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On the class number of the genus of $\mathbb{Z}$-maximal lattices with respect to quadratic form of the sum of squares

By

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Abstract

In this paper we consider the quadratic form $\varphi[x] = \sum_{i=1}^{n} x_i^2$ over the vector space $\mathbb{Q}_n^1$. We take a $\mathbb{Z}$-maximal lattice L in $\mathbb{Q}_n^1$ with respect to φ. Let $\{L^{(i)}\}_{i=1}^{k(n)}$ be a complete set of representatives for the classes belonging to the genus of L. Applying Shimura's mass formula, we determine these representatives $L^{(i)}$ explicitly for $n = 11, 13$, and 14. Consequently we obtain class numbers $k(11) = 3$, $k(13) = 4$, and $k(14) = 4$.

1. Introduction

We denote by $\mathbb{Q}_n^1$ the n-dimensional row vector space over the rational number field $\mathbb{Q}$. Let $\varphi : \mathbb{Q}_n^1 \times \mathbb{Q}_n^1 \to \mathbb{Q}$ be a nondegenerate symmetric $\mathbb{Q}$-bilinear form. We denote by $\varphi[x]$ the quadratic form $\varphi(x, x)$ on $\mathbb{Q}_n^1$. In this paper we consider the case $\varphi[x] = x1_n \cdot {}^t x = \sum_{i=1}^{n} x_i^2$ for $x = (x_1, \ldots, x_n) \in \mathbb{Q}_n^1$, exclusively. By a $\mathbb{Z}$-*maximal* lattice L with respect to φ, we understand a $\mathbb{Z}$-lattice L in $\mathbb{Q}_n^1$ which is maximal among $\mathbb{Z}$-lattices on which the values $\varphi[x]$ are contained in $\mathbb{Z}$. Let G^φ be the orthogonal group of φ, that is, $G^\varphi = \{\alpha \in GL_n(\mathbb{Q}) \mid \alpha \cdot {}^t\alpha = 1_n\}$, and consider the adelization $G_{\mathbb{A}}^\varphi$ of G^φ. For L and $a \in G_{\mathbb{A}}$, let La be the $\mathbb{Z}$-lattice in $\mathbb{Q}_n^1$ such that $(La)_p = L_p a_p$ for all primes p. Here, L_p is the $\mathbb{Z}_p$-lattice in $(\mathbb{Q}_p)_n^1$ which is spanned by L over $\mathbb{Z}_p$. We call $\{La \mid a \in G_{\mathbb{A}}^\varphi\}$ (resp. $\{L\alpha \mid \alpha \in G^\varphi\}$) the genus (resp. the class) of L with respect to φ. It is known that the genus of L is a disjoint union of finitely many classes. Let $\{L^{(i)}\}_{i=1}^{k(n)}$ be a complete set of representatives for the classes belonging to the genus of a $\mathbb{Z}$-maximal lattice L. We call the number $k(n)$ the class number of the genus of L. Here, we remark that $k(n)$ does not depend on the choice of a $\mathbb{Z}$-maximal lattice L (see §2 below).

It is known $k(n) = 1$ for $1 \le n \le 9$ (see [8, Lemma 1.6]), and $k(n) = 2$ for $n = 10, 12, 15$ and 16, which will be mentioned in §3. The main purpose of this

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paper is to determine the number $k(n)$ for $n = 11$, 13, and 14. The mass $\mathfrak{m}(L)$ of the genus of L is defined by

$$\mathfrak{m}(L) = \sum_{i=1}^{k(n)} [\Gamma(L^{(i)}) : 1]^{-1}. \tag{1.1}$$

Here, $\Gamma(L^{(i)}) = \{\alpha \in G^{\varphi} \mid L^{(i)}\alpha = L^{(i)}\}$. As for the value $\mathfrak{m}(L)$, we have the formula by [6, Theorem 5.8]. Therefore, to determine the number $k(n)$, first we calculate the left-hand side of (1.1) by Shimura's mass formula. On the other hand, we evaluate $[\Gamma(L^{(i)}) : 1]$ in (1.1) by finding a complete set of representatives $\{L^{(i)}\}_{i=1}^{k(n)}$ for the classes belonging to the genus of L, for which the equation (1.1) holds.

Now, we summarize the contents of the paper. In §2, we recall the notion of the genus, the class and the mass of a $\mathbb{Z}$-lattice with respect to φ. In §3, we give a criterion for the "$\mathbb{Z}$-maximality" of $\mathbb{Z}$-lattices which contains $\mathbb{Z}_n^1$. In §4, we explain how to evaluate the order of $\Gamma(L^{(i)})$ and give a set of representatives for the classes. The result is $k(11) = 3$, $k(13) = 4$, and $k(14) = 4$.

As a final remark, it should be noted that our class number $k(n)$ concerns only the genus of $\mathbb{Z}$-maximal lattices with respect to $\varphi[x] = \sum_{i=1}^{n} x_i^2$. As mentioned above our determination of this number relies on the formula due to Shimura, which is given for the case of $\mathbb{Z}$-maximal lattices. As explained in the introduction of [6], the maximal lattice plays, the similar role to which the maximal order in algebraic number theory. The results on the maximal lattices are more fundamental than results on smaller lattices.

Notation. We denote by $\mathbb{Q}$ the rational number field and by $\mathbb{Z}$ the ring of rational integers, by $\mathbb{Z}_+$ the set of all positive integers. For a rational prime p, we denote by $\mathbb{Q}_p$ the field of p-adic numbers and by $\mathbb{Z}_p$ the ring of p-adic integers. We put $\mathbb{Q}_\infty = \mathbb{R}$. For an associative ring R with identity element, if M is a R-module, then we write M_n^m for the R-module of all $m \times n$ matrices with entries in M. We write 1_n for the identity element of R_n^n. For $x \in R_n^m$ we denote by ${}^t x$ the transpose of x. For $x \in R_n^n$ we denote by $\det(x)$ the determinant of x. The set of all invertible elements of R_n we write $GL_n(R)$. For R-module A and B, we denote by $A \oplus B$ the direct sum of A and B as R-module and we denote by $A^{\oplus n}$ the direct sums of n copies of A. If F is the field of quotients of a Dedekind domain $\mathfrak{o}$ and if W is an n-dimensional vector space over F, then by an $\mathfrak{o}$-lattice in W, we understand a finitely generated $\mathfrak{o}$-module in W that spans W over F. By a lattice we mean a finitely generated $\mathfrak{o}$-module in W (not necessarily span W over F). We denote by $\#X$ the number of elements of X. If X is a disjoint union of its subsets $Y_1, \ldots, Y_m$, we write $\bigsqcup_{i=1}^m Y_i$ or $Y_1 \sqcup \cdots \sqcup Y_m$. If B is a subgroup of a group A, we put $[A : B] = \#(A/B)$. We denote by δ_{ij} the Kronecker's delta.

2. Preliminaries

We recall here the notion of [5], [6], [7] and [8] which will be needed in this paper.

2.1. Quadratic form

For a positive integer n, let $V = \mathbb{Q}_n^1$ be an n-dimensional vector space over $\mathbb{Q}$. We fix the standard basis $\{e_i\}_{i=1}^n$ of V. We take a nondegenerate symmetric $\mathbb{Q}$-bilinear form $\varphi : V \times V \to \mathbb{Q}$. We take the matrix $\varphi_0 \in GL_n(\mathbb{Q})$ such that $\varphi(x,y) = x\varphi_0 \cdot {}^t y$ for any $x, y \in V$. We set $\varphi[x] = \varphi(x,x)$ for $x \in V$. Here we assume that φ is *positive definite*. We identify φ with φ_0. We put $G^\varphi = \{\alpha \in GL_n(\mathbb{Q}) \mid \alpha\varphi \cdot {}^t\alpha = \varphi\}$ and call the orthogonal group of φ, and we put $G^\varphi_+ = \{\alpha \in G^\varphi \mid \det(\alpha) = 1\}$. Let $\mathbf{h}$ denote the set of nonarchimedean primes, and ∞ denote the archimedean prime of $\mathbb{Q}$. We put $\mathbf{v} = \{\infty\} \cup \mathbf{h}$. For any $v \in \mathbf{v}$ and a $\mathbb{Z}$-lattice L in V, we put

$$G^\varphi_v = \{\alpha_v \in GL_n(\mathbb{Q}_v) \mid \alpha_v\varphi \cdot {}^t\alpha_v = \varphi\},$$

$$G^\varphi_{\mathbb{A}} = \left\{ (\alpha_v) \in \prod_{v \in \mathbf{v}} G^\varphi_v \mid L_p\alpha_p = L_p \text{ for almost all } p \in \mathbf{h} \right\}.$$

Here, L_p is the $\mathbb{Z}_p$-lattice in $(\mathbb{Q}_p)_n^1$ which is spanned by L over $\mathbb{Z}_p$. $G^\varphi_{\mathbb{A}}$ is independent of the choice of L. G^φ (resp. G^φ_v) acts V (resp. $V_v = V \otimes_{\mathbb{Q}} \mathbb{Q}_v$) on the right.

For a given $\mathbb{Z}$-lattice L in V, we define the class $\mathfrak{k}(L)$ and the genus $\Lambda(L)$ of L by

$$\mathfrak{k}(L) = LG^\varphi = \{L\alpha \mid \alpha \in G^\varphi\}, \quad \Lambda(L) = LG^\varphi_{\mathbb{A}} = \{La \mid a \in G^\varphi_{\mathbb{A}}\}.$$

Here, for an element $a \in G^\varphi_{\mathbb{A}}$, we denote by La a $\mathbb{Z}$-lattice in V such that $(La)_p = L_p a_p$ for all $p \in \mathbf{h}$. Then there exists a finite set of $\mathbb{Z}$-lattices $\{L^{(i)}\}_{i=1}^k$ such that

$$\Lambda(L) = \bigsqcup_{i=1}^k \mathfrak{k}(L^{(i)})$$

(cf. [5, Lemma 8.7 (4)]). We call this k the class number of the genus $\Lambda(L)$. For a positive integer h, we put

$$\nu(L,h) = \{x \in L \mid \varphi[x] = h\}.$$

2.2. Shimura's mass formula

For a $\mathbb{Z}$-lattice L in V, we put $\Gamma(L) = \{\alpha \in G^\varphi \mid L\alpha = L\}$. Let $\{L_i\}_{i=1}^{k(n)}$ be a complete set of representatives for the classes belonging to the genus $\Lambda(L)$. If $\mathfrak{k}(L) = \mathfrak{k}(L')$, then $[\Gamma(L) : 1] = [\Gamma(L') : 1]$. Thus $[\Gamma(L^{(i)}) : 1]$ is independent of the choice of a representative of $\mathfrak{k}(L^{(i)})$. Now we introduce the notion of the mass of the genus of L by the sum

$$\mathfrak{m}(L) = \sum_{i=1}^k [\Gamma(L^{(i)}) : 1]^{-1}.$$

We again note that $\mathfrak{m}(L)$ depends only on the genus of L.

We put

$$C = \{\alpha \in G^{\varphi}_{\mathbb{A}} \mid L\alpha = L\}, \quad C^{+} = (G^{\varphi}_{+})_{\mathbb{A}} \cap C,$$
$$\Gamma^{a} = aCa^{-1} \cap G \quad (a \in G^{\varphi}_{\mathbb{A}}), \quad \Gamma^{a}_{+} = G^{\varphi}_{+} \cap \Gamma^{a},$$
$$\mathfrak{m}(G^{\varphi}, C) = \sum_{G^{\varphi}aC \in \mathcal{B}} [\Gamma^{a} : 1]^{-1}, \quad \mathfrak{m}(G^{\varphi}_{+}, C^{+}) = \sum_{G^{\varphi}_{+}aC^{+} \in \mathcal{B}_{+}} [\Gamma^{a}_{+} : 1]^{-1}.$$

Here, $\mathcal{B}$ (resp. $\mathcal{B}_{+}$) is a complete set of representatives of the classes of $G^{\varphi} \backslash G^{\varphi}_{\mathbb{A}}/C$ (resp. $G^{\varphi}_{+} \backslash (G^{\varphi}_{+})_{\mathbb{A}}/C^{+}$). By taking a bijective map

$$G^{\varphi} \backslash G^{\varphi}_{\mathbb{A}}/C \ni G^{\varphi}aC \mapsto \mathfrak{k}(La^{-1}) \in \Lambda(L)/G^{\varphi} (= \{\mathfrak{k}(L^{(i)}) \mid 1 \leq i \leq k\}),$$

we see that $\mathfrak{m}(G^{\varphi}, C) = \mathfrak{m}(L)$. Moreover, by [6, Lemma 5.6(1)], we have $\mathfrak{m}(G^{\varphi}, C) = 2^{-1}\mathfrak{m}(G^{\varphi}_{+}, C^{+})$.

If L and L' are $\mathbb{Z}$-maximal in V, then L belongs to the same genus of L'. This follows from [5, Lemma 5.9] and [5, Lemma 8.10].

Now we assume that $\varphi = 1_n$. Let L be a $\mathbb{Z}$-maximal lattice in V with respect to φ. We use the formula in [6, Examples 5.16], which is given as follows:

$$\mathfrak{m}(G^{\varphi}_{+}, C^{+}) = \prod_{k=1}^{m} (4k)^{-1}|B_{2k}| \cdot \begin{cases} 2 & \text{if } n \pm 1 \in 8\mathbb{Z}, \\ 3^{-1}(2^{m} - 1) & \text{if } n \pm 3 \in 8\mathbb{Z}, \end{cases} \tag{2.1}$$

$$\begin{aligned} \mathfrak{m}(G^{\varphi}_{+}, C^{+}) = (2m)^{-1}|B_{m}| \prod_{k=1}^{m-1} (4k)^{-1}|B_{2k}| \\ \cdot \begin{cases} 2 & \text{if } n \in 8\mathbb{Z}, \\ 3^{-1}(2^{m} - 1)(2^{m-1} - 1) & \text{if } n - 4 \in 8\mathbb{Z}, \end{cases} \end{aligned} \tag{2.2}$$

$$\begin{aligned} \mathfrak{m}(G^{\varphi}_{+}, C^{+}) = 2^{n-1}(m-1)!(2\pi)^{-m}L(m, \psi) \\ \cdot \prod_{k=1}^{m-1} (4k)^{-1}|B_{2k}| \quad \text{if } n \pm 2 \in 8\mathbb{Z}. \end{aligned} \tag{2.3}$$

Here $m = [n/2]$ and ψ is the primitive Dirichlet character modulo 4 and B_{2k} is the $2k$-th Bernoulli number. Then we obtain $\mathfrak{m}(L)$ for $n = 11, 13$, and 14.

$$\mathfrak{m}(L) = \begin{cases} 31 \cdot (2^{19} \cdot 3^{6} \cdot 5^{2} \cdot 7)^{-1} & \text{if } n = 11, \\ 691 \cdot (2^{23} \cdot 3^{7} \cdot 5^{2} \cdot 7 \cdot 11)^{-1} & \text{if } n = 13, \\ 61 \cdot 691 \cdot (2^{25} \cdot 3^{8} \cdot 5^{3} \cdot 7^{2} \cdot 11 \cdot 13)^{-1} & \text{if } n = 14. \end{cases} \tag{2.4}$$

3. $\mathbb{Z}$-maximality and known results

Hereafter and until the end of this paper, we assume that $\varphi = 1_n$. To obtain the class number of the genus we have to find an explicit representatives of the classes. Therefore, we consider the properties which is required for the maximality.

3.1. $\mathbb{Z}$-maximality

Let L be a $\mathbb{Z}$-lattice in $\mathbb{Q}_n^1$ which contains $\mathbb{Z}_n^1$. Let $\varepsilon_1, \ldots, \varepsilon_n$ be a basis of L. We define a matrix ε by

$$\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix} \in GL_n(\mathbb{Q}).$$

Then we can express $L = \sum_{i=1}^n \mathbb{Z}\varepsilon_i = \mathbb{Z}_n^1 \varepsilon$.

Now, put $\widetilde{L} = \{x \in V \mid 2\varphi(x, L) \subset \mathbb{Z}\}$. If $\varphi[L] \subset \mathbb{Z}$, then we have

$$\mathbb{Z}_n^1 \subset L \subset \widetilde{L} \subset \widetilde{\mathbb{Z}_n^1} = 2^{-1}\mathbb{Z}_n^1 \tag{3.1}$$

and $[\widetilde{\mathbb{Z}_n^1} : \widetilde{L}] = [L : \mathbb{Z}_n^1] = |\det(\varepsilon^{-1})|$, and hence

$$[\widetilde{L} : L] = 2^n \cdot \det(\varepsilon)^2. \tag{3.2}$$

Lemma 3.1. *Let $L = \sum_{i=1}^n \mathbb{Z}\varepsilon_i$ be a $\mathbb{Z}$-lattice in $\mathbb{Q}_n^1$ which contains $\mathbb{Z}_n^1$. Then the following conditions are equivalent:*
(a) L is $\mathbb{Z}$-maximal with respect to φ,
(b) $\{\varepsilon_i\}_{i=1}^n$ satisfies

$$\begin{cases} \varphi[\varepsilon_i] \in \mathbb{Z}, \\ 2\varphi(\varepsilon_i, \varepsilon_j) \in \mathbb{Z} \text{ for } 1 \le i, j \le n,\ i \ne j\ , \end{cases} \tag{3.3}$$

and

$$|\det(\varepsilon)| = \begin{cases} 2^{(a-n)/2} & \text{if } n \equiv \pm a \pmod 8, \quad 0 \le a \le 3, \\ 2^{(2-n)/2} & \text{if } n \equiv 4 \pmod 8. \end{cases} \tag{3.4}$$

Proof. Now we assume that $L = \sum_{i=1}^n \mathbb{Z}\varepsilon_i$ is a $\mathbb{Z}$-maximal lattice which contains $\mathbb{Z}_n^1$. We first note that $\mathbb{Z}$-maximal lattice L satisfies $\varphi[L] \subset \mathbb{Z}$, and the condition $\varphi[L] \subset \mathbb{Z}$ is equivalent to (3.3). Under the condition (3.3), the maximality of L implies the condition

$$[\tilde{L} : L] = \begin{cases} 2^a & \text{if } n \equiv \pm a \pmod 8, \quad 0 \le a \le 3, \\ 2^2 & \text{if } n \equiv 4 \pmod 8. \end{cases} \tag{3.5}$$

by the equality of [8, (1.6)], which is equivalent to (3.4) by (3.2).

Conversely, we assume that $L = \sum_{i=1}^{n} \mathbb{Z}\varepsilon_i$ is a $\mathbb{Z}$-lattice containing $\mathbb{Z}_n^1$ and satisfying (3.3) and (3.4). Then L also satisfies (3.5). There exists a $\mathbb{Z}$-maximal lattice H in $\mathbb{Q}_n^1$ such that $L \subset H$ (cf. [5, Lemma 4.8(1)]). Since H is $\mathbb{Z}$-maximal, $[\widetilde{H} : H]$ is equal to the right hand side of (3.5), and hence $[\widetilde{H} : H] = [\widetilde{L} : L]$. Since $L \subset H \subset \widetilde{H} \subset \widetilde{L}$, we have $L = H$. Therefore L is $\mathbb{Z}$-maximal. Hence we obtain the assertion of our lemma. □

3.2. The class number $k(n)$ for $n = 10$, 12, 15, and 16

Now, we introduce the results for the class number $k(n)$ of the genus of $\mathbb{Z}$-maximal lattices in the cases of $n = 10, 12, 15$, and 16, which are followed from the results of classification of even integral lattices in [2] and [9]. As for the notation and terminology concerning lattice theory, see [1]. For root lattices $M = A_1$, E_7, E_8, and D_{2n}, when we express $L = \mathbb{Z}_n^1\delta$ with $\delta \in \mathbb{Q}_n^m$, it can be seen that there exists a $\mathbb{Z}$-lattice $L = \mathbb{Z}_n^1\varepsilon$ such that $\delta \cdot {}^t\delta = 2^{-1}\varepsilon \cdot {}^t\varepsilon$, which is unique up to isomorphism. We fix such L which contains $\mathbb{Z}_n^1$ and put $M(2^{-1}) = L$. Especially for $M = D_{2n}$, we take

$$D_{2n}(2^{-1}) = \sum_{i=1}^{2n} \mathbb{Z}\varepsilon_i,$$

where

$$\varepsilon_i = \begin{cases} e_i & \text{if } i \text{ is odd or } i = 2, \\ f_i = 2^{-1}(e_i + e_{i-1} + e_{i-2} + e_{i-3}) & \text{otherwise.} \end{cases} \tag{3.6}$$

We put

$$\begin{aligned} &L_1 = A_1(2^{-1}), && L_2 = A_1^{\oplus 2}(2^{-1}), && L_3 = A_1^{\oplus 3}(2^{-1}), \\ &L_4 = D_4(2^{-1}), && L_5 = D_4(2^{-1}) \oplus A_1(2^{-1}), && L_6 = D_6(2^{-1}), \\ &L_7 = E_7(2^{-1}), && L_8 = E_8(2^{-1}), && L_9 = E_8(2^{-1}) \oplus A_1(2^{-1}), \end{aligned}$$

then L_n is $\mathbb{Z}$-maximal by Lemma 3.1. Since $k(n) = 1$ for $1 \le n \le 9$, by [8, Lemma 1.6], L_n is a set of representatives for the classes belonging to the genus of $\mathbb{Z}$-maximal lattices for $1 \le n \le 9$.

By the result of the classification of even integral lattices in [2], we can take a set of representatives for $n = 10$ as $\{D_{10}(2^{-1}),\ E_8(2^{-1}) \oplus A_1^{\oplus 2}(2^{-1})\}$. Hence we see that $k(10) = 2$. Similarly, we can take a set of representatives for $n = 12$ as $\{E_8(2^{-1}) \oplus D_4(2^{-1}),\ D_{12}(2^{-1})\ \}$ and for $n = 15$ as $\{E_8(2^{-1}) \oplus E_7(2^{-1}),\ D_{14}(2^{-1}) + \mathbb{Z}h_{15}\}$. Here, $h_{15} = 2^{-1}(e_1 + e_3 + e_5 + e_7 + e_9 + e_{11} + e_{13} + e_{15})$. Consequently, we have $k(12) = 2$ and $k(15) = 2$. When $n = 16$, Witt proved in [9] that the class number of the genus of even positive unimodular lattices is 2. Hence we see that $k(16) = 2$, and a representatives is given as $\{E_8(2^{-1}) \oplus E_8(2^{-1}),\ D_{16}^{+}(2^{-1})\ \}$, where $D_{16}^{+} = D_{16} \cup (D_{16} + v)$ and $v = 2^{-1}(1, \ldots, 1) \in \mathbb{Q}_{16}^1$.

4. Determination of the class numbers

In this section we determine the class number $k(n)$ of the genus of $\mathbb{Z}$-maximal lattice with respect to $\varphi = 1_n$ by taking a complete set of representatives for the classes in the genus explicitly in the cases of $n = 11, 13$, and 14. For a lattice $L = \mathbb{Z}_n^1 \delta$ with $\delta \in \mathbb{Q}_m^n$, we put $\det(L) = \det(\delta \cdot {}^t\delta)$ and call the determinant of L.

4.1. 11-dimensional case

Let $n = 11$. We put

$$L^{(1)} = E_8(2^{-1}) \oplus A_1^{\oplus 3}(2^{-1}), \quad L^{(2)} = D_{10}(2^{-1}) \oplus A_1(2^{-1}),$$
$$L^{(3)} = E_7(2^{-1}) \oplus D_4(2^{-1}).$$

$L^{(1)}$, $L^{(2)}$, and $L^{(3)}$ are $\mathbb{Z}$-maximal lattices by Lemma 3.1, since $E_8 \oplus A_1^{\oplus 3}$, $D_{10} \oplus A_1$, $E_7 \oplus D_4$ are even integral and whose determinants are equal to 2^3. Moreover by [4, Lemma 1.4.6 and §4], we have

$$\begin{aligned}[\Gamma(L^{(1)}) : 1] &= [\Gamma(E_8) : 1] \cdot [\Gamma(A_1^{\oplus 3}) : 1] \\ &= (2^{14} \cdot 3^5 \cdot 5^2 \cdot 7)(2^4 \cdot 3) \\ &= 2^{18} \cdot 3^6 \cdot 5^2 \cdot 7,\end{aligned}$$

$$\begin{aligned}[\Gamma(L^{(2)}) : 1] &= [\Gamma(D_{10}) : 1] \cdot [\Gamma(A_1) : 1] \\ &= (2^{10} \cdot 10!) \cdot 2 \\ &= 2^{19} \cdot 3^4 \cdot 5^2 \cdot 7,\end{aligned}$$

$$\begin{aligned}[\Gamma(L^{(3)}) : 1] &= [\Gamma(E_7) : 1] \cdot [\Gamma(D_4) : 1] \\ &= (2^{10} \cdot 3^4 \cdot 5 \cdot 7)(2^7 \cdot 3^2) \\ &= 2^{17} \cdot 3^6 \cdot 5 \cdot 7.\end{aligned}$$

From these, we see that $\mathfrak{k}(L^{(i)}) \neq \mathfrak{k}(L^{(j)})$ for $i \neq j$. On the other hand, by (2.4), we have

$$\begin{aligned}\mathfrak{m}(L^{(1)}) &= 31 \cdot (2^{19} \cdot 3^6 \cdot 5^2 \cdot 7)^{-1} \\ &= [\Gamma(L^{(1)}) : 1]^{-1} + [\Gamma(L^{(2)}) : 1]^{-1} + [\Gamma(L^{(3)}) : 1]^{-1}.\end{aligned}$$

Hence we obtain $k(11) = 3$.

4.2. $\mathbb{Z}$-maximal lattices N_{13} and N_{14}

We define two lattices N_{13} and N_{14} by

$$N_{13} = \sum_{i=1}^{10} \mathbb{Z}\varepsilon_i + \mathbb{Z}e_{11} + \mathbb{Z}e_{12} + \mathbb{Z}h_{13} = D_{10}(2^{-1}) \oplus A_1^{\oplus 2}(2^{-1}) + \mathbb{Z}h_{13},$$
$$N_{14} = \sum_{i=1}^{13} \mathbb{Z}\varepsilon_i + \mathbb{Z}h_{14} = D_{12}(2^{-1}) \oplus A_1(2^{-1}) + \mathbb{Z}h_{14},$$

where ε_i is as in (3.6),

$$h_{13} = 2^{-1}(e_1 + e_3 + e_5 + e_7 + e_9 + e_{11} + e_{12} + e_{13}),$$
$$h_{14} = 2^{-1}(e_1 + e_3 + e_5 + e_7 + e_9 + e_{11} + e_{13} + e_{14}).$$

Then N_{13} and N_{14} have the required properties with respect to the $\mathbb{Z}$-maximality in Lemma 3.1.

Lemma 4.1. *We have*

$$\begin{aligned} N_{13} = \{2^{-1}(x_1, \ldots, x_{13}) \in (2^{-1}\mathbb{Z})^1_{13} \mid & \textstyle\sum_{i=1}^{5} x_{2i} \equiv 0 \pmod 2, \\ & x_{2i-1} + x_{2i} \equiv x_{13} \pmod 2 \ (1 \le i \le 5), \\ & x_{11} \equiv x_{12} \equiv x_{13} \pmod 2\}, \end{aligned}$$

$$\begin{aligned} N_{14} = \{2^{-1}(x_1, \ldots, x_{14}) \in (2^{-1}\mathbb{Z})^1_{14} \mid & \textstyle\sum_{i=1}^{6} x_{2i} \equiv 0 \pmod 2, \\ & x_{2i-1} + x_{2i} \equiv x_{14} \pmod 2 \ (1 \le i \le 6), \\ & x_{13} \equiv x_{14} \pmod 2\}. \end{aligned}$$

Proof. Let $\varphi = 1_m$. Since $\mathbb{Z}^1_m \subset N_m$ and $\varphi[N_m] \subset \mathbb{Z}$, we have $N_m \subset (2^{-1}\mathbb{Z})^1_m$ by (3.1). Now we put

$$\varepsilon^{(13)} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ f_4 \\ e_5 \\ f_6 \\ e_7 \\ f_8 \\ e_9 \\ f_{10} \\ e_{11} \\ e_{12} \\ h_{13} \end{bmatrix}, \qquad \varepsilon^{(14)} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ f_4 \\ e_5 \\ f_6 \\ e_7 \\ f_8 \\ e_9 \\ f_{10} \\ e_{11} \\ f_{12} \\ e_{13} \\ h_{14} \end{bmatrix}. \tag{4.1}$$

Then we have $N_m = \mathbb{Z}^1_m \varepsilon$. For $x = 2^{-1}(x_1, \ldots, x_m) \in (2^{-1}\mathbb{Z})^1_m$, the condition $x \in N_m$ is equivalent to $x(\varepsilon^{(m)})^{-1} \in \mathbb{Z}^1_m$, which shows the assertions of our lemma. □

For $m = 13$ and 14, we define the subgroup H_m of $GL_m(\mathbb{Q})$ by

$$H_m = \left\{ \left[\begin{array}{c|c} a & \\ \hline & d \end{array}\right] \;\middle|\; a \in \Gamma(D_{2(m-8)}(2^{-1})), d \in \Gamma(A_1^{\oplus 16-m}(2^{-1})) \right\}.$$

Then by [4, Lemma 1.4.4], there exists a injective homomorphism of $\Gamma(N_m)$ into H_m. Hence we have $\Gamma(N_m) \subset H_m$.

Proposition 4.1. *For $m = 13$ and 14, we have $[H_m : \Gamma(N_m)] \geq 2$.*

Proof. We put

$$\eta_n = \left[\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & \\ \hline & & 1_{n-2} \end{array}\right].$$

Then $\eta_n \cdot {}^t\eta_n = 1_n$. It can be seen that

$$D_{2n} = \{2^{-1}(x_1, \ldots, x_{2n}) \in (2^{-1}\mathbb{Z})^1_{2m} \mid x_{2i-1} \equiv x_{2i} \pmod 2 \quad (1 \leq i \leq n), \\ \sum_{i=1}^{n} x_{2n} \equiv 0 \pmod 2\}.$$

Thus $D_{2n}(2^{-1})\eta_{2n} = D_{2n}(2^{-1})$, and hence $\eta_{2n} \in \Gamma(D_{2n}(2^{-1}))$. Therefore

$$\eta_m = \left[\begin{array}{c|c} \eta_{2(m-8)} & \\ \hline & 1_{16-m} \end{array}\right] \in H_m.$$

On the other hand,

$$h_{13}\eta_{13} = 2^{-1}(e_2 + e_3 + e_5 + e_7 + e_9 + e_{11} + e_{12} + e_{13}),$$
$$h_{14}\eta_{14} = 2^{-1}(e_2 + e_3 + e_5 + e_7 + e_9 + e_{11} + e_{13} + e_{14}).$$

Thus $h_m\eta_m \notin N_m$ by Lemma 4.1. Hence $\eta_m \notin \Gamma(N_m)$. □

4.3. 13-dimensional case

Let us consider the case in which $n = 13$. We put

$$L^{(1)} = E_8(2^{-1}) \oplus D_4(2^{-1}) \oplus A_1(2^{-1}), \quad L^{(2)} = D_{12}(2^{-1}) \oplus A_1(2^{-1}),$$
$$L^{(3)} = E_7(2^{-1}) \oplus D_6(2^{-1}).$$

Then $L^{(1)}$, $L^{(2)}$ and $L^{(3)}$ are $\mathbb{Z}$-maximal by Lemma 3.1, since $E_8 \oplus D_4 \oplus A_1$, $D_{12} \oplus A_1$, and $E_7 \oplus D_6$ are even integral and whose determinants are equal to 2^3. Then, by [4, Lemma 1.4.6 and §4], we have

$$\begin{aligned}[\Gamma(L^{(1)}) : 1] &= [\Gamma(E_8) : 1][\Gamma(D_4) : 1][\Gamma(A_1) : 1] \\ &= (2^{14} \cdot 3^5 \cdot 5^2 \cdot 7)(2^7 \cdot 3^2) \cdot 2 \\ &= 2^{22} \cdot 3^7 \cdot 5^2 \cdot 7,\end{aligned}$$

$$\begin{aligned}[\Gamma(L^{(2)}) : 1] &= [\Gamma(D_{12}) : 1][\Gamma(A_1) : 1] \\ &= (2^{12} \cdot 12!) \cdot 2 \\ &= 2^{23} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11,\end{aligned}$$

$$\begin{aligned}[\Gamma(L^{(3)}) : 1] &= [\Gamma(E_7) : 1][\Gamma(D_6) : 1] \\ &= (2^{10} \cdot 3^4 \cdot 5 \cdot 7)(2^6 \cdot 6!) \\ &= 2^{20} \cdot 3^6 \cdot 5^2 \cdot 7.\end{aligned}$$

From these, we see that $\mathfrak{k}(L^{(i)}) \neq \mathfrak{k}(L^{(j)})$ for $i \neq j$. On the other hand, by (2.4), $\mathfrak{m}(L^{(1)}) = 691 \cdot (2^{23} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11)^{-1}$. Since

$$\mathfrak{m}(L^{(1)}) - \textstyle\sum_{i=1}^{3}[\Gamma(L^{(i)}) : 1]^{-1} = (2^{21} \cdot 3^5 \cdot 5^2 \cdot 7) > 0, \tag{4.2}$$

we have $k(13) > 3$.

Now we put $L^{(4)} = N_{13}$, which is $\mathbb{Z}$-maximal lattice containing $\mathbb{Z}^1_{13}$ by §4.2. That is, $L^{(4)} \in \Lambda(L^{(1)})$. By the definition of the group H_{13}, we see that

$$[H_{13} : 1] = [\Gamma(D_{10}) : 1][\Gamma(A_1^{\oplus 3}) : 1] = (2^{10} \cdot 10!)(2^4 \cdot 3) = 2^{22} \cdot 3^5 \cdot 5^2 \cdot 7. \tag{4.3}$$

Combining (4.3) with Proposition 4.1, we have

$$[\Gamma(L^{(4)}) : 1] = \frac{[H_{13} : 1]}{[H_{13} : \Gamma(L^{(4)})]} \leq \frac{[H_{13} : 1]}{2} = 2^{21} \cdot 3^5 \cdot 5^2 \cdot 7, \tag{4.4}$$

which shows $[\Gamma(L^{(4)}) : 1] < [\Gamma(L^{(i)}) : 1]$ for $i = 1, 2, 3$. This means that $\mathfrak{k}(L^{(4)}) \neq \mathfrak{k}(L^{(i)})$ for $i = 1, 2, 3$. Then by (4.4) and (4.2), we have $\mathfrak{m}(L^{(1)}) - \sum_{i=1}^{4}[\Gamma(L^{(i)}) : 1]^{-1} \leq 0$, and hence

$$\mathfrak{m}(L^{(1)}) = \sum_{i=1}^{4}[\Gamma(L^{(i)}) : 1]^{-1},$$
$$[\Gamma(L^{(4)}) : 1] = 2^{21} \cdot 3^5 \cdot 5^2 \cdot 7.$$

Therefore we obtain $k(13) = 4$.

4.4. 14-dimensional case

Let $n = 14$. We put

$$L^{(1)} = E_8(2^{-1}) \oplus D_6(2^{-1}), \quad L^{(2)} = D_{14}(2^{-1}), \quad L^{(3)} = E_7(2^{-1}) \oplus E_7(2^{-1}).$$

Then $L^{(1)}$, $L^{(2)}$ and $L^{(3)}$ are $\mathbb{Z}$-maximal by Lemma 3.1, since $E_8 \oplus D_6$, D_{14}, and $E_7 \oplus E_7$ are even integral and whose determinants are equal to 2^2. By [4, Lemma 1.4.6 and §4], we have

$$[\Gamma(L^{(1)}) : 1] = [\Gamma(E_8) : 1][\Gamma(D_6) : 1] = (2^{14} \cdot 3^5 \cdot 5^2 \cdot 7)(2^6 \cdot 6!) = 2^{24} \cdot 3^7 \cdot 5^3 \cdot 7,$$
$$[\Gamma(L^{(2)}) : 1] = [\Gamma(D_{14}) : 1] = (2^{14} \cdot 14!) = 2^{25} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13,$$
$$[\Gamma(L^{(3)}) : 1] = 2 \cdot [\Gamma(E_7) : 1]^2 = 2 \cdot (2^{10} \cdot 3^4 \cdot 5 \cdot 7)^2 = 2^{21} \cdot 3^8 \cdot 5^2 \cdot 7^2.$$

From these, we see that $\mathfrak{k}(L^{(i)}) \neq \mathfrak{k}(L^{(j)})$ for $i \neq j$. On the other hand, by (2.4), $\mathfrak{m}(L^{(1)}) = (61 \cdot 691) \cdot (2^{25} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13)^{-1}$. Since

$$\mathfrak{m}(L^{(1)}) - \textstyle\sum_{i=1}^{3}[\Gamma(L^{(i)}) : 1]^{-1} = (2^{24} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11)^{-1} > 0, \tag{4.5}$$

we have $k(14) > 3$.

Now we put $L^{(4)} = N_{14}$, which is $\mathbb{Z}$-maximal lattice containing $\mathbb{Z}_{14}^1$ by §4.2. By the definition of the group H_{14}, we see that

$$\begin{aligned}[H_{14} : 1] &= [\Gamma(D_{12}) : 1][\Gamma(A_1^{\oplus 2}) : 1] \\ &= (2^{12} \cdot 12!) \cdot 2^3 \\ &= 2^{25} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11.\end{aligned} \tag{4.6}$$

Combining (4.6) with Proposition 4.1, we have

$$[\Gamma(L^{(4)}) : 1] = \frac{[H_{14} : 1]}{[H_{14} : \Gamma(L^{(4)})]} \leq \frac{[H_{14} : 1]}{2} = 2^{24} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11, \tag{4.7}$$

which shows $[\Gamma(L^{(4)}) : 1] < [\Gamma(L^{(i)}) : 1]$ for $i = 1, 2, 3$. This means that $\mathfrak{k}(L^{(4)}) \neq \mathfrak{k}(L^{(i)})$ for $i = 1, 2, 3$. Then by (4.7) and (4.5), we have $\mathfrak{m}(L^{(1)}) - \sum_{i=1}^{4} [\Gamma(L^{(i)}) : 1]^{-1} \leq 0$, and hence

$$\begin{gathered}\mathfrak{m}(L^{(1)}) = \sum_{i=1}^{4} [\Gamma(L^{(i)}) : 1]^{-1}, \\ [\Gamma(L^{(4)}) : 1] = 2^{24} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11.\end{gathered}$$

Therefore we obtain $k(14) = 4$.

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