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The 5-primary homotopy exponent of the exceptional Lie group E_8

By

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Abstract

We construct a new homotopy fibration at the prime 5, involving E_8 and Harper's rank two finite mod-5 H-space. We then use this to show that the 5-primary homotopy exponent of E_8 is bounded above by 5^{31}, which is at most one power of 5 from being optimal.

1. Introduction

Let p be an odd prime. A *torsion Lie group* is a Lie group which has p-torsion in its integral cohomology. Among the classical Lie groups, the only torsion Lie groups are F_4, E_6, E_7, and E_8 at the prime 3, and E_8 at the prime 5. The *homotopy exponent* of a space X is the least power of p which annihilates the p-torsion in $\pi_*(X)$. We write this as $\exp(X) = p^r$. In [T1] upper bounds were calculated for the homotopy exponents of F_4 and E_6 at 3 which equalled known lower bounds, thereby determining exact values for the homotopy exponents. The purpose of this paper is to consider the case of E_8 at 5.

Typically, the first step in exponent calculations is to decompose the space into a product of indecomposable factors and consider the exponents of each factor. In our case, Wilkerson [W] showed there is a 5-local equivalence $E_8 \simeq X \times Y$, where $H^*(X; \mathbf{Z}/5\mathbf{Z}) \cong \mathbf{Z}/5\mathbf{Z}[x_{12}]/(x_{12}^5) \otimes \Lambda(x_3, x_{11}, x_{27}, x_{35})$ and $H^*(Y; \mathbf{Z}/5\mathbf{Z}) \cong \Lambda(x_{15}, x_{23}, x_{39}, x_{47})$. This splitting of E_8 cannot be improved upon: Davis [D2] showed that X is indecomposable and Gonçalves [G] showed that Y is indecomposable. The usual second step in exponent calculations is to try to find suitable homotopy fibrations $F \xrightarrow{g} E \xrightarrow{f} B$ in which E is the space in question, exponent information is known both about the spaces F and B and the maps f and g. In our case, Y turns out to be spherically resolved and so is straightforward to deal with, but X is more subtle.

For some time it was thought that there was a further splitting, $X \simeq K_5 \times B(27, 35)$, where: (1) K_5 is Harper's rank two mod-5 H-space, satisfying

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$H^*(K_5;\mathbf{Z}/5\mathbf{Z}) \cong \mathbf{Z}/5\mathbf{Z}[x_{12}]/(x_{12}^5) \otimes \Lambda(x_3, x_{11})$, and (2) $B(27,35)$ is spherically resolved by a homotopy fibration $S^{27} \longrightarrow B(27,35) \longrightarrow S^{35}$, satisfying $H^*(B(27,35);\mathbf{Z}/5\mathbf{Z}) \cong \Lambda(x_{27}, x_{35})$ with $\mathcal{P}^1(x_{27}) = x_{35}$. While X does not satisfy this further splitting, it is possible that there is a homotopy fibration $B(27,35) \longrightarrow X \longrightarrow K_5$. The existence of the right map is not known and we do not prove this. Instead, we construct the left map (fairly straightforward) and identify its homotopy fiber as ΩK_5 (which is not obvious). Specifically, we show:

Theorem 1.1. *The following hold*:

(a) *there is a homotopy fibration* $\Omega K_5 \longrightarrow B(27,35) \longrightarrow X$*,*

(b) *the space* Y *is spherically resolved, that is, there are homotopy fibrations*

$$Y_1 \longrightarrow Y \longrightarrow S^{47}, \quad Y_2 \longrightarrow Y_1 \longrightarrow S^{39}, \quad \textit{and} \quad S^{15} \longrightarrow Y_2 \longrightarrow S^{23}.$$

Using the homotopy fibrations in Theorem 1.1 and information about the homotopy exponents of spheres and of K_5, we show $\exp(X) \leq 5^{31}$ (Proposition 6.1) and $\exp(Y) \leq 5^{26}$ (Proposition 7.3). Consequently, we have:

Theorem 1.2. $\exp(E_8) \leq 5^{31}$. □

The upper bound in Theorem 1.2 is at most one power of 5 from being optimal. Davis [D2] has shown that there are elements of order 5^{30} in the homotopy groups of E_8. He did this by calculating the 5-primary v_1-periodic homotopy groups of E_8, which represent a certain subcollection of all of the homotopy groups of E_8. It would be interesting to know if there is any element in $\pi_*(E_8)$ whose order is 5^{31}, exceeding the orders of the elements represented by v_1-periodic homotopy.

An additional remark should be made. Ideally, one would like a homotopy fibration $B(27,35) \longrightarrow X \longrightarrow K_5$, so in this sense Theorem 1.1 (a) is a little unsatisfactory. On the other hand, the existence of such a fibration does not seem to help improve the exponent bound on X. The key to calculating the exponent bound on X is to advantageously factor (some loop of) the associated map $\Omega K_5 \longrightarrow B(27,35)$ (see Section 6 for details). The construction we use to prove Theorem 1.1 (a) produces such a factorization.

This paper is organized as follows. Sections 2, 3, and 4 are preliminary. The first describes a method for calculating upper bounds on exponents for certain fibrations, the second reviews the relevant properties of Harper's space K_p, and the third reviews the construction of an H-structure on $B(27,35)$. Section 5 constructs the homotopy fibration of Theorem 1.1 (a). Section 6 uses the homotopy fibration in Theorem 1.1 (a) to calculate an upper bound on the 5-primary homotopy exponent of X. Section 7 constructs the homotopy fibration of Theorem 1.1 (b) and calculates an upper bound on the 5-primary homotopy exponent of Y.

2. A method for computing upper bounds on exponents

In this section we outline a general method for calculating an upper bound on the homotopy exponent of spaces which arise as the total space in certain homotopy fibrations. The method is also described and applied in [T1].

If B is an H-space, the identity map can be multiplied by p^r to give a map $B \xrightarrow{p^r} B$. Let $B\{p^r\}$ be the homotopy fiber of this map. By [N], the homotopy exponent of $B\{p^r\}$ is p^r.

Lemma 2.1. *Suppose there is a homotopy fibration*

$$F \xrightarrow{f} E \xrightarrow{g} B,$$

where E and B are H-spaces. Suppose as well there is a map $B \xrightarrow{i} E$ such that $g \circ i \simeq p^r$. Then there is a homotopy fibration

$$\Omega F \times \Omega B \xrightarrow{\Omega f \cdot (-\Omega i)} \Omega E \longrightarrow B\{p^r\}.$$

Consequently, $\exp(E) \leq p^r \cdot \max(\exp(F), \exp(B))$.

Proof. The homotopy $g \circ i \simeq p^r$ results in a homotopy pullback

$$\begin{tikzcd} B\{p^r\} \arrow[r] \arrow[d] & B \arrow[r, "p^r"] \arrow[d, "i"] & B \arrow[d, equal] \\ F \arrow[r, "f"] & E \arrow[r, "g"] & B. \end{tikzcd}$$

Since E is an H-space we can multiply the maps f and $-i$. The pullback in the diagram above then results in a homotopy fibration

$$B\{p^r\} \longrightarrow F \times B \xrightarrow{f \cdot (-i)} E$$

which is analogous to a Mayer-Vietoris sequence. Continuing the homotopy fibration sequence two steps to the left gives the fibration stated in the lemma. The exponent bound immediately follows. □

Remark 1. Lemma 2.1 is typically applied when $B = S^{2n+1}$ or $B = \Omega S^{2n+1}$.

We now consider an example of Lemma 2.1 which later plays a role in our exponent calculations. Recall from [CMN] that at odd primes we have $\exp(S^{2n+1}) = p^n$.

Example 2.1. Let $q = 2(p-1)$. Let $\alpha_1 \in \pi^S_{q-1}(S^0)$ be a generator of the stable stem. Following Mimura and Toda [MT], for $m \geq 1$ define a space $B(2m+1, 2m+q+1)$ as the homotopy pullback

$$\begin{tikzcd} S^{2m+1} \arrow[r] \arrow[d, equal] & B(2m+1, 2m+q+1) \arrow[r, "q"] \arrow[d] & S^{2m+q+1} \arrow[d, "\alpha_1"] \\ S^{2m+1} \arrow[r] & S^{4m+3} \arrow[r, "w"] & S^{2m+2}, \end{tikzcd}$$

where w is the Whitehead product of the identity map on S^{2m} with itself. As α_1 has order p there is a characteristic map $i : S^{2m+q+1} \longrightarrow B(2m+1, 2m+q+1)$ satisfying $q \circ i \simeq p$. So by Lemma 2.1 we have $\exp(B(2m+1, 2m+q+1)) \leq p \cdot \exp(S^{2m+q+1}) = p^{m+p}$.

It is sometimes possible to improve on the upper bound given by Lemma 2.1.

Corollary 2.1. *With notation as in Lemma* 2.1*, suppose that* $\Omega f \cdot (-\Omega i)$ *has order* p^s*. If* $s < \max(\exp(X), \exp(Z))$ *then* $\exp(Y) \leq p^{r+s}$*.* $\square$

3. Harper's finite H-spaces

In [H] Harper constructs rank 2 finite H-spaces K_p, one for each odd prime p, satisfying

$$H^*(K_p; \mathbf{Z}/p\mathbf{Z}) = \Lambda(x_3, x_{2p+1}) \otimes \mathbf{Z}/p\mathbf{Z}[x_{2p+2}]/(x_{2p+2}^p)$$

with $\mathcal{P}^1(x_3) = x_{2p+1}$ and $\beta(x_{2p+1}) = x_{2p+2}$. The three-connected cover $K_p\langle 3 \rangle$ of K_p satisfies $H^*(K_p\langle 3 \rangle; \mathbf{Z}/p\mathbf{Z}) \cong \Lambda(x_{2p^2+1}, x_{2p^2+2p-1}) \otimes \mathbf{Z}/p\mathbf{Z}[x_{2p^2}]$, where $\beta(x_{2p^2}) = x_{2p^2+1}$ and $\mathcal{P}^1(x_{2p^2+1}) = x_{2p^2+2p-1}$.

This section review some properties of K_p which we will later make use of. Let $J_{p-1}(S^{2n})$ be the $(p-1)^{st}$-stage of the James construction on S^{2n}. Let $T : \Omega J_{p-1}(S^{2n}) \longrightarrow \Omega S^{2np-1}$ be the Toda map. Davis [D1], proving an unpublished result of Harper, showed that there is a homotopy fibration

$$B(3, 2p+1) \longrightarrow K_p \xrightarrow{\pi} J_{p-1}(S^{2p+1}),$$

where π^* is a monomorphism in cohomology and $H^*(B(3, 2p+1); \mathbf{Z}/p\mathbf{Z}) \cong \Lambda(x_3, x_{2p+1})$. Taking three-connected covers, looping, and composing with the Toda map gives a homotopy pullback

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & S^{2p+1} \arrow[r] \arrow[d] & B(3,2p+1)\langle 3 \rangle \arrow[d, equal] \\
\Omega K_p\langle 3 \rangle \arrow[r, "\Omega\pi"] \arrow[d, "\overline{T}"] & \Omega J_{p-1}(S^{2p+2}) \arrow[r] \arrow[d, "T"] & B(3,2p+1)\langle 3 \rangle \\
\Omega S^{p^2+2p-1} \arrow[r, equal] & \Omega S^{2p^2+2p-1},
\end{tikzcd}$$

where $\overline{T}$ is defined as the composite $T \circ \Omega\pi$, and the pullback defines the space M. The following three statements were proven in [T1].

Proposition 3.1. *The following hold*:

(a) *There is a homotopy commutative square*

$$\begin{tikzcd} \Omega S^{2p^2+2p-1} \arrow[r, "\Omega t"] \arrow[d, "p"] & \Omega K_p\langle 3\rangle \arrow[d, "\overline{T}"] \\ \Omega S^{2p^2+2p-1} \arrow[r, equal] & \Omega S^{2p^2+2p-1} \end{tikzcd}$$

for some map t,

(b) $\exp(K_p) = p^{p^2+p}$,

(c) $\exp(M) = p$. □

The space K_p also has the rare property that its homotopy type is determined by its cohomology.

Lemma 3.1. *Suppose L is a space with the property that $H^*(L;\mathbf{Z}/p\mathbf{Z}) \cong H^*(K_p;\mathbf{Z}/p\mathbf{Z})$. Then L is homotopy equivalent to K_p.*

Proof. We need to briefly recall Harper's construction of K_p [H]. Let N be an unstable module over the mod-p Steenrod algebra and let $U(N)$ be the free unstable module over the mod-p Steenrod algebra generated by N. Suppose L is a space such that $H^*(L;\mathbf{Z}/p\mathbf{Z}) \cong U(N)$. Then there is a Massey-Peterson tower which coverges to L. Here the (horizontal) tower has the form

$$\begin{tikzcd} \cdots \arrow[r] & P_3 \arrow[r] \arrow[d] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[d] \\ & G_3 & G_2 & G_1 & G_0, \end{tikzcd}$$

where the the G_i's are products of Eilenberg-MacLane spaces, and the sequences $P_i \longrightarrow P_{i-1} \longrightarrow G_{i-1}$ are homotopy fibrations. The sequence of homotopy fibrations are constructed based on a projective resolution of N, and the inverse limit of the tower is L.

In our case, let $N = \{x_3, x_{2p+1}, x_{2p+2}\}$ with $\mathcal{P}^1(x_3) = x_{2p+1}$ and $\beta(x_{2p+1}) = x_{2p+2}$. Let $T_p = U(N)$, so we have

$$T_p = \mathbf{Z}/p\mathbf{Z}[x_{2p+2}]/(x_{2p+2}^p) \otimes \Lambda(x_3, x_{2p+1}).$$

Note that T_p has dimension $2p^2+2p+2$. Harper first observes there is an epimorphism $\theta : H^*(K(\mathbf{Z},3);\mathbf{Z}/p\mathbf{Z}) \longrightarrow T_p$ which is an isomorphism through dimension $2p^2$. The tower then begins with $P_0 = K(\mathbf{Z},3)$ and G_0 the Eilenberg-MacLane space which represents the cohomology class of lowest dimension in the kernel of θ. It turns out that the $(2p^2+2p+2)$-skeleton of P_3 has cohomology isomorphic to T_p and the G_i's for $i \geq 3$ are more than $(2p^2+2p+2)$-connected. So the tower may as well cease at this point and the space K_p can be defined as the $(2p^2+2p+2)$-skeleton of P_3.

Now suppose L is a space with $H^*(L;\mathbf{Z}/p\mathbf{Z}) \cong H^*(K_p;\mathbf{Z}/p\mathbf{Z})$. Begin with a map $L \longrightarrow K(\mathbf{Z},3)$ which represents the three-dimensional generator in $H^*(L;\mathbf{Z}/p\mathbf{Z})$. This gives an initial map into the tower above at level P_0. The existence of a lift through each stage of the tower is determined by a cohomology calculation. The tower itself is determined by the unstable module T_p, so the isomorphism $H^*(L;\mathbf{Z}/p\mathbf{Z}) \cong H^*(K_p;\mathbf{Z}/p\mathbf{Z}) \cong T_p$ implies that the initial map $L \longrightarrow K(\mathbf{Z},3)$ will iteratively lift to a map $L \longrightarrow P_3$. Since L is $(2p^2+2p+2)$-dimensional, this lift factors through the $(2p^2+2p+2)$-skeleton of P_3, which is K_p. We therefore have a map $L \longrightarrow K_p$ which is an isomorphism in cohomology and is therefore a homotopy equivalence. $\square$

Finally, we record the homology of ΩK_p as calculated in [K].

Lemma 3.2. *There is an algebra isomorphism*

$$H_*(\Omega K_p) \cong \mathbf{Z}/p\mathbf{Z}[d_2, d_{2p}, d_{2p^2+2p-2}]/(d_2^p).$$

The action of the dual Steenrod algebra satisfies $\mathcal{P}^1_* d_{2p} = d_2$ and $\mathcal{P}^1_* d_{2p^2+2p-2} = d_{2p}^p$. $\square$

4. Cohen and Neisendorfer's construction of finite H-spaces

In [CN] Cohen and Neisendorfer give a construction of finite p-local H-spaces. As we will make use of this in detail for a particular example, we state their result in full and then apply it to the case of interest.

Theorem 4.1. *Fix an odd prime p. Let A be a p-local complex of l odd dimensional cells, where $l < p-1$. Then there is a homotopy fibration $B \longrightarrow R' \longrightarrow \Sigma A$ satisfying:*

(a) $\Omega\Sigma A \simeq B \times \Omega R'$,

(b) $H_*(B;\mathbf{Z}/p\mathbf{Z}) \cong \Lambda(x_{d_1},\ldots,x_{d_l})$,

(c) *the composite $A \xrightarrow{E} \Omega\Sigma A \longrightarrow B$ includes $\widetilde{H}_*(A;\mathbf{Z}/p\mathbf{Z})$ into $H_*(B;\mathbf{Z}/p\mathbf{Z})$ as the generating set of the exterior algebra.* $\square$

Note that Theorem 4.1 (a) implies B is an H-space, and dualizing part (b) gives $H^*(B;\mathbf{Z}/p\mathbf{Z}) \cong \Lambda(x_{d_1},\ldots,x_{d_l})$. Another feature of Theorem 4.1 is that if A is a suspension then R' is as well, say $R' \simeq \Sigma R$. In this case, [T2] shows that more can be said about the H-structure of B.

Theorem 4.2. *If the space A in Theorem 4.1 is a suspension and $l < p-2$, then B is a homotopy associative, homotopy commutative H-space and the homotopy fibration connecting map $\Omega\Sigma A \xrightarrow{r} B$ can be chosen to be an H-map.* $\square$

We need to say something about the number of cells of in R' and the dimensions in which they occur. A homological model for the homotopy fibration

$\Omega R' \longrightarrow \Omega\Sigma A \longrightarrow B$ is constructed in [CN] as follows. For a graded vector space V, let $L = L\langle V\rangle$ be the free Lie algebra generated by V. Let UL be the universal enveloping algebra; demanding that the elements of V are primitive gives UL the structure of a Hopf algebra. A sub-Lie algebra of a free Lie algebra is free, so $[L, L]$ is a free sub-Lie algebra of L. Let $L_{ab} = L/[L, L]$. With $V = \widetilde{H}_*(A)$, there is a short exact sequence of Hopf algebras

$$0 \longrightarrow U[L, L] \longrightarrow UL \longrightarrow UL_{ab} \longrightarrow 0$$

and a splitting $UL \cong UL_{ab} \otimes U[L, L]$ which is an isomorphism of right $U[L, L]$-modules and left UL_{ab}-comodules. This short exact sequence of universal enveloping algebras is a model for the sequence

$$H_*(\Omega R') \longrightarrow H_*(\Omega\Sigma A) \longrightarrow H_*(B).$$

In particular, the cells of R' are in one-to-one correspondence with a Lie basis for $[L, L]$, where the dimension of the cell in R' is one more than the dimension of the corresponding Lie element.

Example 4.1. The specific case we are interested in is when $p = 5$ and $l = 2$. Let $\alpha_1 \in \pi_7^S$ represent a generator of the stable 7-stem. Let $A = S^{27} \cup_{\alpha_1} e^{35}$. Applying Theorem 4.1, and observing that A is a suspension, gives a homotopy fibration

$$B(27, 35) \longrightarrow \Sigma R \longrightarrow \Sigma A,$$

where $B(27, 35)$ is an H-space satisfying $H^*(B(27, 35); \mathbf{Z}/5\mathbf{Z}) \cong \Lambda(x_{27}, x_{35})$. Since α_1 is detected by the Steenrod operation $\mathcal{P}^1$ we have $\mathcal{P}^1(x_{27}) = x_{35}$. We have $L = L\langle u_{27}, v_{35}\rangle$, and [CN] show that a Lie basis for the free Lie algebra $[L, L]$ is $\{[u, u], [u, v], [v, v], [u, [u, v]], [u, [v, v]]\}$. Thus R is a five-cell complex whose homology has a vector space basis $\{x_{54}, x_{62}, x_{70}, x_{89}, x_{97}\}$ with $\mathcal{P}^1(x_{54}) = x_{62}$, $\mathcal{P}^1(x_{62}) = x_{70}$, and $\mathcal{P}^1(x_{89}) = x_{97}$. Finally, Theorem 4.2 implies that the homotopy fibration connecting map $r : \Omega\Sigma A \longrightarrow B(27, 35)$ can be chosen to be an H-map.

5. The fibration $\Omega K_5 \longrightarrow B(27, 35) \longrightarrow X$

This is the central section of the paper. We prove the existence of the homotopy fibration in Theorem 1.1 (a) which we will later use in Section 6 to calculate an upper bound on the homotopy exponent of X. Throughout we use the spaces described in Example 4.1.

The strategy of the proof is as follows. We first show there is a map $f : A \longrightarrow X$ which is an epimorphism in cohomology. Since X is an H-space, we can take the pullback of Σf and the Hopf fibration $X \longrightarrow X * X \longrightarrow \Sigma X$ to obtain a homotopy fibration sequence $\Omega\Sigma A \xrightarrow{\partial} X \longrightarrow W \longrightarrow \Sigma A$. Since $\Omega\Sigma A \simeq B(27, 35) \times \Omega\Sigma R$ we obtain a composite $\overline{\partial} : B(27, 35) \longrightarrow \Omega\Sigma A \xrightarrow{\partial} X$ which extends f. The existence of $\overline{\partial}$ will not be enough to allow us to identify its homotopy fiber as ΩK_5. For that we need the much more powerful statement that ∂ factors through $\overline{\partial}$. The homotopy fiber F of $\overline{\partial}$ is then a retract of ΩW. In addition, we show there is a map $K_5 \longrightarrow W$ which is an epimorphism in cohomology and then use this to show that the composite $\Omega K_5 \longrightarrow \Omega W \longrightarrow F$ is a homotopy equivalence.

The bulk of the work is in proving the factorization of ∂, which appears in Corollary 5.1. The factorization is a consequence of the triviality of the composite $R \xrightarrow{\tilde{\lambda}} \Omega\Sigma A \xrightarrow{\partial} X$, where $\tilde{\lambda}$ is the adjoint of λ. The proof of this null homotopy involves several low-dimensional calculations. These are partly based on the fact that R is a five-cell complex with cells in dimensions $\{54, 62, 70, 89, 97\}$. A glance ahead will show how often these dimensions (or those for ΣR) appear in the statements of the coming Lemmas.

We begin by setting some notation. For a space Z, let $Z\langle 3\rangle$ be the three-connected cover of Z. We will denote the t-skeleton of a space Z by $(Z)_t$ (the parantheses are added to avoid confusion with the notation for the space K_5, for example). Throughout, all cohomology calculations are with $\mathbf{Z}/5\mathbf{Z}$-coefficients.

We record some cohomology calculations which will be repeatedly referred to. The calculation for $B(27, 35)$ is trivial, those for K_5 and $K_5\langle 3\rangle$ were already mentioned in Section 3, and those for X and $X\langle 3\rangle$ are derived from $H^*(E_8)$ and the 5-local splitting $E_8 \simeq X \times Y$.

Lemma 5.1. *The following hold:*

(a) $H^*(B(27, 35)) \cong \Lambda(x_{27}, x_{35})$ *with* $\mathcal{P}^1(x_{27}) = x_{35}$,

(b) $H^*(K_5) \cong \mathbf{Z}/5\mathbf{Z}[x_{12}]/(x_{12}^5) \otimes \Lambda(x_3, x_{11})$ *with* $\beta(x_{11}) = x_{12}$ *and* $\mathcal{P}^1(x_3) = x_{11}$,

(c) $H^*(K_5\langle 3\rangle) \cong \mathbf{Z}/5\mathbf{Z}[y_{50}] \otimes \Lambda(y_{51}, y_{59})$ *with* $\beta(y_{50}) = y_{51}$ *and* $\mathcal{P}^1(y_{51}) = y_{59}$,

(d) *the map* $H^*(K_5) \longrightarrow H^*(K_5\langle 3\rangle)$ *sends* x_{27}, x_{35} *to* y_{27}, y_{35},

(e) $H^*(X) \cong H^*(K_5) \otimes H^*(B(27, 35))$,

(f) $H^*(X\langle 3\rangle) \cong H^*(K_5\langle 3\rangle) \otimes H^*(B(27, 35))$,

(g) *the isomorphisms in parts* (e) *and* (f) *are as modules over the Steenrod algebra.* □

Lemma 5.2. *There is a map* $f : A \longrightarrow X$ *which is an epimorphism in cohomology. Its three-connected cover* $f\langle 3\rangle : A \longrightarrow X\langle 3\rangle$ *is also an epimorphism in cohomology.*

Proof. Lemma 5.1 (f) shows that the 35-skeleton of $X\langle 3\rangle$ is homotopy equivalent to A. Let f be the composite

$$f : A \xrightarrow{\simeq} (X\langle 3\rangle)_{35} \longrightarrow X\langle 3\rangle \longrightarrow X.$$

Then parts (d) and (e) of Lemma 5.1 imply that f is an epimorphism in cohomology. The three-connected cover of f is the inclusion $A \longrightarrow X\langle 3\rangle$. $\square$

Since X is a retract of E_8, it is an H-space. The Hopf construction then gives a homotopy fibration $X \longrightarrow X * X \longrightarrow \Sigma X$. Combining this with the map Σf from Lemma 5.2 gives a homotopy pullback

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & W \arrow[r] \arrow[d] & \Sigma A \arrow[d, "\Sigma f"] \\ X \arrow[r] & X * X \arrow[r] & \Sigma X \end{tikzcd}$$

which defines the space W. Let $\partial : \Omega\Sigma A \longrightarrow X$ be the connecting map for the homotopy fibration along the top row of the pullback.

This paragraph is intended to motivate the series of Lemmas leading up to Corollary 5.1. Since $\Omega\Sigma A \simeq B(27,35) \times \Omega\Sigma R$ we have $H^*(\Omega\Sigma A) \cong H^*(B(27,35)) \otimes H^*(\Omega\Sigma R)$. By Lemma 5.1 (e) we have $H^*(X) \cong H^*(K_5) \otimes H^*(B(27,35))$. It should be the case that the fibration connecting map $\Omega\Sigma A \xrightarrow{\partial} X$ has the property that ∂^* factors through $H^*(B(27,35))$, suggesting that we have $H^*(W) \cong H^*(K_5) \otimes H^*(\Sigma R)$. It might even be the case that there is a homotopy equivalence $W \simeq K_5 \times \Sigma R$. Going one step further, it might be possible to choose this homotopy equivalence so the composite $\Sigma R \longrightarrow W \longrightarrow \Sigma A$ is homotopic to λ. If this were the case we could jump immediately to Corollary 5.1 and it would be relatively straightforward to prove the existence of the homotopy fibration in Theorem 1.1 (a). However, it is not clear at this point that such a factorization of ∂^* exists, let alone such a homotopy decomposition of W. In trying to prove such properties it would be advantageous if we knew that X were homotopy associative. But the H-structure on X comes from its retraction off the loop space E_8, and this retraction may not give a multiplication on X which is homotopy associative. Faced with these disadvantages, we resort to low dimensional calculations and prove statements only through a skeletal range. This, though, will suffice since the key dimension, that of ΣR, is 98. We do the cohomology calculations in Lemmas 5.3 through 5.8, and then use them in the homotopy calculations of Lemma 5.9 through Proposition 5.1.

Lemma 5.3. *There is an isomorphism $H^*(W) \cong H^*(K_5 \times \Sigma R)$ for $* \leq 62$, which is valid as modules over the Steenrod algebra.*

Proof. Consider the Serre spectral sequence for the homotopy fibration $X \longrightarrow W \longrightarrow \Sigma A$ which converges to $H^*(W)$. Recall from Lemma 5.1 (e) that $H^*(X) \cong \mathbf{Z}/5\mathbf{Z}[x_{12}]/(x_{12}^5) \otimes \Lambda(x_3, x_{11}, x_{27}, x_{35})$. Let $\{y_{28}, y_{36}\}$ be a vector space basis for $H^*(\Sigma A)$. We have $E_2(W) \cong H^*(X) \otimes H^*(\Sigma A)$. The definition of W involving the map Σf from Lemma 5.2 implies that x_{27} and x_{35} transgress to y_{28} and y_{36} respectively. It is then seen that a complete list of the elements of the E_2-term in dimensions < 63 which survive to E_∞ is given by: (1) the subalgebra $\mathbf{Z}/5\mathbf{Z}[x_{12}]/(x_{12}^5) \otimes \Lambda(x_3, x_{11})$ of $H^*(X)$ which appears along the vertical axis, and (2) the two additional elements $x_{27} \otimes y_{28}$ and $x_3x_{27} \otimes y_{28}$. Note that the subalgebra in (1) is isomorphic to $H^*(K_5)$. The element $x_{27}\otimes y_{28}$ in (2) forms a module over the Steenrod algebra which is isomorphic to $H^*(S^{55}) \cong \Lambda(z_{55})$. The element $x_3x_{27}\otimes y_{28}$ in (2) then corresponds to the product element $x_3 \otimes z_{55}$ in $H^*(K_5 \times S^{55})$. Thus $H^*(W) \cong H^*(K_5 \times S^{55})$ for $* \leq 62$. Finally, observe that $(\Sigma R)_{62} \simeq S^{55}$. □

Lemma 5.4. *There is a map $g : K_5 \longrightarrow W$ with the property that g^* is an epimorphism.*

Proof. Recall that the two lowest dimensional cells of ΣR are in dimensions 55 and 63. So $(\Sigma R)_{62} = S^{55}$. We will show there is a map $(W)_{62} \longrightarrow (\Sigma R)_{62} = S^{55}$. Suppose for now this map has been constructed. Let F be its homotopy fiber. Then the Serre spectral sequence shows that $H^*(F) \cong H^*(K_5)$ for $* \leq 64$. In particular, K_5 is 62-dimensional so $(F)_{62}$ has cohomology isomorphic to that of K_5. Lemma 3.1 says that the homotopy type of K_5 is determined by its cohomology, so $(F)_{62} \simeq K_5$. The composite $K_5 \xrightarrow{\simeq} (F)_{62} \longrightarrow F \longrightarrow (W)_{62} \longrightarrow W$ then gives the asserted map.

It remains to show there is a map $(W)_{62} \longrightarrow S^{55}$. A vector space basis for $H^*(W)$ in dimensions $55 \leq k \leq 62$ has one element in each of the dimensions $\{55, 59, 62\}$ and two elements in dimension 58. The 55-dimensional element corresponds to the bottom cell of ΣR in the isomorphism of Lemma 5.3. Begin with the pinch map onto this cell, $q : (W)_{55} \longrightarrow S^{55}$. We wish to extend q to a map $(W)_{62} \longrightarrow S^{55}$. To do so we need to extend q over the 58, 59, and 62-dimensional cells in $(W)_{62}$. The obstructions to doing so lie in $\pi_{57}(S^{55})$, $\pi_{58}(S^{55})$, and $\pi_{61}(S^{55})$. But each of these groups is zero. □

It becomes easier at this point to work with three-connected covers, simply because $X\langle 3\rangle$, $K_5\langle 3\rangle$, and $W\langle 3\rangle$ are much more sparse in cohomology through dimension 98 than X, K_5, and W. The three-connected cover of the homotopy fibration $X \longrightarrow W \longrightarrow \Sigma A$ gives a homotopy fibration $X\langle 3\rangle \longrightarrow W\langle 3\rangle \longrightarrow \Sigma A$.

Instead of phrasing the coming statements in terms of a product $K\langle 3\rangle \times \Sigma R$ as in Lemma 5.3, we use a wedge $K_5\langle 3\rangle \vee \Sigma R$. We can then consider the wedge sum of maps into $W\langle 3\rangle$. Note that we do not miss out on anything because the bottom cells of $K_5\langle 3\rangle$ and ΣR are in dimensions 50 and 55 respectively, so the first nontrivial cross-product in $H^*(K_5 \times \Sigma R)$ occurs in dimension 105. In other words, there is a homotopy equivalence $(K_5\langle 3\rangle \vee \Sigma R)_t \simeq (K_5\langle 3\rangle \times \Sigma R)_t$ for $t \leq 104$.

Lemma 5.5. *There is an isomorphism $H^*(W\langle 3\rangle) \cong H^*(K_5\langle 3\rangle \vee \Sigma R)$ for $* \leq 98$, which is valid as modules over the Steenrod algebra.*

Proof. Consider the Serre spectral sequence for the homotopy fibration $X\langle 3\rangle \longrightarrow W\langle 3\rangle \longrightarrow \Sigma A$ which converges to $H^*(W\langle 3\rangle)$. By Lemma 5.1 (f), $H^*(X\langle 3\rangle) \cong \mathbf{Z}/5\mathbf{Z}[x_{50}] \otimes \Lambda(x_{27}, x_{35}, x_{51}, x_{59})$. Let $\{y_{28}, y_{36}\}$ be a vector space basis for $H^*(\Sigma A)$. We have $E_2(W\langle 3\rangle) \cong H^*(X\langle 3\rangle) \otimes H^*(\Sigma A)$. The definition of W involving the map Σf from Lemma 5.2 implies that x_{27} and x_{35} transgress to y_{28} and y_{36} respectively. It is then seen that a complete list of the elements of the E_2-term in dimensions ≤ 98 which survive to E_∞ is

$$\{x_{50}, x_{51}, x_{59}, x_{27} \otimes y_{28}, x_{27} \otimes y_{36}, x_{35} \otimes y_{36}, x_{27}x_{35} \otimes y_{28}, x_{27}x_{35} \otimes y_{36}\}.$$

The monomials of length 1 in this list correspond to the three cells of $K_5\langle 3\rangle$ below dimension 98. The monomials of lengths 2 and 3 form a module over the Steenrod algebra which is isomorphic to $H^*(\Sigma R)$. This proves the asserted isomorphism. $\square$

Lemma 5.6. *The map $K_5\langle 3\rangle \xrightarrow{g\langle 3\rangle} W\langle 3\rangle$ obtained from Lemma 5.4 by taking three-connected covers has the property that $(g\langle 3\rangle)^*$ is an epimorphism through dimension 98.*

Proof. Lemma 5.1 (c) shows that $K_5\langle 3\rangle$ has no cells in dimensions 60 through 98, so $(K_5\langle 3\rangle)_{98} \simeq (K_5\langle 3\rangle)_{59}$. It therefore suffices to prove that $(g\langle 3\rangle)^*$ is an epimorphism through dimension 59. Lemma 5.1 (c) also shows that a vector space basis for $H^*((K_5\langle 3\rangle)_{59})$ is given by $\{y_{50}, y_{51}, y_{59}\}$ with $\beta(y_{50}) = y_{51}$ and $\mathcal{P}^1(y_{51}) = y_{59}$. The Steenrod operations then imply that it suffices to show that $(g\langle 3\rangle)^*$ is an epimorphism just in dimension 50.

Consider the homotopy pullback

$$\begin{tikzcd}
F \arrow[r, equal] \arrow[d] & F \arrow[d] & \\
K_5\langle 3\rangle \arrow[r] \arrow[d, "g\langle 3\rangle"] & K_5 \arrow[r] \arrow[d, "g"] & K(\mathbf{Z},3) \arrow[d, equal] \\
W\langle 3\rangle \arrow[r] & W \arrow[r] & K(\mathbf{Z},3).
\end{tikzcd}$$

Lemmas 5.3 and 5.4 imply that F is 53-connected. Thus $g\langle 3\rangle$ is a homotopy equivalence through dimension 53, and so in particular $(g\langle 3\rangle)^*$ is an epimorphism in dimension 50. $\square$

Lemma 5.7. *Let $t \in \{55, 63, 71, 90, 98\}$. Suppose the restriction of λ to the t-skeleton of ΣR lifts to some map λ':*

$$\begin{tikzcd}
 & (\Sigma R)_t \arrow[dl, "\lambda'"'] \arrow[d, "(\lambda)_t"] \\
W\langle 3\rangle \arrow[r] & \Sigma A.
\end{tikzcd}$$

Then $(\lambda')^*$ is an epimorphism.

Proof. First, we prove the Lemma in the case when λ lifts to $W\langle 3\rangle$ without restriction. This corresponds to the $t = 98$ case since ΣR has dimension 98.

From the existence of a lift we obtain a homotopy pullback

$$\begin{tikzcd}
B(27,35) \arrow[r] \arrow[d, "j"] & \Sigma R \arrow[r, "\lambda"] \arrow[d, "\lambda'"] & \Sigma A \arrow[d, equal] \\
X\langle 3\rangle \arrow[r] & W\langle 3\rangle \arrow[r] & \Sigma A
\end{tikzcd}$$

for some map j. Continuing the homotopy fibration sequences one step to the left we see that the connecting map $\Omega\Sigma A \xrightarrow{\partial\langle 3\rangle} X\langle 3\rangle$ factors as the composite $\Omega\Sigma A \xrightarrow{r} B(27,35) \xrightarrow{j} X\langle 3\rangle$. Since the composite $A \xrightarrow{E} \Omega\Sigma A \xrightarrow{\partial\langle 3\rangle} X\langle 3\rangle$ is homotopic to the map f appearing in Lemma 5.2, we see that j extends f.

Now consider the Serre spectral sequences for the horizontal fibrations which converge to $H^*(\Sigma R)$ and $H^*(W\langle 3\rangle)$. Since j extends f, j^* is an epimorphism and therefore there is an epimorphism $E_2(W\langle 3\rangle) = H^*(X\langle 3\rangle) \otimes H^*(\Sigma A) \longrightarrow E_2(\Sigma R) = H^*(B(27,35)) \otimes H^*(\Sigma A)$. The differentials in both spectral sequences are determined by the respective transgressions of $x_{27}, x_{35} \in H^*(X\langle 3\rangle)$ and $\bar{x}_{27}, \bar{x}_{35} \in H^*(B(27,35))$ to $y_{28}, y_{35} \in H^*(\Sigma A)$. The transgressions in these two spectral sequences are determined by $f \simeq \partial\langle 3\rangle \circ E$ and $r \circ E$ respectively. Since $\partial\langle 3\rangle \simeq r \circ j$, we see that the epimorphism of E_2-terms above respects the differentials in each page of the spectral sequence and so the map $E_\infty(W\langle 3\rangle) \longrightarrow E_\infty(\Sigma R)$ is also an epimorphism. Hence $H^*(W\langle 3\rangle) \xrightarrow{(\lambda')^*} H^*(\Sigma R)$ is an epimorphism.

Now consider the cases when $t < 98$. We have a homotopy pullback

$$\begin{tikzcd}
G \arrow[r, equal] \arrow[d] & G \arrow[d] & \\
F \arrow[r] \arrow[d] & (\Sigma R)_t \arrow[r, "(\lambda)_t"] \arrow[d] & \Sigma A \arrow[d, equal] \\
B(27,35) \arrow[r] & \Sigma R \arrow[r, "\lambda"] & \Sigma A.
\end{tikzcd}$$

Since the cells of ΣR are in the same dimensions as the possible values of t, the inclusion $(\Sigma R)_t \longrightarrow \Sigma R$ is actually at least $(t+7)$-connected. The five-lemma then implies that the map $F \longrightarrow B(27,35)$ is also at least $(t+7)$-connected. This extra boost of 7 in connectivity ensures that the argument above in the unrestricted case is also valid in dimensions $\leq t$ when the homotopy fibration $B(27,35) \longrightarrow \Sigma R \xrightarrow{\lambda} \Sigma A$ is replaced by $F \longrightarrow (\Sigma R)_t \xrightarrow{(\lambda)_t} \Sigma A$. $\square$

Lemma 5.8. *Let $t \in \{55, 63, 71, 90, 98\}$. If there is a lift λ' as in Lemma 5.7, then the wedge sum*

$$(K_5 \vee \Sigma R)_t \xrightarrow{(g\langle 3\rangle \perp \lambda')_t} (W\langle 3\rangle)_t$$

is a homotopy equivalence.

Proof. By Lemma 5.6, $(g\langle 3\rangle)^*$ is an epimorphism through dimension 98. By Lemma 5.7, $(\lambda')^*$ is an epimorphism. Now observe that through dimension 98, K_5 has cells in dimensions $\{50, 51, 59\}$ while ΣR has cells in dimensions $\{55, 63, 71, 90, 98\}$. As these two sets are disjoint, $((g\langle 3\rangle \perp \lambda')_t)^*$ is an epimorphism. The isomorphism in Lemma 5.3 then implies that $((g\langle 3\rangle \perp \lambda')_t)^*$ is an isomorphism. Hence $(g\langle 3\rangle \perp \lambda')_t$ is a homotopy equivalence. □

We now turn from cohomology to homotopy. We wish to geometrically realize the homology isomorphism $H^*(W\langle 3\rangle) \cong H^*(K_5\langle 3\rangle \vee \Sigma R)$ in Lemma 5.5 as a homotopy equivalence $(W\langle 3\rangle)_{98} \simeq (K_5\langle 3\rangle \vee \Sigma R)_{98}$. In addition, noting that ΣR has dimension 98, we wish to choose the homotopy equivalence so the composite $\Sigma R \longrightarrow (W\langle 3\rangle)_{98} \longrightarrow \Sigma A$ is homotopic to λ. In other words, we want to show that the map $\Sigma R \xrightarrow{\lambda} \Sigma A$ lifts through the map $W\langle 3\rangle \longrightarrow \Sigma A$. To do so we proceed by incrementally increasing dimension. The first step is in Lemma 5.11, and the succeeding steps in Proposition 5.1 advance in dimensional steps corresponding to the dimensions of the cells of ΣR.

We need a couple preliminary lemmas. The first records some trivial (5-local) homotopy groups of $B(27, 35)$.

Lemma 5.9. $\pi_t(B(27, 35)) = 0$ *for* $t \in \{54, 62, 70, 89, 97\}$.

Proof. Since there is a homotopy fibration $S^{27} \longrightarrow B(27, 35) \longrightarrow S^{35}$, the Lemma will follow if $\pi_t(S^{27}) = \pi_t(S^{35}) = 0$ for $t \in \{54, 62, 70, 89, 97\}$. These homotopy groups of spheres are all in the stable range and it is well known that the corresponding 5-stems in each case are trivial. □

Lemma 5.10. *Let C be a finite CW-complex with cells in dimensions $\{t_1, \ldots, t_n\}$. Suppose Z is a space and there is a map $f : C \longrightarrow Z$. If $\pi_{t_1}(Z) = \cdots = \pi_{t_n}(Z) = 0$, then f is null homotopic.*

Proof. To simplify notation, we assume that C has a single cell in each dimension t_j, the general case being similar. Iteratively pinching out the bottom cell gives a sequence of cofibrations $S^{t_1} = (C)_{t_1} \longrightarrow C \longrightarrow C/(C)_{t_1}$, $S^{t_2} \longrightarrow C/(C)_{t_1} \longrightarrow C/(C)_{t_2}$, $\ldots$, $S^{t_{n-1}} \longrightarrow C/(C)_{t_{n-2}} \longrightarrow C/(C)_{t_{n-1}} = S^{t_n}$. Since $\pi_{t_1}(Z) = 0$, f extends to a map $f_1 : C/(C)_{t_1} \longrightarrow Z$. Since $\pi_{t_2}(Z) = 0$, f_1 extends to a map $f_2 : C/(C)_{t_2} \longrightarrow Z$. Continuing in this way we obtain an extension $f_{n-1} : C/(C)_{t_{n-1}} = S^{t_n} \longrightarrow Z$. Since $\pi_{t_n}(Z) = 0$, f_{n-1} is null homotopic, and hence f is null homotopic. □

Lemma 5.11. *There is a homotopy equivalence*

$$(W\langle 3\rangle)_{59} \simeq (K_5\langle 3\rangle \vee \Sigma R)_{59}.$$

Proof. By Lemma 5.6, the restriction of $K_5\langle 3\rangle \xrightarrow{g\langle 3\rangle} W\langle 3\rangle$ to 59-skeletons is an epimorphism in cohomology. The description of $H^*(W\langle 3\rangle)$ in Lemma 5.5 says that $(W\langle 3\rangle)_{59}$ has only one cell in dimension 55 not accounted for by $(g\langle 3\rangle)_{59}$. This cell is attached by a cofibration $S^{54} \xrightarrow{h} (W\langle 3\rangle)_{54} \longrightarrow (W\langle 3\rangle)_{55}$.

Lemma 5.5 also tells us that $(W\langle 3\rangle)_{54}$ is homotopy equivalent to the mod-5 Moore space $P^{51}(5)$. Since $\pi_{54}(P^{51}(5)) = \pi_{54}(S^{51}) \oplus \pi_{54}(S^{50}) = 0$, the attaching map h is trivial and so there is a homotopy equivalence $(W\langle 3\rangle)_{55} \simeq P^{51}(5) \vee S^{55}$. Let s be the composite of inclusions $s : S^{55} \longrightarrow (W\langle 3\rangle)_{55} \longrightarrow W\langle 3\rangle$. Then the wedge sum $K_5\langle 3\rangle \vee S^{55} \xrightarrow{g\perp s} W\langle 3\rangle$ is a cohomology isomorphism through dimension 59 and therefore a homotopy equivalence when restricted to 59-skeletons. Finally, observe that $(\Sigma R)_{59}$ is homotopy equivalent to S^{55}. $\square$

The next Lemma is key to making our incremental approach work. Special attention should be paid to the fact that the hypotheses are in terms of the t-skeleton of ΣR while the conclusion is in terms of the $(t+1)$-skeleton.

Lemma 5.12. *Let $t \in \{54, 62, 70, 89, 97\}$. Suppose there is a homotopy equivalence $(W\langle 3\rangle)_t \simeq (K_5\langle 3\rangle \vee \Sigma R)_t$. Suppose as well that the composite $\epsilon : (X\langle 3\rangle)_t \longrightarrow (W\langle 3\rangle)_t \longrightarrow (K_5\langle 3\rangle)_t$ is a monomorphism in cohomology, where the right map is the pinch onto the wedge summand. Then there is a lift of $(\lambda)_{t+1}$ to some map λ':*

$$\begin{tikzcd} & (\Sigma R)_{t+1} \arrow[dl, "\lambda'"'] \arrow[d, "(\lambda)_{t+1}"] \\ W\langle 3\rangle \arrow[r] & \Sigma A. \end{tikzcd}$$

Proof. Let $\tilde{\lambda} : R \longrightarrow \Omega\Sigma A$ be the adjoint of λ. The existence of the lift in the statement of the lemma is equivalent to the composite $(R)_t \xrightarrow{(\tilde{\lambda})_t} \Omega\Sigma A \xrightarrow{\partial\langle 3\rangle} X\langle 3\rangle$ being null homotopic. We prove the equivalent statement.

Suppose for the moment there exists a homotopy fibration $F \longrightarrow X\langle 3\rangle \xrightarrow{e} K_5\langle 3\rangle$ where e^* sends $H^*(K_5\langle 3\rangle)$ isomorphically onto the corresponding tensor factor in $H^*(X\langle 3\rangle) \cong H^*(K_5\langle 3\rangle) \otimes H^*(B(27,35))$. Then the Serre spectral sequence shows that $H^*(F) \cong H^*(B(27,35))$. But by [MNT, 7.1], the homotopy type of $B(27,35)$ is determined by its cohomology, so there is a homotopy equivalence $F \simeq B(27,35)$.

Now, the map e may not exist, but by hypothesis a restricted version ϵ does exist for a given value of t. Consider the homotopy fibration $G \longrightarrow (X\langle 3\rangle)_t \xrightarrow{\epsilon} (K_5\langle 3\rangle)_t$. Tracing through the Serre spectral sequence calculation above for $H^*(F)$, we see that there is an isomorphism $H^*(G) \cong H^*(B(27,35))$ for $* \leq t$. Note that this depends on the specific values of t. Hence there is a homotopy equivalence $(G)_t \simeq (B(27,35))_t$.

Next, because ϵ factors through $(W\langle 3\rangle)_t$, the composite $(\Omega\Sigma A)_t \xrightarrow{(\partial\langle 3\rangle)_t} (X\langle 3\rangle)_t \xrightarrow{\epsilon} (K_5\langle 3\rangle)_t$ factors through the t-skeletons of two consecutive maps in a homotopy fibration and so is null homotopic. Thus $(\partial\langle 3\rangle)_t$ lifts to a map $(\Omega\Sigma A)_t \longrightarrow G$. Consider the composite

$$\varphi : (R)_t \xrightarrow{(\tilde{\lambda})_t} (\Omega\Sigma A)_t \longrightarrow G.$$

For dimensional reasons, φ factors through the t-skeleton of G, which we just saw is homotopy equivalent to $(B(27,35))_t$ (which is simply $B(27,35)$ if $t > 59$). The cells of R occur in the same dimensions as the possible values of t, and Lemma 5.9 states that the homotopy groups of $B(27,35)$ are trivial in those same dimensions. So by Lemma 5.10, φ is null homotopic. Since φ is a lift of $(\partial\langle 3\rangle \circ \tilde{\lambda})_t$, we see that the latter map is null homotopic, as required. ☐

Proposition 5.1. *There is a lift of λ to some map λ':*

$$\begin{tikzcd} & \Sigma R \arrow[dl, "\lambda'"'] \arrow[d, "\lambda"] \\ W\langle 3\rangle \arrow[r] & \Sigma A. \end{tikzcd}$$

Proof. As a reminder, ΣR is a five-cell complex with cells in dimensions $\{55, 63, 71, 90, 98\}$.

Lemma 5.11 shows that there is a homotopy equivalence $(W\langle 3\rangle)_{59} \simeq (K_5\langle 3\rangle \vee \Sigma R)_{59}$. The cohomology description of $(W\langle 3\rangle)_{98}$ in Lemma 5.5 shows that $W\langle 3\rangle$ has no additional cells in dimensions 60 through 62. The same is true of $K_5\langle 3\rangle$ and ΣR. So in fact we have a homotopy equivalence $(W\langle 3\rangle)_{62} \simeq (K_5\langle 3\rangle \vee \Sigma R)_{62}$. By Lemma 5.12 this implies the restriction of λ to $(\Sigma R)_{63}$ lifts to $W\langle 3\rangle$; call this lift $\lambda' : (\Sigma R)_{63} \longrightarrow W\langle 3\rangle$. Then by Lemma 5.8, the wedge sum

$$(K_5\langle 3\rangle \vee \Sigma R)_{63} \xrightarrow{(g \perp \lambda')_{63}} (W\langle 3\rangle)_{63}$$

is a homotopy equivalence.

Now repeat the procedure in the first paragraph. Having dealt with the 63-dimensional cell in ΣR, the next cell occurs in dimension 71. Arguing as above, we have $(W\langle 3\rangle)_{63} \simeq (W\langle 3\rangle)_{70}$ and $(W\langle 3\rangle)_{70} \simeq (K_5\langle 3\rangle \vee \Sigma R)_{70}$. Lemma 5.12 then implies the restriction of λ to $(\Sigma R)_{71}$ lifts to $W\langle 3\rangle$, and so Lemma 5.8 gives a homotopy equivalence $(W\langle 3\rangle)_{71} \simeq (K_5\langle 3\rangle \vee \Sigma R)_{71}$. We can continue in this way, lifting the restriction of λ to the 90 and 98-skeletons of ΣR to $W\langle 3\rangle$. But as ΣR has dimension 98, we have in fact lifted λ (without restriction) to $W\langle 3\rangle$. ☐

Corollary 5.1. *The composite $\Omega\Sigma R \xrightarrow{\Omega\lambda} \Omega\Sigma A \xrightarrow{\partial} X$ is null homotopic.*

Proof. The homotopy fibration $\Omega W\langle 3\rangle \longrightarrow \Omega\Sigma A \xrightarrow{\partial\langle 3\rangle} X\langle 3\rangle$ together with the lift in Proposition 5.1 shows that $\partial\langle 3\rangle \circ \Omega\lambda$ is null homotopic. The Corollary immediately follows. ☐

Recall that the homotopy fibration $\Omega\Sigma R \xrightarrow{\Omega\lambda} \Omega\Sigma A \xrightarrow{r} B(27,35)$ splits as $\Omega\Sigma A \simeq B(27,35) \times \Omega\Sigma R$. Let $s : B(27,35) \longrightarrow \Omega\Sigma A$ be a right homotopy inverse of r. Let $\overline{\partial}$ be the composite

$$\overline{\partial} : B(27,35) \xrightarrow{s} \Omega\Sigma A \xrightarrow{\partial} X.$$

Lemma 5.13. *There is a factorization*

$$\begin{tikzcd} \Omega\Sigma A \arrow[r, "\partial"] \arrow[d, "r"] & X \arrow[d, equal] \\ B(27,35) \arrow[r, "\overline{\partial}"] & X. \end{tikzcd}$$

Proof. The homotopy fibration connecting map $\Omega\Sigma A \xrightarrow{\partial} X$ has an action $\theta_X : \Omega\Sigma A \times X \longrightarrow X$ which makes the right square in the following diagram homotopy commute:

$$\begin{tikzcd} B(27,35) \times \Omega\Sigma R \arrow[r, "s\times\Omega\lambda"] \arrow[d, "\pi_1"] & \Omega\Sigma A \times \Omega\Sigma A \arrow[r, "\mu"] \arrow[d, "1\times\partial"] & \Omega\Sigma A \arrow[d, "\partial"] \\ B(27,35) \arrow[r, "s\times *"] & \Omega\Sigma A \times X \arrow[r, "\theta_X"] & X. \end{tikzcd}$$

As for the left square, π_1 is the projection onto the first factor and the left square homotopy commutes by Corollary 5.1. Observe that the top row is a homotopy equivalence e, while the bottom row is homotopic to $\overline{\partial}$. So the homotopy commutative square shows that $\partial \circ e \simeq \overline{\partial} \circ \pi_1$. Composing on the right with e^{-1} then gives $\partial \simeq \overline{\partial} \circ \pi_1 \circ e^{-1}$. If we knew that $\pi_1 \circ e^{-1}$ was homotopic to r then the proof would be complete.

To show $\pi_1 \circ e^{-1} \simeq r$ we need to use the fact that r is also a homotopy fibration connecting map (by its definition) and so there is a homotopy action $\theta_B : \Omega\Sigma A \times B(27,35) \longrightarrow B(27,35)$. The same diagram as above with (X, ∂, θ_X) replaced by $(B(27,35), r, \theta_B)$ shows that $r \circ e \simeq \theta_B \circ (s \times *) \circ \pi_1$. But $\theta_B \circ (s \times *)$ is homotopic to the identity map on $B(27,35)$ so we have $r \circ e \simeq \pi_1$. Composing on the right with e^{-1} then gives $r \simeq \pi_1 \circ e^{-1}$, as required. $\square$

Extending the diagram in Lemma 5.13 gives a homotopy pullback

$$\begin{tikzcd} \Omega W \arrow[r] \arrow[d, "r'"] & \Omega\Sigma A \arrow[r, "\partial"] \arrow[d, "r"] & X \arrow[d, equal] \\ F \arrow[r, "\varphi"] & B(27,35) \arrow[r, "\overline{\partial}"] & X \end{tikzcd}$$

which defines the space F and the maps φ and r'.

Lemma 5.14. *F is an H-space, φ and r' are H-maps, and r' has a right homotopy inverse.*

Proof. Consider the iterated homotopy pullback diagram

$$\begin{tikzcd} F \arrow[r, "s'"] \arrow[d, "\varphi"] & \Omega W \arrow[r, "r'"] \arrow[d] & F \arrow[d, "\varphi"] \\ B(27,35) \arrow[r, "s"] \arrow[d, "\overline{\partial}"] & \Omega\Sigma A \arrow[r, "r"] \arrow[d, "\partial"] & B(27,35) \arrow[d, "\overline{\partial}"] \\ X \arrow[r, equal] & X \arrow[r, equal] & X. \end{tikzcd}$$

Here, s' is defined by the left pullback. Observe that the outer diagram is also a homotopy pullback. Thus, as $r \circ s$ is homotopic to the identity on $B(27,35)$, we have $r' \circ s'$ homotopic to the identity on F. This shows that r' has a right homotopy inverse and F is a retract of an H-space and so is an H-space itself.

The pullbacks above, together with the fact that $\Omega W \longrightarrow \Omega\Sigma A$ is a loop map, give a homotopy commutative diagram

$$\begin{tikzcd} F \times F \arrow[r, "s' \times s'"] \arrow[d, "\varphi \times \varphi"] & \Omega W \times \Omega W \arrow[r, "\mu"] \arrow[d] & \Omega W \arrow[r, "r'"] \arrow[d] & F \arrow[d, "\varphi"] \\ B(27,35) \times B(27,35) \arrow[r, "s \times s"] & \Omega\Sigma A \times \Omega\Sigma A \arrow[r, "\mu"] & \Omega\Sigma A \arrow[r, "r"] & B(27,35). \end{tikzcd}$$

The top and bottom rows are the definitions of the mulitplications μ_F and μ_B on F and $B(27,35)$ respectively. Thus φ is an H-map.

Finally, recall from Example 4.1 that r can be chosen to be an H-map. The pullback of r and φ (immediately preceeding the Lemma) is then a pullback of H-spaces and H-maps, so r' can be chosen to be an H-map. $\square$

We wish to calculate $H_*(F)$ and show it is isomorphic to $H_*(\Omega K_5)$. To do so, we need to improve the isomorphism $H_*(X) \cong H_*(K_5) \otimes H_*(B(27,35))$ of Lemma 5.1 (e) from being one only of modules over the Steenrod algebra.

Lemma 5.15. *There is a short exact sequence of Hopf algebras*

$$0 \longrightarrow H^*(K_5) \longrightarrow H^*(X) \xrightarrow{\overline{\partial}^*} H^*(B(27,35)) \longrightarrow 0$$

and a splitting $H^*(X) \cong H^*(K_5) \otimes H^*(B(27,35))$ of left $H^*(K_5)$-modules and right $H^*(B(27,35))$-comodules.

Proof. Let $A = H^*(K_5)$, $B = H^*(X)$, and $C = H^*(B(27,35))$. Since each of K_5, X, and $B(27,35)$ are H-spaces, A, B, and C are Hopf algebras. Let ψ_A, ψ_B, and ψ_C be the comultiplications on A, B, and C respectively. Lemma 5.1 (b) says that, as an algebra $A \cong \mathbf{Z}/5\mathbf{Z}[x_{12}] \otimes \Lambda(x_3, x_{11})$. Since ψ_A is an algebra homomorphism, it is determined by its value on the algebra generators x_3, x_{11}, and x_{12}. But for degree reasons, the reduced diagonal on each of these elements is zero and so each is primitive. Hence the isomorphism for A above is actually as (primitively generated) Hopf algebras. Similarly, $C \cong \Lambda(x_{27}, x_{35})$ is an isomorphism of (primitively generated) Hopf algebras. Lemma 5.1 (e) says there is an algebra isomorphism $B \cong A \otimes C$. Again, for degree reasons, ψ_B is primitive on x_3, x_{11}, and x_{12}. Thus there is a Hopf algebra inclusion $i : A \longrightarrow B$. On the other hand, B_{27} and B_{35} have vector space bases $\{x_{27}, x_3 x_{12}^2\}$ and $\{x_{35}, x_{11} x_{12}^2\}$ respectively. So degree arguments alone are not sufficient to show that $x_{27}, x_{35} \in B$ are primitive. But their reduced diagonals, if not zero, are in $A \otimes A$.

Next, the map of spaces $B(27, 35) \xrightarrow{\overline{\partial}} X$ gives an algebra map $\overline{\partial}^* : B \longrightarrow C$ which is the projection of $B \cong A \otimes C$ onto C. We now wish to compare $(\overline{\partial}^* \otimes \overline{\partial}^*) \circ \psi_B$ and $\psi_C \circ \overline{\partial}^*$. Since ψ_B and ψ_C are algebra homomorphisms, it suffices to check what happens to the algebra generators in B. But $\overline{\partial}^*$ is zero on x_3, x_{11}, and x_{12}, and we have just seen that $\overline{\psi}_B(x_{27}), \overline{\psi}_B(x_{35}) \in A \otimes A$, showing that $(\overline{\partial}^* \otimes \overline{\partial}^*) \circ \psi_B = \psi_C \circ \overline{\partial}^*$. Hence $\overline{\partial}^*$ is actually a Hopf algebra homomorphism. Combining the previous two paragraphs, we obtain a short exact sequence of Hopf algebras

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\overline{\partial}^*} C \longrightarrow 0$$

and a splitting $B \cong A \otimes C$ as left A-modules and right C-comodules. □

Lemma 5.16. *There is an algebra isomorphism $H_*(F) \cong H_*(\Omega K_5)$.*

Proof. We begin with the Eilenberg-Moore spectral sequence for the homotopy fibration $F \longrightarrow B(27, 35) \xrightarrow{\overline{\partial}} X$ which converges to $H^*(F)$. We have $E_2(F) = \mathrm{Tor}^{H^*(X)}(H^*(B(27, 35)), \mathbf{Z}/5\mathbf{Z})$, and the spectral sequence converges to $H^*(F)$ as a coalgebra. The isomorphism of left $H^*(K_5)$-modules in Lemma 5.15 implies $\mathrm{Tor}^{H^*(X)}(H^*(B(27, 35)), \mathbf{Z}/5\mathbf{Z}) \cong \mathrm{Tor}^{H^*(K_5)}(\mathbf{Z}/5\mathbf{Z}, \mathbf{Z}/5\mathbf{Z})$. The right side in this equation converges, as a coalgebra, to $H^*(\Omega K_5)$. Hence there is a coalgebra isomorphism $H^*(F) \cong H^*(\Omega K_5)$. Dualizing proves the Lemma. □

Lemma 5.17. *The composite $\Omega K_5 \xrightarrow{\Omega g} \Omega W \xrightarrow{r'} F$ is a homotopy equivalence.*

Proof. Let $t = r' \circ \Omega g$. We are considering $H_*(\Omega K_5) \xrightarrow{t_*} H_*(F)$. To show t is a homotopy equivalence it suffices to show that t_* is an isomorphism. By Lemma 5.16, there is an algebra isomorphism $H_*(F) \cong H_*(\Omega K_5)$. As well, by Lemma 5.14, r' is an H-map and so t is also an H-map. Thus to show t_* is an isomorphism it suffices to do so on algebra generators. By Lemma 3.2, $H_*(\Omega K_5)$ has three algebra generators and they are all connected by Steenrod operations, so it suffices to show that t_* is an isomorphism on the lowest dimensional algebra generator – that is, on the bottom cell. But this follows since both $(\Omega g)_*$ and $(r')_*$, by their definitions, are isomorphisms on the bottom cell. □

Let a be the composite

$$a : K_5 \xrightarrow{g} W \longrightarrow \Sigma A.$$

Let γ be the composite

$$\gamma : \Omega K_5 \xrightarrow{\Omega a} \Omega\Sigma A \xrightarrow{r} B(27,35).$$

Proof of Theorem 1.1 (a). Lemma 5.17 and the pullback immediately preceding Lemma 5.14 imply that there is a homotopy fibration $\Omega K_5 \xrightarrow{\gamma} B(27,35) \xrightarrow{\overline{\partial}} X$. □

6. The exponent of X

We begin by factoring certain maps to get more control over the coming exponent calculations. In Section 5 we constructed a homotopy fibration $\Omega K_5 \xrightarrow{\gamma} B(27,35) \xrightarrow{\overline{\partial}} X$ where γ was defined as the composite $\Omega K_5 \xrightarrow{\Omega a} \Omega\Sigma A \xrightarrow{r} B(27,35)$. Let $\delta : \Omega X \longrightarrow \Omega K_5$ be the homotopy fibration connecting map.

Recall from Section 3 that there is a homotopy fibration $B(3,11) \xrightarrow{b} K_5 \longrightarrow J_4(S^{12})$. The dimension of $B(3,11)$ is 14 while the connectivity of ΣA is 26, so the composite $B(3,11) \xrightarrow{b} K_5 \xrightarrow{a} \Sigma A$ is null homotopic. Let C be the homotopy cofiber of b. Then $a \circ b \simeq *$ implies there is a homotopy commutative square

$$\begin{tikzcd}
K_5 \arrow[r] \arrow[d, "a"] & C \arrow[d, "c"] \\
\Sigma A \arrow[r, equal] & \Sigma A
\end{tikzcd}$$

for some map c. As well, there is a homotopy fibration diagram

$$\begin{tikzcd}
\Omega K_5 \arrow[r] \arrow[d] & \Omega J_4(S^{12}) \arrow[r] \arrow[d, "d"] & B(3,11) \arrow[r, "b"] \arrow[d] & K_5 \arrow[d] \\
\Omega C \arrow[r, equal] & \Omega C \arrow[r] & * \arrow[r] & C
\end{tikzcd}$$

for some map d. The previous two diagrams show that the composite $\Omega K_5 \longrightarrow \Omega J_4(S^{12}) \xrightarrow{d} \Omega C \xrightarrow{\Omega c} \Omega\Sigma A \xrightarrow{r} B(27,35)$ is homotopic to γ. Define $e : \Omega J_4(S^{12}) \longrightarrow B(27,35)$ by the composite $e = r \circ \Omega c \circ d$.

Next consider the *EHP* fibration $S^{11} \longrightarrow \Omega J_4(S^{12}) \xrightarrow{T} \Omega S^{59}$. By connectivity, the composite $S^{11} \longrightarrow \Omega J_4(S^{12}) \xrightarrow{e} B(27,35)$ is trivial. This gives a homotopy fibration diagram

$$\begin{tikzcd}
\Omega^2 J_4(S^{12}) \arrow[r, "\Omega T"] \arrow[d, "\Omega e"] & \Omega^2 S^{59} \arrow[r] \arrow[d, "\epsilon"] & S^{11} \arrow[r] \arrow[d] & \Omega J_4(S^{12}) \arrow[d, "e"] \\
\Omega B(27,35) \arrow[r, equal] & \Omega B(27,35) \arrow[r] & * \arrow[r] & B(27,35)
\end{tikzcd}$$

for some map ϵ. We have shown:

Lemma 6.1. *In the homotopy fibration $\Omega^2 K_5 \xrightarrow{\Omega\gamma} \Omega B(27,35) \xrightarrow{\Omega\overline{\partial}} \Omega X$ there is a factorization of $\Omega\gamma$ as a composite*

$$\Omega^2 K_5 \longrightarrow \Omega^2 J_4(S^{12}) \xrightarrow{\Omega T} \Omega^2 S^{59} \xrightarrow{\epsilon} \Omega B(27,35)$$

for some map ϵ. $\square$

Lemma 6.2. *The composite $S^{57} \xrightarrow{E^2} \Omega^2 S^{59} \xrightarrow{\epsilon} \Omega B(27,35)$ has order $\leq 5^2$.*

Proof. Applying [MNT, 6.3] in the case of $B(27,35)$ shows $\pi_{58}(B(27,35)) \cong \mathbf{Z}/5^2\mathbf{Z}$. $\square$

Remark 2. It should be the case that the map ϵ in Lemma 6.1 is a loop map, $\epsilon \simeq \Omega\epsilon'$, where $\epsilon' : \Omega S^{35} \longrightarrow B(27,35)$ is a multiplicative extension of a map $\epsilon'' : S^{58} \longrightarrow B(27,35)$. In his proof of the indecomposability of X, Davis [D2] mentions that it is likely that the composite $S^{58} \xrightarrow{\epsilon''} B(27,35) \longrightarrow S^{35}$ is homotopic to α_3, the generator of the stable 23-stem. Any such lift of α_3 to $B(27,35)$ generates the element of order 5^2 in $\pi_{58}(B(27,35))$ referred to in the proof of Lemma 6.2. Another such lift would be $\epsilon \circ E^2$. Thus ϵ'', if it exists, or the map $\epsilon \circ E^2$ in Lemma 6.2, should have order exactly 5^2.

For an odd prime p, Cohen, Moore, and Neisendorfer [CMN] show there is a map $\phi : \Omega^2 S^{2n+1} \longrightarrow S^{2n-1}$ satisfying: (1) the composition $S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\phi} S^{2n-1}$ is homotopic to the degree p map, and (2) the composition $\Omega^2 S^{2n+1} \xrightarrow{\phi} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$ is homotopic to multiplication by p on $\Omega^2 S^{2n+1}$. We are about to use both of these properties of ϕ when $p = 5$ and $n = 29$.

The $p = 5$ case of Proposition 3.1 (a) gives maps $S^{59} \xrightarrow{t} K_5$ and $\Omega K_5 \xrightarrow{\overline{T}} \Omega S^{59}$ with the property that $\overline{T} \circ \Omega t$ is homotopic to multiplication by 5.

Lemma 6.3. *There is a homotopy commutative diagram*

$$\begin{tikzcd}
S^{57} \arrow[r, "E^2"] \arrow[d, "s"] & \Omega^2 S^{59} \arrow[r, "5"] & \Omega^2 S^{59} \arrow[d, "\Omega^2 t"] \\
\Omega^2 X \arrow[rr, "\Omega\delta"] & & \Omega^2 K_5
\end{tikzcd}$$

for some map s.

Proof. Consider the homotopy fibration $\Omega^2 X \xrightarrow{\Omega\delta} \Omega^2 K_5 \xrightarrow{\Omega\gamma} \Omega B(27,35)$. If $\Omega\gamma \circ (\Omega^2 t \circ 5 \circ E^2)$ is null homotopic then there is a lift of $\Omega^2 t \circ 5 \circ E^2$ to a map $s : S^{57} \longrightarrow \Omega^2 X$ which gives the asserted homotopy commutative diagram. To show the null homotopy, consider the diagram

$$\begin{tikzcd}
S^{57} \arrow[r, "E^2"] & \Omega^2 S^{59} \arrow[r, "5"] & \Omega^2 S^{59} \arrow[r, "\phi"] \arrow[d, "\Omega^2 t"] & S^{57} \arrow[d, "E^2"] & \\
& & \Omega^2 K_5 \arrow[r, "\Omega\overline{T}"] & \Omega^2 S^{59} \arrow[r, "\epsilon"] & \Omega B(27,35).
\end{tikzcd}$$

The commutative square in Proposition 3.1 (a) together with the factorization of the 5^{th}-power map on $\Omega^2 S^{59}$ through the double suspension imply that the middle square above homotopy commutes. Observe that the top row is homotopic to the degree 5^2 map on S^{57}. Lemma 6.2 then implies that the upper direction around the diagram is null homotopic. On the other hand, the definition of $\overline{T}$ and the factorization in Lemma 6.1 says that the bottom row in the diagram is homotopic to $\Omega\gamma$. Thus $\Omega\gamma \circ (\Omega^2 t \circ 5 \circ E^2)$ is null homotopic, as required. $\square$

Consider the map $S^{57} \xrightarrow{s} \Omega^2 X$ appearing in Lemma 6.3. Since $\Omega^2 X$ is a double loop space, s extends to a map $\bar{s} : \Omega^2 S^{59} \longrightarrow \Omega^2 X$. However, it is not clear whether $\bar{s}$ can be chosen so there is a homotopy commutative diagram

$$\begin{tikzcd}
\Omega^2 S^{59} \arrow[r, "5"] \arrow[d, "\bar{s}"] & \Omega^2 S^{59} \arrow[d, "\Omega^2 t"] \\
\Omega^2 X \arrow[r, "\Omega\delta"] & \Omega^2 K_5.
\end{tikzcd}$$

On the other hand, let u be the composite $u : \Omega^2 S^{59} \xrightarrow{\phi} S^{57} \xrightarrow{s} \Omega^2 X$. We do have the following.

Lemma 6.4. *There is a homotopy commutative diagram*

$$\begin{tikzcd}
\Omega^2 S^{59} \arrow[r, "5^2"] \arrow[d, "u"] & \Omega^2 S^{59} \arrow[d, "\Omega^2 t"] \\
\Omega^2 X \arrow[r, "\Omega\delta"] & \Omega^2 K_5,
\end{tikzcd}$$

where the map u is divisible by 5*.*

Proof. By definition of u, Lemma 6.3, and the factorization of multiplication by 5 on $\Omega^2 S^{59}$, we have $\Omega\delta \circ u = \Omega\delta \circ s \circ \phi \simeq (\Omega^2 t \circ 5 \circ E^2) \circ \phi \simeq \Omega^2 t \circ 5 \circ 5 \simeq \Omega^2 t \circ 5^2$. This proves the homotopy commutativity of the asserted diagram. The divisibility of u is given by the sequence of homotopies $u = s \circ \phi \simeq \bar{s} \circ E^2 \circ \phi \simeq \bar{s} \circ 5$. $\square$

Proposition 6.1. $\exp(X) \leq 5^{31}$.

Proof. Recall from Section 3 the homotopy fibration $M \longrightarrow \Omega K_5 \xrightarrow{\overline{T}} \Omega S^{59}$. Consider the homotopy pullback

$$\begin{tikzcd}
N \arrow[r] \arrow[d, "h"] & M \arrow[r] \arrow[d] & B(27,35) \arrow[d, equal] \\
\Omega X \arrow[r, "\delta"] \arrow[d, "\widetilde{T}"] & \Omega K_5 \arrow[r, "\gamma"] \arrow[d, "\overline{T}"] & B(27,35) \\
\Omega S^{59} \arrow[r, equal] & \Omega S^{59},
\end{tikzcd}$$

where $\widetilde{T}$ is defined as the composite $\overline{T} \circ \delta$ and the pullback defines the space N and the map h. Juxtaposing the homotopy commutative diagrams in Lemma 6.4 and Proposition 3.1 (a) gives a homotopy commutative diagram

$$\begin{tikzcd}
\Omega^2 S^{59} \arrow[r, "5^3"] \arrow[d, "u"] & \Omega^2 S^{59} \arrow[d, equal] \\
\Omega^2 X \arrow[r, "\Omega\widetilde{T}"] & \Omega^2 S^{59},
\end{tikzcd}$$

where u is divisible by 5. By Lemma 2.1 this results in a homotopy fibration

$$\Omega^2 N \times \Omega^3 S^{59} \xrightarrow{\Omega^2 h \cdot (-\Omega u)} \Omega^3 X \longrightarrow \Omega^2 S^{59}\{5^3\}.$$

Recall that $\exp(S^{59}) = 5^{29}$. Suppose $\exp(N) \leq 5^{29}$. Lemma 2.1 then says that $\exp(X) \leq 5^3 \cdot 5^{29} = 5^{32}$. But we can do a bit better by considering the order of the map $\Omega^2 h \cdot (-\Omega u)$ on torsion homotopy groups. Since u is divisible by 5, $\pi_*(u)$ has order 5^{28} on torsion homotopy groups. Now supoose we have a slightly stronger exponent bound, $\exp(N) \leq 5^{28}$. Then $\pi_*(\Omega^2 h \cdot (-\Omega u))$ has order $\leq 5^{28}$ on torsion homotopy groups. So (as in Corollary 2.1) we have $\exp(X) \leq 5^3 \cdot 5^{28} = 5^{31}$.

It remains to show that $\exp(N) \leq 5^{28}$. The pullback defining N gives a homotopy fibration

$$\Omega B(27,35) \longrightarrow N \longrightarrow M.$$

Thus $\exp(N) \leq \exp(B(27,35)) \cdot \exp(M)$. By Example 2.1, $\exp(B(27,35)) \leq 5^{15}$. By Proposition 3.1 (c), $\exp(M) = 5$. Thus $\exp(N) \leq 5^{16}$. ☐

7. The spherical resolution and exponent of Y

In this section we prove Theorem 1.1 (b), that Y is spherically resolved. We then iteratively use the methods in Section 2 to show that $\exp(Y) \leq 5^{26}$. Again, we assume that all homology calculations are with $\mathbf{Z}/5\mathbf{Z}$ coefficients.

Lemma 7.1. *There are homotopy fibrations*

(a) $Y_1 \longrightarrow Y \longrightarrow S^{47}$,

(b) $Y_2 \longrightarrow Y_1 \longrightarrow S^{39}$, *and*

(c) $S^{15} \longrightarrow Y_2 \longrightarrow S^{23}$.

Proof. Recall that $H^*(Y) \cong \Lambda(x_{15}, x_{23}, x_{39}, x_{47})$. The action of the Steenrod algebra is given by $\mathcal{P}^1(x_{15}) = x_{23}$ and $\mathcal{P}^1(x_{39}) = x_{47}$.

The proof of each part is by blunt force, part (a) being by far the longest. Let $(Y)_m$ be the m-skeleton of Y. Noting that Y has one cell in dimension 47, consider the pinch map $f : (Y)_{47} \longrightarrow S^{47}$ onto the top cell. We wish to show that f can be extended to a map $g : Y \longrightarrow S^{47}$. The Serre spectral sequence will then imply that the fiber Y_1 of g satisfies $H^*(Y_1) \cong \Lambda(x_{15}, x_{23}, x_{39})$.

The cells of Y in dimensions larger than 47 occur in dimensions

$$S = \{54, 62, 70, 77, 85, 86, 101, 109, 124\}.$$

Let $m_0 = 47$ and $m_1, \ldots, m_9$ be, in order from left to right, the elements of S. For $i \geq 1$, the obstruction to extending $(Y)_{m_{i-1}} \longrightarrow S^{47}$ to $(Y)_{m_i} \longrightarrow S^{47}$ is an element $t \in \pi_{m_i - 1}(S^{47})$. Note that each $m_i - 1$ is in the stable range of S^{47} so $t \in \pi^S_{m_i - 48}(S^0)$. Comparing the dimensions $m_i - 48$ to the dimensions of the stable elements in $\pi_*(S^0)$ we see there is only one match, when $i = 6$ and we have $\beta_1 \in \pi_{38}(S^0)$. Thus the only potential obstruction to extending f to g is $\beta_1 \in \pi_{85}(S^{47})$.

We show this potential obstruction is not an actual obstruction. Recall that β_1 is detected by the secondary operation

$$\psi : \text{Ker } \mathcal{P}^2 H^{47}(Y) \longrightarrow H^{86}(Y)/\mathcal{P}^3 H^{62}(Y)$$

associated to the Adem relation $\mathcal{P}^3\mathcal{P}^2 = 0$. If we can show ψ has no indeterminacy in these dimensions while $\psi(x_{47})$ is defined and equals zero, then β_1 is not detected and so is not an obstruction in extending f to g, proving part (a).

Observe that $H^{47}(Y)$ has basis $\{x_{47}\}$ and $H^{62}(Y)$ has basis $\{x_{23}x_{39}, x_{15}x_{47}\}$. Since $\mathcal{P}^1(x_{23}) = \mathcal{P}^1(x_{47}) = 0$, $\mathcal{P}^1(x_{15}) = x_{23}$, and $\mathcal{P}^1(x_{39}) = x_{47}$, we have $\text{Ker}\mathcal{P}^2 H^{47}(Y) = H^{47}(Y)$ and $\mathcal{P}^3 H^{62}(Y) = 0$. Thus $\psi : H^{47}(Y) \longrightarrow H^{86}(Y)$ is well-defined and has no indeterminacy. A similar calculation shows that

$$\psi : \text{Ker } \mathcal{P}^2 H^{47}(Y \times Y) \longrightarrow H^{86}(Y \times Y)/\mathcal{P}^3 H^{62}(Y \times Y)$$

has no indeterminacy and gives a well-defined map $\psi : H^{47}(Y \times Y) \longrightarrow H^{86}(Y \times Y)$. Since Y is a retract of the H-space E_8, it has a multiplication $\mu : Y \times Y \longrightarrow Y$. This gives a comultiplication $\mu^* : H^*(Y) \longrightarrow H^*(Y) \otimes H^*(Y)$. The two maps $\psi : H^{47}(\ ) \longrightarrow H^{86}(\ )$ above together with the naturality of ψ implies that $\psi(x_{45})$ is primitive. But a basis for $H^{86}(Y)$ is $\{x_{39}x_{47}\}$ which is not primitive. Hence $\psi(x_{47}) = 0$, as required.

For part (b), apply the same method in trying to extend the pinch map $f : (Y_1)_{39} \longrightarrow S^{39}$ to a map $g : Y_1 \longrightarrow S^{39}$. Here $H^*(Y_1) \cong \Lambda(x_{15}, x_{23}, x_{39})$ so Y_1 has cells of dimension greater than 39 in dimensions $\{54, 62, 77\}$. The potential obstructions in extending f to g are in $\pi_{m+39}(S^{39}) = \pi_m^S(S^0)$ for $m \in \{14, 22, 37\}$. But all such groups are zero. Thus f extends to g and the Serre spectral sequence implies the fiber Y_2 of g satisfies $H^*(Y_2) \cong \Lambda(x_{15}, x_{23})$.

For part (c), the one obstruction to extending the pinch map $(Y_2)_{23} \longrightarrow S^{23}$ to $Y_2 \longrightarrow S^{23}$ is an element of $\pi_{37}(S^{23}) = \pi_{14}^S(S^0) = 0$. Thus the extension exists and the Serre spectral sequence implies its fiber has the cohomology of S^{15} and so is homotopy equivalent to S^{15}. □

Lemma 7.2. *There are compositions*

(a) $S^{47} \longrightarrow Y \longrightarrow S^{47}$ *of degree* 5^3,

(b) $S^{39} \longrightarrow Y_1 \longrightarrow S^{39}$ *of degree* 5^2, *and*

(c) $S^{23} \longrightarrow Y_2 \longrightarrow S^{23}$ *of degree* 5.

Proof. We first prove (c), then (b), and then (a). It suffices to show the right adjoints of the composites exist. The homotopy fibration $S^{15} \longrightarrow Y_2 \xrightarrow{q_2} S^{23}$ of Lemma 7.1 (c) results in a homotopy fibration $\Omega Y_2 \xrightarrow{\Omega q_2} \Omega S^{23} \longrightarrow S^{15}$. Consider the composite $S^{22} \xrightarrow{E} \Omega S^{23} \longrightarrow S^{15}$, where E is the suspension. Since $\pi_{23}(S^{15}) = \mathbf{Z}/5\mathbf{Z}$ (generated by α_1) we have $5 \cdot E$ lifting through Ωq_2. This proves (c).

Next, the homotopy fibration $Y_2 \longrightarrow Y_1 \xrightarrow{q_1} S^{39}$ of Lemma 7.1 (b) results in a homotopy fibration $\Omega Y_1 \xrightarrow{\Omega q_1} \Omega S^{39} \longrightarrow Y_2$. Consider the composite $S^{38} \xrightarrow{E} \Omega S^{39} \longrightarrow Y_2$. In the fibration $S^{15} \longrightarrow Y_2 \longrightarrow S^{23}$ we have $\pi_{38}(S^{15}) = \mathbf{Z}/5\mathbf{Z}$ and $\pi_{38}(S^{23}) = \mathbf{Z}/5\mathbf{Z}$ (generated by α_3 and α_2 respectively). Thus $5^2 \cdot \pi_{38}(Y_2) = 0$. So $5^2 \cdot E$ lifts through Ωq_1, proving (b).

Part (c) will follow as in the proof of (b) once we know $5^3 \cdot \pi_{46}(Y_1) = 0$. In the fibration $S^{15} \longrightarrow Y_2 \longrightarrow S^{23}$ we have $\pi_{46}(S^{15}) = \mathbf{Z}/5\mathbf{Z}$ and $\pi_{46}(S^{23}) = \mathbf{Z}/5\mathbf{Z}$ (generated by α_4 and α_3 respectively) so $5^2 \cdot \pi_{46}(Y_2) = 0$. Combine this with the fibration $Y_2 \longrightarrow Y_1 \longrightarrow S^{39}$ and the fact that $\pi_{46}(S^{39}) = \mathbf{Z}/5\mathbf{Z}$ (generated by α_1) and we have $5^3 \cdot \pi_{46}(Y_1) = 0$. □

Lemma 7.3. *The following exponent bounds hold*:

(a) $\exp(Y) \leq 5^{26}$,

(b) $\exp(Y_1) \leq 5^{21}$, *and*

(c) $\exp(Y_2) \leq 5^{13}$.

Proof. Again, we first prove (c), then (b), and then (a). In what follows, we apply Lemma 2.1 assuming Y_1 and Y_2 are H-spaces; if they are not, apply Lemma 2.1 to ΩY_1 and ΩY_2.

Use part (c) of both Lemmas 7.1 and 7.2 as input into Lemma 2.1 to obtain a homotopy fibration $\Omega S^{15} \times \Omega S^{23} \longrightarrow \Omega Y_2 \longrightarrow S^{23}\{5\}$, implying $\exp(Y_2) \leq 5 \cdot \exp(S^{23}) = 5^{13}$.

Do the same for part (b) of Lemmas 7.1 and 7.2 to obtain a homotopy fibration $\Omega Y_2 \times \Omega S^{39} \longrightarrow \Omega Y_1 \longrightarrow S^{39}\{5^2\}$. Since $\exp(Y_2) < \exp(S^{39}) = 5^{19}$, we have $\exp(Y_1) \leq 5^2 \cdot \exp(S^{39}) = 5^{21}$.

Similarly, from part (a) of Lemmas 7.1 and 7.2 we obtain a homotopy fibration $\Omega Y_1 \times \Omega S^{47} \longrightarrow \Omega Y \longrightarrow S^{47}\{5^3\}$, and as $\exp(Y_1) < \exp(S^{47}) = 5^{23}$, we have $\exp(Y) \leq 5^3 \cdot \exp(S^{47}) = 5^{26}$. □

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