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On harmonic dimensions and bilinear relations on open Riemann surfaces

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§ 1. Introduction.

In this paper we shall deal with the harmonic dimension and the bilinear relations of open Riemann surfaces. The harmonic dimension was introduced by M. Heins [10] as a remarkable property of open Riemann surfaces of infinite genus. Afterward some authors studied and modified it ([7], [11], [19]-[21]). Recently, S. Segawa [26] has shown certain relation between the harmonic dimension and the space of bounded harmonic functions.

Next, the extension of the classical bilinear relations to open Riemann surfaces has been a significant problem, and many authors obtained several types of generalized bilinear relations ([1], [2], [12], [17], [22] etc.). Especially, Y. Kusunoki [12] showed the bilinear relations for any pair of square integrable harmonic differentials of the classes O', O'' defined in terms of the extremal length. Recently from the viewpoint of the quasiconformal deformations on open Riemann surfaces, this research is in the limelight ([15], [16]).

Here, we shall investigate further the above subjects and extend some known results. At first we shall consider in Sec. 2, 3 the cluster sets of bounded harmonic functions and their convex hulls, and then we shall give a sufficient condition in order that the Heins' end should have finite harmonic dimension (Theorem 3. 1). This result is an extension of the one obtained by S. Segawa ([26]).

In Sec. 4, by using extremal length we shall first define new classes of open Riemann surfaces which contain the Kusunoki's class O'', and next, prove the bilinear relations for square integrable harmonic differentials on Riemann surfaces of our classes. Finally, we shall show some applications of the bilinear relations.

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§ 2.

Let G be a non-compact subregion of a parabolic open Riemann surface with only one ideal boundary component such that the relative boundary ∂G consists of a finite number of analytic closed curves. We call such G a *Heins' end* or simply *H-end*. Let P_G be a family of positive harmonic functions on $\overline{G}$ which vanish identically on ∂G, and Q_G be the set of v in P_G satisfying the normalization $\int_{\partial G}(\partial v/\partial n)\,ds=1$, where n is the inner normal on ∂G.

Then we define the *harmonic dimension* of an H-end G as the minimum nummber of the generators of P_G provided that such a finite set exists, otherwise as ∞. It is known that the harmonic dimension of G is equal to the cardinal number of mutually non-proportional minimal positive harmonic functions in Q_G([10]). Therefore the harmonic dimension is equal to the cardinal number of Martin's minimal boundary points ([8]). It is known that there exist examples of H-ends which have finite (countably infinite or continuum) minimal points at the ideal boundary ([7], [10], [11]).

Definition 2. 1. Let $HB(G)$ be the family of complex valued bounded harmonic functions on G. And the *cluster set* of u at the ideal boundary, which is denoted by $Cl(u)$, means the set

$$Cl(u)=\bigcap_{n=1}^{\infty}\overline{u(V_n)}, \tag{2.1}$$

where $\{V_n\}_1^{\infty}$ is a sequence of subends of G converging to the ideal boundary of G.

Theorem 2. 1. *For each u in $HB(G)$, we put $P_u=\left\{x;\ x=\int_{\partial G}u(\partial v/\partial n)\,ds,\ v\in Q_G\right\}$, then P_u is a closed convex domain and*

(i) $P_u\supset Cl(u)$,

(ii) *Let $Ex(P_u)$ be the set of all extreme points of P_u, then $Ex(P_u)\subset Cl(u)$.*

Proof. It is known that for any p in G,

$$u(p)=\frac{1}{2\pi}\int_{\partial G}u(\partial g_p/\partial n)\,ds$$

where g_p is the Green's function of G with a pole at p.

For each x in $Cl(u)$, there is a sequence $\{p_k\}_1^{\infty}$ in G such that $\lim_{k\to\infty}u(p_k)=x$. And we may choose the subsequence of $\{g_{p_k}\}_1^{\infty}$ which converges to some $2\pi v\,(v\in Q_G)$ uniformly on any compact subset on G. Otherwise, by Harnack's inequality, $\{g_{p_k}\}_1^{\infty}$ is unbounded on any compact subset. On the other hand

$\frac{1}{2\pi}\int_{\partial G}(\partial g_{p_k}/\partial n)\,ds=1$, thus we may easily show the contradiction.

Hence $x=\lim_{k\to\infty}\int_{\partial G}u(\partial g_{p_k}/\partial n)\,ds=\int_{\partial G}u(\partial v/\partial n)\,ds$ by the above formula, and $Cl(u)\subset P_u$. Since Q_G is convex, P_u is also convex and obviously bounded. From the compactness of Q_G (whose proof is similar to that of $\{g_{p_k}\}_1^\infty$) it follows that P_u is closed.

To prove (ii) we shall show the following lemma.

Lemma 2.1. *Let ν be a mapping of P_G into $\boldsymbol{R}^N(N\geqq 1)$ satisfying the following conditions;*

(1) $\nu(v_1+v_2)=\nu(v_1)+\nu(v_2)$ *for any* $v_1, v_2\in P_G$,

(2) $\nu(kv)=k\nu(v)$ *for any* $v\in P_G$ *and any positive number* k,

(3) *If* $\{v_n\}_1^\infty$ *is a sequence of* P_G *converging to* v *in* P_G *uniformly on every compact set in* G, *then* $\lim_{n\to\infty}\nu(v_n)=\nu(v)$.

Then for any x *in* $Ex(\nu(Q_G))$, *there is* $\hat{v}$ *in* Q_G *such that* $\hat{v}$ *is a minimal positive harmonic function and* $\nu(\hat{v})=x$.

Therefore for any z *in* $\nu(Q_G)$, *there exist at most* $(N+1)$ *minimal positive harmonic functions in* Q_G *say* $\hat{v}_1,\cdots,\hat{v}_k\,(k\leqq N+1)$, *such that for certain non-negative numbers* $c_1,\cdots,c_k$, *satisfying* $\sum_{j=1}^k c_j=1$, $\nu(\sum_{j=1}^k c_j\hat{v}_j)=z$.

Proof. (cf. [9]) Let $\{s_k\}_1^\infty$ be a countable dense subset in G, then we define a sequence $\{v_n\}_1^\infty$ in Q_G inductively as follows;

(1) $\nu(v_1)=x$,

(2) For any $k\geqq 1$, put $E_k=\{v\in Q_G;\ \nu(v)=x,\ v(s_j)=v_k(s_j),\ j=1,\cdots,k-1\}$, then v_{k+1} is in E_k and satisfies

$$q_{k+1}(s_k)=\sup_{v\in E_k}v(s_k).$$

From the compactness of Q_G, $\nu(Q_G)$ is closed, and so $E_k\neq\emptyset$. $(k=1,2,\cdots)$. Clearly $\{v_k\}_1^\infty$ possesses a limit $\hat{v}$ in Q_G, and $\nu(\hat{v})=x$. Further $\hat{v}$ is a minimal positive harmonic function. To see this, we show that $\hat{v}$ is an extreme point of Q_G. If $\hat{v}=(1-t)w_1+tw_2$ where $0<t<1$ and $w_1,w_2\in Q_G$, then $\nu(w_1)=\nu(w_2)=x$. If $\hat{v}\neq w_1$, there exists a least natural number m such that $\hat{v}(s_m)\neq w_1(s_m)$. But then by the hypothesis, $\hat{v}(s_m)<\max\{w_1(s_m), w_2(s_m)\}$ and $w_1,w_2\in E_m$. Hence the $\hat{v}(s_m)=v_{m+1}(s_m)=\sup_{v\in E_m}v(s_m)$ is violated. So, $\hat{v}=w_1=w_2$.

Since an extreme point is a minimal positive harmonic function in Q_G, the first assertion is proved.

As for the second assertion, by the elementary theory of convex sets in finite dimensional Euclidean spaces, each point of $\nu(Q_G)$ is the barycenter of at most $(N+1)$ extreme points of $\nu(Q_G)$ (cf. [6] p. 15). Hence we conclude Lemma 2.1.

On proving (ii) of Theorem 2. 1, we set for $v\in Q_G$,

$$\nu(v)=\int_{\partial G}u(\partial v/\partial n)ds=\lambda_u(v).$$

Then by Lemma 2. 1, for any x in $Ex(P_u)=Ex(\lambda_u(Q_G))$ there is a minimal positive harmonic function $\hat{v}$ such that $\lambda_u(\hat{v})=x$.

On the other hand $\hat{v}=ck_b$, where b is a point in the Martin's minimal boundary, k_b is the Martin kernel at b and c is a constant (cf. [8] Hilfssatz 12. 3, Satz 13. 1). Thus we take $\{p_k\}\subset G$ converging to b in the Martin's compactification, then certain subsequence of $\{g_{p_n}/2\pi\}$ converges to $c'k_b$, and we find that $c=c'$ by calculating the flux along ∂G. Consequently, $\lim_{n\to\infty}u(p_n)$ $=\int_{\partial G}u(\partial ck_b/\partial n)\,ds=\int_{\partial G}u(\partial\hat{v}/\partial n)\,ds=x$. Hence $Ex(P_u)\subset Cl(u)$.

Corollary 2. 1. ([10]) *If u is in $HB(G)$ and is real valued,*

$$Cl(u)=P_u\,. \tag{2. 2}$$

Proof. Obviously, both sides of (2. 2) are the segments on the real axis. On the other hand, two end points of P_u are contained in $Cl(u)$. Therefore $P_u\subset Cl(u)$, so $P_u=Cl(u)$ by Theorem 2. 1, (i).

Recently S. Segawa [26] has shown the following result.

Lemma 2. 2. *Let B_0 be the set of $u\in HB(G)$ which possesses the limit 0 at the ideal boundary, and let $B_G=HB(G)/B_0$. If either the harmonic dimension of G or* dim B_G *is finite, then*

$$h.d.(G)=\dim B_G, \tag{2. 3}$$

where $h.d.(G)$ denotes the harmonic dimension of G.

When $h.d.(G)=N$, and $\hat{v}_1\cdots,\hat{v}_N$ are the distinct minimal positive harmonic functions in Q_G, clearly Q_G is the convex hull determined by $\hat{v}_1,\cdots,\hat{v}_N$. So, from Theorem 2. 1, we have

Lemma 2. 3. *If $h.d.(G)=N$, then for any u in $HB(G)$, P_u is a convex polygon with at most N vertices.*

Furthermore from Lemma 2. 2, we have

Theorem 2. 2. *Let G be an H-end of harmonic dimension N, then for any convex polygon P with at most N vertices there exists u in $HB(G)$ such that $P_u=P$.*

Proof. From Lemma 2.2 there are $u_1, \cdots, u_N$ in $HB(G)$ which are the representations of the basis of B_G. Set $\lambda_{u_j}(\hat{v}_i)=c_{ij}(i, j=1, \cdots, N)$. Then $N\times N$ matrix (c_{ij}) is regular since $u_1, \cdots, u_N$ are independent in B_G. Hence the equations

$$\sum_{i=1}^{N} a_i c_{ji}=z_j \quad (j=1, \cdots, N) \tag{2.4}$$

have solutions for any $z_1, \cdots, z_N$.

Set $u=\sum_{i=1}^{N} a_i u_i$. (2.4) means

$$\lambda_u(\hat{v}_j)=z_j \quad (j=1, \cdots, N). \tag{2.5}$$

Namely, P_u is a convex polygon with vertices $z_1, \cdots, z_N$.

§ 3. Sufficient conditions for h.d.$(G)<+\infty$.

In this section we give sufficient conditions for $h.d.(G)<+\infty$. One condition is stated in terms of the extremal length, and another is in terms of the divergence of a sum of modules. The latter condition is given in [26].

Let R be an open Riemann surface in O_G with only one boundary component and G be an H-end in R. We take a regular exhaustion $\{R_n\}_1^\infty$ of R such that $R-R_1\subset G$.

Lemma 3.1. *For u in $HB(G)$ we set $S_u=\bigcap_{l=1}^{\infty} \overline{\bigcup_{n=1}^{\infty} u(\partial R_n)}$. Then*

$$S_u\supset Ex(P_u),$$

where P_u is as same as in Theorem 2.1.

Proof. If some $x\in Ex(P_u)$ is not in S_u, then for large n $u(\partial R_n)$ is contained in some neighbourhood of P_u and for V_x, some neighbourhood of x, $V_x\cap u(\partial R_n)=\emptyset$.

Without loss of generality we may assume $|x|>|y|$ for any y in $Ex(P_u)-V_x$. Then by the maximum principle, $Cl(u)$ is contained in the disk with the radius $\max\{|u(p)|; p\in\partial R_n\}$. Hence $Cl(u)\cap V_x=\emptyset$, but $x\in Cl(u)$ from Theorem 2.1. This is a contradiction.

Theorem 3.1. *Let $\mathfrak{F}^*$ be the family of all 1-cycle β such that β is a sum of at most N closed curves in G and β separates ∂G from the ideal boundary of G. If the extremal length $\lambda(\mathfrak{F}^*)=0$, $h.d.(G)\leq N$.*

Proof. From Lemma 2.2 we must show that any elements $u_1, \cdots, u_{N+1}$ in $HB(G)$ are not linearly independent in B_G.

We choose ρ as the density on G such that

$$\rho|dz|=\begin{cases}\sum_{j=1}^{N+1}|\mathrm{grad}\, u_j||dz| & \text{on } G-R_1\\ 0 & \text{on } G\cap R_1.\end{cases} \tag{3.1}$$

Since $\rho^2=(\sum_{j=1}^{N+1}|\mathrm{grad}\, u_j|)^2\leqq(N+1)\sum_{j=1}^{N+1}|\mathrm{grad}\, u_j|^2$ on $G-R_1$, we have

$$\iint_G \rho^2 dxdy\leqq(N+1)\sum_{j=1}^{N+1}D_{G-R_1}(u_j)<+\infty.$$

Because of $\lambda(\mathfrak{F}^*)=0$, we may choose a subsequence $\{c_k\}_1^\infty$ of $\mathfrak{F}^*$ such that $\{c_k\}_1^\infty$ converges to the ideal boundary of G and

$$\lim_{k\to\infty}\int_{c_k}\rho|dz|=0. \tag{3.2}$$

Hence by (3.1) and (3.2)

$$\lim_{k\to\infty}\int_{c_k}|u_j||dz|=0 \quad (j=1,\cdots,N+1). \tag{3.3}$$

Put $c_k=\beta_1^k\cup\cdots\cup\beta_{N(k)}^k$ (each β_j^k is a closed curve, and $N(k)\leqq N$), then there exist a subsequence $\{c_{k_p}\}$ and a_{ij} $(1\leqq i\leqq N+1,\ 1\leqq j\leqq N)$ such that for any i, $u_i(\beta_j^{k_p})$ converges to a_{ij} as $k_p\to+\infty$. Obviously the vectors $(a_{i1},\cdots,a_{iN})$ $(i=1,\cdots,N+1)$ are not linearly independent, so for certain linear combination $u=\sum_{i=1}^{N+1}r_iu_i$, $u(c_{k_p})$ converges to zero. From Lemma 3.1 and Theorem 2.1 this means that u has the limit zero at the ideal boundary. Hence $\{u_i\}_1^{N+1}$ are not linealy independent in B_G.

Corollary 3.1 ([26]). *Let G be an H-end and $\{A_n\}_1^\infty$ be a sequence of a union of at most N disjoint annuli with analytic Jordan boundaries on G satisfying the condition that for each n A_{n+1} separates A_n from the ideal boundary, A_1 separates ∂G from the ideal boundary, and $\{A_n\}_1^\infty$ converges to the ideal boundary.*

If the sum of the modules of A_n diverges, then h.d.$(G)\leqq N$.

Proof. Let F_n^* be the family of all β such that β is contained in A_n and separates ∂G from the ideal boundary, then it is well known (cf. [5] IV) that

$$\lambda(F_n^*)=2\pi/\mathrm{mod}\, A_n. \tag{3.4}$$

Since $A_i\cap A_j=\emptyset$ $(i\neq j)$,

$$\lambda(\bigcup_{n=1}^m F_n^*)^{-1}=\sum_{n=1}^m\lambda(F_n^*)^{-1} \tag{3.5}$$

(cf. [14] p. 264). On the other hand, by the monotonicity of the extremal length,

$$\lambda(\mathfrak{F}^*)\leqq\lambda(\bigcup_{n=1}^{m}\mathfrak{F}_n^*), \tag{3.6}$$

where $\mathfrak{F}^*$ is in Theorem 3.1.

From (3.4)-(3.6), $\lambda(\mathfrak{F}^*)=0$. Hence $h.d.(G)\leqq N$, by Theorem 3.1.

§ 4.

In this section we shall show an extension of the bilinear relation obtained by Y. Kusunoki [12] on some classes of Riemann surfaces defined by the conditions similar to that of the preceding section.

To begin with we recall the definitions and the results given in [12]. Let R be an open Riemann surface and $\mathfrak{F}$ be the family consisting of all 1-cycle β such that β is a sum of at most finite analytic disjoint *dividing curves* (A dividing curve is a closed curve separating R into two components.) and separates a fixed compact set from the ideal boundary of R.

And let $\{A_n\}_1^\infty$ be a sequence of a union of at most finite number of disjoint annuli with analytic Jordan boundaries on R such that for each n any annulus of A_n separates R into two components, any components of the complement of A_n is non-compact except only one component $\bar{R}_n$ and $\{R_n\}_1^\infty$ become a *canonical* exhaustion of R.

Definition 4.1. We shall denote by O' or O'' the classes of Riemann surfaces for which $\lambda(\mathfrak{F})=0$ resp. $\sum_{n=1}^{\infty}\operatorname{mod} A_n=+\infty$.

From these definitions we may show that $O''\subset O'\subset O_G$, and the inclusion $O'\subset O_G$ is generally strict, while $O''=O'=O_G$ in the case of finite genus ([12]).

Proposition 4.1. *For each Riemann surface R in O', given ω_1 and ω_2 in $\Gamma_h(R)$, then there exists a canonical exhaustion $\{R_n\}_1^\infty$ of R such that*

$$(\omega_1, {}^*\omega_2)=\lim_{n\to\infty}\sum_{j=1}^{p(n)}\left(\int_{A_j}\omega_1\int_{B_j}\bar{\omega}_2-\int_{B_j}\omega_1\int_{A_j}\bar{\omega}_2\right), \tag{4.1}$$

where $p(n)$ is the genus of R_n and $\{A_j, B_j\}_1^{p(n)}$ is the canonical homology basis of R_n modulo ∂R_n.

Proposition 4.2. *For each R in O'', there exists a canonical exhaustion $\{R_n\}_1^\infty$ of R such that for any ω_1, ω_2 in $\Gamma_h(R)$, (4.1) is valid for certain subsequence $\{R_{n_\nu}\}_1^\infty$.*

These proofs are given in [12], so we omit them.

Further, we note the following result about the extremal length on compact bordered Riemann surfaces.

Let R be a compact bordered Riemann surface, and the boundary of R be partitioned into $\alpha^0, \alpha^1, \gamma^0, \gamma^1$, where α^0, α^1 are nonempty.

Now, we shall consider the Kerékjárto-Stoïlow compactification R^* of R, then the sets of contours $\alpha^0, \alpha^1, \gamma^0, \gamma^1$ correspond to finite points sets $\alpha^{0*}, \alpha^{1*}, \gamma^{0*}, \gamma^{1*}$ of R^*.

Let $\mathfrak{F}$ be the class of arcs in $R^*-\gamma^{0*}$ which go from α^{0*} to α^{1*}. Let $\mathfrak{F}^*$ consist of all β such that β is a sum of closed curves in $R^*-\gamma^{1*}$ and separates α^{0*} from α^{1*}.

The following relation between the extremal length $\lambda(\mathfrak{F})$ and $\lambda(\mathfrak{F}^*)$ is well known (cf. [25] p. 124):

Proposition 4.3. *On a compact bordered Riemann surface R,*

$$\lambda(\mathfrak{F}^*)=\lambda(\mathfrak{F})^{-1}=D_R(u), \tag{4.2}$$

where u is the harmonic function on R which is 0 on α^0, 1 on α^1, $\partial u/\partial n=0$ on γ^0 and constant on each contour of γ^1 with zero flux on each contour of γ^1.

Considering a subregion T on an open Riemann surface, we call it a *trousers with n-legs* or *n-trousers* simply, if T satisfies the conditions;

i) T is planar, and relatively compact,

ii) The relative boundary ∂T consists of mutually disjoint $(n+1)$ closed analytic Jordan curves.

Definition 4.2. We shall denote by O_k'' $(k=1, 2, \cdots)$ the class of Riemann surface R for which there exists a sequence $\{T_n\}_1^\infty$ of sets on R satisfying the conditions;

i) $T_n=\bigcup_{j=1}^{p_n} T_{nj}$, where T_{nj} is a $k_n(j)$-trousers, $k_n(j)\leqq k$, and $T_{nj}\cap T_{ni}=\emptyset\,(i\neq j)$.

ii) Each T_{nj} divides R into two components, any component of $R-T_n$ is non-compact except only one component $\bar{R}_n$ and $\{R_n\}_1^\infty$ become a *canonical* exhaustion of R.

iii) Putting $\partial T_{nj}=\alpha_{nj}^0\cup\alpha_{nj}^1\cdots\cup\alpha_{nj}^{k_n(j)}$ as the sum of closed curves such that α_{nj}^0 is $\partial T_{nj}\cap\partial R_n$, we consider $\mathfrak{F}^{nj}$ and $\mathfrak{F}_0^{nj}$ such that $\mathfrak{F}^{nj}$ consists of all β which is a sum of closed curves and separates α_{nj}^0 from $\alpha_{nj}^1\cup\cdots\cup\alpha_{nj}^{k_n(j)}$, and $\mathfrak{F}_0^{nj}$ consists of all β which is a closed curve in $\mathfrak{F}^{nj}$. Then

$$\sum_{n=1}^{\infty} 1/\sum_{j=1}^{p_n}\sqrt{\lambda(\mathfrak{F}_0^{nj})\lambda(\mathfrak{F}^{nj})}=+\infty\,. \tag{4.3}$$

Lemma 4. 1. *If $R\in O_k''$ and G is an H-end on R, then h.d. $(G)\leqq k$.*

Proof. Let $\{T_n^G\}_1^\infty$ be a sequence of trousers on G contained in T_n in Definition 4. 2, and U_n^G be the harmonic function on T_n^G such that U_n^G is 0 on $\alpha_n^{G,0}(=\partial T_n^G\cap\partial R_n)$, and 1 on $\alpha_n^{G,1}\cup\cdots\cup\alpha_n^{G,k_n^G}(=\partial T_n^G-\alpha_n^{G,0}$, and $k_n^G\leqq k)$. Then from Proposition 4. 3, $\lambda(\mathfrak{F}_G^n)=D_{T_n^G}(U_n^G)$, where $\mathfrak{F}_G^n$ consists of all β such that β is a sum of closed curves separating $\alpha_n^{G,0}$ from $\alpha_n^{G,1}\cup\cdots\cup\alpha_n^{G,k_n^G}$. We put $V_n^G=\mu_n U_n^G$ such that $\int_{\alpha_n^{G,0}}{}^*dV_n^G=2\pi$. We may easily show that $\lambda(\mathfrak{F}_n^G)=2\pi/\mu_n$, and $\lambda(\mathfrak{F}_G^n)\leqq\lambda(\mathfrak{F}_{G,0}^n)$, thus it follows that $\sum_{n=1}^\infty\mu_n=+\infty$, from (4. 3).

On the other hand, T_n^G is mapped by $V_n^G+\sqrt{-1}{}^*V_n^G$ to the rectangle $\{a+\sqrt{-1}b;\ 0\leqq a\leqq\mu_n,\ 0\leqq b\leqq 2\pi\}$ with holizontal slits, where ${}^*V_n^G$ is the harmonic conjugate function of V_n^G. The number of these slits is less than $2t-1$, where t is the number of zeros of $\mathrm{d}V_n^G+\sqrt{-1}\mathrm{d}V_n^G$ on T_n^G. Since $\mathrm{d}V_n^G+\sqrt{-1}{}^*V_n^G$ is considered as a holomorphic differential on $\hat{T}_n^G$, the double of T_n^G with respect to ∂T_n^G, and the genus of $\hat{T}_n^G$ is k_n^G, $t=k_n^G-1$, so $2t-1=2k_n^G-3$. We denote these slits by $s_1,\cdots,s_{N(n)}\ (N(n)\leqq 2k_n^G-3)$ and $s_j=\{a+\sqrt{-1}b;\ d_j\leqq a\leqq\mu_n,\ b=\text{const.}\}$ $(j=1,\cdots,N(n))$ supposing $d_1\leqq\cdots\leqq d_{N(n)}$. Considering the rectangles $\{a+\sqrt{-1}b;\ d_{j-1}\leqq a\leqq d_j,\ 0\leqq b\leqq 2\pi\}$ $(j=1,\cdots,N(n)$, and $d_0=0)$, we may find the rectangle in them such that its holizontal side is longer than $\mu_n/2k$. And the pre-image of the rectangle on T_n^G consists of at most k annuli. Hence from $\sum_{n=1}^\infty\mu_n=+\infty$, the H-end G satisfies the condition in Corollary 3. 1, so we conclude $h.d.(G)\leqq k$.

Now, we shall show some examples in O_k''.

Example 1. Since 1-trousers is an annulus and for 1-trousers T_{nj}, $\lambda(\mathfrak{F}^{nj})=\lambda(\mathfrak{F}_0^{nj})=2\pi/\mathrm{mod}\,T_{nj}$, (4. 3) implies $\sum_{n=1}^\infty\mathrm{mod}\,T_n=+\infty$. Thus, if $R\in O_1''$, $R\in O''$ and vice versa.

Example 2. Let $\{a_n\}_1^\infty$ be a monotone increasing sequence of positive numbers satisfying $\lim_{n\to\infty}a_n=+\infty$, and $\{b_n\}_1^\infty$ be another sequence such that $a_n<b_n<a_{n+1}$ for all n. Set $I_n=(a_n,b_n)$, and $W=\{|z|<+\infty\}-I_n$. We choose $a_{n+1}-b_n$ so largely that

$$\sum_{n=1}^\infty\sqrt{m_nM_n}=+\infty\,, \tag{4.4}$$

where $m_n=\log(b_n/a_n)$, and $M_n=\log(a_{n+1}/b_n)$.

Now, we take k copies of W, say $W_1,\cdots,W_k$ and identify, along I_n, the upper edge on W_j with the lower edge of W_{j+1}, where $W_{k+1}=W_1$. We obtain a k-sheeted covering surface R.

For the k-trousers T_n lying over $\{a_n<|z|<a_{n+1}\}$, $\lambda(\mathfrak{F}^n)^{-1}\geqq M_n/2k\pi$, and

$\lambda(\mathfrak{F}_0^n)^{-1} \geqq m_n/2k\pi$. Hence by (4.4), it follows that R is in O_k''.

Theorem 4.1. *The following relations are valid;*

$$O'' = O_1'' \subsetneqq O_2'' \cdots \subsetneqq O_k'' \subsetneqq O_{k+1}'' \cdots \subset \bigcup_{k=1}^{\infty} O_k'' \subsetneqq O_G, \tag{4.5}$$

$$O_k'' \not\subset O' \quad \text{for} \quad k \geqq 2, \tag{4.6}$$

and in the case of finite genus,

$$O_1'' = O_2'' = \cdots = O_k'' = \cdots = O_G. \tag{4.7}$$

Proof. The inclusion $O_k'' \subset O_{k+1}''$ is immediate from the definition and $O_k'' \subset O_G$ is obvious from the condition of the parabolicity in terms of the extremal length (cf. [14]). And $O'' = O_1''$ by Example 1., further (4.7) is trivial from the fact that $O'' = O_G''$ in the case of finite genus ([12]).

It suffices to construct the examples belonging to $O_{k+1}'' - O_k''$, $O_G - \bigcup_{k=1}^{\infty} O_k''$, and $O_k'' - O'$ for $k \geqq 2$.

The example in $O_{k+1}'' - O_k''$. We use the surface in Example 2.. We may take each I_n so shortly on $\{|z|>1\}$ that there exists a positive continuous function u in $\{1 \leqq |z| < +\infty\}$, satisfying; (1) u is zero on $\{|z|=1\}$, (2) $u(x) = \log x$, for any $x \in \bigcup_{n=1}^{\infty} I_n$, (3) u is harmonic in $\{|z|>1\} - \bigcup_{n=1}^{\infty} \bar{I}_n$, and (4) u is bounded on $(-\infty, -1]$. As Example 2., we take $(k+1)$ copies and $(k+1)$ covering surface R in O_{k+1}''. Then from the argument in [10] Sec. 10, the H-end lying on $\{|z|>1\}$ is with harmonic dimension $k+1$. Hence R is not in O_k'' from Lemma 4.1.

The example in $O_G - \bigcup_{k=1}^{\infty} O_k''$. By the method of Z. Kuramochi [11] or C. Constantinescu und A. Cornea [7], we may construct an open Riemann surface R in O_G with only one boundary component which has infinite harmonic dimension. Then for any k, R is not in O_k'' because of Lemma 4.1. Thus R is in $O_G - \bigcup_{k=1}^{\infty} O_k''$.

The example in $O_k'' - O'$ *for* $k \geqq 2$. It suffices from (4.5) to show the example in $O_2'' - O'$. We consider the surface R in $O_2'' - O_1''$ given at the first case for $k=1$. Then R has only one H-end whose harmonic dimension is two. From Theorem 3.1, it follows that R is not in O'. Q.E.D.

Let R be an open Riemann surface in O_k'', and $\{T_n\}_1^{\infty}$ $(T_n = \bigcup_{j=1}^{p_n} T_{nj})$ is a sequence of the union of trousers with at most k-legs in Definition 4.2. We consider a harmonic function u_{nj} on each T_{nj} such that u_{nj} is 0 on α_{nj}^0 and is constant μ_{nj} on $\alpha_{nj}^1 \cup \cdots \cup \alpha_{nj}^{k_n(j)}$ $(k_n(j) \leqq k)$ so that $\int_{\alpha_{nj}^0} {}^*du_{nj} = 2\pi$. The

the quantity μ_{nj} is known as the harmonic modulus of T_{nj}, and $\mu_n^{-1}=\sum_{j=1}^{p_n}\mu_{nj}^{-1}$, μ_n is the harmonic modulus of T_n. Further as in the proof of Lemma 4.1. $f_{nj}=u_{nj}+\sqrt{-1}{}^*u_{nj}$ maps T_{nj} conformally onto the rectangle $\{a+\sqrt{-1}b;\ 0\leqq a\leqq\mu_{nj},\ 0\leqq b\leqq 2\pi\}$ with at most $2k_n(j)-3$ horizontal slits, and $\mu_{nj}=2\pi/\lambda(\mathfrak{F}^{nj})$. We denote these slits in the rectangle by $s_1^{nj},\cdots,s_{N(nj)}^{nj}$ $(N(nj)\leqq 2k_n(j)-3)$, and $s_r^{nj}=\{a+\sqrt{-1}b;\ d_r\leqq a\leqq\mu_{nj},\ b=\text{const.}\}$ $(r=1,\cdots,N(nj))$ supposing $d_1^{nj}\leqq d_2^{nj}\leqq\cdots\leqq d_{N(nj)}^{nj}$. Considering the rectangles $\{a+\sqrt{-1}b;\ d_{r-1}^{nj}\leqq a\leqq d_r^{nj},\ 0\leqq b\leqq 2\pi\}$ $(r=1,\cdots,N(nj))$ and $d_0^{nj}=0)$, we may find the rectangle in them such that its horizontal side is longer than $\mu_{nj}/2k$. And the pre-image of the rectangle on T_{nj} consists of at most k annuli. We denote it by A_{nj}.

Lemma 4.2. *Let ω_1, ω_2 be differentials in $\Gamma_h(R)$, then there exist 1-cycle ℓ_{nj} on A_{nj}, a close curve ℓ_{nj}^0 in T_{nj}, and a subsequence $\{n_\nu\}$ such that*

$$\lim_{n_\nu\to\infty}\sum_{j=1}^{p_{n_\nu}}\int_{\ell_{n_\nu j}}|\omega_1|\int_{\ell_{n_\nu j}}|\omega_2|=0, \tag{4.8}$$

and

$$\begin{cases}\displaystyle\lim_{n_\nu\to\infty}\sum_{j=1}^{p_{n_\nu}}\int_{\ell_{n_\nu j}^0}|\omega_1|\int_{\ell_{n_\nu j}}|\omega_2|=0, \\ \displaystyle\lim_{n_\nu\to\infty}\sum_{j=1}^{p_{n_\nu}}\int_{\ell_{n_\nu j}}|\omega_1|\int_{\ell_{n_\nu j}^0}|\omega_2|=0.\end{cases} \tag{4.9}$$

Proof. We may choose the density ρ on R such that

$$\rho=(|\text{grad }h_1|+|\text{grad }h_2|), \tag{4.10}$$

where $\omega_1=dh_1$ and $\omega_2=dh_2$. Suppose that A_{nj} is the preimage of the rectangle $\{a+\sqrt{-1}b;\ d_{r_0}^{nj}\leqq a\leqq d_{r_0+1}^{nj},\ 0\leqq b\leqq 2\pi\}$, then from the choice of A_{nj}, $\mu_{nj}/(d_{r_0+1}^{nj}-d_{r_0}^{nj})\leqq 2k$. We consider the annuli, the components of A_{nj}, say $U_{nj}^1,\cdots,U_{nj}^{i(nj)}$ $(i(nj)\leqq k(nj))$, then obviously $\operatorname{mod}U_{nj}^i=2\pi(d_{r_0+1}^{nj}-d_{r_0}^{nj})/L_{nj}$, where L_{nj} is the height of $f_{nj}(U_{nj}^i)$, therefore $\sum_{i=1}^{i(nj)}L_{nj}^i=2\pi$.

For each U_{nj}^i, we take β_{nj}^i in the set of the level curves of u_{nj} contained in U_{nj}^i such that

$$\int_{\beta_{nj}^i}\rho|dz|=\inf\left\{\int_{u_n=c}\rho|dz|\right\}, \tag{4.11}$$

where infimum is taken in the set of all level curves in U_{nj}^i. Set $\ell_{nj}=\bigcup_{i=1}^{i(nj)}\beta_{nj}^i$. And we take ℓ_{nj}^0 in $\mathfrak{F}_0^{nj}$ satisfying for given $\eta>1$,

$$\int_{\ell_{nj}^0}\rho|dz|\leqq\eta\inf\left\{\int_c\rho|dz|;\ c\in\mathfrak{F}_0^{nj}\right\}. \tag{4.12}$$

Then by (4.10), (4.11), and Schwarz's inequality, we have

$$\int_{\ell_{nj}}\rho|dz|\leqq 2\sqrt{2k\pi\mu_{nj}^{-1}(\|\omega_1\|_{T_{nj}}^2+\|\omega_2\|_{T_{nj}}^2)}\,.$$

Obviously $\int_{\ell_{nj}}|\omega_i|\leqq\int_{\ell_{nj}}\rho|dz|\ (i=1,2)$, thus

$$\int_{\ell_{nj}}|\omega_i|\leqq 2\sqrt{2k\pi\mu_{nj}^{-1}(\|\omega_1\|_{T_{nj}}^2+\|\omega_2\|_{T_{nj}}^2)}\,. \tag{4.13}$$

On the other hand, by the definition of the extremal length and (4.12)

$$\int_{\ell_{nj}^0}|\omega_i|\leqq\int_{\ell_{nj}^0}\rho|dz|\leqq\eta\sqrt{2\lambda(\mathfrak{F}_0^{nj})(\|\omega_1\|_{T_{nj}}^2+\|\omega_2\|_{T_{nj}}^2)}\,. \tag{4.14}$$

Hence by (4.13), (4.14), and $\mu_{nj}=2\pi/\lambda(\mathfrak{F}^{nj})$, we have

$$\left\{\begin{aligned}
&\int_{\ell_{nj}}|\omega_1|\int_{\ell_{nj}^0}|\omega_2|\leqq 2\eta\sqrt{2k\lambda(\mathfrak{F}_0^{nj})\lambda(\mathfrak{F}^{nj})}\,\Delta_n(j),\\
&\int_{\ell_{nj}^0}|\omega_1|\int_{\ell_{nj}}|\omega_2|\leqq 2\eta\sqrt{2k\lambda(\mathfrak{F}_0^{nj})\lambda(\mathfrak{F}^{nj})}\,\Delta_n(j),\\
&\int_{\ell_{nj}}|\omega_1|\int_{\ell_{nj}}|\omega_2|\leqq 4k\lambda(\mathfrak{F}^{nj})\,\Delta_n(j)\\
&\qquad\qquad\leqq 4k\sqrt{\lambda(\mathfrak{F}_0^{nj})\lambda(\mathfrak{F}^{nj})}\,\Delta_n(j),
\end{aligned}\right. \tag{4.15}$$

where $\Delta_n(j)=\|\omega_1\|_{T_{nj}}^2+\|\omega_2\|_{T_{nj}}^2$.

Summing up from $j=1$ to $j=p_n$, we have

$$\left\{\begin{aligned}
&\sum_{j=1}^{p_n}\int_{\ell_{nj}}|\omega_1|\int_{\ell_{nj}^0}|\omega_2|\leqq 2\eta\sqrt{2k}\sum_{j=1}^{p_n}\sqrt{\lambda(\mathfrak{F}_0^{nj})\lambda(\mathfrak{F}^{nj})}\,\Delta_n(j)\\
&\qquad\leqq 2\eta\sqrt{2k}\sqrt{\sum_{j=1}^{p_n}\lambda(\mathfrak{F}_0^{nj})\lambda(\mathfrak{F}^{nj})}\sqrt{\sum_{j=1}^{p_n}\Delta_n(j)}\\
&\qquad\leqq 2\eta\sqrt{2k}\sum_{j=1}^{p_n}\sqrt{\lambda(\mathfrak{F}_0^{nj})\lambda(\mathfrak{F}^{nj})}\,(\|\omega_1\|_{T_n}^2+\|\omega_2\|_{T_n}^2),\\
&\sum_{j=1}^{p_n}\int_{\ell_{nj}^0}|\omega_1|\int_{\ell_{nj}}|\omega_2|\leqq 2\eta\sqrt{2k}\sum_{j=1}^{p_n}\sqrt{\lambda(\mathfrak{F}_0^{nj})\lambda(\mathfrak{F}^{nj})}\,(\|\omega_1\|_{T_n}^2+\|\omega_2\|_{T_n}^2),\\
&\sum_{j=1}^{p_n}\int_{\ell_{nj}}|\omega_1|\int_{\ell_{nj}}|\omega_2|\leqq 4k\sum_{j=1}^{p_n}\sqrt{\lambda(\mathfrak{F}_0^{nj})\lambda(\mathfrak{F}^{nj})}\,(\|\omega_1\|_{T_n}^2+\|\omega_2\|_{T_n}^2).
\end{aligned}\right. \tag{4.16}$$

Hence we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\Bigl(\sum_{j=1}^{p_n}\Bigl(\int_{\ell_{nj}}|\omega_1|\int_{\ell_{nj}}|\omega_2|+\int_{\ell_{nj}^0}|\omega_1|\int_{\ell_{nj}}|\omega_2|\\
&\qquad+\int_{\ell_{nj}}|\omega_1|\int_{\ell_{nj}}|\omega_2|\Bigr)\Bigr)\Bigl(\sum_{j=1}^{p_n}\sqrt{\lambda(\mathfrak{F}_0^{nj})\lambda(\mathfrak{F}^{nj})}\Bigr)^{-1}\\
&\qquad\leqq 10\eta k(\|\omega_1\|_R^2+\|\omega_2\|_R^2)<+\infty\,.
\end{aligned} \tag{4.17}$$

From (4.3) and (4.17), we may find the subsequence $\{n_\nu\}$ satisfying

(4. 8) and (4. 9). Q.E.D.

Remark. By noting (4. 15) and (4. 16), we show that if

$$\sup_n \max_j \lambda(\mathfrak{F}_0^{nj})\lambda(\mathfrak{F}^{nj}) < +\infty$$

or

$$\sum_{n=1}^{\infty}\Big(\sum_{j=1}^{p_n}\lambda(\mathfrak{F}_0^{nj})\lambda(\mathfrak{F}^{nj})\Big)^{-1/2} = +\infty$$

is satisfied instead of (4. 3), the above assertions are also true.

Let $\{L_{nj}^i\}$ is the mentioned above, then there is an index i such that $L_{nj}^i \geqq 2\pi/k$ because of $\sum_{i=1}^{i(nj)} L_{nj}^i = 2\pi$ and $i(nj) \leqq k$. Set $i=1$.

Lemma 4. 3. *Let differentials ω_1, ω_2 and a subsequence $\{n_\nu\}$ be the same as in Lemma 4. 2, then there exist a level curve σ_{nj}^i of ${}^*u_{nj}$ on $T_{nj}(i=1, \cdots, i(nj))$*, such that *$\sigma_{nj}^i$ intersects with β_{nj}^i, and*

$$(4.18)\qquad \begin{cases} \lim\limits_{n_\nu\to\infty} \sum\limits_{j=1}^{p_{n_\nu}}\Big(\sum\limits_{i=1}^{i(nj)} \int_{\beta_{nj}^i}|\omega_1| \int_{\sigma_{nj}^i}|\omega_2|\Big) = 0, \\ \lim\limits_{n_\nu\to\infty} \sum\limits_{j=1}^{p_{n_\nu}}\Big(\sum\limits_{i=1}^{i(nj)} \int_{\sigma_{nj}^i}|\omega_1| \int_{\beta_{nj}^i}|\omega_2|\Big) = 0, \end{cases}$$

$$(4.19)\qquad \begin{cases} \lim\limits_{n_\nu\to\infty} \sum\limits_{j=1}^{p_{n_\nu}}\Big(\sum\limits_{i=1}^{i(nj)} \int_{\beta_{nj}^i}|\omega_1| \int_{\sigma_{nj}^1}|\omega_2|\Big) = 0, \\ \lim\limits_{n_\nu\to\infty} \sum\limits_{j=1}^{p_{n_\nu}}\Big(\sum\limits_{i=1}^{i(nj)} \int_{\sigma_{nj}^1}|\omega_1| \int_{\beta_{nj}^i}|\omega_2|\Big) = 0, \end{cases}$$

Proof. We choose the same density ρ as in the proof of Lemma 4. 2. We may take a level curve σ_{nj}^i of ${}^*u_{nj}$ intersecting with β_{nj}^i and satisfying for given $\eta > 1$,

$$(4.20)\qquad \int_{\sigma_{nj}^i}\rho|dz| \leqq \eta \inf\Big\{\int_{{}^*u_{nj}=c}\rho|dz|\Big\},$$

where the infimum is taken in the set of all level curves intersecting with β_{nj}^i. Then by Schwarz's inequality, we have

$$(4.21)\qquad \int_{\sigma_{nj}^i}\rho|dz| \leqq \eta\sqrt{(2\mu_{nj}/L_{nj}^i)\Delta_n(j)},$$

where $\Delta_n(j) = \|\omega_1\|_{T_{nj}}^2 + \|\omega_2\|_{T_{nj}}^2$.

Furthermore, by (4. 11) we have

$$(4.22)\qquad \int_{\beta_{nj}^i}|\omega_m| \leqq \int_{\beta_{nj}^i}\rho|dz| \quad (m=1, 2)$$
$$\leqq \sqrt{(2L_{nj}^i/(d_{r_0+1}^{nj} - d_{r_0}^{nj})\Delta_n(j)}.$$

From (4.21) and (4.22) we have

$$\int_{\beta^i_{nj}}|\omega_1|\int_{c^i_{nj}}|\omega_2|\leqq 2\eta\sqrt{(\mu_{nj}/(d^{nj}_{r_0+1}-d^{nj}_{r_0})}\,\Delta_n(j)$$

$$\leqq 2\eta\sqrt{2k}\Delta_n(j).$$

Summing up from $i=1$ to $i=i(nj)$ $(\leqq k)$ and from $j=1$ to $j=p_n$,

$$\sum_{j=1}^{pn}\sum_{i=1}^{i(nj)}\int_{\beta^i_{nj}}|\omega_1|\int_{c^i_{nj}}|\omega_2|\leqq 8\eta k^3(\|\omega_1\|_{T_n}+\|\omega_2\|_{T_n}).$$

This implies the first equation of (4.17), and the second is obtained similarly.

Especially for c^1_{nj}, from $L^1_{nj}\geqq 2\pi/k$, we have

$$\sum_{j=1}^{pn}\sum_{i=1}^{i(nj)}\int_{\beta^i_{nj}}|\omega_1|\int_{c^1_{nj}}|\omega_2|\leqq 2\eta k^2\sqrt{\pi}(\|\omega_1\|^2_{T_n}+\|\omega_2\|^2_{T_n}).$$

This implies he first of (4.19), and the second is similarly.

Theorem 4.2. *Let $R\in O''_k$, then for any ω_1, $\omega_2\in\Gamma_h(R)$ there exists a subsequence $\{R_{n_\nu}\}$ such that*

$$(4.23)\qquad (\omega_1, {}^*\omega_2)=\lim_{n_\nu\to\infty}\sum_{j=1}^{p(n_\nu)}\left(\int_{A_j}\omega_1\int_{B_j}\overline{\omega}_2-\int_{B_j}\omega_1\int_{A_j}\overline{\omega}_2\right).$$

Proof. To begin with we take a point p^i_{nj} on β^i_{nj} and arcs $\gamma^2_{nj},\cdots,\gamma^{i(nj)}_{nj}$, where γ^i_{nj} is an open analytic arc from p^1_{nj} to p^i_{nj} in the region bounded by α^0_{nj}, β^1_{nj}, $\cdots$, and $\beta^{i(nj)}_{nj}$, and $\gamma^i_{nj}\cap\gamma^{i'}_{nj}=\emptyset\,(i\neq i')$. Using Stokes' formula for the relatively compact region R^+_n cut along $\bigcup_{j=1}^{p_n}(\bigcup_{i=1}^{i(nj)}\beta^i_{nj}\cup\bigcup_{i=1}^{i(nj)}\gamma^i_{nj})$, we have

$$(\omega_1, {}^*\omega_2)_{R^+_n}=\sum_{j=1}^{p(n)}\left(\int_{A_j}\omega_1\int_{B_j}\overline{\omega}_2-\int_{B_j}\omega_1\int_{A_j}\overline{\omega}_2\right)+\int_{\partial R_n^+}\Phi_1\overline{\omega}_2,$$

where $\Phi_1(z)=\int^z\omega_1$. And $\partial R^+_n=\bigcup_{j=1}^{pn}\ell_{nj}\cup\bigcup_{i=2}^{i(nj)}(\gamma^i_{nj})^+\cup(\gamma^i_{nj})^-$, where $(\gamma^i_{nj})^+$ and $(\gamma^i_{nj})^-$ are two sides of γ^i_{nj}.

For each z on γ^i_{nj}, we denote it by z^+, z^- considered on $(\gamma^i_{nj})^+$, $(\gamma^i_{nj})^-$, respectively $(i=2,\cdots,i(nj))$ (Fig. 1). Then we have

$$\Phi_1(z^+)=\Phi_1(z^-)-\int_{\beta^i_{nj}}\omega_1.$$

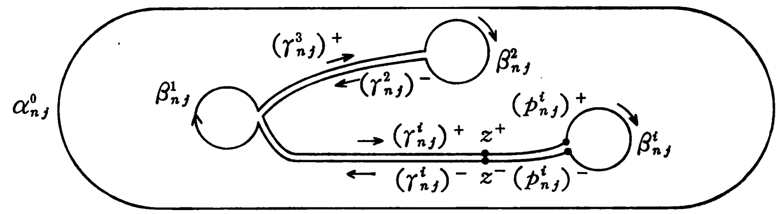

Fig. 1.

Therefore

$$\int_{(\gamma^i_{nj})^+ + (\gamma^i_{nj})^-} \Phi_1 \bar{\omega}_2 = \int_{\beta^i_{nj}} \omega_1 \int_{(\gamma^i_{nj})^-} \bar{\omega}_2 . \tag{4.24}$$

We join β^1_{nj} and β^i_{nj} by a sum of subarcs of o^1_{nj}, o^i_{nj}, ℓ^0_{nj}, and $\ell_{nj} (= \bigcup_{i=1}^{i(nj)} \beta^i_{nj})$, and denote it by d^i_{nj}. Then γ^i_{nj} is homologous to d^i_{nj} plus certain parts of ℓ_{nj} (Fig. 2).

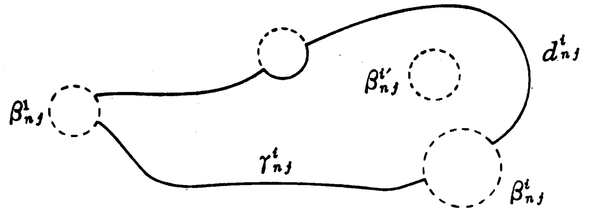

Fig. 2.

Hence

$$\left| \int_{\gamma^i_{nj}} \omega_m \right| \leqq \int_{\ell_{nj} + \ell^0_{nj} + o^1_{nj} + o^i_{nj}} |\omega_m| \quad (m=1, 2). \tag{4.25}$$

By (4.24) and (4.25) we have

$$\left| \int_{(\gamma^i_{nj})^+ + (\gamma^i_{nj})^-} \Phi_1 \bar{\omega}_2 \right| \leqq \int_{\beta^i_{nj}} |\omega_1| \int_{\ell_{nj} + \ell^0_{nj} + o^1_{nj} + o^i_{nj}} |\omega_2| . \tag{4.26}$$

Since $\bar{\omega}_2$ is *semi-exact* and $\Phi_1(p_1^+) - \Phi_1(p_i^+) = \int_{\gamma^i_{nj} + \sum_{t=1}^{i-1} \beta^t_{nj}} \omega_1$, we have

$$\begin{aligned} \left| \int_{\ell_{nj}} \Phi_1 \bar{\omega}_2 \right| &= \left| \sum_{i=1}^{i(nj)} \int_{\beta^i_{nj}} (\Phi_1 - \Phi_1(p_i^+)) \bar{\omega}_2 \right. \\ &\quad \left. - \sum_{i=2}^{i(nj)} \int_{\beta^i_{nj}} (\Phi_1(p_1^+) - \Phi_1(p_i^+)) \bar{\omega}_2 \right| \\ &\leqq \sum_{i=1}^{i(nj)} \int_{\beta^i_{nj}} |\omega_1| \int_{\beta^i_{nj}} |\omega_2| + \sum_{i=2}^{i(nj)} \int_{\beta^i_{nj}} |\omega_2| \int_{\gamma^i_{nj} + \sum_{t=2}^{i-1} \beta^t_{nj}} |\omega_1| \\ &\leqq \sum_{i=1}^{i(nj)} \int_{\beta^i_{nj}} |\omega_1| \int_{\beta^i_{nj}} |\omega_2| + \sum_{i=2}^{i(nj)} \int_{\beta^i_{nj}} |\omega_2| \int_{\gamma^i_{nj} + \ell_{nj}} |\omega_1| . \end{aligned} \tag{4.27}$$

Summing up (4.26) and (4.27) from $j=1$ to $j=p_n$, and using (4.25), we know that Lemma 4.2 and 4.3 are applicable to them, that is, there exists a subsequence $\{n_\nu\}$ such that

$$\lim_{n_\nu \to \infty} \left| \int_{\partial R^*_{n_\nu}} \Phi_1 \bar{\omega}_2 \right| = \lim_{n_\nu \to \infty} \left| \sum_{j=1}^{p_{n_\nu}} \int_{\ell_{n_\nu j} + \sum_{i=2}^{i(n_\nu j)} (\gamma^i_{n_\nu j})^{\pm}} \Phi_1 \bar{\omega}_2 \right| = 0 .$$

Hence we have

$$(\omega_1, {}^*\omega_2) = \lim_{n_\nu\to\infty}(\omega_1, {}^*\omega_2)_{R^*_{n_\nu}}$$

$$= \lim_{n_\nu\to\infty}\sum_{j=1}^{p(n_\nu)}\left(\int_{A_j}\omega_1\int_{B_j}\bar{\omega}_2 - \int_{B_j}\omega_1\int_{A_j}\bar{\omega}_2\right).$$

Remark. A Pfluger [22] has shown the following; Let W be the same as in Example 2., but we need not suppose the condition (4. 4). And we construct the two sheeted covering surface R of W_1 and W_2, two copies of W, as Example 2.. Then fro any ω_1, ω_2 in $\Gamma_h(R)$, the bilinear relation (4. 23) is valid.

We have not shown completely his result from Theorem 4. 2, that is, we may show the bilinear relation when (4. 4) is valid ($R\in O''_2$) or $\{m_n\}_1^\infty$ is divergent ($R\in O''_1$). Since he used the symmetricity of R in his proof and we do not use it, we may show the bilinear relation on non-symmetric surfaces to which his result is not applicable.

Other classes of Riemann surfaces.

Considering Remark of Lemma 4. 2 and the proofs of Lemma 4. 3 and Theorem 4. 2, we may show the following;

Theorem 4. 3. (a) *We define $R\in O'_k\,(k=1, 2, \cdots)$ if there exists a sequence $\{T_n\}_1^\infty$ as in Definition 4. 2 such that instead of (4. 3) it satisfies the condition*

$$\sum_{n=1}^{\infty} 1\Big/\sqrt{\sum_{j=1}^{p_n}\lambda(\mathfrak{F}_0^{nj})\lambda(\mathfrak{F}^{nj})} = +\infty\,. \tag{4. 28}$$

Then for any $\omega_1, \omega_2\in\Gamma_{hse}(R)$, (4. 23) is valid.

(b) *If there exists a sequence $\{T_n\}_1^\infty$ in R as in Definition 4. 2 such that instead of (4. 3) it satisfies the condition*

$$\sup_n \max_j \lambda(\mathfrak{F}_0^{nj})\lambda(\mathfrak{F}^{nj})<\infty\,.$$

Then for any $\omega_1, \omega_2\in\Gamma_{hse}(R)$,

$$(\omega_1, {}^*\omega_2) = \lim_{n\to\infty}\sum_{j=1}^{p(n)}\left(\int_{A_j}\omega_1\int_{B_j}\bar{\omega}_2 - \int_{B_j}\omega_1\int_{A_j}\bar{\omega}_2\right).$$

Proof. It may be omitted.

From the inequalities for $a_j, b_j>0$,

$$\left(\sum_{j=1}^{p_n}\sqrt{a_jb_j}\right)^{-1}\leqq\left(\sqrt{\sum_{j=1}^{p_n}a_jb_j}\right)^{-1}\leqq\sqrt{p_n}\left(\sum_{j=1}^{p_n}\sqrt{a_jb_j}\right)^{-1},$$

we conclude $O''_k\subset O'_k$, and if R has a finite number of ideal boundary components, then $O''_k=O'_k$. And from Theorem 4. 3, $O'_k\subset O_{KD}$. Furthermore,

Theorem 4.4. *The following relations are valid*;

(4.29) $$O_1' \subsetneqq O_2' \cdots \subsetneqq O_k' \subsetneqq O_{k+1}' \cdots \subset \bigcup_{k=1}^{\infty} O_k' \subsetneqq O_{KD},$$

(4.30) $$O_k'' \subsetneqq O_k', \quad O_k' \not\subset O_G,$$

(4.31) $$\bigcup_{k=1}^{\infty} O_k' \cap O_G \subsetneqq O_G.$$

Proof. From the above remark and Theorem 4.1, we may show (4.29) and (4.31). And $O_k' \not\subset O_G$ implies $O_k'' \subsetneqq O_k'$ because of $O_k'' \subset O_G \subset O_{KD}$. Hence it suffices to construct the example belonging to $O_k' - O_G$.

Let $\{q_i\}_1^{\infty}$ be a sequence of real numbers, $0 < q_i < 1$. We denote by $E\{q_i\}$ the generalized Cantor sets, the point set constructed as follows;

Let E_0 be the closed interval $[0, 1]$. We construct inductively, sets $E_n(q_1, \cdots, q_n)$ consisting of 2^n disjoint closed intervals. To pass from $E_n(q_1, \cdots, q_n)$ to $E_{n+1}(q_1, \cdots, q_{n+1})$ we remove from each interval, symmetrically about the midpoint, subintervals the sum of whose length is the ratio q_{n+1} to the original interval, that is, for each interval $[A_{nj}, B_{nj}]$ $(j = 1, \cdots, 2^n)$ in $E_n(q_1, \cdots, q_n)$ we take four points $a_{nj}, a'_{nj}, b_{nj}, b'_{nj}$ such that $A_{nj} < a_{nj} < a'_{nj} < b'_{nj} < b_{nj} < B_{nj}$, $a_{nj} - A_{nj} = B_{nj} - b_{nj} = \frac{1}{2}(b_{nj} - b'_{nj}) = \frac{1}{4} q_{n+1}(B_{nj} - A_{nj})$, and they are symmetric about $\frac{1}{2}(B_{nj} + A_{nj})$, then we set $E_{n+1}(q_1, \cdots, q_{n+1}) = \bigcup_{j=1}^{2^n} [a_{nj}, a'_{nj}] \cup [b'_{nj}, b_{nj}]$. The set $E\{q_i\}$ is defined as $\bigcap_{n=1}^{\infty} \mathrm{E}_n(q_1, \cdots, q_n)$

Then we know analogously to [5] IV 24B. that if

(4.32) $$-\sum_{k=1}^{\infty} 2^{-k} \log q_k (1 - q_k) < +\infty,$$

$E\{q_i\}$ has positive logarithmic capacity, that is $\hat{C} - E\{q_i\}$ is hyperbolic.

On the other hand, we take a sequence of union of 2^n annuli $(n = 1, 2, \cdots)$ such that on the above symbol in $E_{n+1}(q_1, \cdots, q_{n+1})$ we take two annuli on each interval $[A_{nj}, B_{nj}]$ $(j = 1, \cdots, 2^n)$, one annulus is bounded by the circle whose diameter is $\left[A_{nj}, \frac{1}{2}(a'_{nj} + b'_{nj})\right]$ and the circle whose diameter is $[a_{nj}, a'_{nj}]$, and another is bounded by the circle whose diameter is $\left[\frac{1}{2}(a'_{nj} + b'_{nj}), B_{nj}\right]$ and the circle whose diameter is $[b'_{nj}, b_{nj}]$.

Then the sequence satisfies the conditions of O_1' except (4.28). Since $\lambda(\mathfrak{F}_0^{nj}) = \lambda(\mathfrak{F}^{nj}) =$ (a modulus of annulus)$^{-1} \cdot 2\pi$, (4.28) implies

(4.33) $$\sum_{k=1}^{\infty} 2^{-k/2} \log 1/(1 - q_k) = +\infty.$$

We may give a sequence $\{q_k\}_1^{\infty}$ which satisfies (4.32) and (4.33), for example $q_k=1-\exp(-2^{k/2})$. Hence we constructed the desired surface.

§ 5. Applications.

In this section, we shall extend some results on compact Riemann srface the Kusunoki's class O'' to our class O''_k. But the methods are similar to the previous ones. So we shall remark only the results and the references.

To begin with, we notice that the following is shown by the same way as Theorem 4. 2.

Theorem 5. 1. *Let R be in O''_k, then for any Abelian differentals df_1 (1st or 2nd kind) and df_2 with finite number of singularities which have finite Dirichlet norms outside the neighbourhood of their singularities, there exists a subsequence $\{R_{n_\nu}\}$ such that*

$$2\pi\sqrt{-1}\sum_R \operatorname{Res}.F_1df_2=-\lim_{n_\nu\to\infty}\sum_{j=1}^{p(n_\nu)}\left(\int_{A_j}df_1\int_{B_j}df_2 - \int_{B_j}df_1\int_{A_j}df_2\right), \tag{5.1}$$

where $F_1(z)=\int^z df_1$.

By using this theorem we may show *Riemann-Roch* theorem and *Abel's* theorem on $R\in O''_k$ analogously to the classical cases. Furthermore, for the classes referred in the last part of the preceding section, the similar extensions are possible for the restricted differentials and functions (cf. [14], [18], [23] [24]).

Another application is the one on quasiconformal deformations. Let R_0 be a *marked* Riemann surface in O''_k with a canonical homology basis $\{A_j, B_j\}$ modulo ideal boundary as in Theorem 4. 2. We consider a *K-quasiconformal mapping* f_R of R_0 onto R. Then f_R induces on R a canonical homology basis, and we denote it also by $\{A_j, B_j\}$. By the definition R belongs to O''_k.

Let $\theta_0\in\Gamma_a(R_0)$, then we may show that there exists a unique differential $\theta\in\Gamma_a(R)$ having the same A-period with θ_0 (cf. [16] Proposition 2). Hence by the method of Theorem 1 [16], we have

$$\|\theta\circ f_R-\theta_0\|_{R_0}=(K-1)\|\theta_0\|_{R_0}. \tag{5.2}$$

Furthermore, we may show the continuity of norms and the analyticity of periods on the Teichmüller space of R_0 analogously to [16] Corollary 3 and [15].

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