partial: E \ni v \mapsto \partial v(s)=sv-v \in Z(S, \boldsymbol{M})$ [*resp.* $Z(S, E)$] *is called a coboundary map* (*associated with* $\boldsymbol{M}$ [*resp.* $(S, E)$]).

(2) *Let* $\tilde{Z}$ *be a subspace of* $Z(S, \boldsymbol{M})$ *such that* $\partial(E) \subseteq \tilde{Z}$. *Then the map* $\partial$ *of* $E$ *to* $\tilde{Z}$ *is said to be strongly* [*resp. e-strongly*] *surjective if for any* $(f, h) \in Z(\boldsymbol{X}, \boldsymbol{M})$ *where* $f \in \tilde{Z}$ *there exists a* $v \in CO(h)$ [*resp.* $O(h)$] *such that* $f=-\partial v$.

Let $\mathfrak{A}$ be an $\boldsymbol{X}$-linear space and $\boldsymbol{M}=\{S, E, \rho\}$ an $\mathfrak{A}$-admissible a.p. $\boldsymbol{X}$-module. Then we put

$$Z(\boldsymbol{X}, \boldsymbol{M}, \mathfrak{A})=\{(f, h) \in Z(\boldsymbol{X}, \boldsymbol{M});\ h \text{ is } \mathfrak{A}\text{-admissible}\}.$$

$$Z(S, \boldsymbol{M}, \mathfrak{A})=\{f \in Z(S, \boldsymbol{M});\ (f, h) \in Z(\boldsymbol{X}, \boldsymbol{M}, \mathfrak{A}) \text{ for some } h\}.$$

Since the map $X \ni x \mapsto \partial' v(x)=\rho(x)-v \in E$ is $\mathfrak{A}$-admissible for any $v \in E$, we have $\partial(E) \subseteq Z(S, \boldsymbol{M}, \mathfrak{A})$. The following is the main result in this section.

**Theorem 3.3.** (1) *An* $\boldsymbol{X}$-*linear space* $\mathfrak{A}$ *is amenable if and only if for any* $\mathfrak{A}$-*admissible a.p.* $\boldsymbol{X}$-*module* $\boldsymbol{M}=\{S, E, \rho\}$ *the map* $\partial$ *of* $E$ *to* $Z(S, \boldsymbol{M}, \mathfrak{A})$ *is strongly surjective.*

(2) *An* $\boldsymbol{X}$-*algebra* $\mathfrak{A}$ *is extremely amenable if and only if*

*for any $\mathfrak{A}$-admissible a.p. $\boldsymbol{X}$-module $\boldsymbol{M}=\{S, E, \rho\}$ the map $\partial$ of $E$ to $Z(S, \boldsymbol{M}, \mathfrak{A})$ is e-strongly surjective.*

*Proof.* Suppose that $\mathfrak{A}$ is amenable [resp. extremely amenable] and $\varphi \in IM(\mathfrak{A})$ [resp. $\mathfrak{IM}(\mathfrak{A})$]. Let $\boldsymbol{M}=\{S, E, \rho\}$ be an $\mathfrak{A}$-admissible a.p. $\boldsymbol{X}$-module and $(f, h) \in Z(\boldsymbol{X}, \boldsymbol{M}, \mathfrak{A})$. We apply Lemma 2.1 to the map $h$ of $X$ to $CO(h)$. Then there exists a $v \in CO(h)$ [resp. $O(h)$] such that $<v, z>=\varphi(<h(x), z>)$ and $<sv, z>=\varphi(<sh(x), z>)$ for any $s \in S$ and $z \in E^*$. Since $\varphi$ is invariant, we have

$$<v-sv, z>=\varphi(<h(x)-sh(x), z>)=\varphi(<h(x)-h(sx)+f(s), z>)$$

$$=\varphi(<f(s), z>)=<f(s), z>$$

for any $s \in S$ and $z \in E^*$. Hence $f=-\partial v$. Thus the "only if" parts in (1) and (2) are proved. Now let $\boldsymbol{L}^*=\{S, \mathfrak{A}^*, \delta\}$, then $\boldsymbol{L}^*$ is $\mathfrak{A}$--admissible and a.p. (see Lemma 1.2). Suppose that the map $\partial$ of $\mathfrak{A}^*$ to $Z(S, \boldsymbol{L}^*, \mathfrak{A})$ is strongly [resp. $e$-strongly] surjective. Since $(0, \delta) \in Z(\boldsymbol{X}, \boldsymbol{L}^*, \mathfrak{A})$, there exists a $\varphi \in CO(\delta)$ [resp. $O(\delta)$] such that $\varphi-s\varphi=0$ for all $s \in S$. This shows that $\varphi$ is in $IM(\mathfrak{A})$ [resp. $\mathfrak{IM}(\mathfrak{A})$]. Thus the "if" parts in (1) and (2) are proved. q.e.d.

**Theorem 3.4.** *For a $\tau$-semigroup $\boldsymbol{X}=(S, X)$, $B(X)$ is amenable [resp. extremely amenable] if and only if for any a.p. $\boldsymbol{X}$-module $\boldsymbol{M}=\{S, E, \rho\}$ the map $\partial$ of $E$ to $Z(S, \boldsymbol{M})$ is strongly [resp. e-strongly] surjective.*

*Proof.* This follows from Theorem 3.3 and the fact that any a.p. $\boldsymbol{X}$-module is $B(X)$-admissible. q.e.d.

Theorem 3.3 is an analogy of the fact that any locally compact amenable group is strongly amenable in the sense of [9], while the following is a generalization of Theorem 2.5 in [9].

**Theorem 3.5.** *An $\boldsymbol{X}$-algebra $\mathfrak{A}$ is amenable if and only if for any $\mathfrak{A}$-admissible a.p. $\boldsymbol{X}$-module $\boldsymbol{M}=\{S, E, \rho\}$ the map $\partial$ of $E$ to $Z(S, \boldsymbol{M}, \mathfrak{A})$ is surjective.*

*Proof.* The "only if" part is obvious from Theorem 3.3. To see the "if" part we consider an $\mathfrak{A}$-admissible a.p. $\boldsymbol{X}$-module $\boldsymbol{M}_0=\{S, \mathfrak{A}_0^*, \delta_0\}$, where $\mathfrak{A}_0^*=\{\varphi\in\mathfrak{A}^*;\ \varphi(I_X)=0\}$ and $\delta_0: X\ni x\mapsto\delta_0(x)=0\in\mathfrak{A}_0^*$. For a $\varphi\in M(\mathfrak{A})$ put $\tilde{\varphi}(s)=\varphi-s\varphi$ $(s\in S)$ and $\tilde{\varphi}'(x)=\varphi-\delta(x)$ $(x\in X)$. Then $(\tilde{\varphi}, \tilde{\varphi}')\in Z(\boldsymbol{X}, \boldsymbol{M}_0, \mathfrak{A})$. If the map $\partial$ of $\mathfrak{A}_0^*$ to $Z(S, \boldsymbol{M}_0, \mathfrak{A})$ is surjective, there exists a $\psi\in\mathfrak{A}_0^*$ such that $\tilde{\varphi}=\partial\psi$, i.e., $s(\varphi+\psi)=\varphi+\psi$ for all $s\in S$. Since $(\varphi+\psi)(I_X)=\varphi(I_X)=1$, $\varphi+\psi$ is a nonzero invariant element in $\mathfrak{A}^*$. By Theorem 3.3 in Part I, $\mathfrak{A}$ is amenable if it is an $\boldsymbol{X}$-algebra. q.e.d.

From the above theorem we get immediately

**Theorem 3.6.** *For a $\tau$-semigroup $\boldsymbol{X}=(S, X)$, $B(X)$ is amenable if and only if for any a.p. $\boldsymbol{X}$-module $\boldsymbol{M}=\{S, E, \rho\}$ the map $\partial$ of $E$ to $Z(S, \boldsymbol{M})$ is surjective.*

Here we shall note that for the case of a left-introverted $\boldsymbol{S}_l$-linear space Theorems 3.3 and 3.5 can be stated in terms of not $\boldsymbol{S}_l$-modules but $S$-modules. Let $\mathfrak{A}$ be an $\boldsymbol{S}_l$-linear space, $(S, E)$ an a.p. $S$-module and put $Z(S, E, \mathfrak{A})=\{f\in Z(S, E);\ f$ is $\mathfrak{A}$-admissible$\}$. By the definitions of $E$-$\mathfrak{A}$- and $A$-$\mathfrak{A}$-admissibility (see Remark 1.3) we see that $\partial(E)\cap Z(S, E, \mathfrak{A})$ is nonempty if $(S, E)$ is $E$-$\mathfrak{A}$-admissible, and that $\partial(E)\subseteq Z(S, E, \mathfrak{A})$ if $(S, E)$ is $A$-$\mathfrak{A}$-admissible. For any subset $\tilde{Z}$ of $Z(S, E)$ consider the following condition (♯) [resp. (♯♯)]: *For any $f\in\tilde{Z}$ there exists a $v\in CO(f)$ [resp. $O(f)$] such that $f=-\partial v$.*

Let $\tilde{Z}_1$ and $\tilde{Z}_2$ be subsets of $Z(S, E)$ such that $\partial(E)\subseteq\tilde{Z}_1$ and $\partial(E)\cap\tilde{Z}_2\neq\phi$ respectively. Then we say that the map $\partial$ of $E$ to $\tilde{Z}_1$ is *strongly* [resp. *e-strongly*] *surjective* if $\tilde{Z}_1$ satisfies the condition (♯) [resp. (♯♯)], and that the map $\partial$ of $E$ to $Z(S, E)$ is *strongly* [resp. *e-strongly*] *$\tilde{Z}_2$-surjective* if $\tilde{Z}_2$ satisfies the condition (♯) [resp. (♯♯)].

**Theorem 3.7.** *Let $\mathfrak{A}$ be an $\boldsymbol{S}_l$-linear space [resp. $\boldsymbol{S}_l$-algebra]. For the following conditions the implication (1)⇨(2)⇨(3) hold. If $\mathfrak{A}$ is E-left [resp. E-left-m]-introverted then (2) implies (1). Moreover if $\mathfrak{A}$ is left-introverted then all the conditions are mutually equi-*

*valent.*

(1) $\mathfrak{A}$ *is* [*extremely*] *amenable.*

(2) *For any* $E$-$\mathfrak{A}$-*admissible a.p.* $S$-*module* $(S, E)$ *the map* $\partial$ *of* $E$ *to* $Z(S, E)$ *is strongly* [*resp. e-strongly*] $Z(S, E, \mathfrak{A})$-*surjective.*

(3) *For any* $A$-$\mathfrak{A}$-*admissible a.p.* $S$-*module* $(S, E)$ *the map* $\partial$ *of* $E$ *to* $Z(S, E, \mathfrak{A})$ *is strongly* [*resp. e-strongly*] *surjective.*

*Proof.* Let $(S, E)$ be an $E$-$\mathfrak{A}$-admissible a.p. $S$-module. Then there exists a $v \in E$ such that the a.p. $\boldsymbol{S}_l$-module $\boldsymbol{M}_v = \{S, E, \hat{v}\}$ is $\mathfrak{A}$-admissible, where $\hat{v}$ is defined by $S \ni s \mapsto \hat{v}(s) = sv \in E$. Since $(f, f) \in Z(\boldsymbol{S}_l, \boldsymbol{M}_v, \mathfrak{A})$ for any $f \in Z(S, E, \mathfrak{A})$, it follows from Theorem 3.3 that (1) implies (2). (2)$\Rightarrow$(3) is clear. Now suppose that $\mathfrak{A}$ is $E$-left [resp. $E$-left-$m$]-introverted. Then $(S, \mathfrak{A}^*)$ is an $E$-$\mathfrak{A}$-admissible a.p. $S$-module and there exists a $\varphi_0 \in M(\mathfrak{A})$ [resp. $\mathfrak{M}(\mathfrak{A})$] such that $\partial\varphi_0(s) \in Z(S, \mathfrak{A}^*, \mathfrak{A})$. If (2) holds, then we can take a $\psi \in CO(\partial\varphi_0)$ [resp. $O(\partial\varphi_0)$] such that $\partial\varphi_0 = -\partial\psi$. Since $\psi$ is given in the form $\psi = \varphi - \varphi_0$ for some $\varphi \in M(\mathfrak{A})$ [resp. $\mathfrak{M}(\mathfrak{A})$], we have $s\varphi_0 - \varphi_0 = \varphi - \varphi_0 - s\varphi + s\varphi_0$, i.e., $s\varphi = \varphi$ for all $s \in S$. So $\mathfrak{A}$ is [extremely] left amenable. Thus (2) implies (1) if $\mathfrak{A}$ is $E$-left [resp. $E$-left-$m$]-introverted. Moreover if $\mathfrak{A}$ is left-introverted then (1) follows from (3), because the a.p. $S$-module $(S, \mathfrak{A}^*)$ is $A$-$\mathfrak{A}$-admissible. q.e.d.

Combining Theorems 3.7 and 3.5 we get

**Theorem 3.8.** (1) *A left-introverted* $\boldsymbol{S}_l$-*algebra* $\mathfrak{A}$ *is amenable if and only if for any* $A$-$\mathfrak{A}$-*admissible a.p.* $S$-*module* $(S, E)$ *the map* $\partial$ *of* $E$ *to* $Z(S, E, \mathfrak{A})$ *is surjective.*

(2) *An* $E$-*left-introverted* $\boldsymbol{S}_l$-*algebra* $\mathfrak{A}$ *is amenable if and only if for any* $E$-$\mathfrak{A}$-*admissible a.p.* $S$-*module* $(S, E)$ $Z(S, E, \mathfrak{A})$ *is contained in* $\partial(E)$.

## § 4. Representations of amenable $\tau$-semigroups

In this section we shall study the relations between the amenability of any $\boldsymbol{X}$-linear space and the properties (ILM)*, (ILM), (ILE) or (RG) (defined below) concerning representations of $\boldsymbol{X}$. We begin with

some notations and definitions. Let $\boldsymbol{D}=\{T, E, \rho, F\}$ be a representation of $\boldsymbol{X}$ (see Definition 1.5). For any $f\in E$ and $z\in F^*$ we put

$Co(\rho, f)=$the convex hull of $\{\rho(x)f;\ x\in X\}$,

$C(\rho, f)=$the closure of $Co(\rho, f)$,

$C(\rho^*, z)=$the convex $\sigma(E^*, E)$-closure of $\{\rho(x)^*z;\ x\in X\}$,

$K(T, E)=$the linear span of $\{T_sf-f;\ s\in S, f\in E\}$.

Then we say for the sake of brevity that $\boldsymbol{D}$ is *almost periodic* (a.p.) [resp. *-almost periodic (*-a.p.)] if $C(\rho, f)$ [resp. $C(\rho^*, z)$] is compact [resp. $\sigma(E^*, E)$-compact] for any $f\in E$ [resp. $z\in F^*$]. Further we say that $\boldsymbol{D}$ is *uniformly bounded* (u.b.) if $E$ and $F$ are Banach spaces and $\|T_s\|\leqq 1$, $\|\rho(x)\|\leqq 1$ for any $(s, x)\in S\times X$. Note that $\boldsymbol{D}$ is *-a.p. if it is u.b. In what follows we shall consider only a.p., *-a.p. or u.b. representations of $\boldsymbol{X}$.

**Definition 4.1.** *Let $\boldsymbol{D}=\{T, E, \rho, F\}$ be a representation of $\boldsymbol{X}$.*

(1) *$\boldsymbol{D}$ is said to have the property (ILM) [resp. (ILM)*] if there exists a linear map $\Phi$ [resp. $\Phi^*$] of $E$ [resp. $F^*$] to $F$ [resp. $E^*$] satisfying* (4.1) [*resp.* (4.2)]:

$$\Phi T_s=\Phi,\quad \Phi(f)\in C(\rho, f)\qquad \textit{for any}\quad s\in S\quad \textit{and}\quad f\in E, \tag{4,1}$$

$$T_s^*\Phi^*=\Phi^*,\quad \Phi^*(z)\in C(\rho^*, z)\quad \textit{for any}\quad s\in S\quad \textit{and}\quad z\in F^*. \tag{4.2}$$

(2) *Let $E_0$ be a $T$ invariant closed linear subspace of $E$(i.e., $T_s(E_0)\subseteq E_0$ for any $s\in S$), $w_0\in E^*$ and $z_0\in F^*$. Then $(E_0, w_0, z_0)$ is called an invariant system of $\boldsymbol{D}$ if*

$$<f, \rho(x)^*z_0>=<f, w_0>\qquad \textit{for any}\quad f\in E_0\quad \textit{and}\quad x\in X. \tag{4.3}$$

(3) *$\boldsymbol{D}$ is said to have the invariant linear extension property (ILE) if for any invariant system $(E_0, w_0, z_0)$ of $\boldsymbol{D}$ there exists an extension $w\in C(\rho^*, z_0)$ of $w_0$ such that $T_s^*w=w$ for all $s\in S$.*

**Definition 4.2.** *An u.b. representation $\{T, E, \rho, F\}$ of $\boldsymbol{X}$ is said to have the property (RG) if the following inequality, so-called "Reiter Glicksberg inequality", holds for any $f\in E$:*

$$d_E(f, K(T, E)) \geqq d_F(0, Co(\rho, f)) \tag{4.4}$$

*where $d_E$ is the metric induced by the norm in $E$ and $d_F$ similar.*

**Remark 4.3.** (1) Let $\{T, E\}$ be an antirepresentation of $S$. Then $\{T, E, T, E\}$ is regarded as a representation of $\boldsymbol{S}_l=(S, S)$. So we say $\{T, E\}$ is u.b., a.p. and *-a.p. if so is $\{T, E, T, E\}$ respectively. Similarly $\{T, E\}$ is said to have the property $(ILM)_l^*$, $(ILM)_l$, $(ILE)_l$ and $(RG)_l$ if $\{T, E, T, E\}$ has the property $(ILM)^*$, $(ILM)$, $(ILE)$ and $(RG)$ respectively.

(2) The property $(ILE)_l$ of $\{T, E\}$ is stated as follows: *Let $E_0$ be a $T$ invariant closed linear subspace of $E_0$ and $z_0 \in E^*$ satisfy $<T_s f, z_0> = <f, z_0>$ for any $f \in E_0$ and $s \in S$. Then there exists a $z \in C(T^*, z_0)$ such that $T_s^* z = z$ for all $s \in S$. In this case $z$ coincides with $z_0$ on $E_0$.*

(3) The inequality (4.4) for the case of any u.b. antirepresentation $\{T, E\}$ of $S$ can be replaced by the equality:

$$d_E(f, K(T, E)) = d_E(0, Co(T, f)) \tag{4.5}$$

for any $f \in E$. In fact, since $Co(T, f) \subseteq f + K(T, E)$, the "$\leqq$" in (4.5) holds always.

Under the above definitions we can state the main theorem in this section.

**Theorem 4.4.** *For any $\boldsymbol{X}$-linear space $\mathfrak{A}$ the following conditions are mutually equivalent:*

(1) *$\mathfrak{A}$ is amenable.*

(2) *Any $\mathfrak{A}$-admissible *-a.p. representation of $\boldsymbol{X}$ has the property $(ILM)^*$.*

(3) *Any $\mathfrak{A}$-admissible *-a.p. representation of $\boldsymbol{X}$ has the property $(ILE)$.*

(4) *Any $\mathfrak{A}$-admissible u.b. representation of $\boldsymbol{X}$ has the property $(RG)$.*

(5) *Any $\mathfrak{A}$-admissible a.p. representation of $\boldsymbol{X}$ has the property $(ILM)$.*

Since any *-a.p., a.p. or u.b. representation of $\boldsymbol{X}$ is $B(X)$-admissible, the next is an immediate consequence of Theorem 4.4.

**Theorem 4.5.** *For any $\tau$-semigroup $\boldsymbol{X}=(S, X)$ the following conditions are mutually equivalent:*

(1) *$B(X)$ is amenable.*

(2) *Any *-a.p. representation of $\boldsymbol{X}$ has the property $(ILM)^*$.*

(3) *Any *-a.p. representation of $\boldsymbol{X}$ has the property $(ILE)$.*

(4) *Any u.b. representation of $\boldsymbol{X}$ has the property $(RG)$.*

(5) *Any a.p. representation of $\boldsymbol{X}$ has the property $(ILM)$.*

In order to prove Theorem 4.4 we prepare some lemmas.

**Lemma 4.6.** *Let $\mathfrak{A}$ be any $\boldsymbol{X}$-linear space and put $\boldsymbol{L}(\mathfrak{A})=\{L, \mathfrak{A}, \delta, \boldsymbol{C}\}$ (see Lemma 1.7).*

(1) *$\boldsymbol{L}(\mathfrak{A})$ is $\mathfrak{A}$-admissible, u.b. and a.p.*

(2) *$\mathfrak{A}$ is amenable if $L(\mathfrak{A})$ has any one of the properties $(ILM)^*$, $(ILE)$, $(RG)$ or $(ILM)$.*

*Proof.* The assertion (1) is clear (cf. Lemma 1.7). To see (2), suppose at first that $\boldsymbol{L}(\mathfrak{A})$ has $(ILM)$ or $(ILM)^*$. Then there exists linear maps $\Phi: \mathfrak{A}\to\boldsymbol{C}$ or $\Phi^*: \boldsymbol{C}\to\mathfrak{A}^*$ such that $\Phi L_s=\phi$, $\Phi(f)\in C(\delta, f)$ for any $s\in S$ and $f\in\mathfrak{A}$ or $L_s^*\Phi^*=\Phi^*$, $\Phi^*(\lambda)\in C(\delta^*, \lambda)$ for any $s\in S$ and $\lambda\in\boldsymbol{C}$ respectively. It is easily seen that $\Phi$ and $\Phi^*(1)$ are invariant means on $\mathfrak{A}$. Hence $\mathfrak{A}$ is amenable.

Next suppose that $\boldsymbol{L}(\mathfrak{A})$ has the property $(ILE)$. Let $E_0$ be the one-dimensional subspace of $\mathfrak{A}$ consisting of all constant functions and $z_0$ the identity map of $\boldsymbol{C}$. Moreover define the $w_0\in E_0^*$ by $w_0(\lambda I_X)=\lambda$ $(\lambda\in\boldsymbol{C})$. Then $(E_0, w_0, z_0)$ is an invariant system of $\boldsymbol{L}(\mathfrak{A})$. So there exists a $\varphi\in C(\delta^*, z_0)$ such that $L_s^*\varphi=\varphi$ for all $s\in S$. Since $C(\delta^*, z_0)=M(\mathfrak{A})$, $\varphi$ is an invariant mean on $\mathfrak{A}$.

Finally suppose that $\boldsymbol{L}(\mathfrak{A})$ has the property $(RG)$. Let us apply (4.4) to the case for $\boldsymbol{D}=\boldsymbol{L}(\mathfrak{A})$ and $f=I_X$. Then we have

$$1\geqq\inf\{\|I_X-h\|;\ h\in K(L, \mathfrak{A})\}\geqq\inf\{|I_X(x)|;\ x\in X\}=1.$$

So it follows from Theorem 3.2 (4) in Part I that $\mathfrak{A}$ is amenable. Thus (2) is shown. q.e.d.

**Remark 4.7.** By the same as in Lemma 4.6 (2) we can see that an $\boldsymbol{S}_l$-linear space $\mathfrak{A}$ is left amenable if the antirepresentation $\{L, \mathfrak{A}\}$ of $S$ on $\mathfrak{A}$ has any one of $(ILM)_l^*$, $(ILE)_l$, $(RG)_l$ or $(ILM)_l$.

**Lemma 4.8.** *Let $\mathfrak{A}$ be an $\boldsymbol{X}$-linear space and $\varphi \in M(\mathfrak{A})$.*

(1) *For any $\mathfrak{A}$-admissible a.p. representation $\{T, E, \rho, F\}$ of $\boldsymbol{X}$ there corresponds a linear map $\Phi$ of $E$ to $F$ such that*

$$<\Phi(f), z> = \varphi(<\rho(x)f, z>), \quad \Phi(f) \in C(\rho, f)$$

*for any $f \in E$ and $z \in F^*$.*

(2) *For any $\mathfrak{A}$-admissible *-a.p. representation $\{T, E, \rho, F\}$ of $\boldsymbol{X}$ there corresponds a linear map $\Phi^*$ of $F^*$ to $E^*$ such that*

$$<f, \Phi^*(z)> = \varphi(<f, \rho(x)^*z>), \quad \Phi^*(z) \in C(\rho^*, z)$$

*for any $z \in F^*$ and $f \in E$.*

(3) *If $\varphi$ is invariant then $\Phi$ and $\Phi^*$ in the above satisfy $\Phi T_s = \Phi$ and $T_s^* \Phi^* = \Phi^*$ for any $s \in S$ respectively.*

*Proof.* (1) For any $f \in E$ consider the map of $X$ to the compact convex set $C(\rho, f)$ defined by $X \ni x \mapsto \rho(x)f \in C(\rho, f)$. Then this is $\mathfrak{A}$-admissible. So by Lemma 2.1 there exists uniquely a $\Phi(f) \in C(\rho, f)$ satisfying $<\Phi(f), z> = \varphi(<\rho(x)f, z>)$ for any $z \in F^*$. Hence the map $\Phi: E \ni f \mapsto \Phi(f) \in C(\rho, f) \subseteq F$ is well defined and is our desired one. (2) is shown by the same as in (1).

(3) Assume that $\varphi \in IM(\mathfrak{A})$. Since $\rho(x)T_s = \rho(sx)$, we have

$$<\Phi(T_s f), z> = \varphi(<\rho(x)T_s f, z>) = \varphi(<\rho(sx)f, z>)$$
$$= \varphi(<\rho(x)f, z>) = <\Phi(f), z>$$

for any $f \in E$ and $z \in F^*$. Hence $\Phi T_s = \Phi$ for any $s \in S$. Similarly $T_s^* \Phi^* = \Phi^*$ for any $s \in S$. q.e.d.

Now let us prove Theorem 4.4. At first suppose that $\mathfrak{A}$ is amena-

ble and $\varphi \in IM(\mathfrak{A})$. Let $\boldsymbol{D}=\{T, E, \rho, F\}$ be any $\mathfrak{A}$-admissible *-a.p. representation of $\boldsymbol{X}$. Then by Lemma 4.8 (2, 3) there exists a linear map $\Phi^*$ of $F^*$ to $E^*$ satisfying (4.2) and

$$(\sharp) \qquad <f, \Phi^*(z)> = \varphi(<f, \rho(x)^* z>) \qquad \text{for any} \quad z \in F^* \text{ and } f \in E.$$

Hence $\boldsymbol{D}$ has $(ILM)^*$. On the other hand let $(E_0, w_0, z_0)$ be any invariant system of $\boldsymbol{D}$. Then it is easily seen by (4.3) that each element in $C(\rho^*, z_0)$ is an extension of $w_0$. We put $w=\Phi^*(z_0)$. Since $T_s^*\Phi^*=\Phi^*$ for all $s \in S$, this $w$ is our desired extension of $w_0$. Hence $\boldsymbol{D}$ has $(ILE)$. Further assume $\boldsymbol{D}$ is u.b. Then $\Phi^*$ satisfies

$$(\sharp\sharp) \qquad \|\Phi^*(z)\| \leqq \|z\| \quad \text{and} \quad <h, \Phi^*(z)> = 0$$

for any $z \in F^*$ and $h \in K(T, E)$. Let $f \in E$ and put $d=d_F(o, Co(\rho, f))$. We can assume without loss of generality $d>0$. Then there exists a $z_0 \in F^*$, $\|z_0\|=1$ such that $Re(<\rho(x)f, z_0>) \geqq d$ for all $x \in X$, where $Re(c)$ is the real part of the complex number $c$ (e.g. see H. Reiter [15, Lemma A]). So by ($\sharp$) and ($\sharp\sharp$) we have

$$\begin{aligned} d_E(f, K(T, E)) &= \inf\{\|f-h\| ; h \in K(T, E)\} \\ &\geqq \inf\{|<f-h, \Phi^*(z_0)>| ; h \in K(T, E)\} \\ &= |<f, \Phi^*(z_0)>| = |\varphi(<\rho(x)f, z_0>)| \geqq \mathrm{Re}(\varphi(<\rho(x)f, z_0>)) \\ &= \varphi(\mathrm{Re}(<\rho(x)f, z_0>)) \geqq d\varphi(I_X) = d. \end{aligned}$$

Hence (4.4) holds for any $f \in E$ and $\boldsymbol{D}$ has $(RG)$. Thus (1) implies (2), (3) and (4). Further (1)$\Rightarrow$(5) follows from Lemma 4.8 (1, 3).

Conversely suppose that (2), (3), (4) or (5) hold respectively. Then by Lemma 4.6 we can see that $\mathfrak{A}$ is amenable. Thus Theorem 4.4 is proved completely.

**Corollary 4.9.** *An $\boldsymbol{X}$-linear space $\mathfrak{A}$ is amenable if and only if the following inequality holds for any $f \in \mathfrak{A}$:*

$$(4.6) \qquad \inf\{\|f-h\| ; h \in K(L, \mathfrak{A})\} \geqq \inf\{|\lambda| ; \lambda \in Co(f(x) ; x \in X)\}.$$

*Proof.* By Theorem 4.4 and Lemma 4.6 (2), $\mathfrak{A}$ is amenable if and only if the representation $\boldsymbol{L}(\mathfrak{A})=\{L, \mathfrak{A}, \delta, \boldsymbol{C}\}$ of $\boldsymbol{X}$ has the property $(RG)$, that is, (4.6) holds for any $f\in\mathfrak{A}$. q.e.d.

**Remark 4.10.** (1) The equivalence (1)$\Leftrightarrow$(5) in Theorem 4.4 resembles to the result in Remark 3 after Theorem 10 in E. E. Granirer [8] and (1)$\Leftrightarrow$(4) is a generalization of Proposition in P. Eymard [6, p. 23].

(2) By slight modification of the property $(ILM)^*$ we get a characterization of the extreme amenability as follows: *An $\boldsymbol{X}$-algebra $\mathfrak{A}$ is extremely amenable if and only if for any $\mathfrak{A}$-admissible *-a.p. representation $\{T, E, \rho, F\}$ of $\boldsymbol{X}$ there exists a linear map $\Phi^*$ of $F^*$ to $E^*$ such that $T_s^*\Phi^*=\Phi^*$ for all $s\in S$ and $\Phi^*(z)\in O(\rho^*, z)$ for any $z\in F^*$, where $O(\rho^*, z)$ is the $\sigma(E^*, E)$-closure of $\{\rho(x)^*z;\ x\in X\}$.*

(3) If we apply directly Theorem 4.4 to any $\boldsymbol{S}_l$-linear space $\mathfrak{A}$, we obtain necessary and sufficient conditions for $\mathfrak{A}$ to be left amenable. They are stated in terms of not "antirepresentation of $S$" but "representation of $\boldsymbol{S}_l$". So considering only antirepresentations of $S$ we have

**Theorem 4.11.** *Let $\mathfrak{A}$ be an $\boldsymbol{S}_l$-linear space. For the following conditions, (1) implies all the others. If $\mathfrak{A}$ is left-introverted then all the conditions from (1) to (4) are mutually equivalent. Additionally if the antirepresentation $\{L, \mathfrak{A}\}$ of $S$ on $\mathfrak{A}$ is a.p., then all the conditions are mutually equivalent.*

(1) *$\mathfrak{A}$ is left amenable.*

(2) *Any $\mathfrak{A}$-admissible *-a.p. antirepresentation of $S$ has the property $(ILM)_l^*$.*

(3) *Any $\mathfrak{A}$-admissible *-a.p. antirepresentation of $S$ has the property $(ILE)_l$.*

(4) *Any $\mathfrak{A}$-admissible u.b. antirepresentation of $S$ has the property $(RG)_l$.*

(5) *Any $\mathfrak{A}$-admissible a.p. antirepresentation of $S$ has the property $(ILM)_l$.*

*Proof.* If any antirepresentation $\{T, E\}$ of $S$ is $\mathfrak{A}$-admissible, *-a.p. and a.p., then so is $\{T, E, T, E\}$ as a representation of $S_l$ respectively. Hence it follows from Theorem 4.4 that (1) implies all the others. If $\mathfrak{A}$ is left-introverted then $\{L, \mathfrak{A}\}$ is $\mathfrak{A}$-admissible (see Lemma 1.9 (1)), *-a.p. and u.b. Hence it has the properties $(ILM)_l^*$, $(ILE)_l$ or $(RG)_l$ if (2), (3) or (4) hold respectively. In each case $\mathfrak{A}$ is left amenable by Remark 4.7. Finally suppose that (5) holds. If $\{L, \mathfrak{A}\}$ is $\mathfrak{A}$-admissible and a.p. then it has the property $(ILM)_l$ and hence $\mathfrak{A}$ is left amenable by Remark 4.7. q.e.d.

**Remark 4.12.** (1) For the cases of $\mathfrak{A}=B(S)$ or some function spaces on $S$, the results in Theorem 4.11 are studied by many authors, e.g. (1)$\Leftrightarrow$(2) is in M. M. Day [2, 4] or J. Dixmier [5], (1)$\Leftrightarrow$(3) in [4] or J. Silverman [22] and (1)$\Leftrightarrow$(4) in I. Glicksberg [7], H. Reiter [15], E. E. Granirer [8] or M. Rimersma [17].

(2) Note that the antirepresentation $\{L, \mathfrak{A}\}$ of $S$ on an $S_l$-linear space $\mathfrak{A}$ is a.p. if and only if any $f\in\mathfrak{A}$ is a left almost periodic function on $S$.

## § 5. Representations of extremely amenable $\tau$-semigroups

We shall show by the analogous consideration as in § 4 that the extreme amenability of any $\boldsymbol{X}$-algebra is characterized by means of the properties $(IHM)^*$, $(ERG)^*$, $(MIE)$, $(ERG)$ and $(IHM)$ (defined below) concerning $B$- or $C^*$-representations of $\boldsymbol{X}$. We begin with some notations and definitions. Let $\boldsymbol{D}=\{T, A, \rho, B\}$ be a $B$-representation of $\boldsymbol{X}$ (see Definition 1.6). Then we put

$$H(T, A)=\text{the ideal of } A \text{ generated by } \{T_s f-f;\ s\in S, f\in A\},$$

$$O(\rho^*, \psi)=\text{the } \sigma(A^*, A)\text{-closure of } \{\rho(x)^*\psi;\ x\in X\} \qquad (\psi\in B^*).$$

Since $\rho(x)\in \mathrm{Hom}(A, B)$, we see that $O(\rho^*, \psi)\in\mathfrak{M}(A)$ if $\psi\in\mathfrak{M}(B)$. For the sake of brevity we say that a $C^*$-representation $\{T, A, \rho, B\}$ of $\boldsymbol{X}$ is *m-almost periodic* (*m-a.p.*) if $O(\rho, f)$ is $\sigma(B, \mathfrak{M}(B))$-compact for any $f\in A$, where

$$O(\rho, f) = \text{the } \sigma(B, \mathfrak{M}(B))\text{-closure of } \{\rho(x)f;\ x \in X\}.$$

Note that the $\sigma(B, \mathfrak{M}(B))$-topology is a locally convex Hausdorff one for any commutative $C^*$-algebra $B$ with unit.

**Definition 5.1.** *Let $\boldsymbol{D} = \{T, A, \rho, B\}$ be a $B$-representation of $\boldsymbol{X}$.*

(1) *$\boldsymbol{D}$ is said to have the property $(IHM)^*$ if there exists a map $\Phi^*$ of $\mathfrak{M}(B)$ to $\mathfrak{M}(A)$ such that*

(5.1) $T_s^* \Phi^* = \Phi^*$, $\Phi^*(\psi) \in O(\rho^*, \psi)$ *for any* $s \in S$ *and* $\psi \in \mathfrak{M}(B)$.

(2) *$\boldsymbol{D}$ is said to have the property $(ERG)^*$ if the following inequality holds for any $f \in A$:*

(5.2) $$d_A(f, H(T, A)) \geqq \sup_{\psi} \inf_{x} \{|<\rho(x)f, \psi>|;\ x \in X,\ \psi \in \mathfrak{M}(B)\}.$$

(3) *Let $A_0$ be a $T$ invariant closed subalgebra of $A$, $\varphi_0 \in \mathfrak{M}(A_0)$ and $\psi_0 \in \mathfrak{M}(B)$. Then $(A_0, \varphi_0, \psi_0)$ is called an invariant system of $\boldsymbol{D}$ if*

(5.3) $<f, \rho(x)^* \psi_0> = <f, \varphi_0>$ *for any* $f \in A_0$ *and* $x \in X$.

(4) *$\boldsymbol{D}$ is said to have the multiplicative invariant extension property $(MIE)$ if for any invariant system $(A_0, \varphi_0, \psi_0)$ of $\boldsymbol{D}$ there exists an extension $\varphi \in O(\rho^*, \psi_0)$ of $\varphi_0$ such that $T_s^* \varphi = \varphi$ for any $s \in S$.*

**Definition 5.2.** *Let $\boldsymbol{D} = \{T, A, \rho, B\}$ be a $C^*$-representation of $\boldsymbol{X}$.*

(1) *$\boldsymbol{D}$ is said to have the property $(IHM)$ if there exists a $\Phi \in Hom(A, B)$ such that*

(5.4) $\Phi T_s = \Phi$, $\Phi(f) \in O(\rho, f)$ *for any* $s \in S$ *and* $f \in A$.

(2) *$\boldsymbol{D}$ is said to have the property $(ERG)$ if the following inequality holds for any $f \in A$:*

(5.5) $$d_A(f, H(T, A)) \geqq d_B(0, O(\rho, f)).$$

**Remark 5.3.** (1) The properties $(IHM)^*$, $(MIE)$, $(IHM)$ and

$(ERG)$ (or $(ERG)^*$) in the above are analogous to the properties $(ILM)^*$, $(ILE)$, $(ILM)$ and $(RG)$ in §4 respectively.

(2) A $B$-antirepresentation $\{T, A\}$ of $S$ is said to have the properties $(IHM)_l^*$, $(ERG)_l^*$ and $(MIE)_l$ if the representation $\{T, A, T, A\}$ of $\boldsymbol{S}_l$ has the properties $(IHM)^*$, $(ERG)$ and $(MIE)$ respectively. The properties $(IHM)_l$, $(ERG)_l$ and $m$-almost periodicity of any $C^*$-antirepresentations of $S$ are defined similarly.

(3) The property $(MIE)_l$ for a $B$-antirepresentation $\{T, A\}$ of $S$ is defined as follows: *Let $A_0$ be a $T$ invariant closed subalgebra of $A$ and $\varphi_0 \in \mathfrak{M}(A)$ satisfy $<T_s f, \varphi_0> = <f, \varphi_0>$ for any $s \in S$ and $f \in A_0$. Then there exists a $\varphi \in O(T^*, \varphi_0)$ such that $T_s^* \varphi = \varphi$ for any $s \in S$.*

Under these definitions we can state the main theorem in this section.

**Theorem 5.4.** *For any $\boldsymbol{X}$-algedra $\mathfrak{A}$ the following conditions are mutually equivalent:*

(1) *$\mathfrak{A}$ is extremely amenable.*

(2) *Any $\mathfrak{A}$-$m$-admissible $B$-representation of $\boldsymbol{X}$ has the property $(IHM)^*$.*

(3) *Any $\mathfrak{A}$-$m$-admissible $B$-representation of $\boldsymbol{X}$ has the property $(ERG)^*$.*

(4) *Any $\mathfrak{A}$-$m$-admissible $B$-representation of $\boldsymbol{X}$ has the property $(MIE)$.*

(5) *Any $\mathfrak{A}$-$m$-admissible $m$-a.p. $C^*$-representation of $\boldsymbol{X}$ has the property $(IHM)$.*

(6) *Any $\mathfrak{A}$-$m$-admissible $m$-a.p. $C^*$-representation of $\boldsymbol{X}$ has the property $(ERG)$.*

Since any $B$-representation of $\boldsymbol{X}$ is $B(X)$-$m$-admissible, the next is an immediate consequence of Theorem 5.4. (cf. [19]).

**Theorem 5.5.** *For any $\tau$-semigroup $\boldsymbol{X}=(S, X)$ the following conditions are mutually equivalent:*

(1) *$B(X)$ is extremely amenable.*

(2) *Any $B$-representation of $\boldsymbol{X}$ has the property $(IHM)^*$.*

(3) *Any B-representation of* $\boldsymbol{X}$ *has the property* $(ERG)^*$.

(4) *Any B-representation of* $\boldsymbol{X}$ *has the property* $(MIE)$.

(5) *Any m-a.p.* $C^*$*-representation of* $\boldsymbol{X}$ *has the property* $(IHM)$.

(6) *Any m-a.p.* $C^*$*-representation of* $\boldsymbol{X}$ *has the property* $(ERG)$.

In order to prove Theorem 5.4 we prepare some lemmas.

**Lemma 5.6.** *An* $\boldsymbol{X}$*-algebra* $\mathfrak{A}$ *is extremely amenable if the representation* $\boldsymbol{L}(\mathfrak{A})=\{L, \mathfrak{A}, \delta, \boldsymbol{C}\}$ *has any one of the properties* $(IHM)^*$, $(ERG)^*$, $(MIE)$, $(IHM)$ *and* $(ERG)$.

*Proof.* This is shown by the same way as in Lemma 4.6 (2).

q.e.d.

**Lemma 5.7.** *Let* $\mathfrak{A}$ *be an* $\boldsymbol{X}$*-algebra and* $\varphi \in \mathfrak{M}(\mathfrak{A})$.

(1) *For any* $\mathfrak{A}$*-m-admissible B-representation* $\{T, A, \rho, B\}$ *of* $\boldsymbol{X}$ *there corresponds a map* $\Phi^*$ *of* $\mathfrak{M}(B)$ *to* $\mathfrak{M}(A)$ *such that for any* $\psi \in \mathfrak{M}(B)$ *and* $f \in A$,

$$(5.6) \qquad <f, \Phi^*(\psi)> = \varphi(<f, \rho(x)^*\psi>), \qquad \Phi^*(\psi) \in O(\rho^*, \psi).$$

(2) *Any* $\mathfrak{A}$*-m-admissible m-a.p.* $C^*$*-representation* $\{T, A, \rho, B\}$ *of* $\boldsymbol{X}$ *there corresponds a* $\Phi \in Hom(A, B)$ *such that for any* $f \in A$ *and* $\psi \in \mathfrak{M}(B)$,

$$(5.7) \qquad <\Phi(f), \psi> = \varphi(<\rho(x)f, \psi>), \qquad \Phi(f) \in O(\rho, f).$$

(3) *If* $\varphi$ *is invariant then the maps* $\Phi^*$ *in* (1) *and* $\Phi$ *in* (2) *satisfy* $T_s^*\Phi^* = \Phi^*$ *and* $\Phi T_s = \Phi$ *for any* $s \in S$ *respectively.*

*Proof.* (1) For any $\psi \in \mathfrak{M}(B)$ let us apply Lemma 2.1 the $\mathfrak{A}$--admissible map $X \ni x \mapsto \rho(x)^*\psi \in O(\rho^*, \psi)$. Then we can get a map $\Phi^*$ of $\mathfrak{M}(B)$ to $\mathfrak{M}(A)$ with (5.6) (cf. Lemma 4.8 (2)).

(2) For any $f \in A$ consider the map of $X$ to the $\sigma(B, \mathfrak{M}(B))$--compact set $O(\rho, f)$ defined by $X \ni x \mapsto \rho(x)f \in O(\rho, f)$. Then by Lemma 2.2 we can define the map $\Phi$ of $A$ to $B$ with (5.7). Moreover using (5.7), we can see easily that this $\Phi$ is a homomorphism as $C^*$-algebras

(cf. Theorem 10 in [8]). (3) is shown by the same way as in Lemma 4.8 (3). q.e.d.

Let us now prove Theorem 5.4. At first suppose that $\mathfrak{A}$ is extremely amenable and $\varphi \in \mathfrak{TM}(\mathfrak{A})$. Let $\boldsymbol{D}=\{T, A, \rho, B\}$ be an $\mathfrak{A}$-$m$-admissible $B$-representation of $\boldsymbol{X}$. Then by Lemma 5.7 (1, 3) there exists a map $\Phi^*$ of $\mathfrak{M}(B)$ to $\mathfrak{M}(A)$ with (5.1) and (5.6). Let $f \in A$ and $\psi \in \mathfrak{M}(B)$. Since $<h, \Phi^*(\psi)>=0$ for any $h \in H(T, A)$, we have

$$\begin{aligned} d_A(f, H(T, A)) &= \inf\{\|f-h\| : h \in H(T, A)\} \\ &\geqq \inf\{|<f-h, \Phi^*(\psi)>| ; h \in H(T, A)\} \\ &= |<f, \Phi^*(\psi)>| = |\varphi(<f, \rho(x)^*\psi>)| \\ &= \varphi(|<\rho(x)f, \psi>|) \geqq \inf\{|<\rho(x)f, \psi>| ; x \in X\}. \end{aligned}$$

Hence (5.2) holds for any $f \in A$. Thus $\boldsymbol{D}$ has $(IHM)^*$ and $(ERG)^*$. On the other hand let $(A_0, \varphi_0, \psi_0)$ be any invariant system of $\boldsymbol{D}$ and put $\varphi=\Phi^*(\psi_0) \in O(\rho^*, \psi_0)$. Then by (5.1) $T_s^*\varphi=\varphi$ for any $s \in S$. Since each element in $O(\rho^*, \psi_0)$ is an extension of $\varphi$ by (5.3), $\varphi$ is our desired extension of $\varphi_0$. Hence $\boldsymbol{D}$ has $(MIE)$. Let $\boldsymbol{D}=\{T, A, \rho, B\}$ be an $\mathfrak{A}$-$m$-admissible $m$-a.p. $C^*$-representation of $\boldsymbol{X}$. Then by Lemma 5.7 (2, 3) there exists a $\Phi \in \mathrm{Hom}(A, B)$ with (5.4). Since $\Phi$ satisfies $\|\Phi(f)\| \leqq \|f\|$ for any $f \in A$ and $\Phi(h)=0$ for any $h \in H(T, A)$, we have for any $f \in A$

$$d_A(f, H(T, A))=\inf\{\|f-h\| ; h \in H(T, A)\} \geqq \|\Phi(f)\| \geqq d_B(0, O(\rho, f)).$$

Hence $\boldsymbol{D}$ has $(IHM)$ and $(ERG)$. So it is shown that (1) implies all the others. Conversely suppose that (2), (3), (4), (5) and (6) hold respectively. Then since $\boldsymbol{L}(\mathfrak{A})$ is an $\mathfrak{A}$-$m$-admissible $m$-a.p. $C^*$-representation of $S$, it has $(IHM)^*$, $(ERG)^*$, $(MIE)$, $(IHM)$ and $(ERG)$ respectively. Hence by Lemma 5.6 $\mathfrak{A}$ is extremely amenable in each case. Thus Theorem 5.4 is proved completely.

**Corollary 5.8.** *An $\boldsymbol{X}$-algebra $\mathfrak{A}$ is extremely amenable if and only if the following inequality holds for any $f \in \mathfrak{A}$:*

$$\inf\{\|f-h\|;\ h\in H(L,\mathfrak{A})\}\geqq\inf\{|f(x)|;\ x\in X\}. \tag{5.8}$$

*Proof.* By Theorem 5.4 and Lemma 5.6, $\mathfrak{A}$ is extremely amenable if and only if $\boldsymbol{L}(\mathfrak{A})$ has $(ERG)$, that is, (5.8) holds for any $f\in\mathfrak{A}$.

q.e.d.

**Remark 5.9.** (1) Theorem 5.4 can be easily extended to the case of any quasi-extremely amenable $\boldsymbol{X}$-algebras by only slight modifications of the properties $(IHM)^*$, $(ERG)^*$ and so on. For example it suffices to modify the property $(ERG)$ for a $C^*$-representation $\{T, A, \rho, B\}$ of $\boldsymbol{X}$ as follows:

*For some finite subset* $\alpha=\{a_1, a_2,\dots, a_n\}$ *of* $S$ *we have*

$$d_A(f, H(T(\alpha), A))\geqq d_B(0, O(\rho, f)) \qquad \textit{for any} \quad f\in A,$$

*where* $H(T(\alpha), A)$ *is the ideal of* $A$ *generated by* $\left\{\frac{1}{n}\sum_{i=1}^{n}T_{a_i}(T_sf-f);\ s\in S, f\in A\right\}$.

(2) Applying Theorem 5.4 to any $\boldsymbol{S}_l$-algebra, we get the next theorem by the same way as in Theorem 4.11.

**Theorem 5.10.** *Let* $\mathfrak{A}$ *be an* $\boldsymbol{S}_l$*-algebra. For the following conditions,* (1) *implies all the others. If* $\mathfrak{A}$ *is left-m-introverted, then all the conditions from* (1) *to* (4) *are mutually equivalent. Additionally if the antirepresentation* $\{L, \mathfrak{A}\}$ *of* $S$ *on* $\mathfrak{A}$ *is m-a.p., then all the conditions are mutually equivalent.*

(1) $\mathfrak{A}$ *is extremely left amenable.*

(2) *Any* $\mathfrak{A}$*-m-admissible B-antirepresentation of* $S$ *has the property* $(IHM)_l^*$.

(3) *Any* $\mathfrak{A}$*-m-admissible B-antirepresentation of* $S$ *has the property* $(ERG)_l^*$.

(4) *Any* $\mathfrak{A}$*-m-admissible B-antirepresentation of* $S$ *has the property* $(MIE)_l$.

(5) *Any* $\mathfrak{A}$*-m-admissible m-a.p.* $C^*$*-antirepresentation of* $S$ *has the property* $(IHM)_l$.

(6) *Any* $\mathfrak{A}$*-m-admissible m-a.p.* $C^*$*-antirepresentation of* $S$ *has*

*the property* $(ERG)_l$.

**Remark 5.11.** For the case of $B(S)$ E. E. Granirer [8] gives the same or analogous characterizations as in the above theorem. For example the equivalence (1)$\Leftrightarrow$(4) is Theorem 7 in [8]. Moreover the similar characterizations of quasi-extremely left amenable semigroups are studied by A. T. Lau [11].

DEPARTMENT OF MATHEMATICS,
KAGOSHIMA UNIVERSITY

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