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On a generalization of behaviour spaces and the Riemann-Roch theorem on open Riemann surfaces

By

Turgut BASKAN

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Introduction.

The present paper deals with a certain generalization of my former result [2] and Shiba's one [6] concerning the Riemann-Roch theorem on open Riemann surfaces.

For this purpose we introduce a certain subspace Λ_B of harmonic semiexact differentials (see Definition 2.1) and define the notion of Λ_B-behaviour of meromorphic differentials near the ideal boundary (see Definition 2.2). We find that Λ_B-behaviour gives a generalization of Λ_p- and Λ_0-behaviour in [2] and [4], [5] respectively. By using Λ_B-behaviour we can define, as in [6], the singularities at the ideal boundary and show the existence of elementary differentials with prescribed such singularities.

After these preparations we shall show in § 3 an algebraic duality theorem on two mutually dual spaces of differentials (Theorem 3.5), from which we can immediately deduce the Riemann-Roch theorem (Theorem 3.6). Finally we shall mention some specializations of this theorem (Theorems 3.7 and 3.8).

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§ 1. Preliminaries.

We shall be working on an arbitrary open Riemann surface W with genus $g(\leqq\infty)$. The space of differentials which we are dealing with is a real Hilbert space Λ of square integrable complex differentials on W with the inner product defined by

$$\langle\lambda_1, \lambda_2\rangle=\mathrm{Re}(\lambda_1, \lambda_2)=\mathrm{Re}\iint_W \lambda_1\wedge\bar{\lambda}_2^*=\mathrm{Re}\iint_W (a_1\bar{a}_2+b_1\bar{b}_2)dx\,dy$$

where $\lambda_j=a_j dx+b_j dy$ for a local parameter $z=x+iy$. The norm in Λ will be denoted by $\|\cdot\|$. The real Hilbert space of square integrable real differentials with the usual inner product $(\,,\,)$ will be denoted by Γ. By $\{\Omega\}$ we denote a canonical regular exhaustion of W. For terminology and notations we follow [1], [2], [3], [5], [6].

We use the following orthogonal decompositions and relations.

$$\Lambda=\Gamma\oplus i\Gamma,\quad \Lambda=\Lambda_h\oplus\Lambda_{e0}\oplus\Lambda_{e0}^*,$$

$$\Lambda_h=\Lambda_{hse}^*\oplus\Lambda_{hm}=\Lambda_{hse}\cap\Lambda_{hse}^*\oplus\Lambda_{hm}\oplus\Lambda_{hm}^*,$$

$$\Lambda_c=\Lambda_h\oplus\Lambda_{e0},\quad \Lambda_h=\Lambda_c\cap\Lambda_c^*.$$

(cf. [1], [2], [5])

The following classical Lemma is often useful in our work.

Lemma 1.1. *Let Ω be a canonical regular region on W and $\Xi(W)=\{A_j, B_j\}_{j=1}^g$ a canonical homology basis on W modulo dividing cycles such that $\Xi\cap\bar{\Omega}$ is a canonical homology basis on $\bar{\Omega}$ modulo $\partial\Omega$. Suppose ϕ_1 and ϕ_2 are closed C^1-differentials on Ω and ϕ_1 is semiexact, then*

$$(\phi_1, \phi_2^*)_\Omega=-\int_{\partial\Omega}\left(\int\phi_1\right)\bar{\phi}_2+\sum_\Omega\int_{A_j}\phi_1\int_{B_j}\bar{\phi}_2-\int_{B_j}\phi_1\int_{A_j}\bar{\phi}_2,$$

where $\sum_\Omega$ stands for the sum over all A_j, B_j contained in Ω. (cf. [1], [2], [5], [7])

§ 2. Definitions and existence theorems.

Divide the set of positive integers $J=\{1, 2, \cdots, g\}$ into two disjoint set J_1, J_2 and let $\mathcal{L}=\{L_j\}$ $(j\in J_2)$ be a set of straight lines L_j in the complex plane passing through the origin $z=0$.

Definition 2.1. A (closed) subspace Λ_B of $\Lambda_{hse}\cap\Lambda_{hse}^*$ is called a behaviour space if it satisfies the following conditions.

(i) $\Lambda_B=i\Lambda_B^{\perp *}$, where $\Lambda_B^{\perp}$ is the orthogonal complement in $\Lambda_{hse}\cap\Lambda_{hse}^*$ of Λ_B.

(ii) For each $\lambda_b\in\Lambda_B$,

$$\int_{A_j}\lambda_b=0 \text{ for } j\in J_1 \quad\text{and}\quad \int_{A_j}\lambda_b,\ \int_{B_j}\lambda_b\in L_j \quad\text{for}\quad j\in J_2.$$

We denote such a subspace by $\Lambda_B=\Lambda_B(\mathcal{L}, J_1, J_2)$ or just by Λ_B. It is now an easy matter to verify that $\bar{\Lambda}_B=\{\bar{\lambda}_b|\lambda_b\in\Lambda_B\}$ is also a behaviour space if Λ_B is a behaviour space.

Definition 2.2. A meromorphic differential ϕ on W will be said to have Λ_B-behaviour if there is a neighbourhood U of the ideal boundary ∂W of W on which it can be written as

$$\phi=\lambda_b+\lambda_{hm}+\lambda_{e0}$$

where $\lambda_b\in\Lambda_B$, $\lambda_{hm}\in\Lambda_{hm}$ and $\lambda_{e0}\in\Lambda_{e0}\cap\Lambda^1$.

A meromorphic function f (not necessarily single-valued) on W is said to have Λ_B-behaviour if df has Λ_B-behaviour.

Remark. Let Λ_B be as above. Then the space

$$\Lambda_{B'}=\Lambda_B\oplus\Lambda_{hm}$$

satisfies

(0)′ $\Lambda_{B'}\subset\Lambda_{hse}$,

(i)′ $\Lambda_h=\Lambda_{B'}\oplus i\Lambda_{B'}^*$,

(ii)′ $\forall\lambda\in\Lambda_{B'}$, $\int_{A_j}\lambda=0$ if $j\in J_1$ and $\int_{A_j}\lambda$, $\int_{B_j}\lambda\in L_j$ if $j\in J_2$,

for we know the orthogonal decomposition $\Lambda_h=\Lambda_{hse}\cap\Lambda_{hse}^*\oplus\Lambda_{hm}\oplus\Lambda_{hm}^*$ (cf. [1], [5]). Hence $\Lambda_{B'}$ is an immediate generalization of Λ_p in [2] and also of Λ_0 in [5] (cf. Def. 6.1 in [2]). Conversely, every space $\Lambda_{B'}$ satisfying (0)′, (i)′ and (ii)′ induces a subspace Λ_B as in Definition 2.1. To see this, we only need to note that $\Lambda_{B'}=i\Lambda_{B'}^{*\perp}\supset i\Lambda_{hse}^{*\perp}=i\Lambda_{hm}=\Lambda_{hm}$ and hence we can consider the quotient space $\Lambda_{B'}/\Lambda_{hm}$. In other words, the spaces which we now consider correspond to the behavior space $\Lambda_p(J_2, J_1)$ in [2], § 6 in a one-to-one manner.

Definition 2.3. Two behavior spaces Λ_B and Λ_B' with the same partition $J=(J_1, J_2)$ are called dual to each other if and only if $L_0=L_j\circ L_j'=\{z=z_j\cdot z_j' \mid z_j\in L_j, z_j'\in L_j'\}$ $(j\in J_2)$, and

$$(\lambda_b, \overline{\lambda_b'^*})\in L_0, \quad \text{for } \lambda_b\in\Lambda_B \text{ and } \lambda_b'\in\Lambda_B'.$$

In the following we assume that $L_0=\boldsymbol{R}$. Let P be a regular partition of the ideal boundary ∂W, and take the following real linear space of differentials.

$A^P=\{\phi\mid\phi$ is an analytic differential on some $U\in\mathcal{E}(W)$ and (P)-semiexact$\}$. Here by $\mathcal{E}(W)$ we mean a collection of neighbourhoods of the ideal boundary ∂W. Let $A_{\mathcal{L}}^P=\{\phi\in A^P\mid\int_{A_j}\phi=0$ for $j\in J_1$ and $\int_{A_j}\phi$, $\int_{B_j}\phi\in L_j$ for $j\in J_2$ and $A_j, B_j\in\Xi(U)\}$, $A_{\Lambda_B}^P=\{\phi\in A_{\mathcal{L}}^P\mid\phi$ has Λ_B-behaviour$\}$. And so we consider the quotient space

$$V_{\Lambda_B}^P=A_{\mathcal{L}}^P/A_{\Lambda_B}^P.$$

Definition 2.4. The elements of $V_{\Lambda_B}^P$ will be called $(P)\Lambda_B$-singularities, and the subspaces of $V_{\Lambda_B}^P$ will be called $(P)\Lambda_B$-divisors.

Definition 2.5. Let $V=V(P, \Lambda_B)$ be a $(P)\Lambda_B$-divisor. A regular analytic differential λ on W is said to be a multiple of V if there exist $\sigma\in V$, $\lambda_b\in\Lambda_B$, $\lambda_{hm}\in\Lambda_{hm}$ and $\lambda_{e0}\in\Lambda_{e0}\cap\Lambda^1$ such that

$$\lambda=\sigma+\lambda_b+\lambda_{hm}+\lambda_{e0}$$

on some $U\in\mathcal{E}(W)$.

In this case we say λ has $(P)\Lambda_B$-singularity σ. The following linear space will be a basis for our work.

$$D(V)=\{\lambda|\lambda \text{ is a multiple of } V\}.$$

Now we are ready to give the uniqueness and existence theorems.

Theorem 2.6. **(Uniqueness).** *Let $\phi\in D(V)$ be free of $(P)\Lambda_B$-singularities. If $\int_{A_j}\phi=0$ for $j\in J_1$ and $\int_{A_j}\phi$, $\int_{B_j}\phi\in L_j$ for $j\in J_2$, then $\phi\equiv 0$.*

Proof. Since ϕ does not have $(P)\Lambda_B$-singularity, we can write

$$\phi=\lambda_b+\lambda_{hm}+\lambda_{e0}$$

on some $U\in\mathcal{E}(W)$. Let Ω be a canonical regular region such that $\partial\Omega\subset U$. Then by Lemma 1.1

$$\|\phi\|_{\Omega}^2=(\phi,\ \phi)_{\Omega}=-i(\phi,\ \phi^*)$$

$$=i\int_{\partial\Omega}\left(\int\phi\right)\bar{\phi}-i\sum_{\Omega}\left(\int_{A_j}\phi\int_{B_j}\bar{\phi}-\int_{B_j}\phi\int_{A_j}\bar{\phi}\right).$$

If we apply Lemma 1.1 to the first term in the right side

$$\int_{\partial\Omega}\left(\int\phi\right)\bar{\phi}=\int_{\partial\Omega}\left(\int\lambda_b+\lambda_{hm}+\lambda_{e0}\right)(\bar{\lambda}_b+\bar{\lambda}_{hm}+\bar{\lambda}_{e0})=-(\lambda_b+\lambda_{hm}+\lambda_{e0},\ \lambda_b^*+\lambda_{hm}^*+\lambda_{e0}^*)_{\Omega}$$

$$+\sum_{\Omega}\left(\int_{A_j}\lambda_b\int_{B_j}\bar{\lambda}_b-\int_{B_j}\lambda_b\int_{A_j}\bar{\lambda}_b\right),$$

since $\phi=\lambda_b+\lambda_{hm}+\lambda_{e0}$ on $\partial\Omega$ and $\int_{A_j}\lambda_{hm}=\int_{B_j}\lambda_{hm}=\int_{A_j}\lambda_{e0}=\int_{B_j}\lambda_{e0}=0$. On the other hand, by the hypothesis in the theorem,

$$\int_{A_j}\phi=\int_{A_j}\lambda_b=0 \qquad \text{for } j\in J_1 \text{ and}$$

$$\mathrm{Im}\left[\int_{A_j}\phi\int_{B_j}\bar{\phi}-\int_{B_j}\phi\int_{A_j}\bar{\phi}\right]=\mathrm{Im}\left[\int_{A_j}\lambda_b\int_{B_j}\bar{\lambda}_b-\int_{B_j}\lambda_b\int_{A_j}\bar{\lambda}_b\right]=0 \qquad \text{for } j\in J_2.$$

And so

$$\|\phi\|_{\Omega}^2=\mathrm{Im}(\lambda_b+\lambda_{hm}+\lambda_{e0},\ \lambda_b^*+\lambda_{hm}^*+\lambda_{e0}^*)_{\Omega}=\mathrm{Im}\ P_{\Omega}.$$

Here

$$P_{\Omega}=(\lambda_b+\lambda_{hm}+\lambda_{e0},\ \lambda_b^*+\lambda_{hm}^*+\lambda_{e0}^*)_{\Omega}.$$

By considering the orthogonal decompositions in § 1 one can get $\lim_{\Omega\to W}\mathrm{Im}\ P_{\Omega}=\mathrm{Im}\ P_W=0$, i.e., $\|\phi\|_W^2=0$, that is, $\phi\equiv 0$.

Theorem 2.7. *Let $\{\alpha_j\}_{j\in J_1}$ and $\{\alpha_j,\ \beta_j\}_{j\in J_2}$ be given sets of non-zero complex numbers such that $\alpha_j,\ \beta_j\notin L_j$ for $j\in J_2$. Then there exist holomorphic differentials $\phi_{\alpha_j}(B)$ $(j\in J)$ and $\phi_{\beta_j}(A_j)$ $(j\in J_2)$ such that*

(i) *$\phi_{\alpha_j}(B_j)$ and $\phi_{\beta_j}(A_j)$ are free of $(P)\Lambda_B$-singularities, i.e., these have Λ_B-behaviour,*

(ii) $\int_{A_k}\phi_{\alpha_j}(B_j)=\alpha_j(B_j\times A_k)$ *for $k,\ j\in J_1$*

$$\int_{A_k}\phi_{\alpha_j}(B_j),\ \int_{B_k}\phi_{\beta_j}(A_j)\in L_k \quad for\ k,\ j\in J_2,\ (k\neq j)$$

$$\int_{A_j}\phi_{\alpha_j}(B_j)+\alpha_j\in L_j,\ \int_{B_j}\phi_{\beta_j}(A_j)-\beta_j\in L_j \quad for\ j\in J_2.$$

$$\int_{A_k}\phi_{\alpha_j}(B_j),\ \int_{B_k}\phi_{\alpha_j}(B_j)\in L_k, \quad for\ j\in J_1\ and\ k\in J_2$$

$$\int_{A_k}\phi_{\alpha_j}(B_j)=\int_{A_k}\phi_{\alpha_j}(A_j)=0 \quad for\ j\in J_2\ and\ k\in J_1.$$

These are uniquely determined.

Proof. Regard B_j as an oriented analytic Jordan curve. Let R_1 be relatively compact ring domain containing B_j. Define v, C^2-function on R_1-B_j as follows

$$v=\begin{cases} -i\alpha_j: & \text{on the left side of } B_j \\ 0: & \text{on the right side of } B_j, \end{cases}$$

We can extend v as $\hat{v}\in C_0^2(W-B_j)$. Then $d\hat{v}\in\Lambda_c^1(W)$. (cf. [2], [5], [7]). If we consider the orthogonal decompositions

$$\Lambda_c=\Lambda_h\oplus\Lambda_{e0}=\Lambda_{hse}\cap\Lambda_{hse}^*\oplus\Lambda_{hm}\oplus\Lambda_{hm}^*\oplus\Lambda_{e0}$$

and

$$\Lambda_{hse}\cap\Lambda_{hse}^*=\Lambda_B\oplus i\Lambda_B^*=\Lambda_B^*\oplus i\Lambda_B,$$

then

$$d\hat{v}=\lambda_b^*+i\lambda_b'+\lambda_{hm}^*+\lambda_{hm}'+\lambda_{e0},$$

where λ_b, $\lambda_b'\in\Lambda_B$. λ_{hm}, $\lambda_{hm}'\in\Lambda_{hm}$ and $\lambda_{e0}\in\Lambda_{e0}$. If we set

$$\phi_{\alpha_j}(B_j)=(\lambda_b+\lambda_{hm})+i(\lambda_b+\lambda_{hm})^*,$$

then $\phi_{\alpha_j}(B_j)$ is a holomorphic differential on W and

$$\begin{aligned}\phi_{\alpha_j}(B_j)&=\lambda_b+\lambda_{hm}+id\hat{v}+\lambda_b'-i\lambda_{hm}'-i\lambda_{e0}\\ &=id\hat{v}+(\lambda_b+\lambda_b')+(\lambda_{hm}-i\lambda_{hm}')-i\lambda_{e0}.\end{aligned}$$

Obviously $\lambda_b+\lambda_b'\in\Lambda_B$, $\lambda_{hm}-i\lambda_{hm}'\in\Lambda_{hm}$ and $-i\lambda_{e0}\in\Lambda_{e0}$. Since $d\hat{v}$ has compact support, this differential has Λ_B-behaviour.

Let γ be any cycle. Then we can write

$$\int_\gamma\phi_{\alpha_j}(B_j)=\alpha_j(B_j\times\gamma)-\int_\gamma(\lambda_b+\lambda_b').$$

If we take A_k, B_k instead of γ we obtain the period relations in (ii).

As for uniqueness, suppose that ϕ_1 and ϕ_2 are two admissible differentials. Then $\phi_1-\phi_2$ satisfies the conditions in Theorem 2.6 and so $\phi_1-\phi_2\equiv 0$, i.e., $\phi_1\equiv\phi_2$. The proof for the case $\phi_{\beta_j}(A_j)$ is similar.

Before giving the existence of differentials with $(P)\Lambda_B$-singularities, we give some more terminologies. Let $P: \partial W=\bigcup_{\nu\in I'}\beta_\nu$ and take $\beta\in\{\beta_\nu\}$. We say that $(P)\Lambda_B$-singularity σ is *zero outside of* β if we can find a representative ds $(\in A_\Sigma^P)$ of σ such that $ds\equiv 0$ on a neighbourhood of $\partial W-\beta$. In this case we call ds a *nice* representative of σ. Take a $(P)\Lambda_B$-divisor V. If all elements of V are zero outside β then we say V is zero outside of β.

We denote by $V(P, \Lambda_B; \beta, m)$ a $(P)\Lambda_B$-divisor which is zero outside of β and is of dimension m (as a real vector space), $0\leqq m\leqq\infty$. We assume $m\neq 0$ whenever $\beta\neq\varnothing$.

Theorem 2.8. (*existence of differential with* $(P)\Lambda_B$-*singularity*). *Given* $\sigma\in V=V(P, \Lambda_B; \beta, m)$ *there exists a regular analytic differential* ϕ *on* W *such that* ϕ *has* $(P)\Lambda_B$-*singularity* σ. *Moreover under the period conditions*

$$\int_{A_j}\phi=0 \text{ for } j\in J_1 \text{ and } \int_{A_j}\phi,\ \int_{B_j}\phi\in L_j \text{ for } j\in J_2$$

ϕ *is uniquely determined.*

Proof. Let σ be a representative of σ near ∂W. The domain of definition of σ can contain the closure of some $U\in\mathcal{E}(W)$. Since σ is (P)-semiexact then $\sigma|U$ can be extended to a differential $\hat{\sigma}\in\Lambda_c^1(W)$ such that supp. $\hat{\sigma}\cap\overline{W-U}$ is compact (cf. [5]). Since σ is analytic on U then $\sigma-i\sigma^*=0$ there. And so $\hat{\sigma}-i\hat{\sigma}^*=0$ near ∂W. Thus $\hat{\sigma}-i\hat{\sigma}^*\in\Lambda^1(W)\subset\Lambda(W)$. Because of the orthogonal decompositions $\Lambda=\Lambda_{hse}\cap\Lambda_{hse}^*\oplus\Lambda_{hm}\oplus\Lambda_{e0}\oplus\Lambda_{e0}^*$ and $\Lambda_{hse}\cap\Lambda_{hse}^*=\Lambda_B\oplus i\Lambda_B^*$, we can find $\lambda_b, \lambda_b'\in\Lambda_B$; $\lambda_{hm}, \lambda_{hm}'\in\Lambda_{hm}$; $\lambda_{e0}, \lambda_{e0}'\in\Lambda_{e0}$ such that

$$\hat{\sigma}-i\hat{\sigma}^*=\lambda_b'+i\lambda_b^*+\lambda_{hm}'+\lambda_{hm}^*+\lambda_{e0}'+\lambda_{e0}^*.$$

And so we can get a harmonic differential

$$\tau=\hat{\sigma}-\lambda_b'-\lambda_{hm}'-\lambda_{e0}'=i\hat{\sigma}^*+i\lambda_b^*+\lambda_{hm}^*+\lambda_{e0}^*.$$

If we set

$$\phi=\frac{1}{2}(\tau+i\tau^*)=\hat{\sigma}+\frac{1}{2}(\lambda_b-i\lambda_b')-\frac{1}{2}(i\lambda_{hm}+\lambda_{hm}')-\frac{1}{2}(i\lambda_{e0}+\lambda_{e0}'),$$

it is easily seen that this is a required differential. The uniqueness follows from Theorem 2.6.

§ 3. A duality theorem and the Riemann-Roch theorem.

In this section our main object is to obtain an algebraic duality theorem which gives rise to the Riemann-Roch theorem (cf. [1] pp. 325, [2] [3]). For this purpose we need some new terminologies. Let Q stand for canonical partition of ∂W. Let P be a regular partition such that $\partial W=\alpha\cup\beta\cup\gamma$ where $\beta\cup\gamma\neq\varnothing$. The partition P induces the partition P_Ω: $\partial\Omega=\alpha_\Omega\cup\beta_\Omega\cup\gamma_\Omega$ of the relative boundary of each canonical regular region Ω, such that α_Ω, β_Ω and γ_Ω are dividing cycles homologous to α, β and γ respectively.

Given two dual behaviour spaces $\Lambda'_B=\Lambda_B(\mathcal{L}', J_1, J_2)$, $\Lambda''_B=\Lambda_B(\mathcal{L}'', J_1, J_2)$ and two divisors $V_Q=V(Q, \Lambda'_B, \beta, m)$, $V_P=V(P, \Lambda''_B, \gamma, n)$ we define the following real vector space

$$M(V_Q)=\{f=\int\phi \mid \phi\in D(V_Q) \text{ and } \int_{A_j}\phi=0 \text{ for } j\in J_1,\ \int_{A_j}\phi,\ \int_{B_j}\phi\in L'_j \text{ for } j\in J_2\}.$$

To be able to define a well-defined bilinear mapping from $M(V_Q)\times D(V_P)$ to $\boldsymbol{R}$ we need the following lemmas (cf. [1], pp. 325. [2], [3], [5]).

Lemma 3.1. *If* $\sigma=ds\in A^Q_{\mathcal{L}'}$, $w\in D(V_P)$ *and* $f\in M(V_Q)$, $\tau\in A^P_{\mathcal{L}''}$

$$\lim_{\Omega\to W}\mathrm{Im}\int_{\beta_\Omega}sw \quad \text{and} \quad \lim_{\Omega\to W}\mathrm{Im}\int_{\gamma_\Omega}f\tau$$

exist and are finite.

Proof. Let Ω_1, Ω_2 $(\supset\Omega_1)$ be sufficiently large canonical regular regions, and G be a region bounded by $\beta_1=\beta_{\Omega_1}$ and $\beta_2=\beta_{\Omega_2}$.

Applying Lemma 1.1 to G we get

$$\int_{\beta_2-\beta_1}sw=-(\sigma, \bar{w}^*)_G+\sum_{G,J_2}\left(\int_{A_j}\sigma\int_{B_j}w-\int_{B_j}\sigma\int_{A_j}w\right)$$

since $\int_{A_j}\sigma=0$ and $\int_{A_j}w=0$ for $j\in J_1$. Moreover we have $(\sigma, \bar{w}^*)_G=-i(\sigma, \bar{w})_G=0$ since analytic and antianalytic differentials are orthogonal to each other. Also, because of duality conditions in Definition 2.3,

$$\left(\int_{A_j}\sigma\int_{B_j}w-\int_{B_j}\sigma\int_{A_j}w\right)\in L_0\equiv\boldsymbol{R} \text{ and so}$$

$$\mathrm{Im}\sum_{G,J_2}\left(\int_{A_j}\sigma\int_{B_j}w-\int_{B_j}\sigma\int_{A_j}w\right)=0.$$

Hence $\mathrm{Im}\int_{\beta_2-\beta_1}sw=0$, i.e., $\mathrm{Im}\int_{\beta_2}sw=\mathrm{Im}\int_{\beta_1}sw$. This means that $\mathrm{Im}\int_{\beta_\Omega}sw$ is independent of the choice of Ω provided that Ω is sufficiently large. Thus $\lim_{\Omega\to W}\mathrm{Im}\int_{\beta_\Omega}sw$ exists.

A similar proof for the second part of lemma.

Lemma 3.2. *Let* $f\in M(V_Q)$ *and* $w\in D(V_P)$. *If* f *has* $(Q)\Lambda'_B$*-singularity* σ *and* w *has* $(P)\Lambda''_B$*-singularity* τ, *then*

$$\lim_{\Omega\to W}\mathrm{Im}\int_{\partial\Omega}fw=\lim\ \mathrm{Im}\int_{\beta_\Omega}s_0w+\lim_{\Omega\to W}\mathrm{Im}\int_{\gamma_\Omega}f\tau_0.$$

for any nice representatives ds_0 *of* σ *and* τ_0 *of* τ. *Consequently,*

$$\lim_{\Omega\to W}\mathrm{Re}\frac{1}{2\pi i}\int_{\beta_\Omega}s_0w=\frac{1}{2\pi}\lim_{\Omega\to W}\mathrm{Im}\int_{\beta_\Omega}s_0w$$

and

$$\lim_{\Omega} \operatorname{Re}\frac{1}{2\pi i}\int_{\gamma_\Omega} f\tau_0 = \frac{1}{2\pi}\lim_{\Omega\to W} \operatorname{Im}\int_{\gamma_\Omega} f\tau_0$$

are independent of the choice of nice representatives ds_0 of σ and τ_0 of τ.

Proof. Write $df=ds_0+\lambda'_b+\lambda'_{hm}+\lambda'_{e0}$ and $w=\tau_0+\lambda''_b+\lambda''_{hm}+\lambda''_{e0}$ on some U. Let Ω be a canonical region such that $\partial\Omega\subset U$. Since ds_0 and τ_0 are zero outside of β and γ respectively, then

$$\int_{\partial\Omega} fw = \int_{\alpha_\Omega+\beta_\Omega+\gamma_\Omega}\Big(\int(ds_0+\lambda'_b+\lambda'_{hm}+\lambda'_{e0})\Big)(\tau_0+\lambda''_b+\lambda''_{hm}+\lambda''_{e0})$$

$$=\int_{\alpha_\Omega+\beta_\Omega+\gamma_\Omega}\Big(\int(\lambda'_b+\lambda'_{hm}+\lambda'_{e0})\Big)(\lambda''_b+\lambda''_{hm}+\lambda''_{e0})+\int_{\beta_\Omega} s_0(\lambda''_b+\lambda''_{hm}+\lambda''_{e0})$$

$$+\int_{\gamma_\Omega}\Big(\int(\lambda'_b+\lambda'_{hm}+\lambda'_{e0})\Big)\tau_0 .$$

If we apply Lemma 1.1 to the first term and consider the hypothesis on differentials, it follows that

$$\int_{\partial\Omega} fw = -(\lambda'_b, \overline{\lambda''^*_b})+\varepsilon_\Omega+\sum_{\Omega, J_2}\Big(\int_{A_j}\lambda'_b\int_{B_j}\lambda''_b-\int_{B_j}\lambda'_b\int_{A_j}\lambda''_b\Big)+\int_{\beta_\Omega} s_0 w+\int_{\gamma_\Omega} f\tau_0 ,$$

where

$$\varepsilon_\Omega=-[(\lambda'_b, \overline{\lambda''_{hm}{}^*})_\Omega+(\lambda'_b, \overline{\lambda''_{e0}{}^*})_\Omega+(\lambda'_{hm}, \overline{\lambda''^*_b})_\Omega+(\lambda'_{hm}, \overline{\lambda''_{hm}{}^*})_\Omega$$
$$+(\lambda'_{hm}, \overline{\lambda''_{e0}{}^*})_\Omega+(\lambda'_{e0}, \overline{\lambda''^*_b})_\Omega+(\lambda'_{e0}, \overline{\lambda''_{hm}{}^*})_\Omega+(\lambda'_{e0}, \overline{\lambda''_{e0}{}^*})_\Omega]$$

and $\lim_{\Omega\to W}\operatorname{Im}\varepsilon_\Omega=0$ by means of the orthogonal decompositions in § 1. On the other hand, as Λ'_B and Λ''_B are dual w. r. t. $\boldsymbol{R}$,

$$\lim_{\Omega\to W}\operatorname{Im}(\lambda'_b, \overline{\lambda''^*_b})_\Omega=0 \quad\text{and}\quad \sum_{W, J_2}\operatorname{Im}\Big(\int_{A_j}\lambda'_b\int_{B_j}\lambda''_b-\int_{B_j}\lambda'_b\int_{A_j}\lambda''_b\Big)=0 .$$

Hence

$$\lim_{\Omega\to W}\operatorname{Im}\int_{\partial\Omega} fw=\lim_{\Omega\to W}\operatorname{Im}\int_{\beta_\Omega} s_0 w+\lim_{\Omega\to W}\operatorname{Im}\int_{\gamma_\Omega} f\tau_0 , \qquad \text{q. e. d.}$$

From this lemma we give the following definition.

Definition 3.3. We call $-\lim_{\Omega\to W}\operatorname{Re}\frac{1}{2\pi i}\int_{\beta_\Omega} s_0 w$ $\Big($resp. $-\lim_{\Omega\to W}\operatorname{Re}\frac{1}{2\pi i}\int_{\gamma_\Omega} f\tau_0\Big)$ the residue of sw at β (resp. $f\tau$ at γ) and write $\operatorname{Res}_\beta sw$ (resp. $\operatorname{Res}_\gamma f\tau$), where $\sigma=ds$ (resp. τ) is the $(Q)\Lambda'_B$-(resp. $(P)\Lambda''_B$-) singularity of $f\in M(V_Q)$ (resp. $w\in D(V_P)$). Similarly, we can define, for those f and w in Lemma 3.2, $\operatorname{Res}_\alpha fw$, $\operatorname{Res}_\beta fw$, $\operatorname{Res}_\gamma fw$ and $\operatorname{Res}_{\partial W} fw$.

With this definition the above result can be written as

Lemma 3.4. *Let f and w be as in Lemma* 3.2. *Then*

$$\operatorname*{Res}_{\alpha} fw+\operatorname*{Res}_{\beta} fw+\operatorname*{Res}_{\gamma} fw=\operatorname*{Res}_{\beta} sw+\operatorname*{Res}_{\gamma} f\tau .$$

Now we define the following real vector spaces.

$$S(V_Q\| V_P)=\{f\in M(V_Q)\mid f \text{ is single-valued on } W \text{ and } \operatorname*{Res}_{\gamma} f\tau=0,\ \forall\tau\in V_P\}$$

$$D(V_P\| V_Q)=\{w\in D(V_P)\mid \operatorname*{Res}_{\beta} sw=0,\ \forall ds\in V_Q\}.$$

After these definitions we state the duality theorem.

Theorem 3.5. (Duality Theorem). *If*

$$\dim[M(V_Q)/S(V_Q\| V_P)]<+\infty ,$$

then

$$M(V_Q)/S(V_Q\| V_P)\cong D(V_P)/D(V_P\| V_Q)$$

holds.

Proof. If $f\in M(V_Q)$ and $W\in D(V_P)$, then we have

$$-2\pi \operatorname*{Res}_{\partial W} fw=\sum_J \operatorname{Im}\Big(\int_{A_j} df\int_{B_j} w-\int_{B_j} df\int_{A_j} w\Big) \quad \text{(finite sum)}.$$

For $(f, w)\in M(V_Q)\times D(V_P)$ and the $(Q)\Lambda'_B$-singularity $ds=\sigma$ of f, we define

$$h(f, w)=\operatorname*{Res}_{\beta} sw .$$

Because of Lemma 3.2, h is a well-defined bilinear mapping from $M(V_Q)\times D(V_P)$ into $\boldsymbol{R}$. From Lemma 3.4 we can write

$$h(f, w)=-\frac{1}{2\pi}\sum_J \operatorname{Im}\Big(\int_{A_j} df\int_{B_j} w-\int_{B_j} df\int_{A_j} w\Big)-\operatorname*{Res}_{\gamma} f\tau .$$

One can see that $S(V_Q\| V_P)$ is the left-kernel and $D(V_P\| V_Q)$ is the right-kernel of h. ([1], [2], [5]). The duality theorem follows.

Main Results:
From Theorem 3.5, as in [6] we can deduce the following theorems.

Theorem 3.6. *If m is finite, then*

$$\dim S(V_Q\| V_P)=m+2-2\min(\sharp\gamma, 1)-\dim D(V_P)/D(V_P\| V_Q) ,$$

where $\sharp\gamma$ denotes the number of (ideal) boundary components of γ.

This is a generalization of the Riemann-Roch theorem in [2], [6]. Indeed, if J_1 (resp. J_2) is empty, then Theorem 3.6 reduces to Theorem 4, [6] (resp. Theorem 5, [2]).

Theorem 3.7. *If the genus g of W is finite then*

$$\dim S(1/\Delta)-\dim D(\Delta)=\mathrm{Ind.}\,\Delta-2g+2$$

where $\Delta=V_P\| V_Q$, $1/\Delta=V_Q\| V_P$ *and* $\mathrm{Ind.}\,\Delta=m-n-2\min(\sharp\gamma, 1)$ *is the index of* Δ.

Let W_0 be any Riemann surface of finite genus and $\beta=\{p_1, p_2, \cdots, p_r\}$, $\gamma=\{q_1, q_2, \cdots, q_s\}$ disjoint subsets of W_0 such that $\beta\cup\gamma\neq\varnothing$. Take the open Riemann surface $W=W_0-\beta\cup\gamma$ then $\partial W=\alpha\cup\beta\cup\gamma$ where $\alpha=\partial W_0$. Let $(m_1, m_2, \cdots, m_r)$ and $(n_1, n_2, \cdots, n_s)$ be ordered sets of positive integers associated with β and with γ respectively. We set $m_0=\sum_i m_i$, $n_0=\sum_j n_j$. If $\beta=\varnothing$ we take $m_0=0$ and if $\gamma=\varnothing$ we take $n_0=0$.

As in [6] we consider the vector space $V(\beta)$ spanned by the differentials $w_i^{\mu_i}$ and $\tilde{w}_i^{\mu_i}$ which are holomorphic near ∂W and

$$w_i^{\mu_i}\ (\text{resp. } \tilde{w}_i^{\mu_i})=\begin{cases} \dfrac{dz_i}{z_i^{\mu_i+1}}\ \left(\text{resp. } \sqrt{-1}\dfrac{dz_i}{z_i^{\mu_i+1}}\right): \text{near } p_i \\ 0\ (\text{resp. } 0): \text{near } \partial W-\{p_i\} \end{cases}$$

$$(1\leqq i\leqq r,\ 1\leqq\mu_i\leqq m_i).$$

And the vector space $V(\gamma)$ spanned by the differentials

$$\varphi_j^{\nu_j}\ (\text{resp. } \tilde{\varphi}_j^{\nu_j})=\begin{cases} \dfrac{d\zeta_j}{\zeta_j^{\nu_j}}\ \left(\text{resp. } \sqrt{-1}\dfrac{d\zeta_j}{\zeta_j^{\nu_j}}\right): \text{near } q_j \\ 0\ (\text{resp. } 0): \text{near } \partial W-\{q_j\} \end{cases}$$

and

$$\phi_k\ (\text{resp. } \tilde{\phi}_k)=\begin{cases} \dfrac{d\zeta_1}{\zeta_1}\ \left(\text{resp. } \sqrt{-1}\dfrac{d\zeta_1}{\zeta_1}\right): \text{near } q_1 \\ \dfrac{-d\zeta_k}{\zeta_k}\ \left(\text{resp. } -\sqrt{-1}\dfrac{d\zeta_k}{\zeta_k}\right): \text{near } q_k \\ 0\ (\text{resp. } 0): \text{near } W-\{q_1, q_k\} \end{cases}$$

$(1\leqq j\leqq s,\ 2\leqq\nu_j\leqq n_j,\ 2\leqq k\leqq s)$ for $\gamma\neq\varnothing$. If $\gamma=\varnothing$ we take $V(\gamma)=\{0\}$. Then we can write $m=\dim V(\beta)=2m_0$ and $n=\dim V(\gamma)=2n_0-2\min(n_0, 1)$. And so we state the following theorem

Theorem 3.8.

$$\dim S(1/\Delta)-\dim D(\Delta)=2(m_0-n_0)-2g+2,$$

where $\Delta=V_P\| V_Q$, $V_Q=V(Q, \Lambda_B'; \beta, m)$, $V_P=V(P, \Lambda_B'', \gamma, n)$.

Finally we remark that by particular choice of the space Λ_B our result reduces to [3], which initiated the study of Riemann-Roch theorem on open Riemann surfaces.

HACETTEPE UNIVERSITY
ANKARA, TURKEY

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