J. Math. Kyoto Univ. (JMKYAZ)
18-3 (1978) 453–508

The behavior of solutions of some non-linear diffusion equations for large time

By

Kōhei UCHIYAMA

(Communicated by Prof. M. Yamaguti, July 10, 1977)

0. Introduction

Consider the semi-linear diffusion equation

$$u^{\cdot}=\tfrac{1}{2}u''+F(u) \qquad t>0,\ -\infty<x<\infty \tag{1}$$

$$(u=u(t,x),\quad u^{\cdot}=\partial u/\partial t,\quad u''=\partial^2 u/\partial x^2)$$

with the initial condition

$$u(0,\cdot)=f. \tag{2}$$

The function F is always assumed in this paper to satisfy

$$F\in C^1[0,1],\quad F(0)=F(1)=0 \quad \text{and} \quad F(u)>0 \quad 0<u<1 \tag{3}$$

and the initial function f to be measurable and compatible to F, i.e. $0\leqq f\leqq 1$. Our interest is in the behavior of the solution for large time t.

We mean by the *solution* of (1) and (2) such a function $u(t, x)$ defined on the upper half plane $[0,\infty)\times(-\infty,\infty)$ that (i) $0\leqq u\leqq 1$, (ii) u has continuous derivatives $u^{\cdot}$ and u'' and satisfies (1) in $(0,\infty)\times(-\infty,\infty)$, and (iii) $u(t,\cdot)$ converges to f as $t\downarrow 0$ in locally L^1 sense. It is well known that such a solution exists and is unique. We denote it by $u(t, x; f)$. It is clear that $u(t, x; u(s,\cdot\,; f))=u(t+s, x; f)$ (Huygens property) and $u(t, x; f(\cdot+y))=u(t, x+y; f)$. We sometimes consider the equation (1) with different F's and in such cases use the notation $u(t, x; f; F)$ in order to elucidate the dependence on F. There are just two trivial solutions of (1): $u\equiv 0$ and $u\equiv 1$. We always consider our problem for non-trivial initial functions f; $f\not\equiv 0$ and $\not\equiv 1$. Such initial functions are called *data*. We will mainly deal with such data that $f(x)\to 0$ as $x\to\infty$.

The behavior of a solution $u(t, x; f)$ is closely related to solutions of ordinary differential equations

$$\tfrac{1}{2}w''+cw'+F(w)=0 \qquad (0\leqq w\leqq 1) \tag{4}$$

where c is a real constant. This equation is formally obtained if we substitute the wave form $u(t, x)=w(x-ct)$ in (1). A non-trivial solution of (1) with such form is called, if exists, a travelling wave with speed c. An associated function (or, equivalently, global solution of (4)) w is called a front of a travelling wave with speed c or simply a *c-front*, which will be denoted by w_c. Since (4) is transformed to $2^{-1}w''-cw'+F(w)=0$ by inverting the sign of x, we always assume $c \geqq 0$.

In many articles ([1], [6], etc.) it is shown, under the restriction $F'(0)>0$, that there exists the minimal speed, denoted by c_0, such that a c-front exists if and only if $|c| \geqq c_0$. We will give, for completeness, a proof of this assertion under present situation, though the proof is essentially the same as those given in papers cited above. Since the equation (4) is invariant under the translation along the x-axis and w_c has the corresponding ambiguity, we set the normalization: $w_c(0)=1/2$ except in §1 and §2. Under this convention just one w_c corresponds to each $c \geqq c_0$.

General solutions of (1) and (2) are related with c-fronts in the following manner: if a datum f satisfies certain conditions, then

$$u(t, x+m(t)) \longrightarrow w_c(x) \quad \text{as} \quad t \longrightarrow \infty \tag{5}$$

where $u=u(t, x; f)$ and $m(t)=\sup\{x;\, u(t, x)=1/2\}$ (any number, e.g. zero, may be assigned to $m(t)$ when the set expressed with braces is void). This phenomenon was observed by Kolmogorov, Petrovsky and Piscounov [13]; they proved that (5) is valid with $c=c_0$ if we set $f=I_{(-\infty,0)}$ (I_S is the indicator function of a set S). Kametaka [10] or Kanel' [11b] found a certain criteria on a datum for (5) to hold which are satisfied with many data but not easily checked for a given one. The main purpose of this article is to prove (5) for sufficiently general datum, e.g. any data with compact support (Theorems 8.1, 8.2, 8.3 and 8.5).

The method of the proofs is similar to that used in [10] or [13] and summarized in the following. Let $u(t, x)$ be a soluiton of (1) and (2) and suppose that for each positive t, $u'(t, x)<0$ on a right half x-axis, i.e. an infinite interval $\{x;\, x>N\}$. Define

$$M(t)=\sup\{u(t, x);\; u'(t, y)<0 \text{ for all } y>x\}$$

and define for $0 \leqq w \leqq M(t)$

$$x(t, w)=\sup\{x;\, u(t, x)=w\}$$

$$\phi(t, w)=u'(t, x(t, w)).$$

Considering ϕ as functional of datum f, we denote it by $\phi(t, w; f)$. Then, since $u(t, x;\, w_c)=w_c(x-ct)$, $\phi(t, w;\, w_c)$ is independent of t: this function is denoted by $\tau_c(w)$. We will prove (5) by showing that $\phi(t, w)$ converges to $\tau_c(w)$. This will be carried out at first, in §6, for data which is subject to several restrictions (Lemmas 6.1 to 6.4) and then, in §8, for general data by using this result and by applying some comparison theorem on a parabolic equation. The section 7 is devoted to estimate the order of $u(t, x; f)$ decreasing to zero as x tends to infinity which justifies the application of the comparison theorem.

The section 1 is devoted to prove the existence of c-fronts. The case where

$c_0 > \sqrt{2F'(0)}$ is illustlated by examples in which an explicit form of w_{c_0} is given. Also some comparison lemmas about the equation (4) are proved. In the section 2 asymptotic behaviors of w_c for large x are investigated. Results are refinements of those obtained from the standard theory of ordinary differential equations but will play minor roles in the main story of this paper. In the section 3 we introduce comparison theorems concerning parabolic equations which will play important roles in later arguments together with results of § 1 and § 5. In the sections 4 and 5 some properties of $u(t, x; f; F)$ which are well known or readily proved are explained. The main theorems are proved through § 6 to § 8 and formulated in § 8.

In the section 9 we will investigate the speed of $m(t)$ tending to infinity. It will be proved that if (5) occurs then $m^{\cdot}(t)$ converges to c as t tends to infinity. If $F'(0) \geqq F'(u)$ we will get, under additional assumptions, a fine estimate;

$$c_0 t - m(t) \sim (3/2c_0) \log t \qquad \text{as} \quad t \longrightarrow \infty.\text{(*)}$$

The question of when we may replace $m(t)$ by $ct+$const. in (5) will be answered.

In the last section an alternative method, which is a modification of that used in P. C. Fife and J. B. McLeod [3b](**), is applied to the problem described by (5) in case $c_0 > \sqrt{2F'(0)}$.

Notations. We will use throughout the article the following notations in addition to those introduced above:

$$\alpha = F'(0),\ c^* = \sqrt{2\alpha},$$

$$\beta = \sup F(u)/u, \quad \gamma = \sup |F'(u)|, \quad \gamma^* = \sup F'(u)$$

(the supremum of a function is taken with respect to all arguments for which it is defined unless otherwise specified); for a real number A, $A^+ = \max\{0, A\}$, $A^- = \min\{0, A\}$; if A is a real function of z, A^+ is a function defined by $A^+(z) = A(z)^+$; $R = (-\infty, \infty)$ whole real line, $E = (0, \infty) \times R$ open half plane: $E_t = (0, t) \times R$, $l_t = \{t\} \times R$ $(t > 0)$;

$$p(t, x) = (2\pi t)^{-1/2} e^{-x^2/2t} \qquad t > 0,\ x \in R$$

($p(t, x-y)$ is the fundamental solutions for the heat equation $u^{\cdot} = 2^{-1}u''$); for $t > 0$ and a measurable function g on R we write

$$P_t g = P_t g(x) = \int_R p(t, x-y) g(y) dy$$

if this integral converges absolutely.

Some terminologies, which are used throughout this paper, are introduced in the beginning of the section 1.

Most of the results of the present paper were announced in [17].

(*) "$a(t) \sim b(t)$ as $t \to s$" means that $\lim_{t \to s} a(t)/b(t) = 1$.

(**) Their situation is different from ours, where F changes its sign at least one time.

Acknowledgements. I am indebted to N. Ikeda for introducing me to the problem through his lecture given at Tokyo Institute of Technology in June, 1976 and to him and to T. Sirao for the discussions on the problem. I would like to thank P. C. Fife and J. B. McLeod for sending me a preprint of [3b]. Finally I am deeply grateful to M. Nagasawa for reading the manuscript through and pointing out several errors in it.

1. Fronts of travelling waves

In this section we find all non-trivial solutions for the equation (1), called travelling wave, which has the travelling wave form

$$u(t, x)=w(x-ct)$$

where c is a constant called a speed and w is a function on R called a front or a c-front. We denote any c-front by w_c to elucidate the speed. A c-front is characterized as a non-trivial solution of the equation (4) on R. It will be shown that if a global solution exists it is unique up to the translation along x-axis. Thus w_c corresponds at most one to each c except this ambiguity. As mentioned in §0 we treat only the case $c\geqq 0$. This amounts, as far as global solutions are concerned, to set the boundary condition

$$w(\infty)=0 \quad\text{and}\quad w(-\infty)=1 \tag{1.1}$$

to the equation (4).

We often consider the equation (4) in the phase plane:

$$\begin{cases} w'=p \\ p'=-2cp-2F(w). \end{cases} \tag{1.2}$$

The range of (w, p) is restricted to the strip $0\leqq w\leqq 1$. Any solution of (1, 2) which stays in this strip and terminates at its boundary $w=0$ or 1 is called, for convenience, a *c-manifold*. We call a corresponding solution of (4) a *c-solution*. Thus a c-solution is a function defined and satisfying (4) on a (finite or infinite) interval, at the end points of which it attains 0 or 1. A manifold to which a c-front corresponds is also called a *c-front*. Let $c\geqq\sqrt{2\alpha}$ and put

$$\underline{b}=c-\sqrt{c^2-2\alpha} \quad\text{and}\quad \bar{b}=c+\sqrt{c^2-2\alpha}. \tag{1.3}$$

It will be proved that if $c>\sqrt{2\alpha}$ there exist c-manifolds which enter the origin along a line $p=-\underline{b}w$ or a line $p=-\bar{b}w$. A c-manifold entering the origin along $p=-\underline{b}w$ (resp. $p=-\bar{b}w$) is called $(c, \underline{b})$-*manifold* (resp. $(c, \bar{b})$-*manifold*).

We will mean also by a c-manifold a corresponding curve drawn in (w, p)-plane. Parametrizing the part of this curve under w-axis with its w-coordinates, we denote its p-coordinate by $\tau(w)$. Then τ satisfies

$$\tau'=-2c-2F(w)/\tau \tag{1.4}$$

in its domain of definition.

The prospects of the vecter field defined by the right-hand side of (1.2) is important in the arguments of later sections as well as of this section. It will be explained in the proof of Theorem 1.1 and illustrated in Appendix. We often use explicitely or implicitely an argument described below. Let $Q_1(x)=(w_1(x), p_1(x))$ and $Q_2(x)=(w_2(x), p_2(x))$ be smooth curves in R^2. Suppose $(w_0, p_0)=(w_1(0), p_1(0))=(w_2(0), p_2(0))$. Then according as

$$\det\begin{pmatrix} w_1'(0) & w_2'(0) \\ p_1'(0) & p_2'(0) \end{pmatrix}<0 \quad \text{or} \quad >0,$$

the angle measured from $Q_2(x)$ toward $Q_1(x)$ around the point (w_0, p_0) lies in the interval $(0, \pi)$ or in the interval $(-\pi, 0)$ for all sufficiently small x. If the former case (resp. the latter case) occurs we will say that the curve Q_2 crosses the curve Q_1 (at (w_0, p_0)) from the left-(resp. right-) hand side of Q_1. For example let $g(x)$ be defined and twice continuously differentiable on an interval (x_1, x_2) with $0\leqq g\leqq 1$. Then the curve $\{(g(x), g'(x)); x_1<x<x_2\}$ crosses c-manifolds from the left or right according as

$$g'(\tfrac{1}{2}g''+cg'+F(g))<0 \quad \text{or} \quad >0$$

at intersecting points, since for a solution (w, p) of (1.2)

$$\det\begin{pmatrix} w' & g' \\ p' & g'' \end{pmatrix}=g'\{\tfrac{1}{2}g''+cg'+F(g)\} \quad \text{at} \quad (w, p)=(g, g').$$

The next theorem follows from standard arguments concerning with the 2-dimentional autonomous system. The proof is given for completeness.

Theorem 1.1. (i) *There exists a positive constant c_0 such that a c-front exists if and only if $c\geqq c_0$. The c-front is unique up to the translation along x-axis.* (ii) *c_0 satisfies that $\sqrt{2\alpha}\leqq c_0\leqq\sqrt{2\beta}$.* (iii) *Let $c\geqq c_0$. Then for a c-front w_c there exists $\lim_{x\to\infty} w_c'(x)/w_c(x)=-b$, $b=\underline{b}$ if $c>c_0$, and $b=\bar{b}$ if $c=c_0$, where $\underline{b}$ and $\bar{b}$ are defined by* (1.3). (*Especially $w_c(\log x)$ is regularly varying at infinity with exponent $-b$,* in other words $w_c(x+x_0)/w_c(x)\to\exp\{-bx_0\}$ as $x\to\infty$.)

Proof. Step 1. Consider the fields for (1.2) for different c's, say c and c', $c'>c$. Since

$$\det\begin{pmatrix} p & p \\ -2F(w)-2cp & -2F(w)-2c'p \end{pmatrix}=2(c-c')p^2$$

is negative, c'-manifolds never cross each c-manifold from the right hand of the c-manifold (c-manifolds are considered to be directed). (cf. Fig. I) Note that the field points downward on the w-axis and that its w-component directs right in the upper half and left in the lower half of the strip $0\leqq w\leqq 1$ (Fig. I). Clearly the

c-front lies always under the w-axis if it exists.

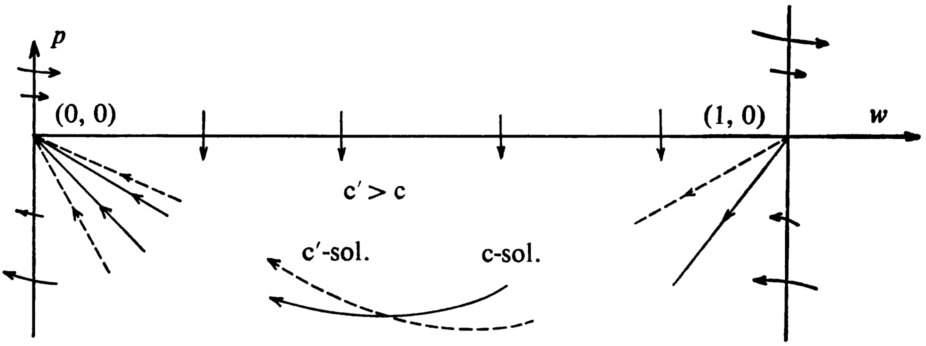


Fig. I

There exists the unique c-manifold issuing from (1, 0). It is routine to prove the existance. To prove the uniqueness assertion let (w^*, p^*) and (w^{**}, p^{**}) be two such manifolds. Regarding the difference $p=p^*-p^{**}$ as a function of w, we see that the derivative $dp/dw=p2F(w)/p^*p^{**}$ has the same sign with p, which implies $p\equiv 0$ since p converges to 0 as $w\to 1$.

The c'-manifold issuing from (1, 0) lies over the c-manifold issuing from (1, 0) if $c'>c$. This is proved by assuming the contray and by tracing back a c'-manifold passing through a point between two manifold until getting to w-axis.

Step 2. Let a c-manifold pass a point $(w_0, -bw_0)$ where $b>0$. The sine of the angle, made by a tangent vector of it at this point and a half line $p=-bw$, $w>0$, which is directed to the origin, and taken from the former toward the latter, is equal to the ratio of

$$\text{(1.5)} \qquad \det\begin{pmatrix} -1 & p \\ b & -2F(w)-2cp \end{pmatrix}_{\substack{w=w_0 \\ p=-bw_0}} = [b^2-2bc+2F(w_0)/w_0]w_0$$

to $\sqrt{1+b^2}\sqrt{b^2+4(F(w_0)/w_0+cb)^2}\,w_0$. This ratio is less than a negative constant, say $-\varepsilon$, if $\bar{b}>b>\underline{b}$ $(c\geqq c^*)$ and larger than a positive constant, say ε, if $b<\underline{b}$, $\bar{b}<b$ or $c^*>c$ $(\geqq 0)$, for $0<w_0<\delta$, where ε or δ, become small unrestrictedly only if $c\geqq c^*$ and b approaches to $\underline{b}$ or $\bar{b}$ (c being fixed). It is easy to see that if $0\leqq c<c^*$ every c-manifold reachs the negative p-axis with finite x and that if $c>c^*$ there exists a c-manifold which enters the origin along $p=-\bar{b}w$ (called $(c, \bar{b})$-manifold) and those along $p=-\underline{b}w$ (called $(c, \underline{b})$-manifold). These exhaust all c-manifolds entering the origin (in case $c>c^*$).(*)

Step 3. From Step 1 it follows that if the c-manifold issuing from (1, 0) enters (0, 0) then the situation is same for any $c'>c$. Let c_0 be the infimum of such c's. By Step 2 $c^*\leqq c_0\leqq\sqrt{2\beta}$ (the right side of (1.5) is negative for $c>\sqrt{2\beta}=b$). Since $\bar{b}$ is increasing with c and every c'-front lies over the c-front for $c'>c>c_0$, the

(*) According to the behavior of F near zero, there occur both cases that a c^*-manifold entering the origine exists and that such one does not exists (see Remark of Lemma 2.2).

c-front enters (0, 0) along $p=-\underline{b}w$ if $c>c_0$. This proves the first half of (iii). The c_0-manifold issuing from (1, 0) enters (0, 0), because c-fronts with $w(0)=1/2$ converges to a c_0-solution with $w(-\infty)=1$ increasingly for $x<0$ and decreasingly for $x>0$ as $c\to c_0$. Thus the c_0-front exists and (i) is proved. The second half of (iii) is trivial if $c_0=c^*$. Let $c_0>c^*$ and $c_0>c>c^*$. The c_0-front is obtained as the limit of c-manifolds issuing from (1, 0) as $c\to c_0$. Since each of these manifolds is bounded from above by the $(c, \bar{b})$-manifold which moves monotonously, as $c\uparrow c_0$, to the $(c_0, \bar{b})$-manifold, the c_0-front is the $(c_0, \bar{b})$-manifold. Thus (iii) is proved.

The proof of the theorem is completed.

Remark. *The* $(c, \bar{b})$*-manifold*, whose existance has been proved in Step 2 of the above proof, *is unique if* $c>c^*$ as is shown below. Parametrizing any two $(c, \bar{b})$-manifolds with w-coordinates, denote by $p=p(w)$ the difference of their p-coordinates. Assume $p>0$. By $2\alpha/\bar{b}^2=2c/\bar{b}-1<1$, we derive from (1.4) that $(w/p)\cdot dp/dw<r<1$ for small w. This implies $p>w^r$ which contradicts to $p=o(w)$, and we have $p\leqq 0$. Similarly $p\geqq 0$. Thus $p=0$.

In order to illustrate that when $\alpha<\beta$ both the case $c_0>c^*$ and the case $c_0=c^*$ occurs according as the shape of F, we give examples which are generalizations of Fisher's population genetic model for the migration of advantageous genes. The results are similar to what K. P. Hadeler and F. Rothe obtained for $F(u)=u(1-u)(1+vu)$, $v>-1$ (cf. [6]).

Let $G(u)$ be a function defined in $0\leqq u\leqq 1$, having the continuous derivative which is continuously differentiable in $0<u\leqq 1$, and satisfying conditions;

$$G(0)=G(1)=0,\ G'(0)>0,\ G''(u)=o(u^{-1}),$$

$$G'(0)\geqq G'(u) \text{ and } G(u)>0 \text{ for } 0<u<1.$$

Put $F(u)=G(u)H(u;\kappa)$, $H(u;\kappa)=1+2\kappa^{-2}(G'(0)-G'(u))$ $\kappa>0$. Then the function $w=w(x;\kappa)$ given in the inverse form

$$x=\int_{1/2}^{w}-\kappa du/2G(u) \tag{1.6}$$

is a front with an associated speed

$$c=\kappa/2+\alpha/\kappa \qquad (\alpha=G'(0)=F'(0)).$$

If $\kappa=\sqrt{2\alpha}$, then $c=\sqrt{2\alpha}$ and hence $c_0=\sqrt{2\alpha}$. As κ increases, $F(u)$ decreases and c_0 does not increase. Since $c_0\geqq\sqrt{2\alpha}$, we get

$$c_0=\sqrt{2\alpha} \qquad \text{for} \quad \kappa\geqq\sqrt{2\alpha}.$$

If $\kappa<\sqrt{2\alpha}$, then

$$\lim_{x\to\infty} w'(x;\kappa)/w(x;\kappa)=-(2/\kappa)\lim_{w\to 0} G(w)/w=-2\alpha/\kappa=-c-\sqrt{c^2-2\alpha}.$$

From (iii) of Theorem 1.1 it therefore follows that $w(x;\kappa)$ is the c_0-front. This means that

$$c_0=\kappa/2+\alpha/\kappa\ (>\sqrt{2\alpha}) \qquad \text{for} \quad \kappa<\sqrt{2\alpha}.$$

If $G'(0)-G'(u)\sim u^pL(u)$ where L is slowly varying at zero and $p>0$, then $\alpha<\beta$ but $c_0=\sqrt{2\alpha}$ for $\sqrt{2\alpha}\leqq\kappa<\sqrt{2\alpha(p+1)}$.

Similar arguments are available in the case $F'(0)=0$. Let $G(u)$ be as above and put

$$F(u)=G(u)M(u;\kappa), \quad M(u;\kappa)=2/\kappa^2(G'(0)-G'(u)).$$

Then $w(x;\kappa)$ given by (1.6) is the c_0-front and $c_0=G'(0)/\kappa$.

The first example may seem to suggest that whether $c_0>c^*$ or $c_0=c^*$ does not depend only on the behavior of $F(u)$ near $u=0$. But there is an exceptional case of Remark to Lemma 2.2 presented later, in which $c_0>c^*$ is implied only by a behavior of F near $u=0$.

Let us state a lemma for use in the next section.

Lemma 1.1. *Let F^* be a function satisfying the same conditions as imposed to F, and denote by w^*, $\bar{b}^*$, $\underline{b}^*$, etc. the corresponding quantities. Assume that $F<F^*$ for $0<u<1$ and that $c>\sqrt{2F^{*\prime}(0)}$. Then,* (i) *the $(c, \bar{b}^*)$-manifold lies over the $(c, \bar{b})$-manifold as far as they are under the w-axis, and* (ii) *for every $(c, \underline{b}^*)$-manifold [resp. $(c, \underline{b})$-manifold] there exists a $(c, \underline{b})$-manifold [resp. $(c, \underline{b}^*)$-manifold] such that the $(c, \underline{b}^*)$-manifold lies under the $(c, \underline{b})$-manifold near the origine.*

Proof. First note that $\underline{b}\leqq\underline{b}^*$, $\bar{b}^*\leqq\bar{b}$. Since

$$\det\begin{pmatrix} w' & w^{*\prime} \\ p' & p^{*\prime} \end{pmatrix}=-2(pF^*(w^*)-p^*F(w))$$

is positive at $(w, p)=(w^*, p^*)$, (ii) is clear. (i) follows from the fact that a c-manifold for F passing a point below the $(c, \bar{b})$-manifold must reach the negative p-axsis (see Fig. II). q.e.d.

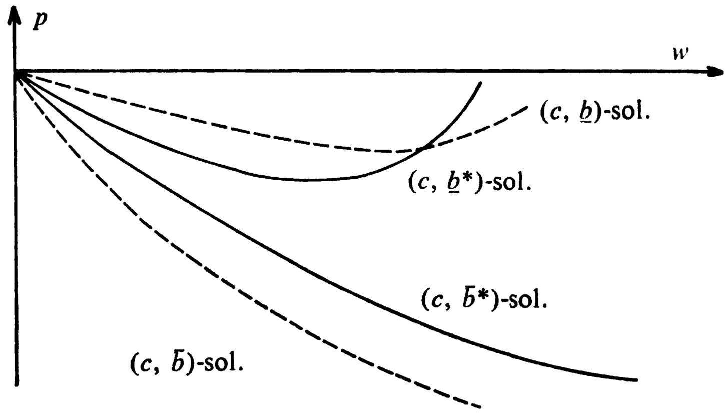


Fig. II ($F\leqq F^*$)

Corollary. *Under the assumptions of Lemma* 1.1 (i) *if* w *and* w^* *are the* $(c, \bar{b})$- *and* $(c, \bar{b}^*)$-*solutions, respectively, with* $w(0) \leqq w^*(0)$, *then* $w(x) < w^*(x)$ *for* $x > 0$, *and* (ii) *if* w [*resp.* w^*] *is a* $(c, \underline{b})$- [*resp.* $(c, \underline{b}^*)$-] *solution, then there exists a* $(c, \underline{b}^*)$- [*resp.* $(c, \underline{b})$-] *solution* w^* [*resp.* w] *such that* $w(x) > w^*(x)$ *for* $x > 0$.

2. Asymptotic behaviors of c-fronts as $x \to +\infty$

The tail of a c-front for large x is nicely approximated by that of a solution for the linear equation $2^{-1}w'' + cw' + \alpha w = 0$, if F behaves regularly (in some sense) near zero. Indeed a theorem in the stability theory says that the error of the aproximation is a small order of e^{-px} with some $p > 0$ if $\alpha u - F(u) = o(u^{1+q})$ with some $q > 0$ (cf. [2]). Here we find (weaker) conditions sufficient and almost necessary for certain estimations about the approximation to hold. Symbols and terminologies introduced in the previous section are used also here (and later sections).

Let us introduce a function $\xi(u)$ defined by

$$F(u) = \alpha u + \tfrac{1}{2}\xi(u) \qquad (\alpha = F'(0)).$$

Theorem 2.1. *Let* $c > \sqrt{2\alpha}$ and $c \geqq c_0$. *Assume*

$$\int_{0+} |\xi(u)| u^{-2} du < \infty. \tag{2.1}$$

Then the c-front satisfies

$$w_c(x) = ae^{-bx}(1 + o(1)) \qquad as \quad x \longrightarrow \infty, \tag{2.2}$$

where $b = -\lim_{x \to \infty} (w_c'(x)/w_c(x))$ *and a is a positive constant and given by*

$$a = \begin{cases} (2\sqrt{c^2 - 2\alpha})^{-1}\{\bar{b}w_c(0) + w_c'(0) - \int_0^\infty e^{bs}\xi(w_c(s))ds\} & if \quad c > c_0 \\ -(2\sqrt{c^2 - 2\alpha})^{-1}\{\underline{b}w_c(0) + w_c'(0) - \int_0^\infty e^{bs}\xi(w_c(s))ds\} & if \quad c = c_0 \end{cases}$$

Theorem 2.2. *Assume* $c_0 = \sqrt{2\alpha}$ *and*

$$\int_{0+} |\xi(u)| u^{-2} |\log u| du < \infty. \tag{2.3}$$

Then the c_0-*front satisfies either* (2.2) *with* $c = c_0$, $b = \sqrt{2\alpha}$ *or*

$$w_{c_0}(x) = a_1 x e^{-\sqrt{2\alpha}x}(1 + o(1)) \qquad as \quad x \longrightarrow \infty, \tag{2.4}$$

where a *or* a_1 *are positive constants and given by*

$$a = w_{c_0}(0) + \int_0^\infty s e^{\sqrt{2\alpha}s}\xi(w_{c_0}(s))ds,$$

$$a_1 = \sqrt{2\alpha}\, w_{c_0}(0) + w_{c_0}'(0) - \int_0^\infty e^{\sqrt{2\alpha}s}\xi(w_{c_0}(s))ds.$$

Remark. *If $\xi(u)$ has a definite sign near $u=0$, the condition* (2.1) *is also necessary for* (2.2) *to hold. The similar statement is asserted to the case of Theorem* 2.2 *as well.* (See Lemmas 2.1 and 2.2)

Theorem 2.3. *Under the assumptions of Theorem* 2.2 *a sufficient condition for* (2.2) *to hold is that there exist no F^*, satisfying* (3) *and not identically equal to F, such that $F^* \geqq F$ and $c_0^* = \sqrt{2\alpha}$ (where c_0^* is the minimal speed corresponding to F^*). If ξ satisfies that*

$$(2.5) \qquad \int_{0+} |\xi'(u)/u| \, |\log u| du < \infty \quad or \quad \xi(u) = o(u^{1+p}) \qquad p>0$$

then this condition is also necessary.

If $\xi \leqq 0$ near zero, then (2.4) *holds under* (2.3).

Applying the last theorem to examples of the section 1, we have that if $\int_{0+} [|G''(u)| + |G'(0) - G'(u)|/u] |\log u| du < \infty$ or $G'(0) - G'(u) = o(u^p)$, $p>0$, then the c_0-front for $F = G \cdot H$ with $\kappa > \sqrt{2G'(0)}$ satisfies (2.4).

The proofs of these theorems follow from Theorem 1.1 and lemmas presented below. The proofs of lemmas are somewhat complicated and may be skipped if the reader is contented with the result for $\xi(u) = o(u^{1+p})$, $p>0$ or is little interested in the problem all its own.

Let us write $z(x) = e^{bx} w(x)$ with $b = -\lim_{x\to\infty} (w'/w)$ for a c-solution which satisfies $w(+\infty) = 0$ and write $z_\infty = \lim_{x\to\infty} z(x)$ if the limit exists.

Lemma 2.1. *Suppose $c > \sqrt{2\alpha}$ and at least one of the following conditions holds*;

$$(2.6) \qquad \int_{0+} \xi(u)^+ u^{-2} du < \infty, ^{(*)}$$

$$(2.7) \qquad \int_{0+} \xi(u)^- u^{-2} du > -\infty. ^{(*)}$$

Then for any c-solution w with $w(+\infty) = 0$ there exists $z_\infty = \lim_{x\to\infty} z(x)$ with $0 \leqq z_\infty \leqq \infty$. If $b = \bar{b}$ the condition (2.6) [*resp.* (2.7)] *is necessary and sufficient that $z_\infty < \infty$* [*resp.* $z_\infty > 0$]. *If $b = \underline{b} > 0$, the condition* (2.6) [*resp.* (2.7)] *is nceessary and sufficient that $z_\infty > 0$* [*resp.* $z_\infty < \infty$].

Proof. Write the equation (1.2) in the form

$$(d/dx + \bar{b})(d/dx + \underline{b}) w = -\xi(w),$$

then apply the formula $\int_0^x e^{bs} (d/dx + b) f(s) ds = e^{bx} f(x) - f(0)$ twice, and you have the integral equation

(*) $x^+ = \max\{x, 0\}$, $x^- = \min\{x, 0\}$.

(2.8) $$w(x)=Ae^{-\bar{b}x}-Be^{-\bar{b}x}+(\bar{b}-\underline{b})^{-1}\int_0^x[e^{-\bar{b}(x-s)}-e^{-\underline{b}(x-s)}]\xi(w(s))ds,$$

where

$$A=[\bar{b}w(0)+w'(0)]/(\bar{b}-\underline{b}),\quad B=[\underline{b}w(0)+w'(0)]/(\bar{b}-\underline{b}).$$

Since $\log w(x)=\int_0^x w'/w\,ds+\text{const.}=-bx+o(x)$ and since $\bar{b}>\underline{b}$, for the $(c,\bar{b})$-solution we have

(2.9) $$w(x)=-Be^{-\bar{b}x}+e^{-\bar{b}x}(\bar{b}-\underline{b})^{-1}\int_0^x e^{\bar{b}s}\xi(w)ds+(\bar{b}-\underline{b})^{-1}\int_x^\infty e^{-\underline{b}(x-s)}\xi(w)ds,$$

and

(2.10) $$\bar{b}w(0)+w'(0)=\int_0^\infty e^{\underline{b}s}\xi(w(s))ds.$$

Let $b=\underline{b}$, i.e. $w'/w\to-\underline{b}$. It follows from (2.8) that

(2.11) $$z(x)=A-Be^{-(\bar{b}-b)x}+(\bar{b}-\underline{b})^{-1}\int_0^x e^{-(\bar{b}-\underline{b})(x-s)}z\frac{\xi(w)}{w}ds$$
$$-(\bar{b}-\underline{b})^{-1}\int_0^x z\frac{\xi(w)}{w}ds,$$

and

(2.12) $$z'(x)=(\bar{b}-\underline{b})Be^{-(\bar{b}-\underline{b})x}-\int_0^x e^{-(\bar{b}-\underline{b})(x-s)}z\frac{\xi(w)}{w}ds$$

Since $z'(x)=o(z(x))$ (which follows from $w'+bw=e^{-bx}z'=o(w)$),

(2.13) $$z(x)=(1+o(1))\left[A-(\bar{b}-\underline{b})^{-1}\int_0^x z(s)\frac{\xi(w(s))}{w(s)}ds\right].$$

First assume $\int_{0+}\frac{\xi(w)^-}{w^2}dw>-\infty$. From the inequality

$$z(x)\leqq A'+D\int_0^x z(s)P(s)ds$$

where A', D are some constants and $P(s)=-\xi(w(s))^-/w(s)$, we can easily deduce the boundedness of $z(x)$ because of the integrability:

$$\int_0^\infty P(s)ds=\int_0^{w(0)}\frac{|\xi(w)^-|}{w^2}\frac{wdw}{|w'|}<\infty.$$

The integral in the right side of (2.13) converges, for the left side must be non-negative and z is bounded. Now we get that there exists $z_\infty=A-(\bar{b}-\underline{b})^{-1}\int_0^\infty\frac{z\xi(w)}{w}ds$, which is positive only if (2.6) holds.

Next assume $\int_{0+}\frac{\xi(w)^+}{w^2}dw<\infty$. Let us prove that $\varliminf_{x\to\infty}z(x)>0$. Without loss of generality, by virtue of Corollary of Lemma 1.1, we may assume $\xi\geqq0$ near $u=0$, which guarantees the existance of $\lim z(x)=z_\infty$. Then $z_\infty=0$, by (2.13), leads to the contradiction:

$$z(x)=(1+o(1))(\bar{b}-\underline{b})^{-1}\int_x^\infty z\frac{\xi(w)}{w}ds=o(z(x)).$$

Thus $\underline{\lim}\, z(x)>0$. Now it suffices only to prove that if $\int_{0+}\frac{\xi(w)^-}{w^2}dw=-\infty$ then $\lim z(x)=\infty$. Let $\int_{0+}\frac{\xi(w)^-}{w^2}dw=-\infty$. Let x_1 and x_2 are two points such that $z(x_1)\geqq z(x)$ for $x_1<x<x_2$. Then, by (2.11), (2.12) and that $z'(x)=o(z(x))$,

$$z(x)\geqq z(x_1)-z(x_1)(\bar{b}-\underline{b})^{-1}\int_{x_1}^x \xi(w)^+w^{-1}ds-o(z(x_1))$$

for $x_1<x<x_2$. If x_1 is large, $\int_{x_1}^x \frac{\xi(w)^+}{w}ds$ is small and $z(x)$ is little less than $z(x_1)$ for $x>x_1$. Since $\overline{\lim}\, z(x)=\infty$, this proves $\lim z(x)=\infty$. The proof of the lemma in the case $b=\underline{b}$ is completed.

In the case $b=\bar{b}$ we can proceed similary as above starting from (2.9) instead of (2.8). q.e.d.

In the case $c=\sqrt{2\alpha}$, $\alpha>0$, we get, instead of (2.8),

$$w(x)=[w(0)+(w'(0)+bw(0))x]e^{-bx} \tag{2.14}$$

$$-\int_0^x(x-s)e^{-b(x-s)}\xi(w(s))ds, \quad b=\sqrt{2\alpha}.$$

Lemma 2.2. *Suppose $\alpha>0$ and at least one of the following conditions holds*:

$$\int_{0+}\xi(u)^+u^{-2}|\log u|du<\infty, \tag{2.15}$$

$$\int_{0+}\xi(u)^-u^{-2}|\log u|du>-\infty. \tag{2.16}$$

Then for every c^*-solution ($c^*=\sqrt{2\alpha}$) w with $w(+\infty)=0$ there exists $y_\infty=\lim_{x\to\infty} x^{-1}e^{bx}w(x)$ ($b=\sqrt{2\alpha}$) with $0\leqq y_\infty\leqq\infty$. If both (2.15) *and* (2.16) *hold, there then occur two and only two cases*: $0<y_\infty<\infty$ ($z_\infty=\infty$); $0<z_\infty<\infty$ ($y_\infty=0$),(*) *and a c^*-solution to the latter case (or corresponding c^*-manifold) is obtained as the limit of $(c, \bar{b})$-solutions (or $(c, \bar{b})$-manifolds) as $c\downarrow c^*$. Conversely if one of these cases occurs to some c^*-solution, then both* (2.15) *and* (2.16) *hold. The cases $y_\infty=\infty$ occur if and only if* (2.16) *does not hold. If* (2.16) [*resp.* (2.15)] *fails to hold, then $y_\infty=0$ implies $z_\infty=0$* [*resp.* $\overline{\lim}\, z(x)=\infty$].

Remark. *In the above lemma if* $\int_{0+}\frac{\xi(u)^+}{u^2}du=\infty$ (which implies (2.15) fails and hence (2.16) holds by the assumption of the lemma) *then there is no c^*-solution with $w(+\infty)=0$. In such case we have $c_0>c^*$.*

Proof. Let w be a c^*-solution with $w(+\infty)=0$. First we note that $x=-b^{-1}\log w+o(\log w)$ as $x\to\infty$ and that

(*) The use of the symbol z_∞ implies the existence of $\lim z(x)$.

$$\int_0^x s\frac{\xi(w)^{\pm}}{w}ds=\int_{w(x)}^{w(0)} b^{-2}\frac{\xi(u)^{\pm}}{u^2}|\log u|du(1+o(1)).$$

From (2.14) it follows that

$$z(x)=w(0)+(w'(0)+bw(0))x-\int_0^x(x-s)z\frac{\xi(w)}{w}ds, \tag{2.17}$$

or, introducing the notation $y(x)=z(x)/x$,

$$y(x)=w(0)/x+w'(0)+bw(0)-\int_0^x\left(1-\frac{s}{x}\right)ys\frac{\xi(w)}{w}ds. \tag{2.18}$$

Assume $y(x)$ to be bounded as $x\to\infty$. (Notice this holds always under (2.16) as is easily seen (see the proof of the next lemma).) Then we have, by the hyposesis of the lemma,

$$\int_0^\infty z\frac{|\xi(w)|}{w}ds<\infty, \tag{2.19}$$

which further implies that $\int_0^x sz\frac{\xi(w)}{w}ds=o(x)$ and hence that

$$y_\infty=w'(0)+bw(0)-\int_0^\infty z\frac{\xi(w)}{w}ds. \tag{2.22}$$

It is clear, by (2.18), that $y_\infty>0$ implies both (2.15) and (2.16).

Let $y_\infty=0$. Then we have

$$z(x)=w(0)+x\int_x^\infty z\frac{\xi(w)}{w}ds+\int_0^x sz\frac{\xi(w)}{w}ds, \tag{2.23}$$

or, by dividing by z

$$1=\frac{w(0)}{z(x)}+\int_x^\infty\frac{z(s)/s}{z(x)/x}s\frac{\xi(w(s))}{w(s)}ds+\int_0^x\frac{z(s)}{z(x)}s\frac{\xi(w(s))}{w(s)}ds. \tag{2.24}$$

Assume (2.15) to be true. Then $z(x)$ is bounded. Because, assuming the contrary, we can chose a sequence $x_1, x_2,\ldots$ such that $z(x)\leqq z(x_n)$ for $x\leqq x_n$, $z(x_n)/x_n>z(x)/x$ for $x>x_n$ and $z(x_n)\to\infty$, which leads to the contradiction, for the right side of (2.24) tends to zero along this sequence. The boundedness of $z(x)$ implies, by (2.23), that $\int_0^\infty sz\frac{|\xi(w)|}{w}ds<\infty$, and hence that

$$z_\infty=w(0)+\int_0^\infty sz\frac{\xi(w)}{w}ds. \tag{2.25}$$

Clearly $z_\infty>0$ only if (2.16) holds. If $z_\infty=0$, we have $z(x)=-\int_x^\infty(s-x)z\frac{\xi(w)}{w}ds$, from which we deduce that (2.16) is spoiled. Thus we have proved that under (2.15) $y_\infty=0$ implies $0\leqq z_\infty<\infty$ where $z_\infty>0$ is equivalent to (2.16).

Now we prove the existance of a c^*-solution with $y_\infty>0$ under (2.15). Let w be a c^*-solution defined on an interval $(0, x_0)$, $0<x_0\leqq\infty$, with $w'(0)/w(0)=-b/2$, $w(0)>0$ $(w(x_0\text{-})=0)$. Note that Corollary of Lemma 1.1 can be readily modified to the present case (where $c=c^*$) and assume $\xi\geqq 0$ near $u=0$. It follows for $0<x\leqq x_0$, from (2.14), that $w(x)\leqq w(0)\left(1+\frac{b}{2}x\right)e^{-bx}$ and then, from (2.18), that

$$y(x)\geqq\frac{b}{2}w(0)-w(0)\int_0^x\left(1+\frac{b}{2}s\right)\frac{\xi(w(s))}{w(s)}ds.$$

Taking $w(0)$ to be small, we have, by the initial condition of w'/w, $-w'(x)/w(x)\geqq b/2$ for $0<x<x_0$, which implies $-\log w(x)\geqq\frac{b}{2}x$ and using this we deduce

$$y(x)\geqq w(0)\left[\frac{b}{2}-\int_0^{w(0)}(1-\log w)\frac{\xi(w)}{w^2}\frac{2}{b}dw\right]\qquad 0<x<x_0.$$

Since the integral in the right hand side converges by (2.15), we have, for a small $w(0)$, $y(x)\geqq\frac{b}{4}w(0)$ for $0<x<x_0$, which shows in particular that $x_0=\infty$ and $y_\infty>0$, i.e. $w(x)$ is a desired c^*-solution.

Under (2.15) a c^*-solution with $y_\infty=0$ is obtained as the limit of $(c, \bar{b})$-solutions $w(x; c)$, with common small $w(0; c)=a_0$, as $c\downarrow c^*$, where the constant a_0 is chosen so that a c^*-manifold starting from $(a_0, 0)$ enters the origin (the existance of such a constant is proved above). The convergence part is clear, since corresponding $(c, \bar{b})$-manifolds increase as $c\downarrow c^*$ and are bounded above by a c^*-manifold which enters the origin (see Fig. I). To prove $y_\infty=0$, it suffices to show that $z(x; c)=e^{\bar{b}x}w(x; c)$ are bounded uniformly in $x>0$ and $c>c^*$. To see this we may assume, as before, that $\xi\geqq 0$ near $u=0$. Then by (2.9) and (2.10)

$$\begin{aligned}z(x; c)&=w(0; c)+(\bar{b}-\underline{b})^{-1}\int_0^\infty(e^{\bar{b}s}-e^{\underline{b}s})\xi(w(s))ds\\&\quad-(\bar{b}-\underline{b})^{-1}\int_0^\infty(e^{\bar{b}(s-x)}-e^{\underline{b}(s-x)})\xi(w(s))ds\\&\leqq a+\int_0^\infty z(s; c)s\frac{\xi(w(s))}{w(s)}ds=z(\infty; c).\end{aligned}$$

Since $w(x; c)$ is bounded above by a c^*-solution, we have $s\leqq -b^{-1}\log w(s; c)+\log s+0(1)$ and, using this,

$$1\leqq\frac{a}{z(\infty; c)}+K\int_0^a\frac{z(s; c)}{z(\infty; c)}\frac{\xi(w)}{w^2}|\log w|dw$$

where a constant K is independent of c. By (2.15) and Lebesgue's convergence theorem, $z(\infty; c)$ is bounded as $c\downarrow c^*$, which was to be proved.

Let (2.15) be spoiled. Since (2.16) implies the boundedness of $y(x)$, by (2.18) we see $y_\infty=0$ and have (2.23). By Corollary of Lemma 1.1 we see that w is bounded from below on $x>0$ by $\hat{w}$ with $\hat{y}_\infty=0$ where $\hat{w}$ is a c^*-solution for $\hat{\xi}\leqq\xi$ with $\int_{0+}|\hat{\xi}(u)|u^{-2}|\log u|du<\infty$. Therefore $\underline{\lim}\, z(x)>0$, which, by (2.23), turns into $\overline{\lim}\, z(x)=\infty$. (Remark follows from these and (2.19))

If $\overline{\lim}\, y(x)=\infty$ (which occurs only if (2.16) fails), then we can prove that $\lim y(x)=\infty$ as in the last part of the proof of Lemma 2.1. Now the proof of Lemma 2.2

is completed.

Lemma 2.3. *Assume* (2.3) *and* (2.5). *Let* $\alpha>0$. *Then there exists uniquely a* c^**-manifold with* $y_\infty=0$.

Proof. When the latter one of (2.5) is assumed, we can apply a theorem in the stability theory (cf. [2]) to get the result by fixing $z_\infty=w(0)+\int_0^\infty sz\frac{\xi(w)}{w}ds$. Therefore we assume the other one of (2.5). A c^*-solution with $y_\infty=0$ satisfies (2.17) and

$$w'(0)+bw(0)=\int_0^\infty \xi(w)e^{bs}ds. \tag{2.26}$$

Assume there are two such solutions with a common $w'(0)$, say δ, and different $w(0)$'s, say ε_1, ε_2. Then we have $y_\infty=0$ for any c^*-solutions with $w'(0)=\delta$ and $\varepsilon_1<w(0)<\varepsilon_2$. Put $z(0)=\varepsilon$ and regard z as a function of x and ε: $z=z(x;\varepsilon)$. By (2.17) we have that $\eta=\partial z/\partial\varepsilon$ satisfies

$$\eta=1+bx-\int_0^x(x-s)\xi'(w)\eta\, ds. \tag{2.27}$$

It follows from this and from $\eta\geqq 0$ (ε_2 is assumed to be small) that

$$\eta\leqq 1+bx+x\int_0^x|\xi'(w)|\eta\, ds. \tag{2.28}$$

By (2.28) η is bounded from the above on $x\geqq 0$ by the solution $\hat{\eta}$ of the linear equation

$$\hat{\eta}=1+bx+x\int_0^x|\xi'(w)|\hat{\eta}\, ds \tag{2.29}$$

which has the unique solution with the bound: $\hat{\eta}(x)\leqq Ax+B$ where A, B are constants chosen independently of ε and δ. Now differentiate the both sides of (2.26) with respect to ε and we have, by the Fubini's theorem, that $b=\int_0^\infty \xi'(w)\eta\, ds$, the right side of which tends to zero if we let ε small. But this is absurd since ε_2 may be arbitrarily small (together with δ). q.e.d.

Lemmas 2.1 to 2.3 and results of the section 1 prove Theorems 2.1, 2.2 and 2.3 except the last statement in Theorem 2.3. But since, for a front, $w'(0)+bw(0)$ can be assumed to be positive, $\xi\leqq 0$ implies, by (2.22), $y_\infty>0$ as desired.

Theorem 2.4. *Let* $\alpha=0$ *and* $c>c_0$. *Then for any small* $\varepsilon>0$ *we can find constants* C_1, C_2 *and* N *such that*

$$q(x/(c-\varepsilon)+C_1)\leqq w_c(x)\leqq q(x/(c+\varepsilon)+C_2) \qquad \textit{for} \quad x>N$$

where $q(x)$ *is the inverse function of*

$$x(w)=\int_w^{1/2}du/F(u).$$

Proof. Put $\tau(w_c)=w_c'$. Then $F(w)/\tau(w)\to -c$ as $w\downarrow 0$. For any $\varepsilon>0$ we can find $\delta>0$ such that

$$(-c-\varepsilon)\frac{1}{F(w)}\leqq\frac{1}{\tau(w)}\leqq(-c+\varepsilon)\frac{1}{F(w)}\qquad 0<w<\delta.$$

By integrating each part of this inequality, we get for $x>w_c^{-1}(\delta)$

$$(c+\varepsilon)\int_{w_c(x)}^{\delta}du/F(u)\geqq x-w_c^{-1}(\delta)\geqq(c-\varepsilon)\int_{w_c(x)}^{\delta}du/F(u),$$

or equivalently

$$q\left(\frac{x}{c+\varepsilon}-\frac{w_c^{-1}(\delta)}{c+\varepsilon}+q^{-1}(\delta)\right)\geqq w_c(x)\geqq q\left(\frac{x}{c-\varepsilon}-\frac{w_c^{-1}(\delta)}{c-\varepsilon}+q^{-1}(\delta)\right).$$

q.e.d.

To illustrate what Theorem 2.3 says we put $F(u)=u^{1+p}L(u)$ with $p>0$ and L slowly varying at zero. Then for $c>c_0$

$$w_c(x)\sim c^{1/p}q(x)\qquad\text{as}\quad x\longrightarrow\infty$$

and $q(x)$ is regularly varying at infinity with exponent $-1/p$. If we take $F(u)=u(-\log u)^{-1-r}L(-\log u)$ with $r>0$ and L slowly varying at infinity, then for $c>c_0$

$$\log w_c(x)\sim c^{-1/(2+r)}\log q(x)\qquad\text{as}\quad x\longrightarrow\infty,$$

and $|\log q(x)|$ is regularly varying at infinity with exponent $1/(2+r)$. (In these cases (with additional conditions on L) w_{c_0} satisfies (2.2).)

The next lemma will be used in the proof of Theorem 9.3.

Lemma 2.4. *Let $c\geqq c^*$ and $\alpha>0$. Assume the condition of Theorem 2.1 if $c>c^*$ and that of Theorem 2.2 if $c=c^*$. Let S be the part of the half strip $0<w\leqq 1$, $p\leqq 0$ sweeped out by all c-manifolds that enter the origin. Let (w, p) be a c-manifold starting from $(w_0, p_0)\in S$ at $x=0$. Consider the quantities*

$$a\;=\lim_{x\to\infty}e^{bx}w(x)\qquad \textit{if}\quad c>c^*$$

$$a_1=\lim_{x\to\infty}e^{c^*x}x^{-1}w(x)\qquad \textit{if}\quad c=c^*$$

to be functions of $(w_0, p_0)\in S$. Then they are continuous. Especially if (w_0, p_0) approaches to a boundary point of S which is not on $\{(w, p);\ w=1$ or $p=0\}$, then a or a_1 tends to zero in each cases.

Proof. When $c>c^*$, the statement is clear by (2.10) and by the first expression of a in Theorem 2.1 which is valid for any $(c, \underline{b})$-solution, because $\{w/w';\ w<\delta/2\}$ and $\exp\{\underline{b}x\}w(x)$ are uniformly bounded as long as $(w(0), w'(0))$ moves in the intersection of S and $w>\delta$. In case $c=c^*$ use (2.22) and the expression of a_1 in Theorem 2.2.

Corollary. *Let w be a c-solution with $(w(x_0), w'(x_0))$ being a inner point of S. Then, under the assumption of Lemma 2.4, if $(w(x_0), w'(x_0))$ approaches to a boundary point of S not on* $\{(w, p); w=1$ or $p=0\}$ *as a (or a_1) and $w(x_0)$ being fixed, x_0 tends to infinity.*

3. Parabolic Equations

We exibit here comparison theorems on the parabolic equation

$$u^{\cdot}=au''+bu'+cu+Q \qquad u=u(t, x) \tag{3.1}$$

where a, b, c and Q are functions of $(t, x)\in \bar{E}=[0, \infty)\times R$. It is assumed throughout this section that $a\geqq 0$ and $Q\geqq 0$. Most of the results presented below are standard and proofs of some of them are omitted (see e.g. [4], [8]). When we say u satisfies (3.1) in an open set, it means that $u^{\cdot}$, u' and u'' exist, are continuous and satisfy (3.1) together with u in it. Let D be an open set of E_T, $T>0$. We denote by $\bar{D}$ the closure of D in R^2 and by ∂D its boundary. We will further impose on solutions in D the continuity on $\bar{D}$.

Proposition 3.1. *Let u satisfy* (3.1) *in an open set D of E_T, $T>0$ and be continuous on $\bar{D}$. Assume there exists a constant M such that*

$$a(t, x)\leqq M,\ |b(t, x)|\leqq M(|x|+1),\ c(t, x)\leqq M(x^2+1) \tag{3.2}$$

and

$$u(t, x)\geqq -Me^{Mx^2} \tag{3.3}$$

for $(t, x)\in D$. Then $u\geqq 0$ in D if $u\geqq 0$ on $\partial D-l_T$.(*)

Proposition 3.2. *Let D be an open set contained in the rectangle* $(0, T)\times(0, 1)$. *Let u satisfy* (3.1) *in D and be continuous on $\bar{D}$. Suppose there exists a constant M such that*

$$a(t, x)\leqq Mx^2(1+|\log x|),\ |b(t, x)|\leqq Mx(1+|\log x|) \tag{3.4}$$
$$\text{and}\quad c(t, x)\leqq M(1+|\log x|)\quad \text{for}\quad (t, x)\in D.$$

Then $u\geqq 0$ in D if $u\geqq 0$ on $\partial D-l_T$.

Proof. *Putting $u_*(t, x)=u(t, \exp\{(1-x^2)/2\})(1\leqq x<\infty)$,* apply Proposition 3.1 to u_*.

Proposition 3.3. *Let D be a rectangle $(0, T)\times(0, L)$ with $0<L\leqq\infty$. Let u satisfy* (3.1) *in D and be continuous and nonnegative on $\bar{D}$. Suppose that* (3.2) *and* (3.3) *are satisfied, that $b_*=\sup_D b$ and $c_*=\inf_D c$ are finite and $a_*=\inf_D a>0$, that $\delta_1=\inf_{0<t<T} u(t, 0)>0$, and that $\delta_2=\inf_{0<t<T} u(t, L)>0$ if $L<\infty$. There then exists a function v defined and continuous on $\bar{D}$, which is positive on $\bar{D}-l_0$ and*

(*) $l_T=\{(T, x); x\in R\}$.

depends only on and continuously on δ_1, δ_2, a_*, b_*, c_*, T *and* L *such that* $u \geqq v$ *on* $\bar{D}$.

Proof. We prove the proposition only when $L<\infty$. Set

$$v(t, x)=\varepsilon e^{c_* t}\int_{-\infty}^{0} p(a_* t,\ x+b_* t-y)y^2 dy$$

where ε is a positive constant chosen so small that

$$v(t, 0)\leqq\delta_1 \quad \text{and} \quad v(t, L)\leqq\delta_2 \qquad \text{for} \quad 0<t<T.$$

Noticing v is a solution of

$$v^{\cdot}=a_* v''+b_* v'+c_* v \quad \text{with} \quad v(0, x)=x^2 I_{(-\infty,0)}(x),$$

we see that $w=u-v$ satisfies (3.1) with Q replaced by $Q_*=(a-a_*)v''+(b-b_*)v'+(c-c_*)v+Q$ and the boundary condition: $w\geqq 0$ on $\partial D-l_T$. It is easily seen that $v''\geqq 0$ and $v'\leqq 0$, and hence $Q^*\geqq 0$. Therefore by Proposition 3.2 we have $w\geqq 0$ in D as desired.

Proposition 3.4. *Let* u *satisfy* (3.1) *with* $Q\equiv 0$ *in* E and be continuous on $\bar{E}$. *Suppose that* (3.2) *and* (3.3) *are satisfied in* E_t *for each* $t>0$ *and that* c *is bounded below on each compact set of* $\bar{E}$. *Suppose* $g(x)=u(0, x)$ *satisfies*

$$g(x)\leqq 0 \quad \text{if} \quad x_1<x<x_2;\ \geqq 0 \quad \text{if} \quad x<x_1 \quad \text{or} \quad x>x_2$$

with some extended real constants x_1 *and* x_2: $-\infty\leqq x_1<x_2\leqq\infty$. *Then there exist extended real functions* $X_1(t)$ *and* $X_2(t)$ *of* $t>0$ *with* $-\infty\leqq X_1(t)\leqq X_2(t)\leqq\infty$ such that

$$u(t, x)\begin{cases}\leqq 0 & \text{if} \quad X_1(t)<x<X_2(t)\\ >0 & \text{if} \quad x<X_1(t) \quad \text{or} \quad x>X_2(t).\end{cases} \tag{3.5}$$

If $x_1=-\infty$ [*resp.* $x_2=\infty$], *we may set* $X_1(t)\equiv-\infty$ [*resp.* $X_2(t)\equiv\infty$].

Proof. By virtue of Proposition 3.1 it suffices to prove that if $u(T, \bar{x}_1)<0$ and $u(T, x_2)<0$ with $\bar{x}_1<\bar{x}_2$, $T>0$ then $u(T, x)\leqq 0$ for $\bar{x}_1<x<\bar{x}_2$. Let D_1 and D_2 are connected components of $\{(t, x);\ u(t, x)<0,\ 0<t<T\}$ whose boundary contains $(T, \bar{x}_1)$ and $(T, \bar{x}_2)$, respectively. Define an open set D contained in E_T by the relation that

$$(t, x)\in D \text{ iff }\begin{cases} y_1<x<y_2 & \text{for some} \quad y_1 \text{ and } y_2 \text{ with}\\ (t, y_1)\in D_1 & \text{and} \quad (t, y_2)\in D_2.\end{cases}$$

Since by Proposition 3.1 both $\bar{D}_1\cap l_0$ and $\bar{D}_2\cap l_0$ contain points of the segment $\{0\}\times[x_1, x_2]$, $g\leqq 0$ on $\bar{D}\cap l_0$ by (3.5) and hence $-u\geqq 0$ on $\partial D-l_T$. Then Proposition 3.1 is applied to $-u$ to get $u\leqq 0$ in D (we may assume $\bar{D}$ is compact by curtailing it if necessary). Thus the proposition is proved.

4. Fundamental Properties of $u(t, x; f; F)$

It is well known that our Cauchy problem (1) and (2) is reduced to finding the solution of the integral equation

$$u(t, x)=P_t f(x)+\int_0^t ds\int_R p(t-s, x-y)F(u(s, y))dy \tag{4.1}$$

such that $0\leqq u\leqq 1$ (this will be proved in the following).

The solution of (4.1) is obtained by the usual method of the successive approximation. The uniqueness of the (bounded) solution is proved by usual method (cf. [13]). Let u be a solution of (4.1) (with $0\leqq u\leqq 1$). Then we have equations for $t>0$

$$u'(t, x) =(P_t f)'(x)+\int_0^t ds\int_R p'(t-s, x-y)F(u(s, y))dy \tag{4.2}$$

$$u'(t, x) =(P_t f)'(x)+\int_0^t ds\int_R p(t-s, x-y)F'(u(s, y))u'(s, y)dy \tag{4.2$'$}$$

and

$$u''(t, x)=(P_t f)''(x)+\int_0^t ds\int_R p'(t-s, x-y)F'(u(s, y))u'(s, y)dy. \tag{4.3}$$

We will use formulas

$$\int_R \frac{|y|}{t}p(t, y)dy=2/\sqrt{2\pi t}, \tag{4.4}$$

$$\int_R \frac{y^2}{t^2}p(t, y)dy=1/t. \tag{4.5}$$

From these equations or formulas it follows that

$$|u'(t, x)|\leqq\sqrt{\pi/2}^{\,-1}\{1/\sqrt{t}+2\|F\|\sqrt{t}\} \tag{4.6}$$

and

$$|u''(t\cdot x)|\leqq 1/t+2\gamma(\|F\|t+1) \tag{4.7}$$

where $\|F\|=\sup_u|F(u)|$. We remark that these inequalities imply in particular, by Huygens property of u, that u' and u'' are bounded on $t>1$, $x\in R$. (Similar boundedness assertion of $u'''$is deduced from (4.8) which follows.) The existance and continuity of $u^{\cdot}$ follows from (4.1) and the inequality (derived from (4.6))

$$\left|\int_R p^{\cdot}(t-s, x-y)F(u(s, y))dy\right| = \left|\int_R p'(t-s, x-y)F'(u)u'dy\right|$$
$$\leqq \text{const.}\{1/\sqrt{s}+\sqrt{s}\}/\sqrt{t-s},$$

Now the derivation of the equation (1) and (2) from (4.1) is immediate. The uniqueness of the continuous solution of (1) and (2) follows from Proposition 3.1 if f is continuous. For a measurable f, putting $f_n=u(1/n,\cdot)$ and $u_n(t, x)=u(t+1/n, x)$ with u a solution of (1) and (2), each u_n is the unique solution of (1) and (2) with f_n in place of f and hence u_n is the solution of (4.1) where f is replaced by f_n. Letting n tend to infinity we see that u satisfies (4.1). Therefore uniqueness assertion for the equations (1) and (2) follows from that for the equation (4.1).

Let u be a solution of (1) and (2): $u(t, x)=u(t, x; f)$. Then u satisfies (4.1) as just proved. By differentiating the both sides of (4.1) with respect to t we obtain

$$(4.8)\qquad u^{\cdot}(t, x)=(P_t f)^{\cdot}+P_t F(f)(x) +\int_0^t ds\int_R p(t-s, x-y)F'(u(s, y))u^{\cdot}(s, y)dy$$

from which the existances of u''' and $u^{\cdot\prime}$ follow. Putting $v=u'$

$$(4.9)\qquad v^{\cdot}=\tfrac{1}{2}v''+F'(u)v.$$

Suppose f is Lipshitz continuous on R. Then by (4.2) u' is bounded on E_T, $T<\infty$, and converges to f' at any points where f' exists, since $(P_t f)'$ has these properties.

Suppose F'' exists and is continuous on $[0, 1]$. Then from (4.8), by considering $u^*(t, x)=u(t+1/n, x)=u(t, x; u(1/n,\cdot))$ if necessary, we see as before that $v=u^{\cdot}$ satisfies (4.9). If we further assume that f' exists and is Lipshitz continuous on R, $u^{\cdot}$ is bounded on each E_T and converges to $\frac{1}{2}f''+F(f)$ at any point at which f'' exists.

The next lemma will be used repeatedly.

Lemma 4.1. *Let $k(t, x)$ and $Q(t, x)$ are bounded measurable functions on E. Then for each bounded measurable function g on R the integral equation*

$$(4.10)\qquad u(t, x)=P_t g(x)+\int_0^t ds P_{t-s}\{k(s,\cdot)u(s,\cdot)+Q(s,\cdot)\}(x)$$

has the unique solution which is bounded and continuous on E_T, $T<\infty$. Such a solution satisfies

$$(4.11)\qquad e^{k_* t}P_t g^-(x)+\int_0^t e^{k_*(t-s)}P_{t-s}Q^-(s,\cdot)(x)ds \leqq u(t, x)\leqq e^{k^* t}P_t g^+(x)+\int_0^t e^{k^*(t-s)}P_{t-s}Q^+(s,\cdot)(x)ds$$

where $k^*=\sup k(t, x)$.

Proof. For a measurable function v on E we write

$$(4.12)\qquad \mathbf{K}_\lambda v=\mathbf{K}_\lambda v(t, x)=\int_0^t e^{\lambda(t-s)}P_{t-s}v(s,\cdot)(x)ds,$$

where λ is a real constant, if the double integral for $|v|$ is finite. Then formulas

(4.13) $$\sum_{n=1}^{\infty}(\mu\mathbf{K}_\lambda)^n v=\mu\mathbf{K}_{\lambda+\mu}v,$$

(4.14) $$\lambda\mathbf{K}_\lambda\{P.g\}(t,\cdot)=e^{\lambda t}P_t g-P_t g,\quad \mathbf{K}_0\{P.g\}(t,\cdot)=tP_t g,$$

(4.15) $$(\lambda-\mu)\mathbf{K}_\lambda\circ\mathbf{K}_\mu=\mathbf{K}_\lambda-\mathbf{K}_\mu$$

are valid as far as the both sides of each equation have the meaning. Rewrite the equation (4.10) in the form $u=P.g+\mathbf{K}_0\{ku+Q\}$, and apply $\mathbf{K}_{-\lambda}$ to the both sides of it, then we obtain, by (4.14), (4.15), the equation

$$u=e^{-\lambda t}P_t g+\mathbf{K}_{-\lambda}\{(k+\lambda)u+Q\}.$$

Iterating this equation, we see that the solution of (4.10) is necessarily given by

$$u=\sum_{n=1}^{\infty}[\mathbf{K}_{-\lambda}\circ(k+\lambda)]^n\{[e^{-\lambda t}P_t g]_{t=\cdot}+\mathbf{K}_{-\lambda}Q\}$$

where $\mathbf{K}_{-\lambda}\circ(k+\lambda)$ is the mapping: $v\mapsto\mathbf{K}_{-\lambda}\{(k+\lambda)v\}$. Chosing λ so large that $k+\lambda\geqq 0$, we have

$$\begin{aligned}u&\leqq\sum_{n=0}^{\infty}[(k^*+\lambda)\mathbf{K}_{-\lambda}]^n\{[e^{-\lambda t}P_t g^+]_{t=\cdot}+\mathbf{K}_{-\lambda}Q^+\}\\&=(k^*+\lambda)\mathbf{K}_{k^*}[e^{-\lambda t}P_t g^+]_{t=\cdot}+\mathbf{K}_{k^*}Q^+\\&=e^{k^*t}P_t g^++\mathbf{K}_{k^*}Q^+.\end{aligned}$$

This is the same as the second inequality of (4.11). The first inequality is similarly proved.

Remark 1. In Lemma 4.1 if k and Q are uniformly Lipshitz continuous in x, the solution of (4.10) gives the unique solution, which is bounded on E_T and converges to g as $t\downarrow 0$, for the equation

$$u^\cdot=\tfrac{1}{2}u''+ku+Q.$$

Remark 2. Let F^* and f^* be a function on [0, 1] satisfying (3) and a datum, respectively. Put $u^*=u(t,x;f^*,F^*)$, $u=u(t,x;f;F)$ and $w=u^*-u$. Then w satisfies (4.10) in which $g=f^*-f$, $k=(F(u^*)-F(u))/(u^*-u)^{(*)}$ and $Q=F^*(u^*)-F(u^*)$. Therefore, by the first inequality of (4.11), *if* $F^*\geqq F$ *and* $f^*\geqq f$ *then* $u^*\geqq u$.

Let g be a bounded nonnegative measurable function on R. Then it follows from the inequality, valid for $|x|<M$,

(4.16) $$P_t g\geqq\|g\|\int_{|y|>N}p(1,y)dy+\sqrt{2\pi t}^{\,-1}\int_{|y|<M+\sqrt{t}N}g(y)dy$$

that

(4.17) *if* $g_n\longrightarrow g$ *in locally* L^1 *sense and boundedly, then* $\sqrt{t}P_t g_n\to\sqrt{t}P_t g$ *uniformly on* $(0,T)\times(-M,M)$ *for each* $T<\infty$ *and* $M<\infty$.

(*) If $u^*=u$, we put $k=F'(u)$.

Similarly we see that, for $|x|<M$

$$(4.18) \qquad |(P_t g)'(x)| \leqq \sqrt{t}^{-1}\|g\|\int_{|y|>N} p(1, y)|y|dy$$

$$+(\sqrt{2\pi}\,t)^{-1}\int_{|y|<M+\sqrt{t}\,N} g(y)dy.$$

Lemma 4.2. *Let f_n, $n=1, 2,\dots$, and f be data and u_n and u corresponding solutions of* (1) *and* (2). *Suppose $f_n \to f$ in locally L^1 sense. Then $\sqrt{t}u_n \to \sqrt{t}u$, $tu'_n \to tu'$, and $t\sqrt{t}u''_n \to t\sqrt{t}u''$ as $n\to\infty$ uniformly on $(0, T)\times(-M, M)$ for each pair of finite constants M and T.*

Proof. Putting $w_n = u_n - u$, we see that w_n satisfies (4.10) with $g=g_n=f_n-f$, $k=(F(u_n)-F(u))/(u_n-u)$ and $Q\equiv 0$. Therefore that $\sqrt{t}w_n\to 0$ in the desired sense follows from (4.11) and (4.17). By (4.2) we have

$$|w'_n| \leqq |(P_n g_n)'| + \int_0^t ds\int_R |p'(t-s, x-y)[F(u_n(s, y))-F(u(s, y))]|dy.$$

The first term multiplied by t tends to zero by virtue of (4.18). The second term is bounded, for $|x|<M$, by

$$2\|F\|\int_{0<s<\varepsilon \text{ or } t-\varepsilon<s<t} ds\int_{-\infty}^{\infty} p(t-s, y)\frac{|y|}{t-s}dy$$

$$+4\|F\|\int_{\varepsilon}^{t-\varepsilon} ds\int_N^{\infty} p(t-s, y)\frac{y}{t-s}dy + 4\gamma\sqrt{t/2\pi}\sup_{\substack{|y|<N+M \\ \varepsilon<s<t}} |w_n(s, y)|.$$

Chose ε so small and N so large that the first two terms are less than an arbitrarily given positive constant and then let n tend to infinity so that the last term tends to zero. This proves that $tw'_n\to 0$. The last convergence assertion is proved similarly by using (4.3).

Lemma 4.3. *Let F_n, $n=1, 2,\dots$, and F be functions on* $[0, 1]$ *satisfying* (3) *and u_n and u corresponding solutions with common initial datum f; $u_n = u(t, x; f; F_n)$, $u=u(t, x; f; F)$. Suppose $F_n\to F$ uniformly. Then $u_n\to u$ and $u'_n\to u'$ uniformly on E_T for each $T<\infty$. Further suppose $F'_n\to F'$ uniformly. Then $\dot{u}_n\to \dot{u}$ and $u''_n\to u''$ in the same sense.*

Proof. Set $w_n=u_n-u$. Then w_n satisfies (4.10) with $g=0$, $k=(F(u_n)-F(u))/(u_n-u)$ and $Q=F_n(u_n)-F(u_n)$. Putting $\delta_n=\|F_n-F\|$, we have $|w_n|\leqq \delta_n\gamma^{-1}(e^{\gamma t}-1)$. This proves $u_n\to u$ in the required sense. Remaining assertions are similarly proved by (4.2) or (4.3).

Lemma 4.4. *Let a datum f have the continuous first derivative on R which is Lipshitz continuous there. Suppose there exists extended real constants x_1 and x_2; $-\infty\leqq x_1\leqq x_2\leqq\infty$, and $c>0$ such that*

$$\text{(4.19)} \qquad \tfrac{1}{2}f''+cf'+F(f)\begin{cases} \geqq 0 & \textit{if} \quad x<x_1 \quad \text{or} \quad x>x_2 \\ \leqq 0 & \textit{if} \quad x_1<x<x_2 \end{cases}$$

where x's are those points at which f'' exist. Then there exist extended real functions X_1 and X_2 of $t>0$ with $-\infty \leqq X_1(t) \leqq X_2(t) \leqq \infty$ such that

$$\text{(4.20)} \qquad z^{\cdot}(t, x)\begin{cases} >0 & \textit{if} \quad x<X_1(t) \quad \textit{or} \quad x>X_2(t) \\ \leqq 0 & \textit{if} \quad X_1(t)<x<X_2(t) \end{cases}$$

where $z(t, x)=u(t, x+ct; f)$. If $x_1=x_2$ we may put $X_1 \equiv X_2$, if $x_1=-\infty$ then $X_1 \equiv -\infty$ and if $x_2=\infty$ then $X_2 \equiv \infty$.

Proof. At first assume F'' exists and is continuous. Then by the equation (4.9) and remarks mentioned just after it the function $v(t, z)=z^{\cdot}(t, x-ct)=u^{\cdot}(t, x)+cu'(t, x)$, where $u=u(t, x; f)$, satisfies (4.9) and that, as $t \downarrow 0$,

$$v(t, x) \longrightarrow g(x)=\tfrac{1}{2}f''(x)+cf'(x)+F(f(x)) \qquad \text{a.s.}$$

and is bounded on E_T, $T<\infty$. It is proved as before that $v(t, x)$ can be approximated uniformly on each finite rectangle $[T^{-1}, T] \times [-M, M]$ by solutions of (4.9) v_n such that v_n are continuous on $t \geqq 0$ and $g_n=v_n(0, \cdot)$ satisfy that $g_n(x) \geqq 0$ if $x<x_1$ or $x>x_2$ and $\leqq 0$ if $x_1<x<x_2$. Therefore we may assume that g is continuous to apply Proposition 3.4 which proves (4.20) (see also Proposition 3.3). In the case that F'' does not exist, use Lemma 4.3 and notice Proposition 3.3 to see the strict inequality in (4.20). q.e.d.

Lemma 4.5. *Suppose two data f and f^* satisfy*

$$f^*(x) \leqq f(x)+O(e^{-bx})$$

where b is a positive constant. Set $v(t, x)=u(t, x; f^*)-u(t, x; f)$. Then for each constant c

$$v(t, x+ct) \leqq O(e^{-\kappa t-bx})^{(*)} \quad \textit{with} \quad \kappa=b(c-b/2-\gamma^*/b),$$

and if $c<b$, for each finite N,

$$v(t, x+ct) \leqq O(\sqrt{t}^{-1}e^{-(c^2/2-\gamma^*)t}) \quad \textit{uniformly in} \quad x>N.$$

Proof. By Lemma 4.1 (see Remark 2 for it) we have

$$v(t, x) \leqq e^{\gamma^* t}P_t[f^*-f]^+(x+ct).$$

Set

$$g(x)=1 \quad \text{if} \quad x<0, \quad =e^{-bx} \quad \text{if} \quad x>0.$$

Then

(*) If $f_*(x) \leqq f(x)+o(e^{-bx})$, then this can be replaced by "$v(t, x+ct) \leqq o(e^{-\kappa t-bx})$ as $t \to \infty$ uniformly in $x>N$."

$$P_t g(x)=\int_{x/\sqrt{t}}^{\infty} p(1, y)dy+e^{-bx+\frac{b^2}{2}t}\int_{-\infty}^{(x-bt)/\sqrt{t}} p(1, y)dy.$$

The lemma follows from these and the formula $\int_x^{\infty} p(1, y)dy \sim \frac{1}{x} p(1, x)$ as $dx\to\infty$.

The following theorem asserts that if $c\geqq\sqrt{2\gamma^*}$ c-fronts are stable in a sense.

Theorem 4.1. *Let $c\geqq\sqrt{2\gamma^*}$ and $f(x)=w_c(x+x_0)+O(e^{-bx})$, with some constants b and x_0. Then*

$$u(t, x+ct; f)=w_c(x+x_0)+O(e^{-\kappa t-bx})$$

where κ is defined in Lemma 4.5; and if $c<b$

$$u(t, x+ct; f)=w_c(x+x_0)+O(\sqrt{t}^{-1}e^{-(c^2/2-\gamma^*)t}) \qquad \textit{uniformly in} \quad x>N>-\infty.$$

Proof. Immediate from Lemma 4.5 and the stationarity of c-fronts: $u(t, x+ct; w_c(\cdot+x_0))=w_c(x+x_0)$.

Let $\alpha>0$. It is proved in McKean [14] (in case $F(u)=u(1-u)$) that *if $f(x)\sim aw_c(x)$ as $x\to\infty$ with $a>0$ and $c\geqq\sqrt{2\gamma^*}$ then $u(t, x+ct; f)\to w_c(x+x_0)$ uniformly in $x>N$, where $x_0=b^{-1}\log a$, $b=-\lim_{x\to\infty}[w_c'/w_c]$.*

Here is a proof of this assertion under our setting. It will be not wasteful to remark that Theorem 4.1 is not directly available for the present problem since $w_c(x)$ decays as $x\to\infty$ little more rapidly than e^{-bx}, $b=c-\sqrt{c^2-2\alpha}\leqq c$ and $\kappa=\alpha-\gamma^*\leqq 0$. Now we return to the proof. By the relation $f(x)\sim w_c(x+x_0)$, for any fixed $\delta>0$, we have $w_c(x+x_0+\delta)\leqq f(x)\leqq w_c(x+x_0-\delta)$ for all sufficiently large x, and by Lemma 4.5 we see

$$w_c(x+x_0+\delta)-Q(t)\leqq u(t, x+ct; f)\leqq w_c(x+x_0-\delta)+Q(t)$$

for $x>N$ with $Q(t)=O(t^{-1/2}\exp\{-(c^2/2-\gamma^*)t\})$. In particular $u(t, x+ct; f)\to w_c(x+x_0)$ as desired.

5. Limits of $u(t, x+ct; f)$

We will investigate in this section the problem: what is the limit of

$$z(t, x)=u(t, x+ct; f)$$

as $t\to\infty$? *The limit $w(x)=\lim_{t\to\infty} z(t, x)$, if exists, must be a solution of (4) on R.* In fact, in the equation

$$z(t+s, x)=u(t, x+ct; z(s, \cdot))$$

letting s tend to infinity, we have, by Lemma 4.2, the equation

$$w(x)=u(t, x+ct; w),$$

from which we see that w satisfies the equation (4) on R. In particular if $0\leqq c<c_0$, $w\equiv 1$ or $w\equiv 0$.

The following lemma is due to Aronson and Weinberger [1] except some additional statements. The proofs given here are based on their idears.

Lemma 5.1. *Let $q(x)$ be a c-solution ($c \geqq 0$) defined on a interval (L_1, L_2), $-\infty \leqq L_1 < L_2 \leqq \infty$, such that $q(L_1)=0$ or $=1$ and that $q(L_2)=0$.*

(i) *Let $q(L_1)=1$ and $L_2=\infty$ (these implies $c \geqq c_0$).* Set

$$f(x)=1 \quad \text{if} \quad x<L_1, \quad =q(x) \quad \text{if} \quad L_1<x \tag{5.1}$$

and set $z(t, x)=u(t, x+ct; f)$. Then $z(t, x)$ decreases with t. The limit $w(x)=\lim z(t, x)$ is zero if $\underline{\lim}_{x\to\infty} q(x)/w_c(x)=0$, and it is a c-front if otherwise and $\alpha>0$.

(ii) *Let $q(L_1)=0$. Put*

$$f(x)=0 \quad \text{if} \quad x<L_1 \quad \text{or} \quad x>L_2, \quad =q(x) \quad \text{if} \quad L_1<x<L_2 \tag{5.2}$$

and set $z(t, x)=u(t, x+ct; f)$. Then $z(t, x)$ increases with t. Its limit is unity if $c<c_0$ or $\overline{\lim}_{t\to\infty} q(x)/w_c(x)=\infty$, and it is a c-front if otherwise and $\alpha>0$.

Proof. We prove only (i), since the proof of (ii) is very similar. Let $q(L_1)=1$, $L_1>-\infty$ and $L_2=\infty$. Noticing $q'(L_1+0)<0$, define for each constant $a>\|F\| =\sup F(u)$

$$f^*(x)=\begin{cases} 1 & x<L_1-\delta \\ 1-a(x-L_1+\delta)^2 & L_1-\delta<x<L_1+\delta \\ q(x) & x>L_1+\delta \end{cases}$$

where δ is a positive constant possibly chosen so that $q'(L_1+\delta)=-4a\delta$ and that $1-a(2\delta)^2=q(L_1+\delta)$. Then f^* is continuous, has the continuous first derivative and satisfies $\frac{1}{2}f^{*\prime\prime}+cf^{*\prime}+F(f^*)\leqq 0$ at any $x\neq L_1\pm\delta$. Thus Lemma 4.4 says that $z^*(t, x)=u(t, x+ct; f^*)$ decreases with t. It is clear that z has the same property by virtue of Lemma 4.2, since f^* converges to f as $a\to\infty$. If $\underline{\lim}\, q(x)/w_c(x)>0$, we have $f(x)\geqq w_c(x+x_0)$ with some constant x_0 and hence $z(t, x)\geqq u(t, x+ct; w_c(\cdot+x_0)=w_c(x+x_0)$. Thus $w(x)=\lim z(t, x)\geqq w_c(x+x_0)$. This proves that w is a c-front. If $\underline{\lim}\, q/w_c=0$, we have $\underline{\lim}\, w/w_c=0$, but this implies $w=0$ because $w_c(x+x_0)/w_c(x)$ converges to e^{-bx_0} as $x\to\infty$. q.e.d.

The information on the behavior of $u(t, x+ct; f)$ may be roughly gathered by Theorem 5.1 stated below. Results will be somewhat sharpened in Theorems 9.3 and 9.4. We will need the following condition on f.

Condition [G]: $u(t, x; f)\to 1$ as $t\to\infty$ locally uniformly.

If $\alpha>0$, this is the case for any data. We will discuss about Condition [G] at the end of this section.

Theorem 5.1. *Let f be a datum and set $z(t, x)=u(t, x+ct; f)$.*

(i) *Let $f(x)=O(e^{-bx})$ as $x\to\infty$. Suppose either that $b>c-\sqrt{c^2-2\alpha}$ and $c>c_0$ or that $b>c^*$, $c=c_0=c^*$, and (2.3) and (2.4) are valid. Then for each $N>$*

$-\infty$

$$z(t, x) \longrightarrow 0 \quad as \quad t \longrightarrow \infty \qquad uniformly\ in \quad x>N.$$

(ii) *Suppose either that* $0 \leqq c < c_0$ *and Condition* [G] *is satisfied or that* $c \geqq c_0$ *and* $\underline{\lim} f(x)e^{bx}>0$ *with* $b<c-\sqrt{c^2-2\alpha}$. *Then for each* $N>-\infty$

$$z(t, x) \longrightarrow 1 \quad as \quad t \longrightarrow \infty \qquad uniformly \quad x>N.$$

Remark. Assertions of Theorem 5.1 are obtained by several authors ([13], [10]) in a special case under some restriction on F and by Aronson and Weinberger in case that f has compact support (mainly) and $c \neq c_0$.

Proof of Theorem 5.1. Let $f(x)=o(e^{-bx})$, $b>c-\sqrt{c^2-2\alpha}$ and $c>c_0$. We can chose a constant c_* such that $c>c_*>c_0$ and $b>c_*-\sqrt{c_*^2-2\alpha}$. Then there exists a c_*-solution q which satisfies the conditions of Lemma 5.1 (i) and for which $f \leqq f^*$ where f_* is defined by the right side of (5.1). In the inequality $z(t, x) \leqq u(t, x+ct; f_*)$ the right hand side tends to zero, since, by Lemma 5.1, $u(t, x+c_*t; f)$ tends to zero or to a c^*-front. Thus $z(t, x) \to 0$. The required uniformity of convergence is obvious. In the case $c=c_0=c^*$ we can similarly proceed, but taking as $q(x)$ a c_0-solution which corresponds to the extremal one in all c_0-manifolds that enter the origin (see the last diagram of Appendix). These prove (i).

Let $\underline{\lim} f(x)e^{bx}>0$ with $b<c-\sqrt{c^2-2\alpha}$, $c \geqq c_0$. We can find a function F_* satisfying (3) such that $F_* \leqq F$ and $F'_* \leqq \alpha$; e.g. set, for small u,

$$F^*(u)=\int_0^u (F'(v) \wedge \alpha)dv. \tag{5.3}$$

Then, putting $u_*=u(t, x+ct; f; F_*)$, we have $z \geqq z_*$ and hence $z(t, \cdot) \to 1$, since, by Lemma 4.5 and Lemma 5.1 (ii), $z_*(t, \cdot) \to 1$. In the case $c<c_0$, we can proceed as in the proof of (i) using Lemma 5.1 (ii). Thus (ii) is proved. q.e.d.

In the case $c=c_0>c_*$, which is excluded from the above theorem, the situation is different from the above (see also § 10):

Lemma 5.2. *Let* $c_0>c^*$. *For any couple of constants* b *and* η *with* $c_0-\sqrt{c_0^2-2\alpha}<b<c_0+\sqrt{c_0^2-2\alpha}$ (*i.e.* $b^2/2-c_0b+\alpha<0$) *and* $b^2/2-c_0b+\alpha<-\eta<0$, *there exist positive constants* A *and* A' *such that if we set*

$$U_*(t, x)=w_{c_0}(x-Ae^{-\eta t})-e^{-\eta t-bx}$$

$$U_*(t, x)=w_{c_0}(x-A'(1-e^{-\eta t}))+e^{-\eta t-bx}$$

and if a datum f *satisfies*

$$U_*(t_1, x+x_1) \leqq f(x) \leqq U^*(t_2, x+x_2) \tag{5.4}$$

for some constants t_1, t_2, x_1, x_2, *then*

$$U^*(t+t_1, x+x_1) \leqq u(t, x+c_0t; f) \leqq U^*(t+t_2, x+x_2) \tag{5.5}$$

for all $t>0$, $x\in R$.

Proof. Extending F to a continuously differentiable function $\bar{F}$ on R so that $F'(u)\leqq\alpha$ for all $u\notin[0, 1]$, and setting $v(t, x)=u(t, x+c_0t; f)-U_*(t+t_1, x+x_1)$, we have, by the mean value theorem,

$$v^{\cdot}=\tfrac{1}{2}v''+c_0v'+F'(\theta)v+Q_*(t+t_1, x+x_1)$$

where

$$Q^*=\tfrac{1}{2}U''_*+c_0U'_*\bar{F}(U_*)-U^{\cdot}_*.$$

Since (5.4) implies $v(0, x)\geqq 0$, for the proof of the left hand side inequality in (5.5) it suffices to show that $Q_*\geqq 0$ on E, by virtue of Proposition 3.1. Set $w(t, x)=w_{c_0}(x-Ae^{-\eta t})$ and $h(t, x)=e^{-\eta t-bx}$. Then, putting $\bar{\alpha}=-b^2/2+c_0b-\eta$,

$$Q_*(t, x)=-\bar{F}(w)+\bar{F}(w-h)+\bar{\alpha}h-A\eta e^{-\eta t}w'.$$

Since $\bar{\alpha}>\alpha$, we can find a positive constant $\delta>0$ so small that

$$0<w<\delta \text{ or } 1-\delta<w<1 \text{ implies } \bar{\alpha}-\frac{1}{h}[\bar{F}(w)-\bar{F}(w-h)]>0$$

for all $h>0$. Then choose constants N and $a>0$ such that

$$\delta<w_{c_0}(x)\leqq 1-\delta \quad \text{implies} \quad w'_{c_0}(x)<-a \quad \text{and}$$

$$w_{c_0}(x)\leqq 1-\delta \quad \text{implies} \quad x>-N.$$

Note that $w(t, x)\leqq 1-\delta$ implies $w_{c_0}(x)\leqq 1-\delta$. Now we may define A by the equation

$$A\eta a-\gamma^*e^{bN}=0$$

so that $Q_*\geqq 0$ (constants δ, N and a are chosen independently of A).

Noticing that $w^*\equiv w_{c_0}(x-A'(1-e^{-\eta t}))\geqq w_{c_0}(x)\geqq h(t, x)$ for large x, and following the procedure similar to that taken in the above, we can find a constant A' such that

$$\begin{aligned} Q^*(t, x)&\equiv\tfrac{1}{2}U^{*\prime\prime}+c_0U^{*\prime}+\bar{F}(U^*)-U^{*\cdot} \\ &=-\bar{F}(w^*)+\bar{F}(w^*+h)-\bar{\alpha}h+A'\eta e^{-\eta t}w^{*\prime} \qquad (U^*=w^*+h) \\ &\leqq 0 \end{aligned}$$

which proves the right hand side inequality of (5.5).

These complete the proof of Lemma 5.2.

As the direct consequence of Lemma 5.2 we have

Lemma 5.3. *Let* $c_0>c^*$ *and* b *and* η *make up a couple of constants in Lemma* 5.2.

(i) *Suppose that a datum* f *satisfies Condition* [G] *and* $f(x)=O(e^{-bx})$. *Then* *for some constants* x_1, x_2, *and* K

$$w_{c_0}(x+x_1)-Ke^{-\eta t-bx}\leqq u(t, x+c_0t; f)\leqq w_{c_0}(x+x_2)+Ke^{-\eta t-bx}$$

for all $t>0$, $x\in R$.

(ii) *For any* $\varepsilon>0$ *there exists a positive constant* δ *such that if* $|f(x)-w_{c_0}(x)|<\delta e^{-bx}$ *for all* $x\in R$ *then* $|u(t, x+c_0t; f)-w_{c_0}(x)|<\varepsilon e^{-bx}$ *for all* $t>0$, $x\in R$.

Remarks to Condition [G]. Functions F satisfying (3) are classified into two classes according as Condition [G] is satisfied for all data or otherwise. In the former [the latter] case we say F belongs to the *class* I [resp. *class* II]. The class II is not empty. Some criteria for F to belong to the class I are obtained by several authors: Fujita [5], Hayakawa [7], Kobayashi-Sirao-Tanaka[12] (they all deal with the problem in the multidimentional case). Hayakawa [7] says that if $\underline{\lim}_{u\downarrow 0} F(u)/u^3>0$(*) then F belongs to the class I and that if $F(u)=o(u^p)$(*) with some $p>3$ then F belongs to the class II. Kobayashi *et al.*'s results are sharpenings of these consequences. Here is a rapid proof of the assertion: *if* $F(u)/u^3\to\infty$ *as* $u\downarrow 0$ *then* F *belongs to the class* I (the proof is good for the multidimentional case). Let f be any datum and set $u=u(t, x; f; F)$. Then there exists $\varepsilon>0$ and $t_0>0$ such that

$$u(1, x)\geqq\varepsilon p(t_0, x) \qquad x\in R.$$

It suffices to prove that $u_*=u(t, x; f_*)\to 1$ where $f_*=\varepsilon p(t_0, \cdot)$. Noticing $u_*(t, x)\geqq P_tf_*(x)=\varepsilon p(t+t_0, x)$, it suffices, in turn, to prove that $g(x)=\varepsilon p(t+t_0, x)$ satisfies, with some $t>0$,

$$\tfrac{1}{2}g''(x)+F(g(x))>0 \qquad x\in R, \tag{5.6}$$

because this inequality implies that $u(t, x; g)$ increases with t and hence tends to unity by what is remarked at the beginning of this section. Since $g''(x)=g(x)(x^2-t_1)/t_1^2$, $t_1=t+t_0$ and since $\varepsilon/\sqrt{2\pi t_1}\geqq g(x)\geqq\varepsilon/\sqrt{2\pi t_1 e}$ for $|x|<\sqrt{t_1}$, we have

$$\tfrac{1}{2}g''+F(g)\begin{cases} >0 & \text{if} \quad |x|\geqq\sqrt{t_1} \\ >F(g)-g/2t_1\geqq[F(g)/g^3-\pi e/\varepsilon^2]g^3 & \text{if} \quad |x|<\sqrt{t_1}. \end{cases}$$

The right hand side of the last inequality is positive for some large t. Thus (5.6) is obtained.

We note that for any $\varepsilon>0$ there exists a datum $f<\varepsilon$ with compact support for which [G] is valid. This follows from Lemma 5.1. It is also obtained that if $\underline{\lim}_{t\to\infty} u(t, x_0)>0$ for some x_0 then [G] is valid, as is proved below. The idea of the proof comes from Kanel' [11a]. We may assume $x_0=0$. It is easily checked that

$$u(t, x)=\int_0^\infty p^*(t, x, y)f(y)dy+\int_0^t ds\int_0^\infty p^*(t-s, x, y)F(u(s, y))dy$$

(*) In n-dimentional case these must be replaced by $\underline{\lim}\, F(u)/u^{1+2/n}>0$ and by $F(u)=o(u^p)$ with $p>1+2/n$, respectively.

$$-\int_0^t u(s, 0)p'(t-s, x)ds \qquad x>0,$$

where $p^*(t, x, y)=p(t, x-y)-p(t, x+y)$. The first two integrals are positive and the last integral, which is equal to $2\int_{x/\sqrt{t}}^{\infty} u(t-x^2/v^2, 0)p(1, v)dv$, is bounded below by $(1/2)\underline{\lim}\, u(t, 0)$ for $0<x<N$ and for all sufficiently large t. Thus the fact remarked at the beggining of this paragraph proves $u(t, x)\to 1$ in the desired sense.

6. Behavior of Front of $u(t, x; f)$ (Special Case)

In this section we treat the problem expressed by (5) when datum f has a certain special form. Suppose $u(t, x)=u(t, x; f)$ decreases as x increases on some right half of the x-axis and tends to zero as $x\to\infty$ for each $t>0$. Put

$$L(t)=\sup\{x;\ u'(t, x)=0\}\ (=-\infty \text{ if } \{\cdot\} \text{ is empty}).$$

$$M(t)=u(t, L(t))$$

$$x(t, w)=\sup\{x;\ u(t, x)=w\} \qquad 0\leqq w\leqq M(t)$$

$$\phi(t\cdot w)=u'(t, x(t, w)).$$

Note that ϕ is determined only by the shape of the tail of u (i.e. invariant under the transform: $f\to f(\cdot+\text{const.})$) and conversely restored to it through the inverse form

$$x(t, w)=\int^w \frac{du}{\phi(t, u)}\,. \tag{6.1}$$

We write $\phi=\phi(t, w; f)$ to express that ϕ is determined by f and $M(t)=M(t;\phi)$ for convenience. We also write $M=M\{\tau\}$ if τ is a nonnegative function defined on an interval $[0, M]$. Thus $M(t;\phi)=M\{\phi(t, \cdot)\}$. For each t the graph of $\phi(t, \cdot)$ is identical to the orbit of $(w, p)=(u(t, x), u'(t, x))$, $L(t)\leqq x<\infty$. We will use abbreviations $\phi^{\cdot}=\partial\phi/\partial t$, $\phi'=\partial\phi/\partial w$, etc. By formulas $\partial x/\partial w=1/u'$, $\partial x/\partial t=-u^{\cdot}/u'$, $\phi'=u''/u'$ and

$$\phi''=\frac{u'u'''-(u'')^2}{(u')^3}, \tag{6.2}$$

we derive from (1) the equation

$$\phi^{\cdot}=\tfrac{1}{2}\phi^2\phi''-F\phi'+F'\phi \qquad 0<w<M(t). \tag{6.3}$$

Let g be another initial datum and set $\psi=\phi(t, w; g)$ and $\omega=\phi-\psi$. Then it follows from (6.3) that ω satisfies

$$\omega^{\cdot}=\tfrac{1}{2}\phi^2\omega''-F\omega'+[F'+\tfrac{1}{2}(\phi+\psi)\psi'']\omega \tag{6.4}$$

in the domain $\{(t, w);\ 0<w<\min\{M(t, \phi), M(t, \psi)\}, t>0\}$. This equation is fundamental in the later arguments. We will consider it as a parabolic equation discussed in §3 by regarding ϕ, ψ, ϕ'' or ψ'' as given functions. Let τ be a solution of (1.4). Then $\omega=\phi-\tau$ satisfies (6.4) where ψ is replaced by τ, since (1.4) implies

$$0=\tfrac{1}{2}\tau^2\tau''-F\tau'+F'\tau \qquad 0<w<M\{\tau\}.$$

We denote by τ_c $(c\geqq c_0)$ the solution of (1.4) corresponding to the c-front, i.e. the unique solution solving (1.4) on the interval $[0, 1]$ with $\tau(0)=\tau(1)=0$. The equation (6.4) will be sometimes cited in the alternative form

$$(6.4)' \qquad \omega^{\cdot}=\tfrac{1}{2}\psi^2\omega''-F\omega'+[F'+\tfrac{1}{2}(\phi+\psi)\phi'']\omega.$$

Lemma 6.1. *Let $c_1\geqq c_0$. Let a datum f satisfy the assumption of Lemma 4.4 with $c=c_1$. Suppose that there exists a constant x_3 such that*

$$(6.5) \qquad f'\geqq 0 \quad \text{if} \quad x\leqq x_3, \quad <0 \quad \text{if} \quad x>x_3,$$

that there exists a function $\varepsilon(t)>0$, $t>0$ such that

$$(6.6) \qquad \phi(t, w; f)<\tau_{c_1}(w) \qquad 0<w<\varepsilon(t),\ t>0^{(*)}$$

and that $u(t, x)=u(t, x; f)$ satisfies

$$(6.7) \qquad u(t, x)\longrightarrow 1 \qquad as \quad t\longrightarrow\infty$$

and

$$(6.8) \qquad u(t, x+c_1t)\longrightarrow 0 \qquad as \quad t\longrightarrow\infty.$$

Then $\phi=\phi(t, w; f)$ satisfies

$$(6.9) \qquad \phi(t, w)\leqq\tau_{c_1}(w)+o(1) \qquad 0\leqq w\leqq M(t;\phi)$$

uniformly as $t\to\infty$.

Proof. Applying Proposition 3.4 to the equation (4.9) satisfied by $v=u'$, we see, by (6.5), that $u'(t,\cdot)<0$ on a right half line and $\geqq 0$ on the other half. Especialy $\phi=\phi(t, w; f)$ is well defined and $M(t;\phi)=\max_{x\in R}u(t, x)$. Put $z(t, x)=u(t, x+c_1t)$. Then by Lemma 4.4 there exists extended real functions $X_1(t)$, $X_2(t)$, $-\infty\leqq X_1\leqq X_2\leqq\infty$, with which (4.20) holds. We will examine the evolution of the orbits of the vector functions $(w(x), p(x))=(z(t, x), z'(t, x))$, $x\in R$ in the half strip $D=\{(w, p); 0<w<1, p\leqq 0\}$. Parts of these orbits contained in D are denoted by S_t. For the proof of the lemma we assumed that $-\infty<X_1(t)<X_2(t)<\infty$ for any $t>0$, since the other case is easy to deal with. Denote by A_t a point in D that has coordinates $(z(t, X_1(t)), z'(t, X_1(t)))=(z(t, X_1(t)), \phi(t, z(t, X_1(t))))$ or coordinates $(M(t;\phi), 0)$ according as $z'(t, X_1(t))<0$ or $\geqq 0$. Denote also by B_t a point with coordinates $(z(t, X_2(t)), z'(t, X_2(t)))$. By the equation $z^{\cdot}=\frac{1}{2}z''+c_1z'+F(z)$ and by what is remarked just before Theorem 1.1, c_1-manifolds cross S_t from the right or the left hand of S_t according as $z^{\cdot}>0$ or <0 at intersecting points (see Fig. III). Notice that $z^{\cdot}\leqq 0$ on the closed arc of S_t between A_t and B_t and $z^{\cdot}>0$ on the other parts of S_t. From these and the hypothesis (6.6) it follows that S_t lies under the c_1-manifold passing through A_t for each $t>0$. We denote this manifold by T_t. A_t

(*) This condition can be removed if $x_2=\infty$ where x_2 appears in (4.19).

and T_t lies over the c_1-front.

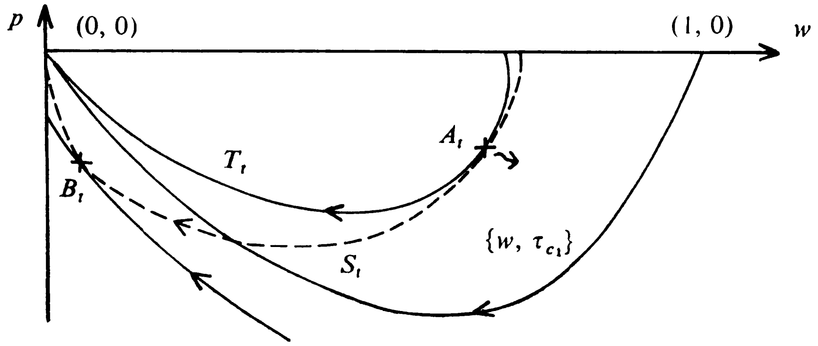


Fig. III

It is shown below that if $z'(t, X_1(t))<0$ then A_s for $s>t$ does not enter into the open domain bounded by T_t and the w-axis untill $z'(s, X_1(s))$ vanishes for the first time after t. Let $\tau(u)$ be a solution of (1.4) corresponding to T_t and set $\omega=\phi-\tau$. Then ω satisfies (6.4) with $\psi=\tau$ and by Proposition 3.1 $\omega(t+s, w)\geqq 0$, $0<w>M\{\tau\}$ for sufficiently small s. This proves the desired assertion. Especially we proved that if $z'(t, X_1(t))<0$ for $t_1<t<t_2$ then T_{t_1} lies over S_t for $t_1<t<t_2$.

To prove the assertion of the lemma first assume that there exists a sequence $\{t_n\}$ such that $t_n\to\infty$ and $z'(t_n, X_1(t_n))\geqq 0$, $n=1, 2,\dots$. By the hypothesis (6.7), for each $\varepsilon>0$ we can find n_0 such that $M(t;\phi)>1-\varepsilon$ for $t>t_{n_0}\equiv t_*$. Let T be a c_1-manifold passing through $(1-\varepsilon, 0)$. If $t>t_*$ and $z'(t, X_1(t))\geqq 0$, then T lies over T_t and hence over S_t. If $t>t_*$ and $z'(t, X_1(t))<0$, then, by what has proved in the previous paragraph and by the continuity of S_t with respect to t, we can find a time t' with $t_*<t'\leqq t$ such that T lies over $T_{t'}$ and $T_{t'}$ lies over S_t. Consequently T lies over T_t for any $t>t^*$. Since T converges to the c_1-front as $\varepsilon\downarrow 0$, we have (6.9).

Next assume the remaining case: $z'(t, X_1(t))<0$ for $t>t_*$ with some constant t_*. Then T_t lies over $S_{t'}$ if $t'\geqq t>t_*$. There exists an unbounded sequence t_n such that w-coordinate of A_{t_n} tends to unity as $n\to\infty$. Because in the opposite case we can find $\delta\in(0, 1)$ such that $z(t_0, x_0)>\delta$ implies $z'(t_0, x_0)>0$, which in turn implies $z(t, x_0)>\delta$ for $t>t_0$ and contradicts to (6.7) and (6.8). Since A_{t_n} tends to the point $(1, 0)$, T_{t_n} converges to the c_1-front. Consequently we have (6.9). The proof of the lemma is completed.

Lemma 6.2. *Let $f_0=I_{(-\infty,0]}$. Then for any data f for which ϕ is well defined, we have*

$$\phi(t, w; f_0)\leqq\phi(t, w; f) \qquad t>0, \quad 0<w<M(t)$$

where $M(t)=M(\phi(t, \cdot\,; f))$.

Proof. Set $u=u(t, x; f)$ and $u_0=u(t\cdot x; f_0)$. We prove a stronger assertion: for any $s>0$

$$u_0'(s, x_0)\leqq u'(s, x_1) \qquad \text{if} \quad u(s, x_1)=u_0(s, x_0).$$

Fix $s>0$. Let $u(s, x_1)=u_0(s, x_0)$. Putting

$$v(t, x)=u(t, x-x_0+x_1)-u_0(t, x),$$

we have $v(s, x_0)=0$ and $v'(s, x_0)=u'(s, x_1)-u_0'(s, x_0)$. Therefore it suffices to prove $v(s, x_0+x)\geqq 0$ for $x\geqq 0$. Since v solves the equation

$$v^{\cdot}=\tfrac{1}{2}v''+F'(u_0+\theta v)v \qquad 0\leqq\theta\leqq 1$$

with the initial condition

$$\lim_{t\downarrow 0} v(t, x)=f(x-x_0+x_1)-f_0(x)\begin{cases} \leqq 0 & \text{if } x<0 \\ \geqq 0 & \text{if } x>0, \end{cases}$$

we see, by Proposition 3.1 and by the method of approximation as used in the proof of Lemma 4.4, that $v(t, \cdot)\geqq 0$ on a right half x-axis and <0 on the other half for each t. This proves the required assertion.

By the same method as used above Kolmogorov *et al.* showed that $\phi(t, w; f_0)$ increase with t. But this fact now clear by Huygens property: $\phi(t+s, w; f)=\phi(t, w; u(s, \cdot\,; f))$ and the lemma just proved. Thus there exists $\tau(w)=\lim_{t\to\infty}\phi(t, w; f_0)$. Since $\tau_{c_0}(w)=\phi(t, w; w_{c_0})$ we have $\tau(w)\leqq\tau_{c_0}(w)$. These prove that $u(t, x+m(t); f_0)$ converges to some function, say $\hat{w}(x)$, which is decreasing with $\hat{w}(\infty)=0$ and $\hat{w}(-\infty)=1$. But by Lemma 4.2 $\phi(t, w; \hat{w})=\lim_{s\to\infty}\phi(t, w; u(s, x+m(s); f_0))=\lim_{s\to\infty}\phi(t+s, w; f_0)=\tau(w)$, from which we see, using (6.3), that τ satisfies (1.4) for some c. Since $\tau\leqq\tau_{c_0}$, we have $\tau=\tau_{c_0}$. Consequently we have

Lemma 6.3. (Kolmogorov *et al.*). *Let $f_0=I_{(-\infty,0]}$. Then*

$$\phi(t, w; f_0)\uparrow\tau_{c_0}(w) \qquad \textit{as} \quad t\uparrow\infty.$$

The next lemma is complementary to Lemmas 6.1, 6.2 and 6.3.

Lemma 6.4. *Let $c_2>c_0$. Suppose a datum f has the Lipshitz continuous first derivative with $f'\leqq 0$ and satisfies that*

$$\tfrac{1}{2}f''+c_2 f'+F(f)\begin{cases} \leqq 0 & \textit{if } x<0 \\ \geqq 0 & \textit{if } x>0 \end{cases}$$

where x's are those points at which f'' exist. Further suppose that $u(t, x+c_2 t; f)\to 1$ as $t\to\infty$. Then $\phi=\phi(t, w; f)$ satisfies

$$\phi(t, w)\geqq\tau_{c_2}(w)+o(1) \qquad 0<w<M(t; \phi) \tag{6.10}$$

uniformly as $t\to\infty$.

Proof. The proof is very similar to that of Lemma 6.1 and here only the outline is given. Let S_t be the orbit of the vector function $(z(t, x), z'(t, x))$ of $x\in R$, where $z(t, x)=u(t, x+c_2 t; f)$. As in the proof of Lemma 6.1 we can take a point A_t on S_t such that $z^{\cdot}<0$ iff z is larger than the w-coordinate of A_t. Then

c_2-manifold T_t passing through A_t bounds $S_{t'}$ below for $t'>t$. There exists a sequence $\{t_n\}$ along which p-coordinate of A_t tends to unity. Since A_t is bounded below by the graph of $\phi(t, \cdot\,; f_0)$ by virtue of Lemma 6.2, A_{t_n} approaches to the point (1.0). Thus T_{t_n} converges to the c_2-front. This implies (6.10).

7. Asymptotic Behavior of $u(t, x; f)$ for large x

In order to apply Proposition 3.3 to the equations (6.4) or (6.4)′ we must know the behavior of $\phi=\phi(t, w; f)$ and ϕ'' as $w\downarrow 0$, which are involved in that of $u=u(t, x; f)$, u', u'' and u''' as $x\to\infty$. Roughly speaking, the behavior of $u(t, x; f)$ and of its derivatives are asymptoticaly same as that of $e^{\alpha t}P_t f$ for data f belonging to certain classes.

Definition. (i) Let μ be a non-negative constant. A datum f is said to belong to the *class* $[E_\mu]$ if

$$f(x)=0 \quad \text{for} \quad x>x_0 \quad \text{with some constant} \quad x_0 \quad \text{in case} \quad \mu=0$$

$$f(x)\sim A(x)p(\mu, x) \quad \text{as} \quad x\longrightarrow\infty \qquad\qquad \text{in case} \quad \mu>0,$$

where p is defined in § 0 and A is such a function that $A(\log x)$ is slowly varying at infinity, i.e. $A>0$ and

$$A(x+x_0)\sim A(x) \quad \text{as} \quad x\longrightarrow\infty \quad \text{for each constant} \quad x_0.$$

(ii) Let λ be a positive constant. A datum f is said to belong to the *class* $[F_\lambda]$ if

$$f(x)\sim A(x)e^{-\lambda x} \qquad \text{as} \quad x\longrightarrow\infty,$$

where A is the same as in (i).

What we want to prove in this section is stated in the next two lemmas.

Lemma 7.1. *Let f be a datum belonging to the class $[E_\mu]$ $(\mu\geqq 0)$.*

(i) *Set $u=u(t, x; f)$. Then following relations hold*

$$\log u(t, x)\sim -x^2/2(\mu+t) \tag{7.1}$$

$$\partial^j u(t, x)/\partial x^j\sim(-x/(t+\mu))^j u(t, x) \qquad j=1, 2, 3 \tag{7.2}$$

as $x\to\infty$ uniformly in $t\in(1/T, T)$ for each (finite) $T>1$.

(ii) *Set $\phi=\phi(t, w; f)$. Then*

$$\phi(t, w)\sim\frac{\sqrt{2}}{\sqrt{t+\mu}}\sqrt{|\log w|}\,w \quad \textit{and} \quad \phi''(t, w)=o\Big(\frac{1}{\sqrt{t+\mu}}\frac{\sqrt{|\log w|}}{w}\Big)$$

as $w\downarrow 0$ uniformly in $t\in(1/T, T)$ for each $T>1$.

If $\mu=0$ all these relations hold uniformly in $t\in(0, T)$.

Lemma 7.2. *Let f be a datum belonging to the class $[F_\lambda]$ $(\lambda>0)$.*

(i) *Set* $u=u(t, x; f)$. *Then the following relations hold*

$$\partial^j u(t, x)/\partial x^j \sim (-\lambda)^j e^{(\lambda^2/2+\alpha)t} A(x) e^{-\lambda x} \qquad j=0, 1, 2, 3 \tag{7.3}$$

as $x\to\infty$ *uniformly in* $t\in(1/T, T)$ *for each* $T>1$.

(ii) *We have*

$$\phi''(t, w; f)=o(w^{-1})$$

as $w\downarrow 0$ *uniformly in* $t\in(1/T, T)$ *for each* $T>1$.

Remark 1. The second parts of Lemmas 7.1 or 7.2 are readily derived from the first parts of them and from (6.2). It is also clear by Lemma 7.1 (ii) combined with Lemma 6.2 that *for any datum f, for which* $\phi(t, w; f)$ *is well defined,*

$$\phi(t, w; f)=O(\sqrt{t}^{-1}\sqrt{|\log w|}\, w) \tag{7.4}$$

as $w\downarrow 0$ *uniformly in* $t\in(0, T)$.

Remark 2. By the fact that $v(t, x)=1-u(t, x; f)$ is a solution of the Cauchy problem

$$v^{\cdot}=\tfrac{1}{2}v''-F(1-v), \quad v(0+, \cdot)=1-f,$$

we can derive similar results on the behaviors of $1-u(t, x; f)$ and its derivatives as $x\to-\infty$ to those obtained above. We will not, however, use them later except the following simplest case: *if* $1-f(-x)$ *belongs to the class* $[E_\mu]$ $(\mu\geqq 0)$, *then*

$$\log(1-u(t, x))\sim -x^2/2(\mu+t)$$

$$u'(t, x)\sim x(\mu+t)^{-1}(1-u(t, x))$$

as $x\to-\infty$ *for each* $t>0$, where $u(t, x)=u(t, x; f)$.

For the proofs of Lemmas 7.1 and 7.2 we prepare several lemmas.

Lemma 7.3. *Let g be a locally bounded measurable function with* $\int_R p(t, x)\cdot |g(x)|dx<\infty$ *for any* $t>0$ *and* $\mathrm{ess.sup}\{x; |g(x)|>0\}=x_1<\infty$. *Then*

$$P_t g(x)=o(\exp\{-(x-x_1)^2/2t\}) \qquad as \quad x\longrightarrow\infty$$

uniformly in $t\in(0, T)$ *for each* $T<\infty$.

Proof. Immediate from

$$\exp\left\{\frac{(x-x_1)^2}{2t}\right\}P_t g(x)=\int_{-\infty}^{x_1} e^{-(x-x_1)(x_1-y)/t}p(t, x_1-y)g(y)dy$$

$$=\int_0^\infty e^{-(x-x_1)w/\sqrt{t}}e^{-w^2/2}g(x_1-\sqrt{t}\,w)dw.$$

Lemma 7.4. *In addition to assumptions imposed on g in* Lemma 7.3, *suppose* $g\geqq 0$ *on the interval* $[x_2, x_1]$ *with some* $x_2<x_1$. *Then for any constant* x_3 *with*

$x_3 < x_1$

$$\exp\{(x-x_3)^2/2t\}P_tg(x) \longrightarrow \infty, \tag{7.5}$$

especially

$$P_tg(x) \sim P_t\{g \cdot I_{(x_3,x_1)}\}(x)$$

and especially

$$(\partial^n/\partial x^n)P_tg(x) \sim (-x/t)^n P_tg(x) \qquad n=1, 2,\ldots \tag{7.6}$$

as $x \to \infty$ *uniformly in* $t \in (0, T)$.

Proof. The divergence in (7.5) follows from

$$\begin{aligned}\exp\{(x-x_3)^2/2t\}P_tg(x) &\geqq \exp\{(x-a)^2/2t\}P_tg(x)\\ &\geqq \int_a^{x_1} e^{(x-a)(y-a)/t}p(t, a-y)g(y)dy \cdot (1+o(1))\end{aligned}$$

as $x \to \infty$, where a is a constant which satisfies $\max\{x_2, x_3\} < a < x_1$, $g(a) > 0$ and $\int_a^x g(y)dy = (x-a)g(a) + o(x-a)$ as $x \downarrow a$ so that $\int_{a+t}^{x_1} p(t, a-y)g(y)dy \to g(a)$ as $t \downarrow 0$ (cf. Widder [17]).

Lemma 7.5. *Let f be a datum with* $\lim_{x\to\infty} f(x) = 0$. *Then*

$$u(t, x; f) = e^{\alpha t}P_tf(x)(1 + t \cdot o(1))$$

where $o(1) \to 0$ *as* $x \to \infty$ *uniformly in* $t \in (0, T)$ *for each* T.

Proof. Define $F(u) = 0$ for $u > 1$. Putting $v(t, x) = e^{\alpha t}P_tf(x)$, $u = u(t, x; f)$ and $w = u - v$, we have

$$w = \mathbf{K}_0\{kw + F(v) - \alpha v\}$$

where $k = (F(u) - F(v))/(u - v)$ and $\mathbf{K}_0$ is defined by (4.12). By Lemma 4.1

$$|w(t, x)| \leqq \int_0^t e^{\gamma(t-s)}P_{t-s}|F(v(s, \cdot)) - \alpha v(s, \cdot)|(x)ds.$$

Since $v(t, x) \to 0$ as $x \to \infty$ uniformly in $t \in (0, T)$, for any $\varepsilon > 0$ we can chose constants M and L so that

$$|F(v(s, x)) - \alpha v(s, x)| \leqq \varepsilon 2^{-1}v(t, x) + MP_{s-\delta}I_{(-\infty,L)}(x)$$

if $\delta < s < T$, where $\delta = \varepsilon/2\beta$. Then, using the inequality $|F(v) - \alpha v| < \beta v$ $(v > 0)$, we see that if $0 < t < T$

$$|w(t, x)| \leqq te^{\gamma t}\{\varepsilon P_tf(x) + MP_{(t-\delta)\vee 0}I_{(-\infty,L)}(x)\}.$$

This proves $w = t \cdot o(P_tf(x))$ uniformly in $t \in (0, T)$, since the second term in the braces is small order of $P_tf(x)$ as $x \to \infty$ uniformly in $t \in (0, T)$ by virtue of Lemmas 7.3 and 7.4.

Lemma 7.6. *Let g be the same as in Lemma 7.4. Let T and n be a positive constant and a non-negative integer, respectively. Then we can find such a constant $K_n(T, g)=K_n(T, x_1)$ depending only on T, n and $x_1=\text{ess.}\sup\{x;\, g\neq 0\}$ that*

$$\int_R |p'(s, x-y)|P_t|g|(y)|y|^n dy \leqq K_n(T, g)(t+s)^{-1}\{|x|^n/\sqrt{s}+|x|^{n+1}\}P_{t+s}|g|(x)$$

for $x>1$, $t\geqq 0$, $s>0$, $t+s<T$.

Proof. Setting

$$J(z, t, x)=\int_R |p'(s, x-y)|p(t, y-z)|y|^n dy$$

we have, for $x>1$,

$$\begin{aligned} J(z, t, x) &= p(t+s, x-z)\int_R \left|\sqrt{\frac{t}{(t+s)s}}\,w+\frac{x-z}{t+s}\right| e^{-w^2/2} \\ &\quad\times\left|x-\sqrt{\frac{ts}{t+s}}\,w-\frac{s}{t+s}(x-z)\right|^n dw \\ &\leqq \frac{K_n(T)}{t+s}\,p(t+s, x-z)\left\{\left(\sqrt{\frac{t(t+s)}{s}}+|z|^n\right)(|x|^n+|z|)+|x|^{n+1}\right\} \end{aligned}$$

where $K_n(T)$ is a constant depending only on n and T, and

$$\int_R J(z, s, x)|g(z)|dz \leqq K_n(T, g)(t+s)^{-1}\{|x|^n/\sqrt{s}+|x|^{n+1}\}P_{t+s}|g|(x),$$

which is the desired inequality.

Proof of Lemma 7.1 *in case $\mu=0$.* Let f belong to the class $[E_0]$. The relation (7.1) is clear by Lemmas 7.3, 7.4 and 7.5. For the estimation of u' we rewrite (4.2)$'$ as follows

$$u'(t, x)=e^{\alpha t}(P_t f)'(x)+\int_0^t ds\int_R p'(t-s, x-y)\,J(s, y)dy$$

where $J(s, y)=F(u(s, y))-\alpha e^{\alpha s}P_s f(y)$. Then, using Lemmas 7.5 and 7.6, we see, as in the proof of Lemma 7.5, that the last term in the above equation is small order of $xt^{-1}P_t f(x)$ as $x\to\infty$ uniformly in $t\in(0, T)$. Since $(P_t f)'(x)\sim -xt^{-1}P_t f(x)\sim -xt^{-1}e^{-\alpha t}u(t, x)$, the case $j=1$ in (7.2) is obtained.

Estimation of u'' is carried out as follows: Set $f_*=u(t, \cdot)$, $u^*(t, x)=u(t, x; f_*)=u(2t, x)$. To prove is that $u''_*(t, x)\sim e^{2\alpha t}(x/2t)^2 P_{2t}f(x)$ uniformly in $t\in(0, T/2)$. By (4.3)

$$u''_*(t, x)=e^{\alpha t}(P_t f_*)''(x)+\int_0^t ds\int_R p'(t-s, x-y)J(s, y)dy \tag{7.7}$$

where

$$J(s, y)=F'(u_*(s, y))u'_*(s, y)-\alpha e^{\alpha s}(P_s f_*)'(y).$$

Since $f'_*(x)\sim -e^{\alpha t}tx^{-1}P_t f(x)\sim e^{\alpha t}(P_t f)'(x)$ uniformly in $t<T/2$, we see, as in the

proof of Lemma 7.5, using (7.6), that

$$(P_s f_*)'(x) = P_s f_*'(x) \sim -e^{\alpha t} x (t+s)^{-1} P_{t+s} f(x)$$
$$\sim e^{-\alpha s} u_*'(s, x)$$

uniformly in $0 < s \leqq t < T/2$. By this relation and by the inequality (4.6), for any $\varepsilon > 0$ we can find constants M and L depending only on ε, T and f such that

$$|J(s, y)| \leqq \varepsilon |y| (t+s)^{-1} P_{t+s} f(y) + M \sqrt{t+s}^{\,-1} P_s I_{(-\infty, L)}(y)$$

for $0 < s \leqq t < T/2$. Therefore Lemma 7.6 says that the last term in (7.7) is bounded by

$$K_1(T, f)(x/t\sqrt{t} + x^2/t) P_{2t} f(x) + K x P_t I_{(-\infty, L)}(x)$$

for $x > 1$, $t \in (0, T/2]$, where K is a constant depending only on L, M, and T, while we see also by Lemma 7.6 that

$$(P_t f_*)''(x) = (P_t f_*')'(x) \sim e^{\alpha t} (x/2t)^2 P_{2t} f(x)$$

uniformly in $t \in (0, T/2]$. Thus we have $u(2t, x) = u_*(t, x) \sim e^{2\alpha t}(x/2t)^2 P_{2t} f(x)$ with required uniformity.

Noticing that $u^{\cdot} \sim \frac{1}{2} u''$ and $u''' \sim 2u^{\cdot\prime}$, and using the equation $u^{\cdot\prime} = \frac{1}{2}(P_t f)''' + (P_t F(f))' + (\mathbf{K}_0\{F'(u)u^{\cdot}\})'$($\mathbf{K}_0$ is defined by (4.12)), we can estimate the tail of u''' at infinity as in the case of u''. Now Lemma 7.1 has been proved in the case $\mu = 0$.

For the proof of Lemma 7.2 and of the rest of Lemma 7.1 we prepare the next

Lemma 7.7. *Let $A(x)$ be a function as appears in Definition of the classes $[E_\mu]$ and $[F_\lambda]$, and T a positive constant.*

(i) *Let $\{g_t(x)\}_{0 \leqq t < T}$ be a family of bounded functions such that $g_t(x) \sim A(x\mu/(\mu+t)) p(\mu+t, x)$ $(\mu > 0)$ as $x \to \infty$ uniformly in $t \in (0, T)$, then*

$$(\partial^n/\partial x^n) P_t g_s(x) = g_{t+s}(x)(-x/(\mu+t+s))^n (1 + o(1)/\sqrt{t}^{\,n}) \qquad n = 0, 1 \tag{7.8}$$

where $o(1) \to 0$ as $x \to \infty$ uniformly in $0 \leqq s < t < T$.

(ii) *If $g(x) \sim A(x) e^{-\lambda x}$ $(\lambda > 0)$ and $g(x)$ is bounded, then*

$$(\partial^n/\partial x^n) P_t g(x) = A(x) e^{-\lambda x}\{(-\lambda)^n + (1/t^{n-1}) \cdot o(1)\} \qquad n = 0, 1, 2$$

where $o(1) \to 0$ as $x \to \infty$ uniformaly in $t \in (0, T)$.

Proof. First we prove (i). Write $g_t(x) = A_t(x) p(\mu+t, x)$. Then $A_t(x) = A(\mu x/(\mu+t))(1 + o(1))$ as $x \to \infty$ uniformly in t and

$$\int_R ((x-y)/t)^n p(t, x-y) g_s(y) dy$$
$$= \int_{-\infty}^x A_s(x-y) p(\mu+s, x-y) p(t, y)(y/t)^n dy + O(x^{n-1} e^{-x^2/2t} / \sqrt{t}^{\,2n-1}) \qquad x > 1.$$

Let J denote the first term in the right hand side of this equation. Then

$$J=p(\mu+t+s,\ x)\int_{-\infty}^{w_1} A_s\Big(\frac{\mu+s}{\mu+t+s}x-\sqrt{\frac{(\mu+s)t}{\mu+t+s}}w\Big)\times$$

$$\times\Big(\sqrt{\frac{\mu+s}{\mu+s+t}}\frac{w}{\sqrt{t}}+\frac{1}{\mu+t+s}x\Big)^n e^{-w^2/2}dw/\sqrt{2\pi},$$

where w_1 is defined by $(\mu+s)(\mu+t+s)^{-1}x-\sqrt{(\mu+s)t/(\mu+t+s)}w_1=0$. Since $A(\log x)$ is slowly varying at infinity, $A(x)$ is expressed in the form

$$A(x)=a(x)\exp\Big\{\int_1^{e^x}\frac{\varepsilon(y)}{y}dy\Big\}\qquad x>0$$

where $a(x)\to a_0$, $0<a_0<\infty$ and $\varepsilon(x)\to 0$ as $x\to\infty$. Especially

$$\frac{A(x-x_0)}{A(x)}=\frac{a(x-x_0)}{a(x)}\exp\Big\{\int_{e^x}^{e^{x-x_0}}\frac{\varepsilon(y)}{y}dy\Big\}=o(e^{-x_0})\quad\text{as}\quad x_0\longrightarrow-\infty$$

uniformly in $x>0$. Then it is not difficult to see

$$J=p(\mu+t+s)\int_{-\infty}^{w_1} A\Big(\frac{\mu}{\mu+t+s}\Big(x-\sqrt{\frac{t(\mu+t+s)}{\mu+s}}w\Big)\Big)$$

$$\times\Big(\frac{x}{\mu+s+t}\Big)^n e^{-w^2/2}dw/\sqrt{2\pi}\cdot(1+o(1)/\sqrt{t}^{\,n})$$

$$=g_{t+s}(x)\Big(\frac{x}{\mu+t+s}\Big)^n\Big(1+o(1)\frac{1}{\sqrt{t}^{\,n}}\Big).$$

These combined with the fact that $t^{-n}p(t,x)=o(g_t(x))$ as $x\to\infty$ uniformly in t proves (7.8).

The second part of the lemma follows from

$$\int_R p(t,\ x-y)g(y)\Big(\frac{x-y}{t}\Big)^n dy$$

$$=\frac{e^{-\lambda x}}{\sqrt{t}^{\,n}}\int_{-\infty}^{x/\sqrt{t}} A(x-\sqrt{t}y)\frac{e^{-y^2/2}}{2}e^{\lambda\sqrt{t}y}y^n dy+O\Big(\frac{x^{n-1}}{t^n/\sqrt{t}}e^{-x^2/2t}\Big)$$

$$=e^{-\lambda x}\frac{1}{t^n}\frac{\partial^n}{\partial\lambda^n}e^{\lambda^2 t/2}A(x)\Big(1+o(1)\frac{1}{\sqrt{t}^{\,n}}\Big).$$

The proof of Lemma 7.7 is completed.

Now we prove Lemma 7.1 in case $\mu>0$. Fix $t_1>0$. By (7.8) and (4.2)$'$ and by Lemma 7.5 it is easy to see that (7.2) with $j=1$ holds uniformly in $t\in(t_1,\ T)$. Using this, (7.8) and the equation

$$u''(t_1+t,\ x)=(P_t u'(t_1,\ \cdot))'(x)+\int_0^t(P_{t-s}g_s)'(x)ds$$

where $g_s(x)=F'(u(t_1+s,\ x))u'(t_1+s,\ x)$, we see (7.2) with $j=2$. The case of $j=3$ is similarly proved. The proof of Lemma 7.1 (i) is completed.

Lemma 7.2 (i) is similarly proved by Lemma 7.7 (ii).

8. Approach of Front of $u(t, x; f)$ to Front of Travelling Wave

Here we will clarify the behavior of the front of $u=u(t, x; f)$, i.e. the function $u(t, \cdot+m(t))$ where

$$m(t)=\sup\{x;\ u(t, x)=\tfrac{1}{2}\},$$

for large t. We will use symbols $\phi(t, u; f)$, τ_c, $M(t; \phi)$, which were introduced in §6, in this section too. We will mainly deal with data which satisfy the following

Condition [W]: there exist $t_0>0$ and a finite number N such that

$$(8.1) \qquad \lim_{x\to\infty} u(t_0, x)=0, \quad u'(t_0, x)<0 \qquad \text{for} \quad x>N$$

and

$$(8.2) \qquad \varliminf_{x\to-\infty} u(t_0, x)>0 \quad \text{or} \quad u'(t_0, x)>0 \qquad \text{for} \quad x<-N$$

where $u(t, x)=u(t, x; f)$: if F belongs to the class II (see Remark to [G] in §5 for the definition) we assume $\varliminf_{x\to-\infty} u(t_0, x)>0$.

This condition scarcely narrows the class of data to be dealt with. For example if f does not increase for large values of x and tends to zero as $x\to\infty$ then (8.1) is satisfied. Data which belong to the class [E_μ] or [F_λ] also satisfy (8.1). The condition (8.1) guarantees the existance of $\phi(t, w; f)$. The condition (8.2) is imposed in order to apply Lemmas 8.1 or 8.2, given later, which prove $M\{\phi(t, \cdot; f)\}\to 1$ as $t\to\infty$ for data f satisfying [W]. Especially [W] implies Condition [G] (see §5) and that $m(t)$ takes a definite value for every sufficiently large t. If F belongs to class I any datum with compact support satisfies [W].

Now we state the main theorems, from which it will be seen that the behavior of the front of $u(t, x; f)$ depends mainly on the behavior of f for large x which is inherited to $u(t, \cdot; f)$ as was seen in the previous section.

Theorem 8.1. *Let a datum f belong to the class [E_μ] ($\mu\geqq 0$) or to the class [F_λ] with $\lambda>c_0-\sqrt{c_0^2-2\alpha}$.*(*) *Suppose Condition [W] is satisfied. Then $u=u(t, x; f)$ satisfies*

$$(8.3) \qquad \lim_{t\to\infty} u(t, x+m(t))=w_{c_0}(x)$$

uniformly in $x>-m(t)$.

Corollary. *Let f be a datum with compact support. Suppose F belongs to the class I. Then $u=u(t, x; f)$ satisfies*

$$u(t, x)-w_{c_0}(x-m(t))I_{(0,\infty)}(x)-w_{c_0}(-x+m^*(t))I_{(-\infty,0)}(x)$$
$$\longrightarrow 0 \quad \textit{as} \quad t\longrightarrow\infty \qquad \textit{uniformly in} \quad x\in R$$

(*) The classes [E_μ] and [F_λ] are defined in §7.

where $m_*(t)=\inf\{x;\ u(t, x)=1/2\}$.

Theorem 8.2. *Let a datum f belong to the class* $[F_\lambda]$ *with* $0<\lambda\leqq c_0-\sqrt{c_0^2-2\alpha}$ $(\alpha>0)$. *Suppose Condition* $[W]$ *is satisfied. Then* $u=u(t, x; f)$ *satisfies*

$$\lim_{t\to\infty} u(t, x+m(t))=w_c(x) \qquad c=\lambda/2+\alpha/\lambda \tag{8.4}$$

uniformly in $x>-m(t)$.

Remark. Let f be a datum such that $f(-x)$ belongs to the class $[E_\mu]$ or $[F_\lambda]$ as well as $f(x)$. Then assertions analogous to Corollary of Theorem 8.1 hold if F belongs to the class I. The condition that F belongs to class I can be replaced by Condition $[G]$ (cf. §10).

In Theorems 8.1 or 8.2 the condition that f belongs to the class $[E_\mu]$ or $[F_\lambda]$ should be weakened. This is suggested by the next theorem.

Theorem 8.3. *Let f be a differentiable datum for which Condition* $[W]$ *holds. Suppose*

$$-f'(x)\geqq f(x)(b+o(1)) \qquad as \quad x\longrightarrow\infty \tag{8.5}$$

and

$$f(x)e^{-b_* x}\longrightarrow\infty \qquad as \quad x\longrightarrow -\infty \tag{8.6}$$

for a positive constants $b\geqq c_0-\sqrt{c_0^2-2\alpha}$ *and* $b_*<b$. *Then for* $u=u(t, x; f)$, (8.3) *holds uniformly in* $x>-m(t)$.

The next theorem is complementary to these theorems.

Theorem 8.4. *Let* $\alpha>0$. *Let f be a datum which is differentiable and positive for large x and satisfies that* $\lim_{x\to\infty} f(x)=0$ *and* $\lim_{x\to\infty} f'(x)/f(x)=0$. *Then under Condition* $[W]$

$$\lim_{t\to\infty} u(t, x+m(t))=\tfrac{1}{2}$$

uniformly on each compact set of R.

For proofs of these theorems we need two more lemmas.

Lemma 8.1. *Let f be a continuously differentiable datum. Suppose a constant* $\delta>0$ *is related with f in such a manner that the equation* $f(x)=\delta$ *has just two roots, say* $x_1, x_2, x_1<x_2$, *and* $f'(x)<0$ *for* $x\geqq x_2$, >0 *for* $x\leqq x_1$. *Let* $\delta(t)$ *be a solution of the equation* $\delta^\cdot(t)=F(\delta(t))$ *with* $\delta(0)=\delta$. *Then for each t,* $\delta(t)$ *is related with* $u(t, \cdot\,; f)$ *in the same manner of how* δ *is related with f, as long as* $\delta(t)<\sup_{x\in R} u(t, x; f)$.

Proof. Let $g(x)$ be a continuously differentiable function such that $0\leqq g\leqq 1$, $g'\leqq 0$ and g is not a constant. Suppose that the equation $f(x)=g(x)$ has just two roots, say $y_1, y_2, y_1<y_2$, and that $f'(x)<0$ and $f(x)\leqq g(x)$ for $x\geqq y_2$. Put $u=$

$u(t, x; f)$, $v=u(t, x; g)$ and $T=\sup\{t;\ u(t, x)>v(t, x)$ for some $x\in R\}$. We show here that $u'(t, x)<0$ for $x\geqq\sup\{x;\ u(t, x)=v(t, x)\}$ if $0<t<T$. Since $w=u-v$ satisfies

$$w^{\cdot}=\tfrac{1}{2}w''+F'(v+\theta w)w' \qquad (0\leqq\theta=\theta(t, x)\leqq 1),$$

the set $\{(t, x);\ T>t\geqq 0,\ u(t, x)<v(t, x)\}$ has just two connected components by virtue of Proposition 3.4. Let D be one of them which contains a right half x-axis. By Lemmas 7.3, 7.4 and 7.5 $\bar{D}$, the closure of D in $\bar{E}$, contains a right half of $l_T=\{T\}\times R$ if $T<\infty$. Assume for simplicity that T is finite and set $\partial D=\bar{D}-D$ and $\Gamma=\partial D-(l_T\cup l_0)$. Then $u'\leqq v'<0$ on Γ because the intersection of D and l_t is connected for each positive $t<T$. Thus $u'<0$ on Γ, which implies, by Proposition 3.3, $u'<0$ in D. This is the same as what was announced to be shown.

Now the lemma is easily proved. Let t be such a time that $\delta(t)<\sup_x u(t, x)$. Clearly we can find a function g for which $g\geqq\delta$, conditions stated at the beginning of this proof are satisfied and $t<T$ where T is defined in the previous paragraph. Since $\delta(t)<u(t, x;\ g)$ $x\in R$, we have $u'(t, x)<0$ for $x\geqq X_2(t)=\sup\{x;\ \delta(t)=u(t, x)\}$. Similarly we get $u'(t, x)>0$ for $x\leqq X_1(t)=\inf\{x;\ \delta(t)=u(t, x)\}$. It is clear that $u(t, x)>\delta(t)$ if $X_1(t)<x<X_2(t)$. Thus $\delta(t)$ and $u(t, \cdot)$ are related in the required manner.

The proof of the lemma is completed.

The similar method proves

Lemma 8.2. *Let f be a continuously differentiable datum. Suppose the set $\{x; f(x)=\delta\}$ consists of just one point and $f'(x)<0$ if $f(x)\leqq\delta$. Let $\delta(t)$ be defined as in Lemma 8.1. Then $\delta(t)$ has the same relation to $u(t, \cdot\,; f)$ as δ does to f for each $t>0$.*

Proof of Theorem 8.1. Step 1. Set $u=u(t, x; f)$ and $\phi(t, w)=\phi(t+t_0, w; f)=\phi(t, w;\ u(t_0, \cdot))$ where t_0 appears in Condition [W]. We will prove

$$\phi(t, w)=\tau_{c_0}(w)+o(1) \qquad 0\leqq w<M(t;\ \phi) \tag{8.7}$$

as $t\to\infty$ uniformly. Since we know that $\phi\geqq\tau_{c_0}+o(1)$ as a direct consequence of Lemmas 6.2 and 6.3, for the proof of (8.7) it suffices to show that

$$\phi(t, w)\leqq\tau_{c_0}(w)+o(1) \qquad 0\leqq w<M(t;\ \phi). \tag{8.8}$$

Let $c_1>c_0$. Condition [W] enables us to find a constant $\delta>0$ which is related with $u(t_0, \cdot)$ in the manner stated in Lemma 8.1 or 8.2. We will show in Step 2 that there exists a datum f^* and a constant $t_1>t_0$ such that f^* satisfies conditions imposed in Lemma 6.1 and inequalities

$$f^*(x)<\min\{g(x),\ \delta\} \qquad x\in R$$

and

$$\phi(0, w; f^*)>\phi(0, w;\ g) \qquad 0<w<M(\phi(0, \cdot\,; f^*))$$

where $g(x)=u(t_1, x)$. Let such f^* and t_1 be found for each $c_1>c_0$ sufficiently near c_0. Set $\phi^*=\phi(t, w; f^*)$ and $\psi=\phi(t, w; g)=\phi(t+(t_1-t_0), w)$. Then we have, by Lemma 8.1 or 8.2,

$$M(t; \phi^*)<M(t; \psi) \qquad t\geqq 0$$

and hence $\omega=\psi-\phi^*$ is defined and satisfies (6.4), where ϕ is replaced by ϕ^*, in the domain $D=\{(t, w); t>0, 0<w<M(t; \phi^*)\}$. Since $M(t; \phi^*)=\sup_x u(t, x; f^*)$ and $\omega(t, M(t; \phi^*))<0$, there exists a continuous function $M(t)$ such that $0<M(t)<M(t; \phi^*)$, $t\geqq 0$ and $\omega<0$ in $D-D_*$ where $D_*=\{(t, w); t>0, 0<w<M(t)\}$. Then $\omega\leqq 0$ on ∂D_*. Check that Proposition 3.2 is applicable to the equation (6.4) in D^* for the present ω, using Lemma 7.1 (ii) or Lemma 7.2 (ii). Then we have $\omega\leqq 0$ in D_*. Consequently $\psi\leqq\phi^*$ in D. Since $\phi^*\leqq\tau_{c_1}+o(1)$ by Lemma 6.1 and since $\tau_{c_1}\downarrow\tau_{c_0}$ as $c_1\downarrow c_0$ we get (8.8).

Step 2. Now we construct f^*. We carry out this only in the case $\varliminf_{x\to-\infty}f(x)=0$ (in the other case the construction is much simpler). Thus we assume that F belongs to the class I. Set $h(x)=\delta\exp\{-x^2\}$ and take $t_1>t_0$ such that $h(x)\leqq\min\{g(x), \delta\}$ $x\in R$ where $g(x)=u(t_1, x; f)$ (see Lemma 7.4).

First we assume that f belongs to the class $[E_\mu]$ or $[F_\lambda]$ with $\lambda>c_0+\sqrt{c_0^2-2\alpha}$. Let $c_1>c_0$ and let $\lambda/2+\alpha/\lambda>c_1>c_0$ if f belongs to the class $[F_\lambda]$. Then, by Lemma 7.1 or Lemma 7.2 there exists a constant x_2 such that $g'<0$ for $x>x_2$ and

$$\tfrac{1}{2}g''+c_1g'+F(g)>0 \quad\text{and}\quad g<\delta/2e \qquad\text{for}\quad x>x_2-1.$$

Set $k(x)=-a(x-x_2)^2+g'(x_2)(x-x_2)+g(x_2)$ where the constant $a>0$ is chosen so large that

$$\tfrac{1}{2}k''+c_1k'+F(k)<0 \qquad\text{if}\quad k>0,$$

$$\max_{x\in R}k(x)<\delta/e \quad\text{and}\quad \sqrt{g'(x_2)^2+4ag(x_2)}/a<1.$$

Since $(1/2)h''+c_1h'+F(h)>0$ for $x<-1$ and $h(-1)=\delta/e$, two trajectories $\{(h(x), h'(x)); x<-1\}$ and $\{(k(x), k'(x)); k(x)>0, k'(x)<0\}$ drawn in the vertical half strip $(0, \delta)\times(0, \infty)$ cross each other at just one point, say (w, p). Let x_1^* and x_1 be values of parameter at which they pass though it: $h(x_1^*)=k(x_1)=w$, $h'(x_1^*)=k'(x_1)=p$. Now we may put

$$f^*(x)=\begin{cases} g(x) & \text{if}\quad x>x_2 \\ k(x) & \text{if}\quad x_1<x<x_2 \\ h(x-x_1+x_1^*) & \text{if}\quad x<x_1. \end{cases}$$

By Theorem 5.1 $u(t, x+c_1t; f^*)\to 0$ as $t\to\infty$ and by Lemmas 7.1 and 7.2 $\varlimsup_{w\downarrow 0}\phi(t, w; f^*)/w<-c_1-\sqrt{c_1^2-2\alpha}$ which implies (6.6) for $\phi=\phi(t, w; f^*)$. Other requirements for f^* are clear by the construction and hypotheses.

When f belongs to the class $[F_\lambda]$ with $c_0-\sqrt{c_0^2-2\alpha}\leqq\lambda\leqq c_0+\sqrt{c_0^2-2\alpha}$, we can find x_2' such that

$$\tfrac{1}{2}g''+c_1g'+F(g)<0 \quad \text{and} \quad g<\delta/2e \qquad \text{for} \quad x>x_2'$$

and then construct f^* as above, but in this case f^* satisfies the condition (4.19) with $x_2=\infty$. Thus f^* is constructed.

Step 3. The inverse function of $u(t, \cdot+m(t))$, denoted by $x(t, \cdot)$, is given by

$$\int_{1/2}^{u}\frac{dw}{\phi(t, w; f)}=x(t, u)-m(t)$$

and converges to $w_{c0}^{-1}(u)$; $w_{c0}^{-1}(w_{c0}(x))=x$. The desired assertion follows from the inequality

$$\begin{aligned}|u(t, x+m(t))-w_{c_0}(x)| &= |u(t, x+m(t))-u(t, x(t, w_{c_0}(x)))| \\ &\leqq K_t|w_{c0}^{-1}(w)-x(t, w)+m(t)| \qquad \text{if} \quad w=w_{c_0}(x)\leqq M(t, \phi)\end{aligned}$$

where $K_t=\sup\{|u'(t, x)|;\ x\in R\}$ is bounded for large t by the remark following (4.6). This completes the proof of Theorem 8.1.

Proof of Theorem 8.2. Set $\phi=\phi(t, w; f)$. It is proved as in the proof of Theorem 8.1 that $\phi\leqq\tau_c+o(1)$. Thus it suffices to prove that $\phi\geqq\tau_c+o(1)$. This follows from Lemma 7.2 and the next lemma.

Lemma 8.3. *Let $\alpha>0$. Let a datum f be positive and differentiable on a right half x-axis. Suppose* $\lim_{x\to\infty}f(x)=0$ *and*

$$(8.9) \qquad 0\leqq -f'(x)\leqq(b+o(1))f(x) \qquad as \quad x\longrightarrow\infty^{(*)}$$

with $0<b<c_0-\sqrt{c_0^2-2\alpha}$. *Then*

$$(8.10) \qquad \phi(t, w; f)\geqq\tau_c(w)+o(1) \qquad as \quad t\longrightarrow\infty$$

uniformly where $c=b/2+\alpha/b$.

Proof. Set $u=u(t, x; f)$. First it is proved that (8.9) implies

$$(8.11) \qquad 0\leqq -u'(t, x)\leqq(b+o(1))u(t, x) \qquad \text{as} \quad x\longrightarrow\infty$$

for each $t>0$. Put $v(t, x)=\exp\{\alpha t\}P_tf(x)$. It is easy to see $0\leqq -v'(t, x)\leqq(b+o(1))v(t, x)$. By Lemma 7.5 $v(t, x)\sim u(t, x)$ as $x\to\infty$. Set $w=v-u$. Then $w'=\mathbf{K}_0\{F'(u)w'+\alpha v'-F'(u)v'\}$ ($\mathbf{K}_0$ is defined by (4.12)) and, by Lemma 4.1,

$$|w'(t, x)|\leqq\mathbf{K}_\gamma\{|\alpha-F'(u)||v'|\}(t, x)=o(v(t, x))$$

as $x\to\infty$. Thus we have (8.11).

Set $\phi(t, w)=\phi(t+1, w; f)$. From (8.11) it follows that $\phi(t, w)\geqq -bw+o(w)$ as $w\downarrow 0$. Take a constant c_2 with $c_0<c_2<c$. It is not difficult to construct a continuous function $\psi_0(w)$, $0\leqq w\leqq 1$ which has a piece-wise continuous derivative bounded on each compact set of the half open interval $[0, 1)$ and satisfies

$$\psi_0(w)<0 \quad 0<w<1, \quad \psi_0(0)=\psi_0(1)=0$$

(*) "$a(t)\leqq b(t)$ as $t\to\infty$" means that $a(t)\leqq b(t)$, $t>N$ for some $N<\infty$.

$$\psi_0(w) < \phi(0, w) \quad 0 < w < M(0; \phi)$$

$$-2c_2 - 2F(w)/\psi_0(w)\begin{cases} \geqq \psi_0'(w) & 0 < w < \frac{1}{2} \\ \leqq \psi_0'(w) & \frac{1}{2} < w < 1 \end{cases}$$

and

$$-\psi_0'(0) < c_2 - \sqrt{c_2^2 - 2\alpha}$$

(ψ_0 may be taken to be equal to a (c', $\underline{b}$)-manifold with $c_2 < c' < c$ near $w = 0$ and equal to $\phi(1/2, w; f_0)$, $f_0 = I_{(-\infty, 0)}$ near $u = 1$). Let $g(x)$ be a non-trivial solution of the differential equation $g' = \psi_0(g)$ on R and set $\psi = \phi(t, w; g)$. By Theorem 5.1 $u(t, x + c_2 t; g) \to 1$ as $t \to \infty$, because the last condition imposed on ψ_0 implies $g(x)\exp\{b^* x\} \to \infty$ as $x \to \infty$ if $-\psi_0'(0) < b^* < c_2 - \sqrt{c_2^2 - 2\alpha}$. It is easily checked that g satisfies conditions imposed on f in Lemma 6.4. Thus $\psi(t, w) \geqq \tau_{c_2}(w) + o(1)$, while (6.4) combined with boundary conditions:

$$\phi(0, w) > \psi(0, w) = \psi_0(w) \qquad 0 < w < M(0; \phi),$$

$$\phi(t, w) > \psi(t, w) \qquad \text{for } w \text{ near } 0 \text{ or near } M(t; \phi)^{(*)}$$

implies $\phi(t, w) > \psi(t, w)$ $0 < w < M(t; \phi)$. Therefore we have $\phi(t, w) \geqq \tau_{c_2}(w) + o(1)$ by virtue of Lemma 6.4. This proves (8.10) because $\tau_{c_2}(w) \uparrow \tau_c(w)$ as $c_2 \uparrow c$.

q.e.d.

Proof of Theorem 8.3. Set $\phi(t, w) = \phi(t + t_0, w; f)$ where t_0 is a constant which appears in [W]. As in the proof of Lemma 8.3 we see that $\phi(t, w) \leqq -bw + o(w)$ and that for each b_1, $b^* < b_1 < b$ there exists a smooth datum g such that

$$g(x) \sim e^{-b_1 x}$$

$$g'(x) < 0 \qquad x < 0$$

and

$$\psi(0, w) > \phi(0, w) \qquad 0 < w < M(0; \psi)$$

where $\psi = \phi(t, w; g)$. Let δ be a positive constant which is related with $u(t_0, \cdot\,; f)$ in the manner stated in Lemmas 8.1 or 8.2. Conditions (8.2) and (8.6) allow us to assume that g satisfies in addition $g(-x) < \min\{u(t_0, x; f), \delta\}$ so that $M(t; \psi) < M(t; \phi)$, $t \geqq 0$ and $M(t; \psi) \to 1$ as $t \to \infty$. Then as before we have $\phi < \psi$, while, as was shown in the proof of Theorem 8.1,

$$\psi(t, w) \leqq \tau_{c'}(w) + o(1) \qquad c' = \max\{c_0, b_1/2 + \alpha/b_1\}.$$

This implies $\phi(t, w) = \tau_{c_0}(w) + o(1)$ and proves (8.1) as before.

Proof of Theorem 8.4. Set $\phi = \phi(t, w; f)$. Condition [W] and Lemma 8.1 implies $M(t; \phi) \to 1$ as $t \to \infty$, while Lemma 8.3 says that $\phi(t, w) = o(1)$ as $t \to \infty$.

(*) If $M(t; \phi) = 1$, to get this strict inequality we may use Remark 2 of Lemmas 7.1 and 7.2.

From these the assertion of the theorem is obvious.

Under an additional restriction on F, Condition $[W]$ can be removed:

Theorem 8.5. *Suppose $F(u)/u$ is non-increasing. Then in each of Theorems 8.1 to 8.4 Condition $[W]$ may be removed from assumptions of each one. In Theorem 8.3 the condition (8.6) may be also removed simultaneously.*

For the proof we use the following lemmas

Lemma 8.4. *Suppose $F(u)/u$ is non-increasing. Let f_1, f_2 and f be so related that $f=f_1+f_2$. Let u, u_1 and u_2 be corresponding solutions of (1) and (2). Then $u \leqq u_1+u_2$.*

Proof. Set $k(t, x)=F(u(t, x))/u(t, x)$ and let u_1^* and u_2^* be solutions of the equation

$$u^{\cdot}=\tfrac{1}{2}u''+ku$$

with $u_1^*(0+, \cdot)=f_1$ and $u_2^*(0+, \cdot)=f_2$, respectively. Then $u=u_1^*+u_2^*$. Since $F(u_i)/u_i \geqq F(u)/u$ for any $(t, x) \in E$, we have $u_i \geqq u_i^*$ $(i=1, 2)$. Thus $u \leqq u_1+u_2$.

Lemma 8.5. *Suppose $F(u)/u$ is non-increasing. Let f_1 and f_2 belong to the class $[E_0]$ i.e. $\sup\{x; f_i(x)>0\}<\infty$ and set $m_i(t)=\sup\{x; u(t, x; f_i)=1/2\}$ $(i=1, 2)$. Then $m_1(t)-m_2(t)$ is bounded for large t.*

Proof. We can assume that $f_1=I_{(-1,0]}$ and $f_2=I_{(-\infty,0]}$. Since $f_1=f_2-f_2(\cdot+1)$, by Lemma 8.4 and Theorem 8.1

$$\varliminf u(t, x+m_2(t); f_1) \geqq w_{c_0}(x)-w_{c_0}(x+1) \sim (1-e^{-c^*})w_{c_0}(x) \qquad (x \longrightarrow \infty).$$

Hence $m_2(t)-m_1(t)$ $(\geqq 0)$ is bounded.

Proof of Theorem 8.5. We deal with only the case that f belongs to the class $[E_\mu]$ for some $\mu>0$. The other cases are analogously treated and omitted here.

Let f belong to the class $[E_\mu]$. For each positive integer n define

$$f_n=1 \quad \text{for} \quad x<-n; \ =f(x) \qquad \text{for} \quad x>-n$$

and set $m_n(t)=\sup\{x; u(t, x; f_n)=1/2\}$. Since $f \leqq f_n \leqq f+f_0(\cdot+n)$, where $f_0=I_{(-\infty,0]}$, by Lemma 8.4

$$\begin{aligned}|u(t, x+m(t))&-w_{c_0}(x+m(t)-m_n(t))| \\ &\leqq u(t, x+m(t)+n; f_0)+|u(t, x+m(t); f_n)-w_{c_0}(x+m(t)-m_n(t))|.\end{aligned}$$

By Theorem 8.1 the last term in this inequality tends to zero as $t \to \infty$ uniformly in $x \in R$. Thus, writing $m_0(t)=\sup\{x; u(t, x; f_0)=1/2\}$,

$$\varlimsup_{t\to\infty} |u(t, x+m(t))-w_{c_0}(x+m(t)-m_n(t))| \leqq w_{c_0}(x+\varliminf_{t\to\infty}(m(t)-m_0(t))+n).$$

Since $\varliminf_{t\to\infty}(m(t)-m_0(t))>-\infty$ by Lemma 8.5, the left side quantity in the above inequality become arbitrarily small uniformly in $x>-N$ for each real N when we let n large. Therefore (8.3) holds uniformly in $x>-N$. The required uniformity in $x>-m(t)$ is obtained if we bound $u(t, x+m(t))$ below by $u(t, x+m(t); f_*)$ where

$$f_*(x)=0 \quad \text{for} \quad x<0; \ =f(x) \qquad \text{for} \quad x>0$$

and apply Theorem 8.1 and Lemma 8.5. q.e.d.

In the proof carried out in the above we needed Theorems 8.1 to 8.4 applied to data with $\underline{\lim}_{x\downarrow-\infty} f(x)>0$. But for such data the proofs of these theorems are much simplified. Indeed we need only the comparison argument based on Proposition 3.1 in the phase space and Theorem 4.1 in addition to Lemmas 6.2, 6.3 and 8.2 (see the proof of Theorem 8.3). Correspondingly Theorem 8.5 can be obtained more easily than Theorems 8.1 or 8.2.

9. Speed of Propagation

We have seen in the previous section that the front of $u(t, x; f)$ propagates with speed $m^{\cdot}(t)$ as forming the shape of the c-front with some constant c, provided that the tail of f at (positive) infinity behaves regularly in a certain sense. The purpose of this section is to get nice estimations of $m(t)$.

Theorem 9.1. *Let f be a datum. Set $u=u(t, x; f)$. Suppose, for some continuous function $k(t)$, there exists*

$$\lim_{t\to\infty} u(t, x+k(t))=g(x) \qquad \textit{in locally } L_1 \textit{ sense,}$$

where g is not a constant. Then g is a c-front with some speed c, $|c|\geqq c_0$. If $m(t)$ is defined by (for large t)

$$u(t, m(t))=\tfrac{1}{2} \quad \textit{and} \quad m(t)-k(t) \quad \textit{being bounded,}$$

then m is continuously differentiable and $m^{\cdot}(t)\to c$ as $t\to\infty$. Furthermore $v(t, x)\equiv u(t, x+m(t))$, v' and v'' converge to w_c, w_c' and w_c'', respectively, as $t\to\infty$ locally uniformly.

Proof. By the remark following (4.6) and (4.7), functions u', u'', u''' are uniformly bounded for $t>1$. It follows that the function g is twice continuously differentiable and, setting $v(t, x)=u(t, x+k(t))$, we have

$$(9.1) \qquad v' \longrightarrow g' \quad \text{and} \quad v'' \longrightarrow g'' \qquad \text{as} \quad t\longrightarrow\infty$$

locally uniformly in x. Let J be a connected component of the set $\{x;\ g'(x)\neq 0\}$. Without loss of generality we can assume $g'<0$ on J. Let $x_1\in J$ be fixed. Then we can define a continuous function $k_*(t)$ (for large t) by

$$u(t, k_*(t); f)=g(x_1) \quad \text{and} \quad \lim_{t\to\infty}(k_*(t)-k(t))=x_1.$$

Set $v_*(t, x)=u(t, x+k_*(t); f)$. Then by (9.1)

(9.2) $\qquad v_* \longrightarrow g,\ v'_* \longrightarrow g' \quad \text{and} \quad v''_* \longrightarrow g'' \quad \text{as} \quad t \longrightarrow \infty$

locally uniformly. Note that $v_*(t, 0)=g(x_1)$ is constant. Letting x be fixed at zero and t tend to infinity in the equation

$$v^{\cdot}_* = \tfrac{1}{2} v''_* + k^{\cdot}_* v'_* + F(v_*),$$

we have

$$k^{\cdot}_*(t) \longrightarrow c \equiv -\frac{\{(1/2)g''(x_1)+F(g(x_1))\}}{g'(x_1)} \quad \text{as} \quad t \longrightarrow \infty.$$

Integrating the both sides of the same equation by t from n to $n+1$ and letting n tend to infinity, we have $0=(1/2)g''+cg'+F(g)$. Hence g is a c-front. q.e.d.

McKean [14] found that if $F(u)$ has the special form $\alpha u(1-u)$ and initial datum f is the indicater function of negative real axis then $m(t) \leqq c^* t - \sigma(t) +$ const., where we write

(9.3) $$\sigma(t) = (2c^*)^{-1} \log t.$$

This is easily extended to the case that $F(u)/u \leqq \alpha$, i.e. $\beta=\alpha$, and is readily derived from

Proposition 9.1. *Let v be a solution of the linear equation*

(9.4) $$v^{\cdot} = \tfrac{1}{2} v'' + \alpha v$$

with $v(0+, \cdot)=g$ a.e. and with $v(t, x)=O(\exp\{x^2\})$ uniformly in $t \in (0, T)$ for each $T<\infty$, where g is measurable and satisfies $\int_R e^{c^* x}|g(x)|dx < \infty$. Then

$$v(t, x+c^* t-\sigma(t)) \longrightarrow \frac{1}{\sqrt{2\pi}} \int_R e^{c^* y} g(y) dy\, e^{-c^* x} \qquad \text{as} \quad t \longrightarrow \infty.$$

Proof. By the equation $(x+c^*t-\lambda \log t - y)^2 = (x-y-\lambda \log t)^2 + 2c^* t(x-y-\lambda \log t) + 2\alpha t^2$, we see

$$\begin{aligned} v(t, x+c^*t-\lambda \log t) &= e^{\alpha t} P_t g(x+c^*t-\lambda \log t) \\ &= \frac{e^{c^*\lambda \log t}}{\sqrt{2\pi t}} \int_R \exp\left\{\frac{-(x-y-\lambda \log t)^2}{2t} + c^* y\right\} g(y) dy\, e^{-c^* x}. \end{aligned}$$

Then substitute $\lambda = 1/2c^*$ to get the result.

Suppose $F(u)/u \leqq \alpha$. Let f be a datum and set $u=u(t, x; f)$, $m(t)=\sup\{x; u(t, x)=1/2\}$. Then

$$m(t) \leqq c^* t - \sigma(t) + \text{const.} \qquad \text{if} \quad \int_R e^{c^* x} f(x) dx < \infty.$$

This is immediate from Proposition 9.1 and the inequality $u(t, x) \leqq e^{\alpha t} P_t f(x)$. Let A be a positive function such that

$$\int_R A(x)dx < \infty \quad \text{and} \quad A(\log x) \quad \text{varies slowly at infinity.}$$

Then under the same restriction on F

$$c^*t - \sigma(t) - m(t) \longrightarrow \infty \qquad \text{if} \quad f(x) = O(A(x)e^{-c^*x}) \quad \text{as} \quad x \longrightarrow \infty.$$

For the proof it suffices to show that if $f(x) \sim A(x)\exp\{-c^*x\}$ as $x \to \infty$ and $f(x) = 1$ for $x < 0$ then $u(t, c^*t - \sigma(t)) \to 0$ as $t \to \infty$. But for such f we have seen, in Theorem 8.1, $u(t, x + m(t)) \to w_{c_0}(x)$, while $\overline{\lim}\, u(t, x + c^*t - \sigma(t)) \leqq \text{const.} \exp\{-c^*x\} = o(w_{c_0}(x))$, since, by Lemma 2.2 (see also (2.22)), $\lim_{x\to\infty} w_{c_0}(x)\exp\{c^*x\}/x > 0$. Thus $\overline{\lim}\, u(t, x + c^*t - \sigma(t)) = 0$.

We get here a more exact estimation under some additional restrictions on F and f.

Theorem 9.2. *Suppose $F(u) \leqq \alpha u$ for $0 < u < 1$ and $\alpha u - F(u) = o(u^{1+\delta})$ with some $\delta > 0$. Let f be a datum with* $\sup\{x; f(x) > 0\} < \infty$. *Then*

$$c^*t - 3\sigma(t) + \text{const.} \leqq m(t) \leqq c^*t - 3\sigma(t) + O(\log\log t).$$

Proof. Step 1. We can fined a function F^* and F_* satisfying (3) such that $F^* \geqq F \geqq F_*$ and $F^*(u)/u$ and $F_*(u)/u$ are decreasing. Therefore there is no loss of generality in assuming $f = f_0 = I_{(-\infty,0)}$ by virtue of Lemma 8.5 and in assuming that $F(u)/u$ is decreasing.

Set $u = u(t, x; f_0)$ and $v(t, x) = u(t, x + m(t))$. Then, by Lemma 6.3, we see $v(t, x) \downarrow w_{c_0}(x)$ for each $x < 0$ and $\uparrow w_{c_0}(x)$ for each $x > 0$ as $t \to \infty$.

Let $\underline{m}(t)$ be the maximal convex function on $t > 0$ that bounds $m(t)$ below. In the remaining part of this step we prove that $m(t) - \underline{m}(t)$ is bounded. Define a function $a(t, s, N)$, $t, s, N > 0$ by

$$m(t+N) - m(t) = m(t+s+N) - m(t+s) + a(t, s, N).$$

Since $c^*t - m(t) \to \infty$ and $m'(t) \to c^*$, there exists an unbounded sequence $\{t_n\}$ such that $m(t_n) = \underline{m}(t_n)$. It is easily seen that

$$0 \leqq m(t) - \underline{m}(t) \leqq \sup_{\substack{s,N>0 \\ r>t_n}} a(r, s, N) \qquad \text{if} \quad t > t_n.$$

Therefore for our present purpose it is sufficient to prove that

$$(9.5) \qquad a(t, s, N) \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty \quad \text{uniformly in} \quad s, N > 0.$$

Set $h(t, x) = [v(t, x) - v(t+N, x)]^+$. Then $h(t, x) \leqq v(t, x + M(t))$ with some function M such that $M(t) \to \infty$ as $t \to \infty$. By Lemma 8.4 and the monotonicity of $F(u)/u$

$$\begin{aligned} \tfrac{1}{2} &= u(s, m(t+s) - m(t); v(t, \cdot)) \\ &\leqq u(s, m(t+s) - m(t); h(t, \cdot)) + u(s, m(t+s) - m(t); v(t+N, \cdot)). \end{aligned}$$

Since $u(s, m(t+s) - m(t); h(t, \cdot)) \leqq v(t+s, M(t)) \to 0$ as $t \to \infty$ uniformly in s, N, we have

$$u(s,\ m(t+s)-m(t);\ v(t+N,\ \cdot))+o(1)$$

$$\geqq \tfrac{1}{2}=u(s,\ m(t+s+N)-m(t+N);\ v(t+N,\ \cdot))$$

and hence $m(t+s+N)-m(t+N)\geqq m(t+s)-m(t)+o(1)$ where $o(1)\to 0$ as $t\to\infty$ uniformly in s, N. This is the same as the statement (9.5).

Step 2. Set $k(t,x)=F(v(t,x))/v(t,x)$. Then we have

$$v^{\cdot}=\tfrac{1}{2}v''+m^{\cdot}v'+kv.$$

In terms of the standard 1-dimentional Brownian motion $\{B_t,\ t\geqq 0;\ P_x,\ x\in R\}$ (cf. [9]), we have by Kac's formula

$$v(t,\ x)=E_x[e^{\int_0^t k(s,B_{t-s}+m(t)-m(s))ds}f_0(B_t+m(t))],$$

where $E_x[\cdot]$ stands for the expectation of Brownian motion B_t starting from a position x. Let q_* be a function on R defined by

$$q_*(x)=Me^{-bx}\qquad M>0,\quad b>0,$$

where constants M and b are chosen so that $k\geqq\alpha-q_*$. This is possible, because $v(t,x)\uparrow w_{c_0}(x)$ as $t\uparrow\infty$ for $x>0$, $\log w_{c_0}(x)\sim -c^*x$ as $x\to\infty$ and $F(u)/u=\alpha+o(u^\delta)$. Then

$$v(t,\ x)\geqq e^{\alpha t}E_x[e^{-\int_0^t q_*(B_s+m(t)-m(t-s))ds}f_0(B_t+m(t))].$$

Write

$$m(t)=c^*t-n(t).$$

Setting $L=\inf_{t>s>0}\{m(t)-m(t-s)-(s/t)m(t)\}$, we see, by the convexity of $\underline{m}(t)$ and the boundedness of $m(t)-\underline{m}(t)$, that L is finite and we have

$$v(t,\ x)\geqq e^{\alpha t}E_x[e^{-\int_0^t q_*(B_s+m(t)\frac{s}{t}+L)ds}f_0(B_t+m(t))]$$

and

$$v(t,\ -1)\geqq e^{\alpha t}E_{-1}[e^{-\int_0^t q_*(B_s+m(t)\frac{s}{t}+L)ds}|B_t+m(t)=-1]$$
$$\times P_{-1}[-1<B_t+m(t)<0].$$

where $E.[\cdot|\cdot]$ stands for the conditional expectation. Since $\{B_s+m(t)s/t;\ 0<s<t\}$ conditioned on $B_t+m(t)=-1$ has the same conditional law as $\{B_s;\ 0<s<t\}$ conditioned on $B_t=-1$, the right hand side of the last inequality is equal to

$$e^{\alpha t}E_{-1}[e^{-\int_0^t q_*(B_s+L)ds}|B_t=-1]\int_0^1 p(t,\ y-m(t))dy$$

$$=E_{-1+L}[e^{-\int_0^t q_*(B_s)ds}|B_t=-1+L]\frac{e^{c^*n(t)}}{\sqrt{2\pi t}}\left(\int_0^1 e^{c^*y}dy+o(1)\right)$$

$$=\frac{e^{c^*}-1}{c^*}p_*(t,\ -1+L,\ -1+L)e^{c^*n(t)}(1+o(1))$$

where $p_*(t, x, y)$ is the fundamental solution of the parabolic equation

$$u^{\cdot}=\frac{1}{2}u''-q_*u \qquad t>0, \quad x\in R.$$

In order to estimate $p^*(t, x, x)$ let $\hat{p}(t, x, y)$ be the fundamental solution of $u^{\cdot}=u''-\hat{q}u$ where $\hat{q}(x)=\exp\{-2x\}$. Then

$$\hat{p}(t, x, y)=\int_0^\infty e^{-t\lambda}\frac{1}{\pi^2}K_{is}(e^x)K_{is}(e^y)\sinh(s\pi)d\lambda \qquad s^2=\lambda,$$

where $i=\sqrt{-1}$ and K_μ are modified Bessel functions of the second kind:

$$K_{is}(z)=\int_0^\infty e^{-z\cosh t}\cos(st)dt$$

(cf. [15]). It is easy to obtain the corresponding integral representation of $p_*(t, x, y)$ from which we have the asymptotic formula:

$$p_*(t, x, x)=\frac{4\sqrt{2}}{\sqrt{\pi}b^2}\left\{K_0\left(\frac{2\sqrt{2M}}{b}e^{-bx/2}\right)\right\}^2\frac{1}{\sqrt{t^3}}\left(1+O\left(\frac{1}{\sqrt{t}}\right)\right).^{(*)}$$

Therefore

$$v(t, -1)\geqq \text{const.}\frac{e^{c^*n(t)}}{\sqrt{t^3}}(1+o(1))$$

and

$$n(t)\leqq\frac{3}{2c^*}\log t+\text{const.}$$

This proves the first inequality in the theorem.

Step 3. We may assume without loss of generality that $F(u)/u<\alpha-\eta$ for $1/2<u<1$ with some positive constant $\eta\,(<\alpha)$ (if this is not the case, consider $\sup\{x; u(t, x)=1-\varepsilon\}$ instead of $m(t)$). Then, setting

$$q^*(x)=\eta \quad \text{if} \quad x<0 \quad \text{and} \quad =0 \quad \text{if} \quad x>0,$$

we have $k\leqq\alpha-q^*$ for all $t>0$, $x\in R$, and

$$v(t, x)\leqq e^{\alpha t}E_x[e^{-\int_0^t q^*(B_s+m(t)-m(t-s))ds}f_0(B_t+m(t))].$$

Since for large t, by Step 2, $m(t)-m(t-s)-m(t)s/t=n(t-s)-n(t)(1-s/t)\leqq 4\sigma(t)$, we see, as before,

$$\begin{aligned}\frac{1}{2}=v(t, 0)&\leqq e^{\alpha t}E_0[e^{-\int_0^t q^*(B_s+m(t)\frac{s}{t}+4\sigma(t))ds}|B_t+m(t)=0]\\&\qquad\qquad\times P_0[B_t+m(t)<0]\\&=p^*(t, 4\sigma(t), 4\sigma(t))e^{c^*n(t)}(1/c^*+o(1)),\end{aligned}$$

(*) We need here only "$0<C_1\leqq\sqrt{t}^{\,3}p_*(t, x, x)\leqq C_2<\infty$ $(t\uparrow\infty)$". This is obtained under the assumption that $\int^{+\infty}q_*(x)xdx<\infty$, $q_*\geqq 0$, $q_*\not\equiv 0$ and q_* is locally bounded.

where $p^*(t, x, y)$ is the fundamental solution of

$$u^{\cdot}=\frac{1}{2}u''-q^*u.$$

The Laplace transform of $p^*(t, x, x)$ is explicitely calculated for $x>0$ as

$$G_\lambda(x, x)=\frac{1}{\sqrt{2}}\left\{\frac{1}{\sqrt{\lambda}}(1-e^{-2\sqrt{2\lambda}\,x})-\frac{2}{\eta}(\sqrt{\lambda}-\sqrt{\lambda+\eta})e^{-2\sqrt{2\lambda}\,x}\right\},$$

and inverting the transform we have, setting $\hat{\sigma}=4\sigma$,

$$\begin{aligned} &p^*(t, \hat{\sigma}(t), \hat{\sigma}(t)) \\ &=\frac{1}{\sqrt{2\pi t}}\{1-e^{-8\hat{\sigma}^2/4t}\}+\frac{1}{\eta\sqrt{2}}\left\{\frac{1}{2\sqrt{\pi t}\,t}(1-e^{-\eta t})\right\}*\left\{\frac{\sqrt{2}\,\hat{\sigma}}{\sqrt{\pi t}\,t}e^{-2\hat{\sigma}^2/t}\right\}^{(*)} \\ &=\frac{1}{\sqrt{2\pi t}}\frac{2\hat{\sigma}^2}{t}(1+o(1)) \qquad \text{as} \quad t\longrightarrow\infty. \end{aligned}$$

Thus we have

$$\frac{1}{2}\leqq\frac{2\hat{\sigma}(t)^2}{\sqrt{2\pi}\sqrt{t^3}}e^{c^*n(t)}\left(\frac{1}{c^*}+o(1)\right)$$

or

$$n(t)\geqq(3/2c^*)\log t-(2/c^*)\log\log t+\text{const.}$$

and the second inequality of the theorem has been proved. The proof of Theorem 9.2 is completed.

Remark. Assuming $F'(1)<0$ in addition to the conditions of Theorem 9.2, the speed of the convergence expressed in (8.3) is generally not more rapid than that of $(t\log t)^{-1}$ to tend to zero. In fact we have for $v(t, x)=u(t, x+m(t); I_{(-\infty,0)})$

$$\sup_{\substack{|x|<C\log s \\ s>t}}|v(s, x)-w_{c_0}(x)|\geqq C_1/t\log t$$

where C and C_1 are some positive constants.

The proof is outlined here. Set $M(t)=-\int_{-\infty}^{\infty}u'(t, x; I_{(-\infty,0)})dx$. Then using Lemma 6.1 we have, after elementary calculations, that

$$m(t)=M(t)+\int_{-\infty}^{\infty}w'(x)xdx+o(1),\ M^{\cdot}(t)=\int_{-\infty}^{\infty}F(v(t, x))dx.$$

Since $c_0=\int_{-\infty}^{\infty}F(w_{c_0}(x))dx$, it follows that

$$\begin{aligned} |M^{\cdot}(t)-c_0| &\leqq\gamma\int_{-\infty}^{\infty}|v(t, x)-w_{c_0}(x)|\,dx \\ &\leqq 2\gamma[C\log t\sup_{|x|<C\log t}|v(t, x)-w_{c_0}(x)| \end{aligned}$$

(*) $a(t)*b(t)=\int_0^t a(t-s)b(s)ds.$

$$+\int_{x>C\log t} w_{c_0}(x)dx+\int_{x<-C\log t}(1-w_{c_0}(x))dx],$$

from which we deduce the inequality

$$\sup_{|x|<C\log t}|w_{c_0}(x)-v(t,x)|\geqq C_1\Big(\frac{1}{t\log t}+\frac{k^{\cdot}(t)}{\log t}\Big).$$

Here we put $k(t)=M(t)-c_0t+(3/2c_0)\log t$ and chose positive constants C and C_1 appropriately. Since $M(t)=m(t)+O(1)$, we see $k(t)=O(\log\log t)$ and then that the above inequality implies the desired one.

It is interesting to compare the results as in Theorem 9.2 with the result obtained in case $F(u)=u(u-a)(1-u)$ where $0<a<1$ (as typical example): Fife and McLeod [3a] says that with such F there exists the unique speed c for which the differential equation (4) has a global solution w such that $0\leqq w\leqq 1$, $w(-\infty)=1$ and $w(\infty)=0$ and that for any continuous f with $\underline{\lim}_{x\to-\infty}f(x)>a$ and $\lim_{x\to\infty}f(x)<a$ it holds that

$$(*)\qquad |u(t,x+ct)-w(x+x_0)|\longrightarrow 0 \quad\text{as}\quad t\longrightarrow\infty$$

where u is the solution of (1) and (2) with present F and x_0 is some constant.

Next three theorems give an answer to the question of when $m(t)-ct$ is bounded and such formula as described by $(*)$ holds.

Theorem 9.3. *Let $\alpha>0$ and $c>c_0$. Let f be such a datum that there exists $\lim_{x\to\infty}e^{\underline{b}x}f(x)=a\leqq\infty$ where $\underline{b}=c-\sqrt{c^2-2\alpha}$. Suppose $\int_{0+}|\alpha-F'(u)|u^{-1}du<\infty$. Set $u=u(t,x;f)$. Then $m(t)-ct$ is bounded if and only if $0<a<\infty$. If this is the case and if Condition [W] is satisfied, then*

$$(9.6)\qquad u(t,x+ct)\longrightarrow w_c(x+x_0)\quad as\quad t\longrightarrow\infty\quad uniformly\ in\quad x>N,$$

where $x_0=\underline{b}^{-1}\log(a_0/a)$, $a_0=\lim_{x\to\infty}w_c(x)e^{\underline{b}x}$, for each $N>-\infty$.

Proof. It suffices to prove (9.6) assuming that $0<a<\infty$. Set $w_*=\underline{\lim}_{t\to\infty}u(t,\cdot+ct)$ and $w^*=\overline{\lim}_{t\to\infty}u(t,\cdot+ct)$. We will prove that if $0<a<\infty$

$$(9.7)\qquad w^*(x)\sim w_*(x)\sim f(x)\qquad\text{as}\quad x\longrightarrow\infty.$$

Note that these are immediate consequences of Lemma 4.5 if $c^2/2\geqq\gamma^*$. First we prove $w_*(x)\geqq f(x)(1+o(1))$. For this purpose, as in the proof of Theorem 5.1 (ii), take a function $\hat F$ satisfying (3) such that $\int_{0+}|\hat F(u)-\alpha u|u^{-2}du<\infty$, $\hat F'(0)=\alpha$, $\hat F'\leqq\alpha$ and $\hat F\leqq F$. Then $\hat u(t,x)\equiv u(t,x;f;\hat F)\leqq u(t,x)$ and, since $c^2/2>\alpha=\sup\hat F'$, $\lim\hat u(t,x+ct)=\hat w_c(x+\hat x_0)$ where $\hat w_c$ is a c-front corresponding to $\hat F$ and $\hat x_0$ is determined by $\hat w_c(x+\hat x_0)\sim f(x)$ as $x\to\infty$. Thus $w_*(x)\geqq f(x)(1+o(1))$. Next we prove $w^*(x)\leqq w_c(x+x_0)$. By Corollary of Lemma 2.4, for any $\delta>0$ we can find a continuous datum g such that with some constant L

$$\tfrac{1}{2}g''+cg'+F(g)=0\quad\text{for}\quad x>L\quad\text{and}\quad g=1\quad\text{for}\quad x<L$$

and such that $g(x) \sim f(x)$ as $x \to \infty$ and $f(x) \leqq g(x-\delta)$, $x \in R$. Let $\hat{u}=u(t, x; g)$. Then by Lemma 5.1 $\lim \hat{u}(t, x+ct)=w_c(x+x_0)$. Since $u(t, x) \leqq \hat{u}(t, x-\delta)$, we have $w^*(x) \leqq w_c(x+x_0-\delta)$. Hence $w^* \leqq w_c(x+x_0)$. Since $\lim w_c(x+x_0)e^{\underline{b}x}=a_0 e^{-\underline{b}x_0}$, $w^* \leqq f \cdot (1+o(1))$. Consequently (9.7) has been proved.

If Condition [W] is satisfied, we have $u(t, x+m(t)) \to w_c(x)$, and hence (9.7) implies (9.6). The proof of Theorem 9.3 is completed.

Similarly we obtain

Theorem 9.4. *Let* $c_0=c^*$. *Assume* (2.4) *and that*

$$\int_{0+} |F'(u)-\alpha| \, |\log u| u^{-1} du < \infty \quad or \quad F(u)-\alpha u=o(u^p) \qquad p>1.$$

Suppose $\lim f(x)x^{-1}e^{c^*x}=a$ *exists and is positive and finite. Then under Condition* [W]

$$u(t, x+c_0 t; f) \longrightarrow w_{c_0}(x+x_0) \tag{9.8}$$

where $x_0=\log(a_0/a)/c^*$, $a_0=\lim w_{c_0}(x)x^{-1}e^{c^*x}$.

In case $c=c_0>c^*$ we have

Theorem 9.5. *Assume the hypotheses of Lemma* 5.3 (i). *Then* $m(t)-c_0 t$ *is bounded. If we assume in addition the hypotheses of Theorem* 8.1, *then* (9.8) *holds where* x_0 *is some constant.*

Proof. The boundedness of $m(t)-c_0 t$ is clear by Lemma 5.3 (i). Let f belong to the class $[F_\lambda]$ with $\lambda > c_0-\sqrt{c_0^2-2\alpha}$. Then if $c_0-\sqrt{c_0^2-2\alpha}<b<\min\{\lambda, c_0+\sqrt{c_0^2-2\alpha}\}$ we have

$$\sup_{t>0} |u(t, x+m(t); f)+w_{c_0}(x)| e^{bx} \longrightarrow 0 \quad \text{as} \quad x \longrightarrow \infty$$

and hence, by Theorem 8.1,

$$\sup_{x \in R} |u(t, x+m(t); f)-w_{c_0}(x)| e^{bx} \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty.$$

which combined with Lemma 5.3 (ii) deduces (9.8). When f belongs to the class $[E_\mu]$, we can similarly proceed with any $c_0-\sqrt{c_0^2-2\alpha}<b<c_0+\sqrt{c_0^2-2\alpha}$.

q. e. d.

10. Supplement to The Case $c_0>c^*$

Here is given an alternative proof of Theorem 8.1 in the case $c_0>c^*$, which provides a better consequence. The proof is a modification of a proof given in P. C. Fife and J. B. McLeod [3.b] to the assertion cited in § 9 and simpler than that given through § 6 to § 8.

Theorem 10.1. *Assume* $c_0>c^*$. *Let* f *be a datum such that* $f(x)=O(e^{-bx})$ *for a constant* $b>c_0-\sqrt{c_0^2-2\alpha}$. *It is in addition assumed that Condition* [G] *in*

§5 *is satisfied (this is automatic when* $\alpha>0$). *Then* (9.8) *holds.*

Proof. Set $u=u(t, x; f)$ and $z(t, x)=u(t, x+c_0t)$. Obserbing that $v=u$ or $v=u'$ satisfies that, with $k=F(u)/u$ or $k=F'(u)$, respectively,

$$v(t+1, x+y)=-\int_0^1 p'(1-s, y)v(t+s, x)ds+\int_0^\infty p^*(1, y, r)v(t, x+r)dr$$

$$+\int_0^1 ds\int_0^\infty p^*(1-s, y, r)k(t+s, x+r)v(t+s, x+r)dr$$

for $x\in R$, $y>0$ and $t>0$, where p^* is defined just after the equation appeared in the last paragraph of §5, differentiating the both sides of this equation with respect to y, and then putting $y=1$, we deduce the estimates: for $t>0$, $x\in R$

$$|u'(t+1, x+1)|\leqq K|u|_{t,x}^{t+1}, \quad |u''(t+1, x+1)|\leqq K|u'|_{t,x}^{t+1}$$

where $|v|_{t,x}^{t+1}=\sup_{t<s<t+1, y>x}|v(s, y)|$ and K is a constant independent of t and x. We can assume $b<c+\sqrt{c_0^2-2\alpha}$. Then, by the equality $u^{\cdot}=2^{-1}u''+F(u)$ and by Lemma 5.3 (i), we see that for $t>1$, $x\in R$

$$z(t, x),\ |z'(t, x)|,\ |z^{\cdot}(t, x)|<K_1\min\{e^{-b_1x}+e^{-\eta t}e^{-bx}, 1\} \tag{10.1}$$

where $b_1=c_0+2^{-1}\sqrt{c_0^2-2\alpha}$ and K_1 is a constant independent of t and x. Let ε be a positive constant so small that $(c_0-b)\varepsilon<\eta$, and set

$$E(t)=\int_{-\varepsilon t}^{\varepsilon t} e^{2c_0x}[\tfrac{1}{4}z'(t, x)^2-\int_0^{z(t,x)}F(r)dr]dx.$$

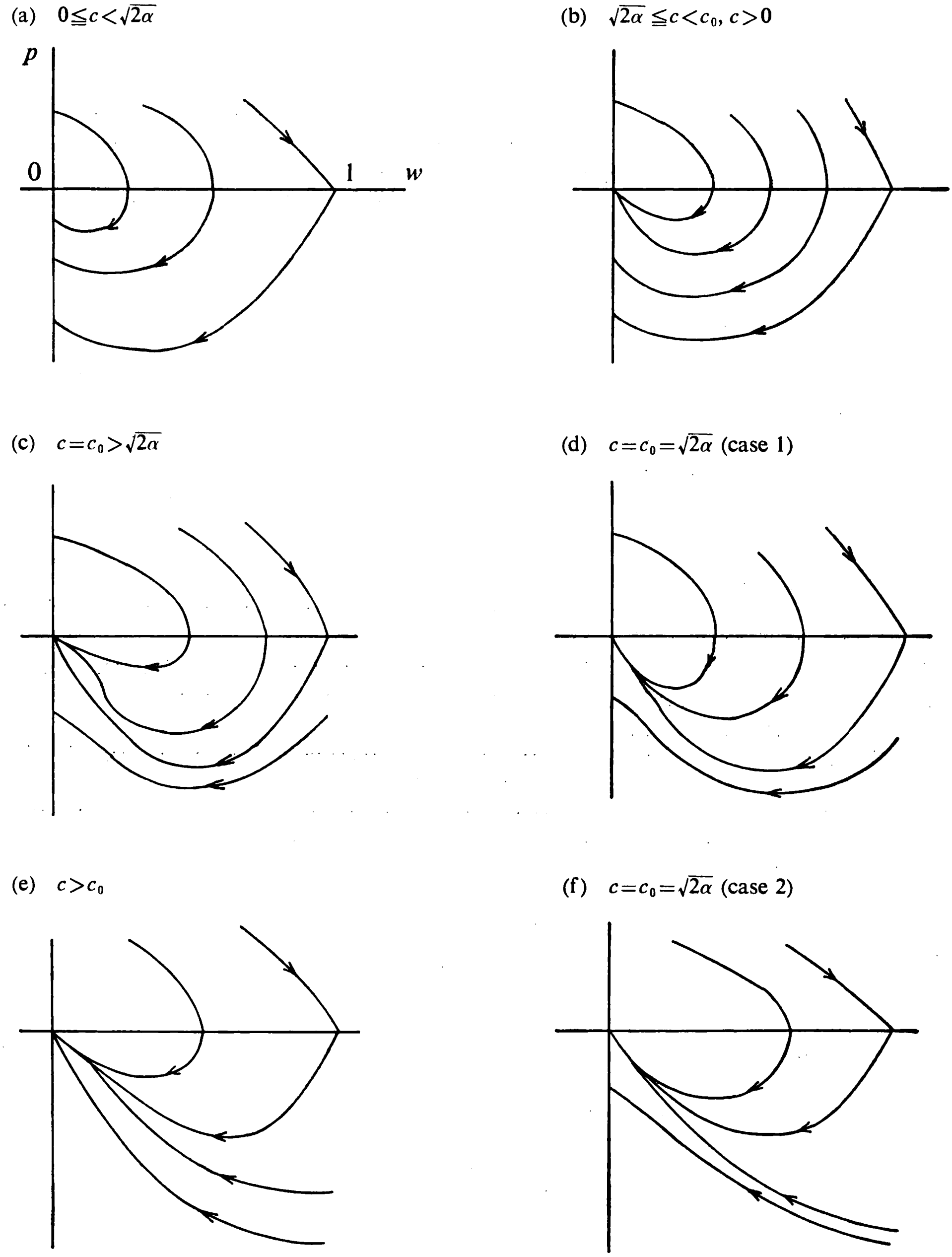
Then, by (10.1), $E(t)$ is bounded as t tends to infinity and

$$E^{\cdot}(t)=o(1)-\int_{-\varepsilon t}^{\varepsilon t} e^{2c_0x}[\tfrac{1}{2}z''+c_0z'+F(z)]^2dx. \tag{10.2}$$

From these it follows that there exists an unbounded sequence $\{t_n\}$ along which $E^{\cdot}(t_n)\to 0$. Since z, z', z'' and z''' are bounded for $x\in R$, $t>1$ (see the remark following (4.6) and (4.7)), we can find a subsequence $\{t_{n'}\}\subset\{t_n\}$ such that $z(t_{n'}, x)$ converges in the norm of $C^2(-N, N)$ for each $N>0$. Let $w(x)=\lim z(t_{n'}, x)$. Then, by (10.2) and $\lim E^{\cdot}(t_{n'})=0$, $2^{-1}w''+c_0w'+F(w)=0$ and by Lemma 5.3 (i), w does not degenerate. Therefore w is a c_0-front. Since any c_0-front is stable in the sense of Lemma 5.3 (ii), we have actually $\lim z(t, x)=w(x)$ (see the proof of Theorem 9.5). Thus the theorem is proved.

Appendix

Following diagrams illustlate solutions of the equation (1.2).



Department of Applied Physics
Tokyo Institute of Technology

References

[1] Aronson, D. G. and Weinberger, H. F., *Nonlinear diffusion in population genetics, combustion, and nerve propagation*, Proc. Tulane Program in Partial Differential Equations, Lecture Notes in Math. No. 446, Springer-Verlag, New York, 1975.

[2] Coddington, E. A. and Levinson, N., *Theory of Ordinary Differential Equations*, McGraw-Hill, 1955.

[3a] Fife, P. C. and McLeod, J. B., *The approach of solutions of nonlinear diffusion equations*, Bull. A. Math. Soc. 81, 1975, 1076–1078.

[3b] Fife, P. C. and McLeod, J. B., *The approach of solutions of nonlinear diffusion equations*, Arch. Rat. Mech. Analysis, 65, 1977, 333-361.

[4] Friedman, A., *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.

[5] Fujita, H., *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. I, 13, 1966.

[6] Hadeler, K. P. and Rothe, F., *Travelling fronts in nonlinear diffusion equations*, J. Math. Biol. 2, 1975, 251–263.

[7] Hayakawa, K., *On the limit state of solutions of some semilinear diffusion equations*, Osaka J. Math. 12, 1975, 767–776.

[8] Il'in, A. M., Kalashnikov, A. S. and Oleinik, O. A., *Second order linear equations of parabolic type*, Russ. Math. Surveys 17, 1963, 1–143.

[9] Ito, K. and McKean, H. P., *Diffusion processes and their sample paths*, Springer, 1965.

[10] Kametaka, Y., *On the nonlinear diffusion equations of Kolmogorov-Petrovsky-Piskunov type*, Osaka J. Math. 13, 1976, 11–66.

[11a] Kanel', Ja. I., *On the stability of solutions of the equation of combustion theory for finite initial functions*, Mat. Sbolnik 65, 1964, 398–418. (in Russian)

[11b] Kanel', Ja. I., *The behaviour of the solution of the Cauchy problem when the time tends to infinity, in the case of quasi-linear equations arising in the theory of combustion*, Sov. Math. Dokl., 1, 1960, 533–536.

[12] Kobayashi, K., Sirao, T. and Tanaka, H., *On the growing up problem for semi-linear heat equations*, J. Math. Soc. Japan, 29, 1977, 407–424.

[13] Kolmogorov, A., Petrovsky, I. and Piskunov, N., *Etude de l'équation de la diffusion avec croissance de la quantité de la martière et son application a un probleme biologique*, Moscow Univ. Bull. Math. 1, 1937, 1–25.

[14] McKean, H. P., *Application of Brownian motion to the equation of Kolmogorov-Petrovsky-Piskunov*, Comm. Pure Appl. Math. 28, 1975, 323–331.

[15] Titchmarsh, E. C., *Eigenfunction Expansions I*, Oxford Univ. Press, 1946.

[16] Uchiyama, K., *The Behavior of Solutions of the Equation of Kolmogorov-Petrovsky-Piskunov*, Proc. Japan Acad, 53, Ser. A. 1977, 225–228.

[17] Widder, D. V., *The Heat Equation*, Academic Press, 1955.