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Spectral asymptotics for elliptic second order differential operators

By

Georgi POPOV

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0. Introduction and statement of the results

Let X be an open set in $\boldsymbol{R}^n$, $n \geq 2$, with a smooth boundary Y and $\boldsymbol{R}^n \backslash X \subset B_R = \{x; |x| \leq R\}$ for some $R > 0$. Suppose that $\boldsymbol{R}^n$ is provided with a smooth Riemannian metric $ds^2 = g^{ij}(x)dx_i dx_j$ which is Euclidean outside the ball B_R. Set $g(x) = \det(g_{ij}(x))$, $g_{ij}g^{jk} = \delta_i^k$ (Kronecker's index), $i, k = 1, \ldots, n$ where the summation convention is used. Let Δ_g,

$$\Delta_g u(x) = g(x)^{-1/2} \frac{\partial}{\partial x_i}\left(g(x)^{1/2} g^{ij}(x) \frac{\partial u}{\partial x_j}(x)\right), \; u \in C^\infty(\boldsymbol{R}^n),$$

be the corresponding Laplace-Beltrami operator and $H = -\Delta_g + V(x)$ with some real-valued function $V \in C_0^\infty(B_R)$.

The operator H will be considered as a self-adjoint operator in $L^2(X)$ (the scalar product in $L^2(X)$ is given by $(u, v) = \int_X u(x)\overline{v(x)} g^{1/2} dx$) with boundary conditions of Dirichlet (Neumann) type on Y. Throughout this paper $B_D u = u_{/Y}$ while Neumann boundary condition is of the form $B_N u = \left(\frac{\partial u}{\partial \nu}(x) + \gamma(x)u(x)\right)_{/Y} = 0$, where $\gamma \in C^\infty(Y)$ is a real-valued function and ν is the outward normal to Y, pointing into $\boldsymbol{R}^n \backslash X$.

The spectral function $e(\lambda; x, y)$, $\lambda \in \boldsymbol{R}^1$, of the operator H is determined as the distribution kernel of the spectral projectors E_λ of H. Namely,

$$(E_\lambda u, v) = \int_{X \times X} e(\lambda; x, y) u(y) \overline{v(x)} g(x)^{1/2} g(y)^{1/2} dx dy, \quad u, v \in C_0^\infty(X).$$

This function is closely related to the outgoing (incoming) Green's functions $G^+(\lambda; x, y)(G^-(\lambda; x, y))$ which are given by

$$(0.1) \qquad (-\Delta_g + V(x) - \lambda^2) G^\pm(\lambda; x, y) = \delta_y(x) \quad \text{in} \quad X, \quad y \in X,$$

$$(0.2) \qquad B_j G^\pm = 0 \quad \text{in} \quad y \in X \quad \text{and} \quad j = D \quad \text{or} \quad j = N,$$

and by the outgoing (incoming) Sommerfeld's condition at infinity

(0.3) $$|G^{\pm}(\lambda;\, x,\, y)| \leq Cr^{(1-n)/2}, \quad \left|\left(\frac{\partial G^{\pm}}{\partial r} \mp i\lambda G^{\pm}\right)(\lambda;\, x,\, y)\right| \leq Cr^{-(1+n)/2}$$

as $r=|x-y|=\sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2}\to\infty$.
Here $\delta_y(x)$ is Dirac measure $(\delta_y(x),\, u(x)g(x)^{1/2})=u(y)$, $u\in C^\infty(X)$. Note that the distributions $G^{\pm}(\lambda;\, x,\, y)$ can be obtained by the principle of limiting absorption

$$G^{\pm}(\lambda;\, x,\, y)=\lim_{\varepsilon\downarrow 0} G(\lambda\pm i\varepsilon;\, x,\, y) \quad \text{in} \quad D^1(X\times X), \quad \lambda\in \boldsymbol{R}^1\backslash 0,$$

where $G(z;\, x,\, y)$, $\operatorname{Im} z\neq 0$, is the distribution kernel of the resolvent $(H-z^2)^{-1}$.

This paper is devoted to the asymptotics of the spectral function $e(\lambda;\, x,\, y)$ and the Green's functions $G^{\pm}(\lambda;\, x,\, y)$ as $\lambda\to\infty$.

The function $G(z;\, x,\, y)$ is exponentially decreasing as $|z|\to\infty$ in the region $\{z\in \boldsymbol{C};\ \theta_0\leq \pm\arg z\leq \pi-\theta_0\}$ for any $\theta_0>0$ and $x\neq y$. To obtain the asymptotics of $G^{\pm}(\lambda;\, x,\, y)$ as $\lambda\to\infty$ some additional restrictions on the geometry of X are needed. To formulate our main assumptions consider the generalized geodesics on $\bar{X}=X\cup Y$ determined as the projections on $\bar{X}$ of the generalized bicharacteristics of the operator $\partial_t^2-\Delta_g$ (see [18]). First we impose the nontrapping condition

(N) There exists $T_R>0$ such that there are no generalized geodesics with length T_R within $X\cap B_R$.

Next we fix two points $x_0,\, y_0\in X$. It will be said that x_0 belongs to the illuminated region $Il(y_0)$ with respect to y_0 if there exists a generalized geodesic connecting x_0 and y_0, otherwise x_0 is said to belong to the shadow. Suppose that $x_0\in Il(y_0)$ and

(T) Any generalized geodesic connecting x_0 and y_0 may hit the boundary only transversally.

Let γ: $[0,\, t_0]\ni t\to x(t,\, y_0,\, \eta_0)$ be a generalized geodesic having a length $t_0\neq 0$ (t is the natural parameter on γ) and initial codirection $\eta_0\in T_yX\backslash 0$ and connecting the points y_0 and x_0 ($\gamma(0)=y_0$, $\gamma(t_0)=x_0$). Then the curve $[0,\, t]\ni s\to x(s,\, y_0,\, \eta)$ may hit the boundary Y only transversally for $(t,\, \eta)$ close to $(t_0,\, \eta_0)$ and the map $(t,\, \eta)\to x(t,\, y_0,\, \eta)$ is well defined and smooth in a conic neighbourhoof of $(t_0,\, \eta_0)$. Next we suppose that the differential of the last map is of maximal rank at $(t_0,\, \eta_0)$, i.e.

(C) $\operatorname{rank}(d_{t,\eta}x)(t_0,\, y_0,\, \eta_0)=n$ for any $t_0\neq 0$, $\eta_0\neq 0$, such that $x(t_0,\, y_0,\, \eta_0)=x_0$.

In case where the curve γ: $[0,\, t_0]\in t\to x(t,\, y_0,\, \eta_0)$ does not hit the boundary Y the condition (C) means that y_0 and x_0 are not conjugated points along γ [15]. Note that the set of all the points $x_0\in Il(y_0)$ that do not satisfy either (T) or (C) is closed and nowhere dense in X for any $y_0\in X$.

In view of the implicit function theorem there exist only finitely many generalized geodesics γ_j: $[0,\, T_j(x,\, y)]\to X$ connecting the points x and y ($\gamma_j(0)=y$, $\gamma_j(T_j(x,\, y))=x$), $j=1,\dots,\, J$, whenever $x\in Il(y)$ and (T) and (C) are satisfied at (x, y). Moreover, J is locally a constant and T_j can be chosen to be locally smooth functions.

The distributions $G^{\pm}(\lambda;\, x,\, y)$ have singularities only at the diagonal $x=y$ in $X\times X$, which can be described by means of Hankel functions $H_\nu^{(1)}(z)=J_\nu(z)+$

$iY_\nu(z)$, $H_\nu^{(2)}(z)=J_\nu(z)-iY_\nu(z)$, $\nu\in \boldsymbol{R}^1$. Namely, for x sufficiently close to y, $\lambda>0$ and for any $\nu=0, 1, 2,\dots$ consider the distributions

$$F_\nu^\pm(\lambda;\, x,\, y)=\pm i2^{-1}\sqrt{\pi}\left(\frac{2T(x,\, y)}{\lambda}\right)^{(2-n)/2+\nu} H^{(l)}_{(n-2)/2-\nu}(\lambda T(x,\, y))$$

where $T(x, y)$ is the Riemannian distance between x and y and $l=1$ in "+" case, $l=2$ in the "−" case. Note that the functions $F_\nu^\pm(\lambda;\, x,\, y)$ are smooth outside the diagonal of $X\times X$ and near this set they behave like $|x-y|^{2-n+2\nu}\ln|x-y|$ when $(2-n)/2+\nu$ is a non-negative integer and as $|x-y|^{2-n+2\nu}$ otherwise.

To separate the regular part of $G^\pm(\lambda;\, x,\, y)$ from the singular one we need some cut-off functions. First denote by r_1 the injectivity radius of the exponential map $T_y\boldsymbol{R}^n\ni v\to \exp_y v\in \boldsymbol{R}^n$ related to the metric g_{ij} ($r_1=\sup\{r;\, B_y(r)\ni v\to\exp_y v$ is a diffeomorphism for any $y\in\boldsymbol{R}^n\}$ where $B_y(r)=\{v\in T_y\boldsymbol{R}^n;\, g_{ij}(y)v^iv^j\le r^2\}$). For given $x,\, y\in X$ set $r_0(x,\, y)=\min(r,\, \mathrm{dist}(x,\, Y),\, \mathrm{dist}(y,\, Y))$, where $\mathrm{dist}(x,\, Y)=\inf\{T(x,\, z);\, z\in Y\}$. Now choose the cut-off functions $\chi_1\in C_0^\infty(\boldsymbol{R}^1)$, $\chi_2=1-\chi_1$ so that $\mathrm{supp}\,\chi_1\subset(-r_0/2,\, r_0/2)$ and $\chi_1(s)=1$ for $|s|\le r_0/4$. The main result in this paper is

Theorem 1. *Suppose the condition* (N) *fulfilled and* $(x_0,\, y_0)\in X\times X$. *Then there exists a neighbourhood* K *of the point* $(x_0,\, y_0)$ *such that*

(i) *If* $x_0\notin Il(y_0)$ *then* $G^\pm(\lambda;\, x,\, y)$ *is a smooth function in* $(0,\,\infty)\times K$ *and*

$$|D_\lambda^\alpha D_x^\beta D_y^\gamma G^\pm(\lambda;\, x,\, y)|\le C\lambda^{-N} \quad \text{in} \quad (\lambda_0,\,\infty)\times K,\quad \lambda_0>0,$$

for any α, β, γ, N *and some constants* C *independent of* $(\lambda,\, x,\, y)$.

(ii) *If* $x_0\in Il(y_0)$ *and* (T) *and* (C) *are fulfilled at* $(x_0,\, y_0)$ *then*

$$\text{(0.4)}\quad G^\pm(\lambda;\, x,\, y)=\chi_1(T(x,\, y))\sum_{\nu=0}^{M-1} U_\nu(x,\, y)F_\nu^\pm(\lambda;\, x,\, y)$$
$$+\sum_{j=1}^{J}\sum_{\nu=0}^{M-1}\chi_2(T_j)C_{\nu,j}^\pm(x,\, y)e^{\pm i\lambda T_j(x,y)}\lambda^{(n-2\nu-3)/2}+R_M^\pm(\lambda:\, x,\, y),$$

where $R_M^\pm\in C^{k_M}((0,\,\infty)\times K))$, $k_M=M-[n/2]$ ($[n/2]$ *is the entire part of* $n/2$) *and*

$$\text{(0.5)}\qquad |D_\lambda^\alpha D_x^\beta D_y^\gamma R_M^\pm(\lambda;\, x,\, y)|\le C\lambda^\delta,\quad \delta=[n/2]-M+|\beta|+|\gamma|$$

in $(\lambda_0,\,\infty)\times K$ *for any* $\lambda_0>0$, $[n/2]+|\beta|+|\gamma|<M$. *The functions* U_ν *and* $C_{\nu,j}$ *are smooth in* K, $C_{\nu,j}^-(x,\, y)=\overline{C_{\nu,j}^\pm(x,\, y)}$ *and*

$$\text{(0.6)}\quad U_0(x,\, y)=2^{-(n+2)/2}\pi^{(1-n)/2}|\det(\partial^2T_j^2(x,\, y)/\partial x\partial y)|^{1/2}(q(x)g(y))^{-1/4}$$

$$\text{(0.7)}\qquad C_{0,j}(x,\, y)=(-1)^{\beta_j}i^{l_j}2^{-n-1/2}\pi^{(1-n)/2}T_j(x,\, y)^{(1-n)/2}$$
$$\times|\det(\partial^2T_j^2(x,\, y)/\partial x\partial y)|^{1/2}(g(x)g(y))^{-1/4},$$

where $l_j=(3-n)/2+\alpha_j$ *for some integers* α_j, *and* β_j *coincides with the number of reflections of* γ_j *at the boundary in case of Dirichlet problem,* $\beta_j=0$ *in case of Neumann problem.*

The number α_j can be described as an index of a curve of Lagrange spaces as it was done in [6], [7], [10].

Theorem 2. *Suppose the condition* (N) *fulfilled. Then* $e \in C^\infty((0, \infty) \times X \times X)$ *and there exists a neighbourhood* K *of the point* $(x_0, y_0) \in X \times X$ *so that*

(i) *If* $x_0 \notin Il(y_0)$ *then*

$$\left|\frac{de}{d\lambda}(\lambda, x, y)\right| \leq C_N \lambda^{-N} \quad \text{in} \quad [\lambda_0, \infty) \times K, \lambda_0 > 0 \quad \text{for any} \quad N > 0,$$

(ii) *If* $x_0 \in Il(y_0)$ *and* (T) *and* (C) *are fulfilled at* (x_0, y_0) *then*

$$(0.8) \quad \frac{de}{d\lambda}(\lambda^2; x, y) = \frac{\chi_1(T)}{2\sqrt{\pi}} \sum_{\nu=0}^{M-1} U_\nu(x, y) \left(\frac{2T(x, y)}{\lambda}\right)^{\frac{2-n}{2}+\nu} J_{\frac{n-2}{2}-\nu}(\lambda T(x, y))$$
$$+ \frac{1}{\pi} \sum_{j=1}^{J} \sum_{\nu=0}^{M-1} \chi_2(T_j) \operatorname{Re}(e^{i\lambda T_j(x,y)} C^+_{\nu,j}) \lambda^{(n-2\nu-3)/2} + R_M(\lambda; x, y),$$

where $R_M \in C^\infty(\boldsymbol{R}^1_+ \times K)$ *and* R_M *satisfies* (0.5) *in* $(\lambda_0, \infty) \times K$ *for any* α, β, γ, M, $\lambda_0 > 0$.

Theorems 1 and 2 hold also in the case where $X = \boldsymbol{R}^n$. The corresponding results are formulated in §4.

The asymptotic behavior of the spectral function and Green's functions for second order elliptic differential operators was investigated by many authors. Results close to those of theorem 1 were obtained by Buslaev [2], [3] for Schrödinger operator $H = -\Delta + V$ in a domain $X \subset \boldsymbol{R}^n$, with Neumann boundary conditions on Y, provided $\boldsymbol{R}^n \backslash X$ is compact and strictly convex. More precisely the uniform asymptotics of $G^+(\lambda; x, y)/(x, y) \in Y \times Y$ was investigated in [3] as $\lambda \to \infty$ and $|x-y| \ll 1$. As a consequence an asymptotic expression of $G^+(\lambda; x, y)_{/y \in Y}$ as $\lambda \to \infty$ was obtained when $x \in Il(y)$ and the line passing through the points y and x hits the boundary transversally. A result close to theorem 1 was obtained also by Alber [30], provided $n = 3$ and if there is only one ray $(t \to y + t(x-y), t \geq 0)$ in X connecting the points x and y. In contrast to [2], [3], [30] we allow the points x and y to be connected with a multiple reflected ray. Moreover, the metric g may not be Euclidean in X, thus the construction of an asymptotic solution to the problem (0.1)–(0.2) by the usual method of geometrical optics breaks down when x is far away from y. We also write explicitly the first coefficients in the asymptotic expressions.

An asymptotic expression of Green's functions including only the first sum of (0.4) was given by Babich [1] but the remainder term R_M was not estimated. The asymptotics of the spectral function of the operator H in $\boldsymbol{R}^n$ was investigated in [22]. The non-trapping condition assumed in [22] is a little more restrictive than (N) because it does not allow the geodesics to have self-intersections. Asymptotics close to those of theorem 1 and 2 were announced in [23], [24] by the author.

An asymptotic expression of the spectral function $e(\lambda; x, y)$ in case $X = \boldsymbol{R}^n$ was obtained also by Vainberg [28], [29]. Without assuming the condition (C) he proved that $\frac{de}{d\lambda}(\lambda; x, y)$ behaves as $\lambda \to \infty$ like a global oscillating function given by Maslov's canonical operator. When (C) is fulfilled he obtained the asymptotics

$$\frac{de}{d\lambda}(\lambda;\, x,\, y)=\lambda^{\frac{n-2}{2}}\sum_{k=1}^{N}\int_{S^{n-2}}\Big(\sum_{j=0}^{\infty}\psi_{k,j}(x,\, y,\, \omega)\,\lambda^{-j/2}\Big)e^{i(\sqrt{\lambda}S_k(x,y,\omega)-\frac{\pi}{2}\gamma_k)}d\omega$$

with some amplitudes $\psi_{k,j}$ and phases S_k.

Note that the results obtained in this paper are connected also with the asymptotics of the scattering phase [2], [21] and the scattering amplitude [20].

The article is organized as follows. In §1 the distributions $G^{\pm}$ are written in the form

$$G^{\pm}(\lambda;\, x,\, y)=\pm\int e^{i\lambda t}\varphi(t)Y(\pm t)U(t,\, x,\, y)dt+0(\lambda^{-N})$$

where $\varphi\in C_0^{\infty}(\boldsymbol{R}^1)$, $\varphi(t)=1$ in a neighbourhood of $t=0$, $Y(t)$ is Heaviside function and $U(t,\, x,\, y)$ is the solution of the mixed problem

$$\tag{0.9}\begin{cases}(D_t^2-H)U=0\\ B_jU=0, \qquad j=D,\, N\\ U(0,\, x,\, y)=0,\ U_t(0,\, x,\, y)=\delta(x-y)\end{cases}$$

To do this, we use essentially the decay of the local energy of $U(t,\, x,\, y)$ as $t\to\infty$, which is valid in view of the non-trapping condition (N). Let $\varphi_j\in C_0^{\infty}(\boldsymbol{R}^1)$, $\varphi_1+\varphi_2=\varphi$ and $\operatorname{supp}\varphi_1\subset(-r_0,\, r_0)$, $\varphi_1(t)=1$ for $|t|<r_0/2$. Consider the distributions $G_k^+=\int\exp(i\lambda t)\varphi_k(t)Y(t)U(t,\, x,\, y)dt$ $k=1,\, 2$. The asymptotics of $G_1^+(\lambda;,\, x,\, y)$ as $\lambda\to\infty$ is investigated in §2. Note that for $|t|<r_0$ and $(x,\, y)$ close to $(x_0,\, y_0)$ the distribution $U^+(t,\, x,\, y)=Y(t)U(t,\, x,\, y)$ will be a solution to the problem

$$\tag{0.10}\begin{cases}(D_t^2-H)U^+=-\delta(t)\delta(x-y)\\ U^+_{/t<0}=0\end{cases}$$

To obtain a parametrix for (0.10) we apply Hadamard's construction (see [9], [11]). Denote by Z_ν, $\nu\geq n/2-1$ the functions

$$Z_\nu(t,\, x,\, y)=(t^2-T^2(x,\, y))^{\nu+(2-n)/2}/\Gamma\Big(1+\nu+\frac{2-n}{2}\Big)$$

for $t\geq T(x,\, y)$ and $Z_\nu=0$ for $t<T(x,\, y)$. The distribution Z_ν admits an analytical continuation with respect to ν in $\operatorname{Re}\nu>-1$. Moreover, for $|t|<r_0$ the fundamental solution of (0.10) can be written in the form

$$U^+(t,\, x,\, y)\sim\sum_{\nu=0}^{\infty}U_\nu(x,\, y)Z_\nu(t,\, x,\, y)$$

for some suitably chosen smooth functions $U_\nu(x,\, y)$. The Fourier transform of Z_ν with respect to t is just F_ν^+.

In §3 we consider the Fourier transform $G_2^+(\lambda;\, x,\, y)$ of the distribution $\varphi_2(t)\cdot Y(t)U(t,\, x,\, y)$ which may have singularities only at the points $(T_j(x,\, y),\, x,\, y)$ for

(x, y) close to (x_0, y_0). Near these points one can apply Chazarain's construction of a parametrix for (0.9). In this way we present the distribution $\varphi_2(t)Y(t)U(t, x, y)$ as a sum of Fourier oscillating integrals for (x, y) close to (x_0, y_0). In view of (C) the corresponding Lagrange manifolds coinside with the conormal bundles of the manifolds $t=T_j(x, y)$ near $(T_j(x_0, y_0), x_0, y_0)$. This allows us to obtain a microlocal representation of $\varphi_2 YU(t, x, y)$ near (T_j, x_0, y_0) in the form

$$\int_0^\infty e^{i\theta(T_j(x,y)-t)}\varphi_2(t)a_j(x, y, \theta)\theta^{(n-1)/2}d\theta \tag{0.11}$$

with some amplitudes a_j. In case where $X=\boldsymbol{R}^n$ a similar representation was found by Y. Colin de Verdiére [5]. The first coefficients $C^+_{0,j}(x, y)$ are obtained after a careful analysis of the principal symbol $a_{0,j}$ of a_j.

Note that $G^+_2(\lambda; x, y)$ can be written also in the form

$$G^+_2(\lambda; x, y)=\int e^{i\lambda t}(i_{x,y})^*U(t)\varphi_2(t)Y(t)dt$$

where $i_{x,y}$ is the inclusion map $i_{x,y}(t)=(t, x, y)$ and the corresponding "pull-back" map $(i_{x,y})^*$ is a Fourier Integral Operator. In view of (T) and (C) it can be proved that $(i_{x,y})^*$ and $U(t, x, y)$ have a transversal composition as Fourier integral operators. which also leads to (0.11). In §5 we impose a little more general condition than (C). Namely, we suppose that the roots of the equation $x(t, y_0, \eta)=x_0$ with respect to (t, η) form a smooth conic manifold W whose tangent space $T_{(t,\eta)}W$ coincides with the kernel of the operator $P_{t,\eta}=(d_{t,\eta}x)(t, y_0, \eta)$ for any $(t, \eta)\in W$, $\eta\neq 0$. Such a situation occures for example in case where X is a domain in $\boldsymbol{R}^n$, $Y=\partial X$ contains a part of a circle cone and x_0, y_0 are some points on the cone axis. In this case the operators $(i_{x,y})^*$ and $U(t, x, y)$ have a clean composition which gives an asymptotic expression for $G^+(\lambda; x, y)$ close to (0.4).

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§1. An integral representation of Green's functions

We start this section with some remarks on the spectrum of the operator H. This operator has no positive eigenvalues in view of Rellich's uniqueness theorem and the unique continuation property for second order elliptic differential operators. Moreover, the nonpositive poing spectrum of the operator H is finite in view of the apriori estimate for elliptic problems in unbounded domains

$$\|u\|_{H^2(X)}\leq C(\|Hu\|_{L^2(X)}+\|B_ju\|_{H^{\frac{1}{2}}(Y)}+\|n\|_{L^2(X\cap(|x|<\rho))})$$

$u\in H^2(X)$, $j=D$, N and $\rho>R$, where $H^s(X)$ is the usual Sobolev space. The continuous spectrum of the operator H coincides with $\overline{\boldsymbol{R}^1_+}=[0, \infty)$. Therefore H is a direct sum $H=H_p\oplus H_c$ of an operator H_p of finite ran kand a non-negative operator H_c with continuous (even absolutely continuous) spectrum. Denote $U_p(t)=H_p^{-1/2}$,

$\sin(tH_p^{1/2})$, $U_c(t)=H_c^{-1/2}\sin(tH_c^{1/2})$ and $U(t)=U_p(t)\oplus U_c(t)$, $t\in \boldsymbol{R}^1$. Obviously $U(t)$ is the propagator of

$$\begin{cases} (D_t^2-H)U(t)f=0 \\ B_jU(t)f=0, \quad j=N,\, D \\ U(0)f=0, \quad \dfrac{d}{dt}U(t)f_{/t=0}=f, \quad f\in L^2(X) \end{cases} \tag{1.1}$$

and the distribution kernel $U(t, x, y)$ of the operator $U(t)$ solves (0.9)

Denote by $\int e^{i\lambda t}U(t, x, y)dt$ the partial Fourier transform of the distribution $U\in S'(\boldsymbol{R}^1\times X\times X)$ with respect to t and set $Y(t)=1$ for $t\geq 0$, $Y(t)=0$ for $t\leq 0$.

Proposition 1.1. *Suppose the condition* (N) *is valid. Then for any compact* $K=K_1\times K_2$, $K_j\subset X$, $j=1, 2$, *there exists a number* T_K *such that for any* $\varphi\in C_0^\infty(\boldsymbol{R}^1)$, $\varphi(t)=1$ on $(-T_K, T_K)$ *we have*

$$G^\pm(\lambda; x, y)=\pm\int e^{i\lambda t}\varphi(t)Y(\pm t)U(t, x, y)dt+r_\varphi^\pm(\lambda, x, y) \tag{1.2}$$

where $r_\varphi^\pm$ *is a smooth function in a neighbourhood of* $\boldsymbol{R}_+^1\times K$. *Moreover, the estimate*

$$|D_\lambda^l D_x^\alpha D_y^\beta r_\varphi^\pm(\lambda, x, y)|\leq C\lambda^{-N} \tag{1.3}$$

holds for $\lambda\geq 1$, $(x, y)\in K$ *with a constant* C *independent on* λ, x, y.

Remark. A similar proposition was proved in [22] in the case when $Y=\emptyset$ (see also [29]). The proof given in [22] can be modified to work also in our situation $Y\neq\phi$. Here we use another idea which is based on the decay of the local energy for hyperbolic equations.

Proof of Proposition 1.1. According to the principle of limiting absorption

$$G^\pm(\lambda; x, y)=\lim_{\varepsilon\downarrow 0} G(\lambda\pm i\varepsilon; x, y), \quad \lambda\in R^1\backslash 0, \tag{1.4}$$

where the limit is taken in a distribution sense. For this reason we consider the equality

$$(H-z^2)^{-1}=\pm\int e^{izt}Y(\pm t)U_c(t)dt+(H_p-z^2)^{-1},$$

for $\pm\operatorname{Im} z>0$, $\operatorname{Re} z\neq 0$, given by the functional calculus. Let $\mathscr{H}_j\in C_0^\infty(\boldsymbol{R}^n)$ $j=1, 2$, $\mathscr{H}_j(x)=0$ in a neighbourhood of the boundary Y and $\mathscr{H}_j(x)=1$ for x in a neighbourhood of K_j. Choose a function $\varphi\in C_0^\infty(\boldsymbol{R}^1)$ such that $\varphi(t)=1$ on $(-T, T)$ for some T which will be specified later. Let us write

$$\mathscr{H}_1(H-z^2)^{-1}\mathscr{H}_2=\pm\int e^{izt}\varphi(t)Y(\pm t)\mathscr{H}_1U(t)\mathscr{H}_2dt+R^\pm(z) \tag{1.5}$$

for $\pm\operatorname{Im} z>0$, $\operatorname{Re} z\neq 0$. Here $R^\pm(z)=R_1^\pm(z)+R_2^\pm(z)$,

$$R_1^{\pm}(z)=\mathscr{H}_1(H_p-z^2)^{-1}\mathscr{H}_2\mp\int e^{izt}\varphi(t)Y(\pm t)\mathscr{H}_1U_p(t)\mathscr{H}_2dt$$

$$R_2^{\pm}(z)=\pm\int e^{izt}\psi(t)Y(\pm t)\mathscr{H}_1U_c(t)\mathscr{H}_2dt$$

and $\psi\in C^\infty(\boldsymbol{R}^1)$, $\psi(t)=0$ for any $t\in(-T, T)$, $\psi(t)=1$ for t sufficiently large.

Let us denote by $\mathscr{L}(H^{-s}, H^s)$ the Banach space of bounded linear operators mapping $H^{-s}(X)$ into $H^s(X)$. The distribution kernel of the operator $\varphi(t)Y(\pm t)\cdot\mathscr{H}_1U_p(t)\mathscr{H}_2$ is a smooth function in $\overline{\boldsymbol{R}_+^1}\times X\times X$ since the point spectrum of H is finite and the corresponding eigenvectors are smooth functions in X.

Integrating N times by parts we get the inequality

$$\|D_z^jR_1^{\pm}(z)\|_{\mathscr{L}(H^{-s},H^s)}\leq C(1+|z|)^{-N} \tag{1.6}$$

in $\{z\in\boldsymbol{C};\ |\mathrm{Re}\,z|>1,\ \pm\mathrm{Im}\,z\geq0\}$.

To estimate the norm of the operator $R_2^{\pm}(z)$ we use the decay of the local energy of $U_c(t)$ as $t\to\infty$. According to (N) there exists a number T_K such that any generalized geodesic beginning in a compact neighbourhood $\theta\subset\overline{X\times X}$ of $(\mathrm{supp}\,\mathscr{H}_1)\times(\mathrm{supp}\,\mathscr{H}_2)$ leaves θ by the time T_K. We now set $T=T_K$ so $\varphi(t)=1$ on $[-T_K, T_K]$. According to the theorem for the propagation of the singularities the function $U(t, x, y)$ is smooth in $[(-\infty, -T_K)\times(T_K, \infty)]\times\theta$, i.e. the generalized Huygens' principle holds. Thus the conditions (A')–(C') in [27] are satisfied and theorems 9 and 11 in [27] (see also (25], [19]) give

$$\|D_t^j(\mathscr{H}_1U_c(t)\mathscr{H}_2f)\|_{L^2(X)}\leq Ct^{\delta-j}\|f\|_{L^2(X)},\ f\in L^2(X) \tag{1.7}$$

in $|t|\geq T_K$ and for some $\delta\in\boldsymbol{R}^1$ and any $j\geq0$. Since $(D_t^2-H)U_c(t)f=0$ for any distribution $f\in D^1(X)$ and the functions $\mathscr{H}_j(x)=0$ for x in a neighbourhood of the boundary 0 we obtain

$$\|D_t^j\ (\mathscr{H}_1U_c(t)\mathscr{H}_2)\|_{\mathscr{L}(H^{-s},H^s)}=Ct^{\delta-j},\quad |t|\geq T_K,$$

for any $s\in R^1$ and $j\geq0$. Moreover $\psi(t)=1$ for $|t|>T_K^1$ and some $T_K^1>T_K$, thus the last estimate yields

$$\|D_z^lR_2^{\pm}(z)\|_{\mathscr{L}(H^{-s},H^s)}\leq C(1+|z|)^{-N} \tag{1.8}$$

on $\{z\in\boldsymbol{C};\ \pm\mathrm{Im}\,z\geq0,\ |\mathrm{Re}\,z|>1\}$ and for any k, s, N. Letting $\pm\mathrm{Im}\,z\downarrow0$ in (1.5), (1.6), (1.8) we obtain

$$\mathscr{H}_1(H-(\lambda^2\pm i0))^{-1}\mathscr{H}_2=\pm\int e^{i\lambda t}\varphi(t)Y(\pm t)\mathscr{H}_1U(t)\mathscr{H}_2dt+R^{\pm}(\lambda)$$

where $R^{\pm}(\lambda)$ satisfies the estimate

$$\|D_\lambda^lR^{\pm}(\lambda)\|_{\mathscr{L}(H^{-s},H^s)}\leq C_{l,s,N}\lambda^{-N},\quad \lambda>1. \tag{1.9}$$

To prove (1.3) we use (1.4) and the estimate

$$|D_\lambda^lD_x^\alpha D_y^\beta r_\varphi^{+}(\lambda, x, y)|=|(D_\lambda^lR^{\pm}(\lambda)D_y^\beta\delta,\ D_x^\alpha\delta)|\leq$$
$$\leq C_{l,s,N}\|D_y^\beta\delta(y)\|_{-s}\|D_x^\alpha\delta(x)\|_{-s}\lambda^{-N}$$

for $\lambda > 1$ and $s > n + |\alpha| + |\beta|$ which is valid in view of (1.9). This proves the claim.

Let $\varphi_j \in C_0^\infty(\boldsymbol{R}^1)$, $\varphi_1 + \varphi_2 = \varphi$ and $\operatorname{supp} \varphi_1 \subset (-r_0, r_0)$, $\varphi_1(t) = 1$ for $t \in (-2r_0/3, 2r_0/3)$. Here $r_0 = \min(r_1, \operatorname{dist}(x_0, Y), \operatorname{dist}(y_0, Y))$, r_0 is the injectivity radius of the exponential map in $\boldsymbol{R}^n$ related to the metric g_{ij} and φ is determined as in Proposition 1.1. Consider the operator $H = -\Delta_g + V(x)$ in $\boldsymbol{R}^n$ and denote by $U^\pm(t, x, y)$ the solution of the problem

$$(1.10) \qquad \begin{cases} (D_t^2 - H)U^\pm(t, x, y) = \pm\delta(t)\delta(x-y) \\ U^\pm(t, x, y) = 0 \qquad \text{for} \quad \pm t < 0 \end{cases}$$

Let $W(t, x, y)$ be the distribution kernel of the operator $\cos(tH^{1/2})$. This distribution solves the mixed problem

$$(1.11) \qquad \begin{cases} (D_t^2 - H)W(t, x, y) = 0 \\ B_j W = 0, \quad j = D, N \\ W(0, x, y) = \delta(x-y), \quad W_t(0, x, y) = 0 \end{cases}$$

As a consequence of Proposition 1.1 and the finite propagation speed for the solutions of the wave equation $(D_t^2 - H)u = 0$ we obtain

Corollary 1.1. *Suppose the condition* (N) *holds. Then*

$$(1.12) \qquad G^\pm(\lambda; x, y) = \pm\int e^{i\lambda t}\varphi_1(t)U^\pm(t, x, y)dt$$

$$\pm\frac{i}{\lambda}\int e^{i\lambda t}\varphi_2(t)Y(\pm t)W(t, x, y)dt + r^\pm(\lambda, x, y)$$

and $r^\pm(\lambda, x, y)$ are smooth functions in a neighbourhoof of $\boldsymbol{R}_+^1 \times K$ satisfying (1.3) in $[1, \infty) \times K$.

Let us turn now to the spectral function $e(\lambda; x, y)$. This function can be written in the form

$$(1.13) \qquad \frac{de}{d\lambda}(\lambda^2; x, y) = (2\pi i)^{-1}\{G^+(\lambda; x, y) - G^-(\lambda; X, y)\} \qquad \text{for} \quad \lambda \neq 0$$

since the following identity is satisfied

$$\frac{d}{d\lambda}E_\lambda = (2\pi i)^{-1}\{(H - \lambda - i0)^{-1} - (H - \lambda + i0)^{-1}\}.$$

Moreover $G^-(\lambda; x, y) = \overline{G^+(\lambda; x, y)}$ for $x \neq y$ (the coefficients of H are real functions) and we obtain

$$(1.14) \qquad \frac{de}{d\lambda}(\lambda^2; x, y) = \frac{1}{\pi}\operatorname{Im} G^+(\lambda; x, y) \qquad \text{for} \quad x \neq y, \lambda \neq 0.$$

Note that formula (1.13) and Proposition 1.1 yield together

$$\frac{de}{d\lambda}(\lambda^2; x, y)=\frac{i}{\lambda}\int e^{i\lambda t}\varphi(t)W(t, x, y)dt+r(\lambda, x, y) \tag{1.15}$$

where $r\in C^\infty(\boldsymbol{R}^1_+\times K)$ and $r(\lambda, x, y)$ satisfies the estimate (1.2) in $[1, \infty)\times K$. As a consequences of (1.15) one can prove that $e\in C^\infty(\boldsymbol{R}^1_+\times K)$.

§ 2. Hadamard's fundamental solution

In this section we consider the asymptotics of the distribution $G^+_1(\lambda; x, y)=\int e^{i\lambda t}\varphi_1(t)U^+(t, x, y)dt$ where U^+ solves (1.10). To evaluate $G^+_1(\lambda; x, y)$ as $\lambda\to\infty$ we use a local parametrix to the problem (1.10) the idea of which goes back to Hadamard [11] and M. Riesz. At this point we use essentially the construction of a parametrix which is due to Friedlander (see [9], §6.2).

Let $(x_0, y_0)\in X\times X$ be such that $T(x_0, y_0)<r_0/2$ and set

$$K_1=\{x\times X;\ T(x_0, x)<r_0/16\},\quad K_2=\{x\in X;\ T(y_0, x)<r_0/16\},\quad K=K_1\times K_2.$$

Consider the functions Z_ν defined by

$$Z_\nu(t, x, y)=(t^2-T^2(x, y))^{\nu+(1-n)/2}/\Gamma(\nu+(3-n)/2) \qquad \text{for} \quad t\geq T(x, y), \tag{2.1}$$

$$Z_\nu(t, x, y)=0 \qquad \text{for} \quad t<T(x, y)$$

in $\boldsymbol{R}^1\times K$ and for any $\nu\in \boldsymbol{C}$, $\operatorname{Re}\nu>(n-1)/2$ ($\Gamma(\nu)$ is the usual gamma-function). The distribution Z_ν admits an analytical continuation with respect to ν in the region $\operatorname{Re}\nu>-1$, which is described in ([9], §6.1). Moreover, there exist some real-valued functions $U_\nu\in C^\infty(K)$ such that

$$U^+(t, x, y)-\sum_{\nu=0}^{M-1}U_\nu(x, y)Z_\nu(t, x, y)\in C^{k_M}(\boldsymbol{R}^1\times K), \quad k_M=M-[n/2] \tag{2.2}$$

for $M>n/2$, where $[n/2]$ is the entire part of $n/2$. The coefficients $U_\nu(x, y)$ are determined in K by some transport equations. More precisely $U_0(x, y)$ is given by (0.6) while $U_\nu(x, y)$, $\nu>0$, have the form

$$U_\nu(x, y)=-\frac{1}{4}U_0(x, y)\int_0^1\left(\frac{HU_{\nu-1}}{U_0}\right)\Big/(\gamma(s), y)s^{\nu-1}ds,$$

where $\gamma(s)=\exp_y(sv(x, y))$, $s\in[0, 1]$, and $X\ni x\to v(x, y)\in T_yX$ is the inverse function of the exponential map $v\to\exp_y v$ defined in $\{x\in X;\ T(x, y)<r_1\}$, r_1 being the injectivity radius of the metric g in $\boldsymbol{R}^n$. (see [9], §6.2).

The same construction holds also when the operator H has the form $H=-\Delta_g+h(x, D)+V(x)$, $h(x, D)=\sum_{j=1}^n h_j(x)D_j$, $D_j=-i\partial/\partial x_j$ and $h_j\in C_0^\infty(\boldsymbol{R}^n)$. In this case the function $U_0(x, y)$ can be obtained by multiplying (0.6) with

$$\exp\left(-i\int_0^{T(x,y)}h(\phi^s(y, \eta(x, y)))ds\right)$$

where ϕ^s is the Hamilton flow of $q(y, \eta)=(g^{ij}(x)\eta_i\eta_j)^{1/2}$ and $\eta(x, y)$ is given by

$\eta(x, y)=(g_{1j}(y)v(x, y)^j,\dots, g_{nj}(y)v(x, y)^j)$.

Now we shall investigate the Fourier transform of Z_ν with respect to t. First note that

$$F_\nu^+(\lambda;\ x, y)=\int e^{i\lambda t}Z_\nu(t, x, y)dt, \quad \lambda>0,$$

for $(n-3)/2<\operatorname{Re}\nu<(n-1)/2$ (see [8], pp. 11 and 69) and since both sides are analytical distributions with respect to ν in $\operatorname{Re}\nu>-1$ we obtain the last equality for any ν with $\operatorname{Re}\nu>-1$. Next we write

$$\int e^{i\lambda t}\varphi_1(t)Z_\nu(t, x, y)dt=F_\nu^+(\lambda;\ x, y)+R_\nu(\lambda;\ x, y) \tag{2.3}$$

in $\boldsymbol{R}^1\times K$. For any positive and integer N we have

$$\lambda^N R_\nu=(-1)^N\int e^{i\lambda t}(1-\varphi_1)D_t^N Z_\nu dt+\int e^{i\lambda t}\varphi_3(t)P(t, D_t)Z_\nu dt$$

where $\varphi_3\in C_0^\infty(\boldsymbol{R}^1)$, $\varphi_3(t)=0$ for $|t|<2r_0/3$ and P is a differential operator with smooth coefficients. Note that $(1-\varphi_1)D_t^N Z_\nu$ is a smooth function in $\boldsymbol{R}^1\times K$ which is estimated by

$$|(1-\varphi_1)D_t^N Z_\nu(t, x, y)|\le C(1+|t|)^{-p}, \quad p>0,$$

in $\boldsymbol{R}^1\times K$ for $N>p+2\nu+1-n$, while the function $\varphi_3(t)P(t, D_t)Z_\nu(t, x, y)$ is smooth in $\boldsymbol{R}^1\times K$ and has a compact support with respect to t. Therefore $R_\nu\in C^\infty((0, \infty)\times K)$ and satisfies the estimates

$$|D_\lambda^\alpha D_x^\beta D_y^\gamma R_\nu(\lambda;\ x, y)|\le C\lambda^{-N} \quad \text{in} \quad [\lambda_0, \infty)\times K,\ \lambda_0>0,$$

for any α, β, γ and $N>0$. In view of (2.2) and (2.3) we obtain

$$G_1^+(\lambda;\ x, y)=\sum_{\nu=0}^{M-1} U_\nu(x, y)F_\nu^+(\lambda;\ x, y)+R_{M,1}^+(\lambda;\ x, y) \tag{2.4}$$

where

$$R_{M,1}^+(\lambda;\ x, y)=\sum_{\nu=0}^{M-1} R_\nu(\lambda;\ x, y)+\int e^{i\lambda t}\varphi_1(t)S_M(t, x, y)dt$$

and $S_M\in C^{M-[n/2]}(\boldsymbol{R}^1\times K)$. Therefore $R_{M,1}^+\in C^{M-[n/2]}((0, \infty)\times K))$ and is infinitely differentiable with respect to λ in $\lambda>0$. Moreover, an integration by parts yields the estimate (0.5) for $R_{M,1}^+$ in $[\lambda_0, \infty)\times K$, $\lambda_0>0$ for any α, β, γ and M such that $M-n/2-|\beta|-|\gamma|>0$.

Similarly we get (2.4) for $G_1^-(\lambda;\ x, y)=\int e^{i\lambda t}\varphi_1(t)U^-(t, x, y)dt$, where "+" sign is replaced by "−" sign.

Now we have

$$G_1^+(\lambda;\ x, y)-G_1^-(\lambda;\ x, y)=\sum_{\nu=0}^{M-1} U_\nu(x, y)\{F_\nu^+(\lambda;\ x, y)-F_\nu^-(\lambda;\ x, y)\} \tag{2.5}$$
$$+R_{M,1}(\lambda;\ x, y).$$

Note that the function

$$F_\nu^+(\lambda;\, x,\, y)-F_\nu^-(\lambda;\, x,\, y)=\sqrt{\pi}\, i\left(\frac{2T(x,\, y)}{\lambda}\right)^{\frac{2-n}{2}-\nu} J_{\frac{n-2}{2}-\nu}(\lambda T(x,\, y))$$

is smooth in $\{(x, y)\in X\times X;\, T(x, y)<r_1\}$ for any $\nu\in \boldsymbol{R}^1$. Indeed $T^{-\nu}J_\nu(T)$ is an analytical and even function with respect to $T\in \boldsymbol{R}^1$. Moreover the square of $T(x, y)$ is a smooth function in $\{(x, y)\in X\times X;\, T(x, y)<r_1\}$ since $T(x, y)=|v(x, y)|_g=\sqrt{g_{ij}(y)v(x, y)^i v(x, y)^j}$ ($T(x, j)$ is the geodesic distance between x and y along $[0, 1]\ni s\to\gamma(s)=\exp_y(sv(x, y))$, see [15]) and the vector valued function $v(x, y)$ is a smooth map for x close to y as the inverse function of the exponential map. Therefore the remainder term $R_{M,1}$ is a smooth function in $(0, \infty)\times K$ which satisfies (0.5) in $[\lambda_0, \infty)\times K$, $\lambda_0>0$ for any α, β, γ and M.

§3. Fourier distribution related to $W(t, x, y)$.

The fundamental solution $W(t, x, y)$ of the mixed problem (1.11) can be written for $(x, y)\in X\times X$, $x\in Il(y)$ and satisfying (T) as a sum of Fourier distributions (see [4], [10]). For this reason we shall investigate Fourier distributions $I\in I^{-1/4}(\boldsymbol{R}^{n+1}\times X;\, C)$ which Lagrangean manifold C has the form

$$C=\{(t,\, x,\, y;\, \tau,\, \xi,\, -\eta)\in T^*(\boldsymbol{R}^{n+1}\times X)\backslash 0;\, (x,\, \xi)=r^t(y,\, \eta),\, \tau+q(y,\, \eta)=0\}$$

and $q(y, \eta)=(g^{ij}(y)\eta_i\eta_j)^{1/2}$. Here $r^t(x, \eta)=(x(t, y, \eta), \xi(t, y, \eta))$ is a smooth family of homogeneous canonical transformations which iwll be specified later in §4 where the results of this chapter will be applied to obtain the asimptotics of $G_2^+(\lambda;\, x, y)$.

Suppose that

$$\text{rank}\,(d_{t,\eta}x)(t,\, y_0,\, \eta)=n \tag{3.1}$$

for any $(t, \eta)\in(\boldsymbol{R}^1\backslash 0)\times T^*_{y_0}X$ such that $x(t, y_0, \eta)=x_0$.

Let $S_y^*=\{\eta\in T_y^*X;\, q(y, \eta)=1\}$ and $T>0$. Denote by $(T_j(x, y), \eta_j(x, y))\in(\boldsymbol{R}^1\backslash 0)\times S_y^*$, $j=1,\ldots, J$ all the solutions of the equation $x(t, x, y)=x$ with respect to (t, η) for (x, y) close to (x_0, y_0) and such that $0<T_j(x, y)<T$. These functions are given by the implicit function theorem.

In this section we write $I(t, x, y)$ near the point $(T_j(x, y), x, y)$ as an oscillatory integral with a phase function $\varphi=\theta(T_j(x, y)-t)$, $\theta\in R^1_+$ and compute the principal part of the corresponding amplitude.

The principal symbol of I is a section in the tensor product $\Omega_{1/2}(C)\otimes L(C)$ of the helf-density bundle $\Omega_{1/2}(C)$ and the Keller-Maslov line bundle $L(C)$ over C. The bundle $\Omega_{1/2}(C)$ has an intrinsic trivialisation. Indeed, the projection

$$\pi_c\colon C\longrightarrow \boldsymbol{R}^1\times T^*(X),\quad \pi_c(t,\, x,\, y;\, \tau,\, \xi,\, \eta)=(t,\, y,\, \eta)$$

is a diffeomorphism; so the pull-back

$$\mathfrak{S}_1=\pi_C^*(|dt|^{1/2}\otimes|dy\wedge d\eta|^{1/2})$$

is a nowhere vanishing smooth half-density over C which is obviously invariant under the flow

$$(t, y, \eta) \longrightarrow \{(t, x, y; \tau, \xi, -\eta); (x, \xi)=r^t(y, \eta), \tau+q(y, \eta)=0\}.$$

Denote by $\mathfrak{S}_2$ a nowhere vanishing section in $L(C)$ which will be specified later in the concrete situation ($L(C)$ is a trivial bundle) and set $\mathfrak{S}=\mathfrak{S}_1 \otimes \mathfrak{S}_2$. Then the principal symbol a of I is given by $a=\tilde{a}(\mathfrak{S}_1 \otimes \mathfrak{S}_2)$, where $\tilde{a}$ is a smooth function on C.

Proposition 3.1. *Suppose that* (3.1) *is satisfied at the point* $(x_0, y_0) \in X \times X$. *Then there exist some amplitudes* $a_j \in S^0(\boldsymbol{R}^n \times X; \boldsymbol{R}^1)$, $a_j \sim \sum_{s=0}^{\infty} a_{j,s}(x, y)\theta^{-s}$ *for* $|\theta|>1$ *such that*

$$I(t, x, y)=\sum_{T_j<T} (2\pi)^{-\frac{n+1}{2}} i^{\frac{1-n}{2}} \int_0^{\infty} e^{i[\theta(T_j(x,y)-t)+\gamma_j \pi/2]} \times a_j(x, y, \theta)\theta^{\frac{n-1}{2}} d\theta \tag{3.2}$$

for some $\gamma_j \in \boldsymbol{Z}/4$ *in a neighbourhood of* $(0, T) \times \{x_0\} \times \{y_0\}$. *Moreover*

$$a_{j,0}(x,y)=2^{-n/2}T_j(x, y)^{(1-n)/2}|\partial^2 T_j^2/\partial x \partial y|^{1/2}\tilde{a}_j(x, y) \tag{3.3}$$

where $\tilde{a}_j(x, y)=\tilde{a}(T_j, x, y; -1, \xi(T_j, y, \eta_j), -\eta_j(x, y))$ *and* $|\partial^2 T_j^2/\partial x \partial y|$ *stands for the corresponding determinant.*

Remark. Formula (2.2) was first obtained by Colin de Verdière when r^t coinsides with the Hamiltonian flow of $q(y, \eta)$.

Proof of Proposition 3.1. The proof of (3.2) is based on the following

Lemma 3.2. *Suppose the condition* (3.1) *holds at* (x_0, y_0). *Then C is generated by the phase function*

$$\varphi_j(t, x, y, \theta)=\theta(T_j(x, y)-t), \quad (t, x, y, \theta) \in \boldsymbol{R}^{n+1} \times X \times \boldsymbol{R}^1_+$$

near the point $\rho_j=(T_j(x_0, y_0), x_0, y_0, -1, \xi(T_j, x_0, \eta_j), -\eta_j(x_0, y_0))$.

Proof. In view of (3.1) one can choose (x, y, θ) as local coordinates in C near ρ_j, i.e.

$$C=\{(T_j(x, y), x, y; -\theta, \xi_j(x, y)\theta, -\eta_j(x, y)\theta\}$$

where $\xi_j(x, y)=\xi(T_j, y, \eta_j(x, y))$ and $(x, y, \theta) \in \boldsymbol{R}^n \times X \times \boldsymbol{R}^1_+$. Since the symplectic form $dt \wedge d\tau + dx \wedge d\xi + dy \wedge d\eta$ vanishes on C we have $\xi_j(x, y)=\partial T_j(x, y)/\partial x, \eta_j(x, y) = -\partial T_j(x, y)/\partial y$, which proves the claim.

Now formula (3.2) follows immediately from the definition of Fourier distributions [13], [26] and Lemma 3.2.

To compute the amplitudes $a_{j,0}$ one should find the contribution of the phase function $\varphi=\theta(T(x, y)-t)$ (the index j is dropped for the sake of simplicity). First note that the critical set of φ has the form

$$\Sigma_\varphi=\{(t, x, y, \theta)\in \boldsymbol{R}^{n+1}\times X\times \boldsymbol{R}^1_+;\ t=T(x, y)\}$$

Denote by $Q_1\colon \Sigma_\varphi\to C$ the diffeomorphism
$Q_1(t, x, y, \theta)=(t, x, y; \varphi_t, \varphi_x, \varphi_y)$, $(t, x, y, \theta)\in\Sigma_\varphi$. Then the map $Q=\pi\circ Q_1$ Q: $\Sigma_\varphi\to \boldsymbol{R}^1\times(T^*X)$ is a diffeomorphism in a conic neighbourhood of the point $(T(x, y), x, y, 1)$ which provides Σ_φ with local coordinates (t, y, η). The corresponding density d_Σ on Σ_φ (d_Σ is the pull-back of Dirac measure under the map Q) is given by

$$d_\Sigma=|D(t, y, \eta, \psi_\theta)/D(t, y, x, \theta)|^{-1}Q^*(|dt|\otimes|dy\wedge d\eta|)$$

where $|D(u)/D(v)|$ stands for the Jacobian of $u=u(v)$ (see [13], §3.2). Since $\partial^2\varphi/\partial\theta\partial x=\partial T/\partial X$ and $\eta=-\partial T/\partial y$, a short computation gives

$$d_\Sigma=\theta^{n-1}\left|D\left(-\frac{\partial T}{\partial y}\theta,\ T(x, y)\right)\Big/D(x, \theta)\right|^{-1}_{\backslash\theta=1}Q^*(|dt|\otimes|dy\wedge d\eta|).$$

Moreover $\frac{\partial T}{\partial y}(x, y)=-\eta(x, y)\in S^*_y=\{\eta;\ q(y, \eta)=1\}$ for any (x, y) close to (x_0, y_0). Let $\boldsymbol{R}^{n-1}\ni\omega\to f(\omega)\in S^*_y$ be a local coordinate system on S^*_y near $-\eta(x_0, y_0)$. Then $\boldsymbol{R}^n\backslash 0\ni(\omega, \theta)\to f(\omega)\theta\in T^*_yX$ is locally a diffeomorphism. For any $y\in X$ denote $J(\omega, \theta)=|D(f(\omega)\theta)/D(\omega, \theta)|$ and $\omega(x)=f^{-1}\left(-\frac{\partial T}{\partial y}(x, y)\right)$. Then using the chain rule we obtain

$$\left|D\left(-\frac{\partial T}{\partial y}\theta,\ T\right)/D(x, \theta)|\right|_{/\theta=1}=|D(f(\omega)\theta,\ T)/D(\omega, \theta, T)|_{\substack{/\theta=1\\ \omega=\omega(x)}}\times$$

$$\times|D(\omega(x), \theta, T(x, y))/D(x, \theta)|_{/\theta=1}=J(\omega, 1)\,|D(\omega(x), T(x, y))/D(x)|.$$

On the other hand

$$\left|D\left(\frac{\partial}{\partial y}T^2(x, y)\right)\Big/D(x)\right|=2^n|D(f(\omega)\theta)/D/\omega, \theta)|_{\substack{/\theta=T(x,y)\\ \omega=\omega(x)}}\times$$

$$\times|D(\omega(x), T(x, y))/D(x)|=2^nJ(\omega, T(x, y))\,|D(\omega(x), T(x, y))/D(x)|.$$

Since $J(\omega, T)=T^{n-1}J(\omega, 1)$ it follows that

$$\sqrt{d_\Sigma}=2^{n/2}\theta^{(1-n)/2}T(x, y)^{(n-1/2}|\partial^2(T^2)/\partial x\partial y|^{-1/2}\times$$

$$\times Q^*(|dt|^{1/2}\otimes|dx\wedge d\eta|^{1/2})$$

Translating $a_{j,0}\theta^{(n-1)/2}\sqrt{d_\Sigma}$ via the map Q_1^{-1} one obtains a half-density on C which is equal to

$$a_{j,0}T^{(n-1))2}2^{n/2}|\partial^2(T^2)/\partial x\partial y|^{-1/2}\mathfrak{S}_1$$

According to Hörmander [13] the last density multiplied by $i^{\alpha+(1-n)/2}\mathfrak{S}_2$ is equal to the principal symbol $\tilde{a}(\mathfrak{S}_1\otimes\mathfrak{S}_2)$ of the Fourier distribution I for some $\alpha\in \boldsymbol{Z}/4$, hence (3.3) holds. Thus Proposition 3.1. is proved.

§4. Asymptotics of green's functions

In this section we investigate the asymptotics of $G^{\pm}(\lambda;\, x,\, y)$ as $\lambda\to\infty$ in both cases $X=\boldsymbol{R}^n$ and $X\neq\boldsymbol{R}^n$. First we suppose that H is a self-adjoint operator in $L^2(X)$ ($X=\boldsymbol{R}^n$ is provided with the metric g_{ij}) having the form $H=-\Delta_g+h(x,\, D)+V(x)$, where $h(x,\, D)=\sum_{j=1}^{n} h_j(x)D_j$, $h_j\in C_0(\boldsymbol{R}^n)$, $V\in C_0(\boldsymbol{R}^n)$. Let us assume that the following two conditions are fulfilled. First the metric g_{ij} is supposed to be non-trapping, i.e.

(N) Any geodesic $\boldsymbol{R}^1\in t\to\exp_y(tv)$, $(x,\, v)\in T(X)$ goes to infinity as $t\to\infty$.

The second condition is

(C) The points x_0 and y_0, $(x_0,\, y_0)\in X\times X$, are not conjugated along any geodesic arc connecting them.

The latter condition meas that the differential $d_v(\exp_y v)_{/v=v_0}$ is not degenerated at any point $v_0\in T_yX$ such that $\exp_y v_0=x_0$ ([15]).

Let $\boldsymbol{R}^1\ni t\to\phi^t(y,\, \eta)=(x(t,\, y,\, \eta),\, \xi(t,\, y,\, \eta))$ be the projection of the bicharacteristic strip $(t,\, x(t,\, y,\, \eta),\, \mp q(y,\, \eta),\, \xi(t,\, y,\, \eta))$, $q(y,\, \eta)=\sqrt{g^{ij}(y)\eta_i\eta_j}$ of the operator D_t^2-H into $T^*(X)$, $\phi^0(y,\, \eta)=(y,\, \eta)$. Since for any $(t,\, y,\, v)\in\boldsymbol{R}^1\times T(X)$, $v\in S_y=\{v\in T_yX;g_{ij}v^iv^j=1\}$ we have $x(t,\, y,\, v^*)=\exp_v(tv)$ with v^* given by the can onical isomorphism $T_yX\ni v\to v^*=(g_{1j}(y)v^j,\dots,\, g_{nj}(y)v^j)\in T_yX$, the condition (C) is equivalent to

$$\operatorname{rank}(d_{t,\eta}x)(t,\, y_0,\, \eta)=n \tag{4.1}$$

for any $t\neq 0$, $\eta\neq 0$ such that $x(t,\, y_0,\, \eta)=x_0$.

Denote by $G^{\pm}(\lambda;\, x,\, y)=G(\lambda\pm i0,\, x,\, y)$ the Green's functions of the operator H in $X=\boldsymbol{R}^n$; $G(z;\, x,\, y)$ being the distribution kernel of the resolvent $(H-z)^{-1}$. To obtain the asymptotics of $G^{\pm}(\lambda;\, x,\, y)$ as $\lambda\to\infty$ we use essentially the results of the previous section. Set

$$r^t(y,\, \eta)=\phi^t(y,\, \eta),\quad (y,\, \eta)\in T^*X,$$

and denote by $T_j(x,\, y)$ and $\eta_j(x,\, y)\in S_y^*$, $j=1,\dots,\, J$, respectively the length and the unit codirection of the geodesic $[0,\, T_j(x,\, y)]\ni t\to\gamma_j(t)=x(t,\, y,\, \eta_j(x,\, y))$ connecting the points x and y for $(x,\, y)$ close to $(x_0,\, y_0)$. Choose the functions χ_j, $j=1,\, 2$ as in Theorem 1 with r_0 replaced by r_1 and denote

$$b_j(x,\, y)=\exp\Big(-\frac{i}{2}\int_0^{T_j(x,y)} h(\phi^s(y,\, \eta_j(x,\, y)))ds\Big),\quad j=1,\dots,\, J.$$

Theorem 3. *Suppose the condition* (N) *fulfilled and* (C) *holds at* $(x_0,\, y_0)\in\boldsymbol{R}^{2n}$. *Then* (0.4) *and* (0.5) *are satisfied. The coefficients* $U_0(x,\, y)$ *and* $C_{0,j}^{+}(x,\, y)$ *are given by multiplying* (0.6) *and* (0.7) *by* $b_j(x,\, y)$. *Moreover* $\beta_j=0$ *and the number* α_j *coinsides with the number of the points* $t\in(0,\, T_j(x,\, y))$ *conjugated with* $t=0$ *along the curve* $(0,\, T_j)\ni t\to\gamma_j(t)$ *and counted with their multiplicity.*

Proof. Arguing as in §1 one can prove that (1.12) holds with W solving the Cauchy problem

$$\begin{cases} (D_t^2 - H)W(t, x, y) = 0 \quad \text{in} \quad \boldsymbol{R}^{2n+1} \\ W(0, x, y) = \delta_y(x), \quad W_t(0, x, y) = 0 \end{cases}$$

where $(\delta_y(x), u(x)g(x)^{1/2}) = u(y)$. Since the asymptotics of the first integral in (1.12) is given by (2.4) it is enough to investigate

$$G_2^{\pm}(\lambda; x, y) = i\lambda^{-1} \int_0^{\infty} e^{i\lambda t} \varphi_2(t) W(t, x, t) dt$$

as $\lambda \to \infty$. To apply the results of §3 we suppose that W is a half-density rather than function in X and that the operator H acts on half-densities. Such a half-density $W_{1/2}$ can be obtained by multiplying W by the canonical half-density $g(y)^{1/4} g(y)^{1/4}$ on X. The correspoinding to H operator acting on half-densities is $H_{1/2} = g^{1/4} H g^{-1/4}$. Then $W_{1/2}$ is a sum of two Fourier distributions $W_{1/2} = I^+ + I^-$, $I^{\pm} \in I^{-1/4}(R^1 \times X \times X; C^{\pm})$ where

$$C^{\pm} = \{(t, x, y; \tau, \xi, -\eta); \phi^t(y, \eta) = (x, \xi), \tau \pm q(y, \eta) = 0\}.$$

The integral

$$\int_0^{\infty} e^{i\lambda t} \varphi_2(t) I^-(t, x, y) dt$$

is rapidly decreasing with respect to λ as $\lambda \to \infty$ uniformly with respect to (x, y) in a compact neighbourhood of (x_0, y_0) since $WF(I^-) \subset C^- \subset \{(t, x, y, \tau, \xi, \eta); \tau > 0\}$.

Consider the half-density

$$I_2(\lambda, x, y) = i\lambda^{-1} \int_0^{\infty} e^{i\lambda t} \varphi_2(t) I^+(t, x, y) dt$$

for (x, y) close to (x_0, y_0). According to (3.2)

$$I_2(\lambda, x, y) = (2\pi)^{-(n+1)/2} i^{(3-n/2} \lambda^{(n-1)/2}$$
$$\times \left\{ \sum_{j=1}^{J} e^{i\lambda T_j(x,y)} \int_0^{\infty} \int_0^{\infty} e^{i[\lambda(\theta-1)(T_j-t) - \alpha_j \pi/2]} \varphi_2(t) a_j(x, y, \lambda\theta) \theta^{\frac{n-1}{2}} d\theta \right\}$$
$$+ r_2(\lambda, x, y)$$

where the function $r_2(\lambda, x, y)$ aπd its derivatives are rapidly decreasing as $\lambda \to \infty$. The stationary phase method yields

$$I_2(\lambda, x, y) = (2\pi)^{(1-n)/2} i^{(3-n)/2 - \alpha_j} \lambda^{(n-3)/2} \tag{4.2}$$
$$\times \sum_{j=1}^{J} \sum_{v=0}^{M-1} e^{i\lambda T_j(x,y)} b_{v,j}(x, y) \lambda^{-j} + R_M$$

and R_M satisfies (0.5), which proves (0.4). What we have to do is to compute the first coefficients $b_{0,j}(x, y)$. First we define a nowhere vanishing section $\mathfrak{S}_2$ of

$L(C^+)$ (see [7]). The manifold $C^+ \cap \{t=0\}$ is the conormal bundle $N^*\Delta$ of the diagonal Δ in $X \times X$. Moreover $L(N^*\Delta)$ possesses a canonical constant section $\mathfrak{S}_2$ such that the Keller-Maslov part of the principal symbol of $\delta_y(x)$ is equal to $\mathfrak{S}_2$. Now we extend the section $\mathfrak{S}_2$ to a global one on C^+ requiring it to be a constant along the bicharacteristics $t \to \{(t, x, y; \tau, \xi, -\eta);\ \tau + q(y, \eta) = 0,\ (x, \xi) = \phi^t(y, \eta)\}$. As in §3 we write the principal symbol of I^+ in the form $a = \tilde{a}(\mathfrak{S}_1 \otimes \mathfrak{S}_2)$. Then

$$(\mathscr{L}_{\frac{\partial}{\partial t}} - \mathscr{L}_{H_q} + 2i \operatorname{sub}(H_{1/2}))a = 0$$

where $\mathscr{L}_{H_q}$ is the Lie derivative along the Hamiltonian field H_q and $\operatorname{sub}(H_{1/2}) = h(x, \xi)$ is the sub-principal symbol of H. In view of the choice of $\mathfrak{S}_1$ and $\mathfrak{S}_2$ the transport equation gives

$$(\partial_t - H_q + 2ih(x, \xi))\tilde{a} = 0$$

while the initial conditions for W yield $a(0, y, y, -q, \eta, -\eta) = 1/2$. Therefore $\tilde{a} = \frac{1}{2} \exp\left(-i \int_0^t h(\phi^s(y, \eta))ds\right)$ and according to (3.3) we have

$$b_{0,j}(x,y) = 2^{-\frac{n+2}{2}} T_j(x, y)^{\frac{1-n}{2}} |\partial^2 T_j^2 / \partial x \partial y|^{1/2} \exp\left(-\frac{i}{2} \int_0^{T_j} h(\phi^s(y, \eta_j(x, y)))ds\right).$$

Multiplying (4.2) by the canonical density $g(x)^{-1/4} g(y)^{-1/4}$ of order $-1/2$ in $X \times X$ we get the asymptotics of $G_2^+(\lambda; x, y)$ as $\lambda \to \infty$. The first coefficients in this expression are

$$(2\pi)^{(1-n)/2} i^{-\alpha_j + (3-n)/2} b_{0,j}(x, y)(g(x)g(y))^{-1/4}.$$

The assertion concerning α_j was proved by Colin de Veridère in [5]. Thus (0.4) and (0.5) are proved for $G^+(\lambda; x, y)$ and the first coefficients are found. Arguing in the same way can obtain an asymptotic expansion for $G^-(\lambda; x, y)$ and for $\frac{de}{d\lambda}(\lambda; x, y)$. Thus Theorem 3 is proved.

Remark 4.1. It can be proved that the half-density $I_2(\lambda; x, y) = g(x)^{1/4} G_2^+(\lambda; x, y) g(y)^{1/4}$ is a global oscillatory function ([31], def. 1.3.2) without assuming the condition (C). Then the results in ([31], §4) may be used to obtain asymptotic expansions of $G^+(\lambda; x, y)$ as $\lambda \to \infty$ in some cases when (C) is not fulfilled.

Indeed, consider the Lagrange manifold

$$\wedge = \{(x, y; \xi, -\eta) \in T^*(X \times X);\ \xi \in S_x^*,\ \eta \in S_y^* \text{ and } (x, \xi) = \phi^t(y, \eta) \text{ for some } t \in \boldsymbol{R}^1\}$$

which can be viewed as a flow-out of the isotropic manifold $\wedge_0 = \{(y, y; \eta, -\eta);\ \eta \in S_y^*\}$ along the bicharacteristic strips of $q(y, \eta)$. Denote by $i\colon \wedge \hookrightarrow T^*(X \times X)$ the inclusion map.

Proposition 4.1. *$I_2(\lambda; x, y)$ is a global oscillatory function in $\boldsymbol{R}^{2n}$ of order $(n-3)/2$ defined by the Lagrange immersion i* ([31], §1.3).

Proof. First note that C^+ is generated near any $\rho^0 = (t_0. x_0, y_0; \tau^0, \xi^0, \eta^0) \in C^+$ by a non-degenerated phase function of the form

$$\phi(t, x, y, \theta)=(\psi(x, y, \theta/|\theta|)-t)|\theta|, \quad \theta\in \boldsymbol{R}^{k+1}, k\geq 0,$$

where $\psi(x, y, \alpha)$, $\alpha\in S^k=\{\alpha\in \boldsymbol{R}^{k+1};\ |\alpha|=1\}$ is a non-degenerated phase function such that $i\psi$ defines i near $p^0=(x_0, y_0;\ \xi^0/|\tau^0|, \eta^0/|\tau^0|)$ (for definition see [31], §1). Indeed, suppose that $\phi(t, x, y, \theta)$, $\theta\in \boldsymbol{R}^{k+1}$ is a non-degenerated phase function at some point $q^0=(t_0, x_0, y_0, \theta^0)\in\Sigma_\phi$ $\Sigma_\phi=\{q;\ q$ is close to q^0 and $d_\theta\phi=0\}$ which defines C^+ near ρ^0, i.e. rank $d_{(t,x,y,\theta)}d_\theta\phi(q^0)=k+1$ and $C_\phi\subset C^+$ where $C_\phi=i_\phi(\Sigma_\phi)$ and $i_\phi(t, x, y, \theta)=(t, x, y;\ d_t\phi, d_x\phi, d_y\phi)$, $i_\phi(q^0)=\rho^0$, (Σ_ϕ is an $2n+1$ dimensional manifold and i_ϕ: $\Sigma_\phi\to T^*(R^{2n+1})$ is an embedding, see [31], p. 214). Therefore $dt\phi(q^0)\neq 0$ and there exist some smooth functions $\psi(x, y, \alpha)$ $a(t, x, y, \alpha)$, $a\in S^k$ such that $\phi(q)=\phi_1(q)a_1(q)$ where $\phi_1(t, x, y, \theta)=|\theta|(\psi(x, y, \theta/|\theta|)-t)$ and $a_1(t, x, y, \theta)=a(t, x, y, \theta/|\theta|)>0$ in a conic neighbourhoof of q^0. Moreover $\Sigma_\phi=\Sigma_{\phi_1}$ and $i_\phi(q)=i_{\phi_1}(t, x, y, \theta/a_1(q))$ for any $q=(t, x, y, \theta)\in\Sigma_\phi$ close to q^0. Therefore ϕ_1 is a non-degenerated phase function which defines C^+ near ρ^0. As a consequence $\psi(x, y, \alpha)$, $\alpha\in S^k$ is a non-degenerated phase function near (x_0, y_0, α_0), $\alpha_0=\theta^0/|\theta^0|$. Moreover, i_ψ defines i near p^0 since $i_\psi(x, y, \alpha)=\pi_1(i_{\phi_1}(\psi(x, y, \alpha), x, y, \alpha))\in\Lambda$ for any $(x, y, \alpha)\in\Sigma_\psi$ where $\pi_1(t, x, y;\ \tau, \xi, \eta)=(x, y;\ \xi/|\tau|, \eta/|\tau|)$ and π_1 maps C^+ into Λ.

Let K be a compact in $\boldsymbol{R}^{2n}$. Choose some non-degenerated phase functions $\psi_j(x, y, \alpha)$, $\alpha\in S^{k_j}$, $k_j\geq 0$, $j=1,\dots, J$, such that i_{ψ_j} defines i: $\Lambda\hookrightarrow T^*(\boldsymbol{R}^{2n})$ near some point p^j and $\phi_j(t, x, y, \theta)=|\theta|(\psi_j(t, x, y, \theta/|\theta|)-t)$ defines C^+ near some point $\rho^j\in C^+$, and

$$\Big(C^+\setminus\bigcup_{j=1}^{J} i_{\phi_j}(\Sigma_{\phi_j})\Big)\cap\pi_2^{-1}(\operatorname{supp}(\varphi_1)\times K)=\emptyset, \quad \pi_2(t, x, y;\ \tau, \xi, \eta)=(t, x, y).$$

Since $I^+\in I^{-1/4}(\boldsymbol{R}^1\times X\times X;\ C^+)$ there exist some amplitudes $a_j\in S^{(n-k_j-1)/2}(\boldsymbol{R}^{2n+1}\times\boldsymbol{R}^{k_j+1})$ such that

$$I^+(t, x, y)=\sum_{j=1}^{J}\int_{\boldsymbol{R}^{k_j+1}} e^{i\phi_j(t,x,y,\theta)}a_j(t, x, y, \theta)d\theta+Q(t, x, y)$$

and singsupp $Q\cap(\operatorname{supp}(\varphi_1)\times K)=\emptyset$. Using again the stationary phase method we obtain

$$I_2(\lambda, x, y)=\sum_{j=1}^{J}\left(\frac{\lambda}{2\pi}\right)^{k_j/2}\int_{S^{k_j}} e^{i\lambda\psi_j(x,y,\alpha)}b(x, y, \alpha, \lambda)d\alpha$$

where $b_j(x, y, \alpha, \lambda)\sim\sum_{\nu=0}^{\infty} b_{j,\nu}(x, y, \alpha)\lambda^{(n-3)/2-\nu}$ as $\lambda\to\infty$. Therefore $I_2(\lambda;\ x, y)$ is a global oscillatory function of order $(n-3)/2$ defined by the Lagrange immersion i: $\Lambda\to T^*(\boldsymbol{R}^{2n})$.

Now we turn to the case when X is a domain in $\boldsymbol{R}^n$ with a smooth boundary Y.

Proof of Theorem 1. First suppose $x_0\notin Il(y_0)$. Then there are no generalized geodesics connecting the points x_0 and y_0, in particular $x_0\neq y_0$. According to the propagation of singularities of the solution $U(t, x, y)$ to the mixed problem (0.9) we have $(t, x_0, y_0)\notin$ singsupp U for any $t\in\boldsymbol{R}^1$. Moreover $U(t, x, y)=0$ in a neighbourhood of $(0, x_0, y_0)$ since the finite propagation speed for the solutions of the wave equation $(D_t^2-H)U=0$. Then the function $\varphi(t)Y(t)U(t, x, y)$ is smooth in a neighbourhood of $\boldsymbol{R}^1\times\{x_0\}\times\{y_0\}$ and has a compact support with respect to t. Now (i) follows from Proposition 1.1.

Now suppose $x_0\in Il(y_0)$. The asymptotic behaviour of the first integral in (1.12) is given by (2.4). Thus we have to investigate the asymptotics of $G_2^+(\lambda;, x, y)$ as $\lambda\to\infty$.

Let γ: $[0, T_0]\ni t\to x(t, y_0, \eta_0)$ be a generalized geodesic connecting the points x_0 and y_0. Suppose that γ meets the boundary Y only transversally at the times $t=t_s$, $s=1,\dots, k$, $k=0$, where $0<t_1<t_2<\cdots<t_k<T_0$. Consider the broken Hamiltonian flow $\phi^t(y, \eta)=(x(t, y, \eta), \xi(t, y, \eta))$ associated with $\tau^2-g^{ij}(x)\xi_i\xi_j$ in $T^*(X)$ for (t, x, η) close to (T_0, y_0, η_0). Then $q(\phi^t(y, \eta)=q(y, \eta)$ for $0\le t\le T_0$ and (y, η) in a nieghbourhoof of (y_0, η_0). Let $\tilde{\phi}^t(y, \eta)=(x(t, y, \eta), \xi(t, y, \eta))$ be the Hamiltonian flow associated with $\tau^2-g^{ij}(x)\xi_i\xi_j$ in $T^*(\boldsymbol{R}^n)$ t is the dual variable to τ). For (y, η) close to (y_0, η_0) we denote

$$r_k^t(y, \eta)=\tilde{\phi}^{t-T_0}(x(T_0, y, \eta), \xi(T_0, y, \eta)).$$

It was proved in ([10], Proposition 3.7) that r_k^t is a canonical transformation for any $t\in\boldsymbol{R}^1$. Moreover, $r_k^t(y, \eta)=\phi^t(y, \eta)$ for (t, y, η) sufficiently close to (T_0, y_0, η_0), hence $q(r_k^t(y, \eta))=q(y, \eta)$ and

$$C_k^{\pm}=\{(t, x, y;\, \tau, \xi, \eta)\in T^*(\boldsymbol{R}^{n+1}\times X)\backslash 0;\ \tau\pm q(y, \eta)=0,\ (x, \xi)=r_k^t(y, \eta)\}$$

is a Lagrange manifold in $T^*(\boldsymbol{R}^{n+1}\times X)$.

According to Theorem 4.1 in [16] there exist some Fourier distributions $V^{\pm}\in I^{-1/4}(\boldsymbol{R}^{n+1}\times X;\, C_k^{\pm})$ such that the solution $W(t, x, y)$ of (1.1) can be written as a sum $W=V^++V^-$ microlocally at the points $\rho^{\pm}=(T_0, x_0, y_0;\, \mp q(y_0, \eta_0), \xi(T_0, y_0, \eta_0), -\eta_0)\in C_k^+$, i.e.

$$\rho^{\pm}\notin WF(W-V^+-V^-).$$

Let the conditions (N) and (C) be fulfilled. Then the equation $x(t, y, \eta)=x$ has only finitely many solutions $(T_j(x, y), \eta_j(x, y))\in(\boldsymbol{R}^1\backslash 0)\times S_y^*$ in a neighbourhood of (x_0, y_0), $j=1,\dots, J$. Moreover T_j and η_j are smooth functions near (x_0, y_0). Denote by $r_{k_j}^t$ and $C_{k_j}^+$ the corresponding canonical transformations and Lagrange manifolds. Then there exist some Fourier distributions $V_j^+\in I^{-1/4}(\boldsymbol{R}^{n+1}\times X;\, C_{k_j}^+)$, $j=1,\dots, J$, whose symbols vanish outside a small conic neighoburhood of the points $\rho_j^+=(T_j(x_0, y_0), x_0, y_0;\, -1, \xi_j(T_j, y_0, \eta_j), -\eta_j(x_0, y_0))$ and such that the distribution $W-\sum_{j=1}^{J} V_j^+$ is microlocally smooth at $(t, x_0, y_0;\, -1, \xi, \eta)$ for any $t\in\boldsymbol{R}^1\backslash 0$ and any $\xi, \eta\in\boldsymbol{R}^n$. Moreover the projection of $r_{k_j}^t(y, \eta)$ on the base $\boldsymbol{R}^n$ coincides with the curve $x(t, y, \eta)$ for (t, y, η) close to $(T_j, y_0, \eta_j(x_0, y_0))$, so the condition (3.1) holds for $r_{k_j}^t$. Now we apply Proposition 3.1 in the same way as it was done in the proof of theorem 3. The distribution V_j^+ can be written in the form

$$V_j^+=(2\pi)^{-(n+1)/2}i^{(1-n)/2}\int_0^{\infty} e^{i[\theta(T_j(x,y)-t)+\alpha_j\pi/2]}\, a_j(x, y, \theta)\theta^{\frac{n-1}{2}}d\theta$$

where a_j satisfies (3.3). Moreover corollary 4.3 in [10] gives $\tilde{a}_j=1/2$ for the "mixed" Neumann problem and $\tilde{a}_j=(-1)^{k_j}/2$ for the Dirichlet problem, where k_j is the number of reflections of the geodesic γ_j: $[0, T_j(x_0, y_0)]\ni t\to x(t, y_0, \eta_j(x_0, y_0))$ at the boundary. The number α_j can be computed in the same way as in [7], [10], [20]. Denote $\delta_j(t)=(d_{y,\eta}\phi^t)^{-1}(V)$, $t\in(0, T_j(x_0, y_0))$ where the differential is taσen at the point (x, ξ), $x=x(t, y_0, \eta_j(x_0, y_0))$, $\xi=\xi(t, y_0, \eta_j(x_0, y_0))$ aπd $V=\{(0, \delta\xi)\in T_{(x,\xi)}TX\}$ is the vertical space in $T_{(x,\xi)}(TX)$. Following closely the arguments in [7], [10], [20] one can prove that $\alpha_j=ind(\delta_j)$ (the index $ind(\delta_j)$ of the curve δ_j was defined in [6], [13].

Arguing in the same way one can obtain (0.4) and (0.5) for $G^-(\lambda; x, y)$. The equality $C^-_{0,j}(x, y)=\overline{C^+_{0,j}(x, y)}$ holds since $G^-(\lambda; x, y)=\overline{G^+(\lambda; x, y)}$ for $x\neq y$ and the functions $C^\pm_{0,j}(x, y)$ are smooth for (x, y) close to (x_0, y_0). Thus theorem 1 is proved.

Proof of Theorem 2. The function $\frac{de}{d\lambda}(\lambda^2; x, y)$ is smooth in $\boldsymbol{R}^1_+\times X\times X$ in view of (1.15)and since for any $\varphi\in C^\infty_0(\boldsymbol{R}^1)$ the function $X\times X\ni(x, y)\to\langle W(\cdot, x, y), \varphi\rangle$ belongs to $C^\infty(X\times X)$ (here the partial hypoellipticity of the wave operater D_t^2-H with respect to x is used.) Moreover, (i) follows directly from Theorem 1. To prove (ii) we write

$$\frac{de}{d\lambda}(\lambda^2, x, y)=(2\pi i)^{-1}\{G_1^+-G_1^-\}+(2\pi i)^{-1}\{G_2^+-G_2^-\}.$$

Then $(2\pi i)^{-1}\{G_1^+-G_1^-\}$ gives the first expression in (0.8) in view of (2.5) while $(2\pi i)^{-1}\cdot\{G_2^+-G_2^-\}$ gives the second one. This completes the proof of Theorem 2.

§5. Some other results

In this section we investigate the asymptotics of the spectral function $e(\lambda; x_0, y_0)$ at some points (x_0, y_0) which do not satisfy the condition (C). We still suppose that the condition (T) holds at $(x_0, y_0)\in X\times X$.

Consider the set $W=\{(t, \eta)\in\overline{\boldsymbol{R}^1_+}\times T_yX;\ x(t, y_0, \eta)=x_0\}$ and for any $(t, \eta)\in W$ denote by $P_{t,\eta}$ the linear map

$$P_{t,\eta}=(d_{t,\eta}x)(t, y_0, \eta),\quad P_{t,\eta}\colon T_{(t,\eta)}(\boldsymbol{R}^1\times T^*_{y_0}X)\longrightarrow T_{x_0}X.$$

Instead of (C) we impose the following condition

(5.1) There exist some reals $T_{d,j}\geq 0$ and some smooth conic manifolds $\tilde{W}_{d,j}\subset T_{y_0}X$ of dimention $d+1$, $0\leq d\leq n-1$, $j=1,\dots, J$, such that W is a disjoint union of $W_{d,j}=\{T_{d,j}\}\times\tilde{W}_{d,j}$ and $T_{(t,\eta)}W_{d,j}=\operatorname{Ker}(P_{t,\eta})$ at any point $(t, \eta)\in W_{d,j}$.

For exmaple consider a domain iπ $\boldsymbol{R}^n$ with smooth boundary Y which contains a part of a circle cone $\Gamma=\{(s, \omega);\ \omega\in S^{n-2},\ a\leq s\leq b\}$ and suppose that $x_0=y_0$ lies on the cone axis of Γ. Then W contains the $n-1$ dimensional conic manifold $\tilde{W}_{n-2,0}=\{\tau\omega;\ \omega\in S^{n-2},\ \tau>0\}$.

Theorem 4. *Suppose that* (N) *is valid and* (T) *and* (5.1) *are satisfied at the point* (x_0, y_0). *Then*

$$\frac{de}{d\lambda}(\lambda^2, x_0, y_0) \sim \sum_{j,\alpha} \sum_{v=0}^{\infty} \lambda^{(n+d-3)/2-v} \operatorname{Re}(C_{v,d,j} e^{i\lambda T_{d,j}})$$

for some complex number $C_{v,d,j}$ *and* $C_{0,d,j} \neq 0$ *when* $W_{d,j} \neq \emptyset$.

Proof. Denote by i_{x_0,y_0}: $\boldsymbol{R}^1 \to \boldsymbol{R}^1 \times X \times X$ the inclusion map $i_{x_0,y_0}(t) = (t, x_0, y_0)$. Let $(i_{x_0}, {}_{y_0})^*$ be the corresponding "pull-back" map (see [13]). It is well known that the composition $\mathfrak{S}(t) = (i_{x_0,y_0})^* W(t)$ is a well defined distribution when $W(t, x, y)$ solves (1.11). Moreover (1.13) can be written in the form

$$\begin{aligned}\frac{de}{d\lambda}(\lambda^2, x_0, y_0) &= i\lambda^{-1}(i_{x_0}, {}_{y_0})^*\Big(\int_{-\infty}^{\infty} e^{i\lambda t}\varphi(t)W(t, x, y)dt\Big) + 0(\lambda^{-N}) \\ &= i\lambda^{-1}\int_{-\infty}^{\infty} e^{i\lambda t}\varphi(t)\mathfrak{S}(t)dt + 0(\lambda^{-N}).\end{aligned}$$

Note that $(i_{x_0,y_0})^*$ is a Fourier integral operator with a canonical relation $D = \{(t, \tau; t, x_0, y_0, \tau, \xi, \eta), \tau \neq 0\}$ and $(i_{x_0,y_0})^* \in I^{n/2}(\boldsymbol{R}^1, \boldsymbol{R}^1 \times X \times X; D)$. For given $\eta_0 \in \tilde{W}_{d,j}$ and for (t, y, η) close to $(T_{d,j}, y_0, \eta_0)$ we shall denote $r^t_{d,j}(y, \eta) = \tilde{\Phi}^{t-T_{d,j}}(x(T_{d,j}, y, \eta), \xi(T_{d,j}, y, \eta))$. Let $C^+_{d,j}$ be the corresponding Lagrange manifold defined in §4.

Proposition 5.1. *The canonical relation D and the Lagrange manifold* $C^+_{d,j}$ *have a clean composition* $\Lambda_{d,j} = D_0 C^+_{d,j}$ *with excess equal to d and* $\Lambda_{d,j} = \{(T_{d,j}, \tau), \tau < 0\}$.

Proof. Following the notations in [26] we denote $L = D \times C^+_{d,j}$, $M = (T^*\boldsymbol{R}^1 \backslash 0) \times \Delta^*$, where Δ^* is the diagonal in $T^*(\boldsymbol{R}^1 \times \boldsymbol{R}^n \times X) \times T^*(\boldsymbol{R}^1 \times \boldsymbol{R}^n \times X)$. Let Π be the projection Π: $T^*(\boldsymbol{R}^1) \times T^*(\boldsymbol{R}^1 \times \boldsymbol{R}^n \times X) \times T^*(\boldsymbol{R}^1 \times \boldsymbol{R}^n \times X) \to \boldsymbol{R}^1 \times T^*_y X$ defined by $\Pi((s; \mu), (t, x, y; \tau, \xi, \eta), (\tilde{t}, \tilde{x}, \tilde{y}; \tilde{\tau}, \tilde{\xi}, \tilde{\eta})) = (t, \eta) \in \boldsymbol{R}^1 \times T^*_y X$. It is easy to see that the map Π_*: $T_\rho(L \cap M) \to T_\eta(W_{d,j})$ is an isomorphism at any point $\rho = ((t; \tau), (t, x_0, y_0; \tau, \xi, \eta), (t, x_0, y_0; \tau, \xi, \eta))$ where $t = T_{d,j}$, $x(t, y_0, \eta) = x_0$, $\xi(t, y_0, \eta) = \xi$ and $\tau + q(y, \eta) = 0$. Moreover Π_*: $(T_\rho L) \cap (T_\rho M) \to \operatorname{Ker} P_{t,\eta}$, $t = T_{d,j}$, is an isomorphism and in view of (5.1) we get $T_\rho(L \cap M) = T_\rho(L) \cap T_\rho(M)$. Therefore D and $C^+_{d,j}$ have a clean composition. The excess coincides with the dimension of the fibres of the map $L \cap M \in \rho \to (t, \tau) \in \Lambda_{d,j}$ which is equal to $\dim W_{d,j} - 1 = d$. This proves the proposition.

The theorem about the composition of Fourier integral operators [26] gives now

$$\mathfrak{S}(t) = \sum_{d=0}^{n-1} \sum_{j=1}^{J} I_{d,j}, \quad I_{d,j} \in I^{(n+d)/2+1/4}(R^1; \Lambda_{d,j}).$$

Since the function $\varphi(t, \theta) = \theta(T_{d,j} - t)$, $(t, \theta) \in T^*\boldsymbol{R}^1$, $\theta > 0$ defines $\Lambda_{d,j}$ we have

$$I_{d,j}(t) = \int_0^{\infty} e^{i\theta(T_{d,j}-t)} g_{d,j}(\theta) d\theta$$

where supp $g_{d,j} \subset \boldsymbol{R}^1_+$ and $g_{d,j} \sim \sum_{\nu=0}^{\infty} g_{\nu,d,j} \theta^{(n+d-1-2\nu)/2}$, $\theta \to \infty$, for some complex numbers $g_{\nu,d,j}$. Moreover $g_{0,d,j} \neq 0$ since the principal symbol of W does not vanish on $C^+_{d,j}$(W is a Fourier distribution near $C^+_{d,j}$). This proves the theorem.

Note that the regular part of $G^+(t; x, y)$ can be written as a sum of global oscillatory functions in $E = \{(x, y) \in X \times X;\ x \in Il(y)$ and (T) is fulfilled at $(x, y)\}$. Indeed, arguing as in Remark 4.1 we obtain

$$G^{\pm}(\lambda; x, y) \sim \chi_1(T(x, y)) \sum_{\nu=0}^{\infty} U_\nu(x, y) F^{\pm}_\nu(\lambda; x, y) + \chi_2(T(x, y)) \sum_{k=0}^{\infty} f^{\pm}_k(\lambda; x, y)$$

as $\lambda \to \infty$ in any compact $K \subset E$ if (N) is fulfilled. Here the functions $\chi_j(t)$, $T(x, y)$ are as in the introduction and $g(x)^{1/4} f^{\pm}_k(\lambda; x, y) g(y)^{1/4}$ is a global oscillatory fuction defined by the Lagrange immersion i_k: $\Lambda_k \hookrightarrow T^*(\boldsymbol{R}^n \times X)$ where

$$\Lambda_k = \{(x, y;\ \xi, -\eta) \in T^*(\boldsymbol{R}^n \times X);\ \xi \in S_x, \eta \in S_y, (x, \xi) = r^t_k(y, \eta) \text{ for some } t \in \boldsymbol{R}^1\}.$$

The second sum in the last asymptotic expression is finite in $(x, y) \in K$ since in view of (N) and (T) there exist only finitely many $k \in Z$ such that

$$K \cap \{(x, y);\ (x, y;\ \xi, \eta) \in \Lambda_k \text{ for some } (\xi, \eta)\} \neq \emptyset.$$

The half-density $f^{\pm}_k(\lambda; x, y)$ has an asymptotic expansion as $\lambda \to \infty$ similar to (4.2) near any point (x_0, y_0) which does not belong to the caustic set of Λ_k (the projection map $\Lambda_k \ni (x, y;\ \xi, \eta) \to (x, y) \in X \times X$ is a diffeomorphism near (x_0, y_0)). Moreover, some asymptotic expansions of $f^{\pm}_k(\lambda; x, y)$ near the caustic sets of Λ_k may be obtained using the results in ([31], §4).

Institute of Mathematics
Bulgarian Academy of Sciences
1090, Sofia Bulgaria

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