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A sequence of blowing-ups connecting moduli of sheaves and the Donaldson polynomial under change of polarization

By

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Introduction

Let X be a nonsingular projective surface over $\mathbb{C}$, H an ample line bundle on X, and $M_H(c_1, c_2)$ the moduli scheme of S-equivalence classes of rank-two H-semistable sheaves on X with fixed Chern classes (c_1, c_2). It is projective over $\mathbb{C}$.

Fix two ample line bundles H_1 and H_2 on X. In this article, we connect $M_{H_1}(c_1, c_2)$ with $M_{H_2}(c_1, c_2)$ by a sequence of blowing-ups and blowing-downs

(0.1)

$$\begin{tikzcd}
 & \tilde{M}_1 \arrow[dl, "\phi_1"'] \arrow[dr, "\psi_1"] & & \tilde{M}_2 \cdots \arrow[dl, "\phi_2"'] & \cdots \tilde{M}_l \arrow[dr, "\psi_l"] & \\
M_{H_1}(c_1, c_2) & & M_2 & & & M_{H_2}(c_1, c_2)
\end{tikzcd}$$

using canonical properties of moduli schemes, and study the exceptional divisor E_i of ϕ_i in (0.1). Further, we apply this sequence to the calculation of the Donaldson polynomial of X. We shall algebro-geometrically inquire into the fact the Donaldson polynomials of X are independent of the choice of Riemannian metrics when $b_2^+(X) = 2p_g(X) + 1 > 1$.

Now let us survey the historical background and outline the content of this article. Roughly speaking, two methods have been developed to describe the change of moduli of sheaves under the change of polarization as a sequence of (birational) morphisms. First, Matsuki and Wentworth [MW] succeeded in connecting $M_{H_1}(c_1, c_2)$ and $M_{H_2}(c_1, c_2)$ by a sequence of Thaddeus-type flips. They introduced the notion of twisted stability of sheaves, and reduced the construction of the flip (0.1) to the Mumford-Thaddeus principle, which dealt with the change of GIT quotients under a variation of G-linearization.

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On the other hand, Ellingsrud-Göttsche [EG] and Friedman-Qin [FQ] constructed a diagram of blowing-ups (0.1) by elementary transforms of universal sheaves, mainly in case where the Kodaira dimension of X is 0 or $-\infty$. Comparing this with the above-mentioned construction via Thaddeus-type flip, we see that the good point of this method is its definiteness; the centers of blowing-ups in (0.1) is directly described in terms of moduli problems. One can also get the relation between universal sheaves on two moduli spaces in (0.1) very concretely. Thanks to such definiteness, it should be possible to derive interesting properties of this flip with the help of moduli theory. However, this method has been established only for surfaces with $\kappa(X) \leq 0$. One of main results of this article is that we could complete it for any surfaces with any Kodaira dimensions. Our construction of a flip (0.1) shall proceed as follows. In Section 2, we endow a subset

$$P_1 = \{[E] \mid E \text{ is not } H_2\text{-semistable}\}$$

of $M_{H_1}(c_1, c_2)$ with a natural subscheme structure. Here several improvements are needed since P_1 may admit singularities when $\kappa(X)$ is positive. In Section 5, one can also study some structure of this P_1 over $\mathrm{Pic}(X) \times \mathrm{Hilb}(X) \times \mathrm{Hilb}(X)$.

Let $\phi : \tilde{M} \to M_{H_1}(c_1, c_2)$ be the blowing-up of $M_{H_1}(c_1, c_2)$ along P_1. Roughly speaking, we modify the pull-back $(\mathrm{id}_X \times \phi)^* \mathcal{U}_1$ of the universal family of $M_{H_1}(c_1, c_2)$ via an elementary transform to obtain a new flat family $\mathcal{W}$ on $X \times \tilde{M}$, and get a morphism $\psi : \tilde{M} \to M_{H_2}(c_1, c_2)$ using $\mathcal{W}$ in Section 3. This ψ is in fact blowing-up of $M_{H_2}(c_1, c_2)$, as shall be shown in Section 4. Therefore we obtain a sequence of blowing-ups (0.1) connecting M_{H_1} and M_{H_2}.

Although this idea is primarily based on that of Ellingsrud-Göttsche or Friedman-Qin, we have to proceed more carefully. Denote the exceptional divisor of ϕ by E. On $X \times E$, there is the relative Harder-Narasimhan filtration

$$0 \longrightarrow \tilde{\mathcal{F}} \longrightarrow (\mathrm{id} \times \phi)^* \mathcal{U}_1|_{X \times E} \longrightarrow \tilde{\mathcal{G}} \longrightarrow 0$$

with respect to H_2-stability. Then one can naturally induce another exact sequence

$$0 \longrightarrow \tilde{\mathcal{G}}(-E) \longrightarrow \mathcal{W}|_{X \times E} \longrightarrow \tilde{\mathcal{F}} \longrightarrow 0. \tag{0.2}$$

We have to show that (0.2) is a family of *nontrivial* extensions in order to get a morphism $\psi : \tilde{M} \to M_{H_2}$. In contrast to the case where $\kappa(X) \leq 0$, it is not sufficient for our purpose to look only over tangent spaces of E and $\tilde{M}$ since P_1 and E admit singularities. We shall examine the infinitesimal behaviors of E and $(\mathrm{id}_X \times \phi)^* \mathcal{U}_1$.

Here let us mention another good point of this method. When one compares $M_{H_1}(c_1, c_2)$ with $M_{H_2}(c_1, c_2)$, it is often useful and important to grasp the structure of exceptional divisor of ϕ_i in (0.1). When $p_g(X) = 0$ or K_X is trivial, this divisor was investigated in [EG, Section 4], but little has been known about it in general; this divisor is much more complicated when $\kappa(X) > 0$. In Lemmas 8.2 and 8.4, we shall show that the obstruction theory of universal families may provide us with some useful information about exceptional divisors. This is possible because our construction of a flip is concrete enough. The information obtained in such a way shall play an essential role later in this article.

Now let us turn the subject to the Donaldson polynomials. Refer to [FM] about its basic material. Fix an integer c_2 and a polarization H. Using the moduli scheme $M_H(0, c_2)$, Jun Li [Li] introduced a homomorphism $\gamma_H(c_2) : \mathrm{Sym}^{d(c_2)} \mathrm{NS}(X) \to \mathbb{Z}$.

Proposition 0.1 ([Li, p. 456]). *Suppose that X is simply connected and that $p_g(X) > 0$. Then there is such a constant $A(\mathcal{S})$ depending on a compact subset $\mathcal{S} \subset \mathrm{Amp}(X)$ as satisfies the following*:
If $c_2 \geq A(\mathcal{S})$ and if some rational multiple of an ample line bundle H is contained in $\mathcal{S}$, then $\gamma_H(c_2)$ is equal to the restriction of the Donaldson invariant $q(c_2) : \mathrm{Sym}^{d(c_2)} H_2(X, \mathbb{Z}) \to \mathbb{Z}$ to $\mathrm{Sym}^{d(c_2)} \mathrm{NS}(X)$. In particular $\gamma_H(c_2)$ is independent of an ample line bundle H contained in $\mathbb{Q} \cdot \mathcal{S}$.

The independence of $\gamma_H(c_2)$ is due to the fact that the Donaldson polynomial $q_g(c_2)$ is independent of the choice of generic Riemannian metrics g on X. As an application of the flip constructed in the above, we observe this fact algebro-geometrically in the latter half of this article. Up to now, an attempt to explain this fact via a flip succeeded only in $K3$ case ([EG]). We aim to carry out this attempt in more general situations. Our result in this article is as follows.

Suppose that ample line bundles H_1 and H_2 are in neighboring chambers of type $(0, c_2)$ separated by a wall of type $(0, c_2)$, say W. (See Section 1 for the definition of walls and chambers.) Now denote by $A^+(W)$ the set of all the triples $\mathbf{f} = (f, m, n) \in \mathrm{Num}(X) \times \mathbb{N}^{\times 2}$ which satisfy $f \in 2\mathrm{Num}(X)$, $H_1 \cdot f > 0$, $m + n = c_2 + (f^2/4)$, and the set

$$W^f = \{x \in \mathrm{Num}(X) \mid x \cdot f = 0\}$$

is equal to W. Then, for $\mathbf{f} \in A^+(W)$ one can define a homomorphism $C(c_2, \mathbf{f}) : \mathrm{Sym}^{d(c_2)} \mathrm{NS}(X) \to \mathbb{Z}$ such that

$$\gamma_{H_1} - \gamma_{H_2} = \sum_{\mathbf{f} \in A^+(W)} C(c_2, \mathbf{f}).$$

In Section 2 we shall divide P_1 into $\coprod_{\mathbf{f} \in A^+(a)} P_1^{\mathbf{f}}$ as a disjoint union of components in a natural way, and $C(c_2, \mathbf{f})$ is the contribution of $P_1^{\mathbf{f}}$ to $\gamma_{H_1} - \gamma_{H_2}$. In the following theorem, $\mathrm{Pic}^{f/2}(X)$ designates an open subset of $\mathrm{Pic}(X)$

$$\{L \in \mathrm{Pic}(X) \mid [2L] = f \text{ in } \mathrm{Num}(X)\}.$$

Theorem 0.2. *Suppose that $q(X) = 0$ and that some global section $\kappa \in \Gamma(K_X)$ gives a nonsingular curve $\mathcal{K} \subset X$. Let $\mathcal{S}$ be any compact subset of the ample cone* $\mathrm{Amp}(X)$. *Then there are constants $d_0(\mathcal{S})$, $d_1(X)$ and $d_2(X)$ depending on $\mathcal{S}$ such that the following hold*:

Assume that $\mathbf{f} = (f, m, n) \in A^+(W)$ satisfies that

(i) *the functions* $T^{\mathbf{f}} = \mathrm{Pic}^{f/2}(X) \times \mathrm{Hilb}^m(X) \times \mathrm{Hilb}^n(X) \to \mathbb{Z}$ *defined by*

$$(L, Z_1, Z_2) \mapsto \dim \mathrm{Ext}^1_X(\mathcal{O}(L) \otimes I_{Z_1}, \mathcal{O}(-L) \otimes I_{Z_2}) \qquad \textit{and}$$
$$(L, Z_1, Z_2) \mapsto \dim \mathrm{Ext}^1_X(\mathcal{O}(-L) \otimes I_{Z_2}, \mathcal{O}(L) \otimes I_{Z_1})$$

are locally-constant, and that
(ii) $-f^2 > (4/3)c_2 + d_1(\mathcal{S})\sqrt{c_2} + d_2(\mathcal{S})$.
Then $C(c_2, \mathbf{f})$ *is zero if* $c_2 \geq d_0(\mathcal{S})$.

How strong are these conditions (i) and (ii)? As to (ii), recall that $f \in \mathrm{NS}(X)$ defines a wall of type $(0, c_2)$ if $W^f \cap \mathrm{Amp}(X) \neq \emptyset$, $f \equiv 0 \mod 2\mathrm{Num}(X)$ and $0 < -f^2 \leq 4c_2$. Thus the condition (ii) is reasonably weak when c_2 is sufficiently large with respect to $\mathcal{S}$. The condition (i) is more strict, while this is always valid when X is a $K3$ surface. We prove Theorem 0.2 in Section 6, 7, and 8. In the proof it is important to grasp the structure of exceptional divisors in the flip (0.1), as mentioned earlier.

After completing this work, the author realized by chance Mochizuki had shown the independence of $\gamma_H(c_2)$ from H when $p_g(X) > 0$ by using moduli stacks of semistable mixed objects and master spaces with torus action in his paper [Mo]. Mochizuki's proof seems to be very different from ours, and our construction of the sequence of morphisms connecting M_{H_1} and M_{H_2} must be useful.

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Notation.
(1) A scheme is algebraic over $\mathbb{C}$. For a surface X, $\mathrm{Num}(X)$ is the quotient of $\mathrm{Pic}(X)$ modulo the numerically equivalence. $\mathrm{Amp}(X) \subset \mathrm{Num}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ is the ample cone of X. For a closed subscheme D of S, $I_D = I_{D,S}$ means its ideal sheaf. The stability of coherent torsion-free sheaves is in the sense of Gieseker-Maruyama.
(2) For T-schemes $f : X \to T$ and $g : S \to T$, let X_S denote $X \times_T S$. Let $\mathcal{F}$ be a sheaf on X, and $D \subset T$ a subscheme. We often shorten a sheaf $(\mathrm{id}_X \times g)^* \mathcal{F}$ on X_S to $g^* \mathcal{F}$, and shorten $\mathcal{F}|_{X_D}$ to $\mathcal{F}|_D$. hom and ext^i indicate, respectively, dim Hom and $\dim \mathrm{Ext}^i$.

1. Background materials

In this section let us review some background materials introduced in [EG] and [Q2]. Let X be a nonsingular surface, and fix a line bundle c_1 on X and an integer c_2 such that $4c_2 - c_1^2 > 0$.

Definition 1.1. (1) For $f \in \mathrm{Num}(X)$ we define $W^f \subset \mathrm{Amp}(X)$ by

$$W^f = \{x \in \mathrm{Amp}(X) \mid x \cdot f = 0\}.$$

f is said to be *define a wall of type* (c_1, c_2) if W^f is nonempty, $0 < -f^2 \leq 4c_2 - c_1^2$ and $f - c_1$ is divisible by 2 in $\mathrm{Num}(X)$. Then W^f is called *a wall of type* (c_1, c_2).

(2) *A chamber of type* (c_1, c_2) is a connected component of the complement of the union of all walls of type (c_1, c_2). Two different chambers are said to be *neighboring* if the intersection of their closures contains a nonempty open subset of a wall.

For an ample line bundle H on X we denote by $M_H(c_1, c_2)$ the coarse moduli scheme of H-semistable rank-two sheaves with Chern classes (c_1, c_2).

Lemma 1.2. (1) *For H not contained in any wall of type (c_1, c_2), $M_H(c_1, c_2)$ depends only on the chamber containing H.*

(2) *The set of walls of type (c_1, c_2) is locally finite.*

Proof. (1) is [EG, Proposition 2.7]. (2) is [Q2, Proposition 2.1.6]. □

Let H_+ and H_- be ample line bundles lying in neighboring chambers $\mathcal{C}_+$ and $\mathcal{C}_-$ respectively, and H an ample line bundle contained in the wall W separating $\mathcal{C}_+$ and $\mathcal{C}_-$, and not contained in any wall but W. Such a setting is natural because of the lemma above. We can assume that $M = H_+ - H_-$ is effective by replacing H_+ by its high multiple if necessary.

Lemma 1.3. *There is an integer n_0 such that if E is a rank-two sheaf with Chern classes (c_1, c_2) on X then the following holds for any integer $l \geq n_0$:*

(1) *E is H_--stable (resp. semistable) if and only if $E(-lM)$ is H-stable (resp. semistable).*

(2) *E is H_+-stable (resp. semistable) if and only if $E(lM)$ is H-stable (resp. semistable).*

Proof. [EG, p. 6, Lemma 3.1]. □

Let C denote $(n_0 + 1)M$ in this section, where n_0 is that in the lemma above.

Definition 1.4. Let a be a real number between 0 and 1.

(1) We define $P_a(E)$ by $P_a(E) = [(1-a)\chi(E(-C)) + a\chi(E(C))]/\mathrm{rk}(E)$ for a torsion-free sheaf E.

(2) A torsion-free sheaf E on X is said to be *a-stable* (resp. *a-semistable*) if every subsheaf $F \subsetneq E$ satisfies $P_a(F(lH)) \leq P_a(E(lH))$ (resp. $P_a(F(lH)) < P_a(E(lH))$) for sufficiently large integer l.

(3) E is a-semistable if and only if parabolic sheaf $(E(C), E(-C), a)$ is parabolic semistable with respect to H. Hence from [Yk], there is a coarse moduli scheme of S-equivalence classes of a-semistable rank-two sheaves with Chern classes (c_1, c_2) on X, denoted by $M_a(c_1, c_2)$. This is projective over $\mathbb{C}$. $M_a^s(c_1, c_2) \subset M_a(c_1, c_2)$ denotes the open subscheme of a-stable sheaves.

By Lemma 1.3, $M_0(c_1, c_2)$ (resp. $M_1(c_1, c_2)$) is naturally isomorphic to $M_{H_-}(c_1, c_2)$ (resp. $M_{H_+}(c_1, c_2)$). So we would like to study how $M_a(c_1, c_2)$ changes as a varies.

Definition 1.5. For a real number $0 \leq a \leq 1$, $A^{+}(a)$ is the set of $(f, m, n) \in \mathrm{Num}(X) \times \mathbb{Z}^{2}_{\geq 0}$ satisfying that W^{f} is equal to the wall W dividing H_{+} and H_{-}, $H_{+} \cdot f > 0$, $m+n = c_2 - (c_1^2 - f^2)/4$, and $m - n = \langle f \cdot (c_1 - K_X)\rangle/2 + (2a-1)\langle f \cdot C\rangle$. a is called *a miniwall* if $A^{+}(a)$ is nonempty. Remark that the number of miniwalls is finite. *A minichamber* is a connected component of the complement of the set of all miniwalls in $[0, 1]$. Two minichambers are said to be *neighboring* if their closures intersect.

Lemma 1.6. *Let $a_{-} < a_{+}$ be in neighboring minichambers separated by a miniwall a. For torsion-free rank-two sheaf E with Chern classes (c_1, c_2), the following holds.*

(1) *If E is a_{-}-semistable and not a_{+}-semistable, then E is given by a nontrivial extension*

$$0 \longrightarrow \mathcal{O}_X(F) \otimes I_{Z_1} \longrightarrow E \longrightarrow \mathcal{O}_X(c_1 - F) \otimes I_{Z_2} \longrightarrow 0, \tag{1.1}$$

where Z_1 and Z_2 are zero-dimensional subschemes of X such that

$$(2F - c_1, l(Z_1), l(Z_2)) \in A^{+}(a). \tag{1.2}$$

(2) *Conversely suppose that E is given by a nontrivial extension* (1.1) *satisfying* (1.2). *Then E is a_{-}-stable, strictly a-semistable, and not b-semistable for any $b > a$.*

Proof. [EG, Lemmas 3.10 and 3.11]. □

We fix ample line bundles $H_{\pm}$ and H, and neighboring minichambers $a_{-} < a_{+}$ separated by a miniwall a. We shorten $M_{a_{\pm}}(c_1, c_2)$ to $M_{\pm}(c_1, c_2)$ for simplicity.

2. Subscheme consisting of not a_{+}-semistable sheaves

In this section we shall give a natural subscheme structure to a well-defined subset

$$M_{-} \supset \{[E] \mid E \text{ is not } a_{+}\text{-semistable}\} \tag{2.1}$$

contained in M_{-}^{s}. This closed subscheme shall be the center of a blowing-up later.

We begin with a quick review of the construction of $M_{\pm}(c_1, c_2) = M_{\pm}$ referring to [Yk]. Let $\mathcal{F}_{-}(c_1, c_2)$ (or $\mathcal{F}_{+}(c_1, c_2)$, resp.) denote the family of all a_{-}-semistable (a_{+}-semistable, resp.) rank-two sheaves with Chern classes (c_1, c_2) on X. By the boundedness of $a_{\pm}$-semistablity, there is an integer N_0 such that the following conditions are satisfied for any $E \in \mathcal{F}_{-}(c_1, c_2) \cup \mathcal{F}_{+}(c_1, c_2)$.

(1) If $m \geq N_0$, then both $E(C)(mH)|_{2C}$ and $E(-C)(mH)$ are generated by its global sections.

(2) If $m \geq N_0$, then $h^{i}(E(C)(mH)|_{2C}) = 0$ and $h^{i}(E(-C)(mH)) = 0$ for $i > 0$.

We fix an integer $m \geq N_0$. Then $h^0(E(C)(mH)) = R$ is independent of $E \in \mathcal{F}_+(c_1, c_2) \cup \mathcal{F}_-(c_1, c_2)$. The Quot-scheme $\mathrm{Quot}^{P(l)}_{\mathcal{O}(-C-mH)^{\oplus R}/X}$ is denoted by Q, where $P(l)$ is the Hilbert polynomial $\chi(E(lH))$ of $E \in \mathcal{F}_\pm(c_1, c_2)$. On X_Q there is the universal quotient sheaf $\tau_0 : \mathcal{O}_{X_Q}(-C - mH)^{\oplus R} \to \mathcal{U}$. Now let $Q^s_\pm$ (or $Q^{ss}_\pm$, resp.) be the maximal open subset of Q such that, for every $t \in Q^s_\pm$ ($Q^{ss}_\pm$, resp.),

$$H^0(\tau_0(C + mH) \otimes k(t)) : k(t)^{\oplus R} \to H^0(\mathcal{U}(C + mH) \otimes k(t))$$

is isomorphic, $\mathcal{U} \otimes k(t)$ satisfies the hypothesis (i) and (ii) above, and $\mathcal{U} \otimes k(t)$ is $a_\pm$-stable ($a_\pm$-semistable, resp.). Let us denote the universal quotient sheaf of $Q^{ss}_\pm$ by $\mathcal{U}_\pm \in \mathrm{Coh}(X_{Q^{ss}_\pm})$. $\overline{G} = \mathrm{PGL}(R, \mathbb{C})$ naturally acts on $Q^{ss}_\pm$ and $Q^s_\pm$. By [Yk] we can construct a good quotient of $Q^{ss}_\pm$ (or $Q^s_\pm$, resp.) by $\overline{G}$ when m is sufficiently large. This quotient turns out to be the moduli scheme $M_\pm(c_1, c_2)$ ($M^s_\pm(c_1, c_2)$, resp.). Moreover, because a $a_\pm$-stable sheaf is simple, one can prove that the quotient map $\pi_\pm : Q^s_\pm \to M^s_\pm(c_1, c_2)$ is a principal fiber bundle with group $\overline{G}$ ([M2]) in a similar fashion to the proof of [Ma, Proposition 6.4].

Now we try to give a closed-subscheme structure to the subset (2.1). For $\mathbf{f} = (f, m, n) \in A^+(a)$, we can define a functor

$$\mathcal{Q}^{\mathbf{f}} : (\mathrm{Sch}\,/Q^{ss}_-)^\circ \to (\mathrm{Sets})$$

as follows: $\mathcal{Q}^{\mathbf{f}}(S \to Q^{ss}_-)$ is the set of all S-flat quotient sheaves $\mathcal{U}_- \otimes_{Q^{ss}_-} \mathcal{O}_S \to \mathcal{G}'$ such that, for every geometric point $t \in S$, the induced exact sequence

$$0 \longrightarrow \mathrm{Ker} \longrightarrow \mathcal{U}_- \otimes k(t) \longrightarrow \mathcal{G}' \otimes k(t) \longrightarrow 0$$

satisfies that $(c_1 - 2c_1(\mathcal{G}' \otimes k(t)), c_2(\mathrm{Ker}), c_2(\mathcal{G}' \otimes k(t))) = (f, m, n)$. This functor $\mathcal{Q}^{\mathbf{f}}$ is represented by a relative Quot-scheme $Q^{\mathbf{f}}$, that is projective over Q^{ss}_-. On $X_{Q^{\mathbf{f}}}$ there is the universal quotient $\tau_{\mathbf{f}} : \mathcal{U}_- \otimes \mathcal{O}_{Q^{\mathbf{f}}} \to \mathcal{G}$.

Lemma 2.1. *$\mathcal{G} \otimes k(s)$ is torsion-free for every closed point $s \in Q^{\mathbf{f}}$.*

Proof. The proof is by contradiction. Assume that $\mathcal{G} \otimes k(s)$ is not torsion-free, and denote its torsion part by $T \neq 0$. Then we have a new quotient sheaf

$$\mathcal{U}_- \otimes k(s) \to \mathcal{G} \otimes k(s) \to G' = \mathcal{G} \otimes k(s)/T.$$

Then $P_a(\mathcal{G} \otimes k(s)(lH)) > P_a(G'(lH))$ if l is sufficiently large. From the definition of $\mathbf{f}$ and $Q^{\mathbf{f}}$ one can show that $P_a(\mathcal{G} \otimes k(s)(lH)) = P_a(\mathcal{U}_- \otimes k(s)(lH))$ for all l. So the quotient sheaf $\mathcal{U}_- \otimes k(s) \to G'$ satisfies that

$$P_a(\mathcal{U}_- \otimes k(s)(lH)) > P_a(G'(lH)) \tag{2.2}$$

if l is sufficiently large. On the other hand

$$P_{a_-}(\mathcal{U}_- \otimes k(s)(lH)) \leq P_{a_-}(G'(lH)) \tag{2.3}$$

if l is sufficiently large since $\mathcal{U}_- \otimes k(s)$ is a_--semistable. From (2.2), (2.3) and the Riemann-Roch theorem, there should be an integer $a_- \leq b < a$ such that $P_b(\mathcal{U}_- \otimes k(s)(lH)) = P_b(G'(lH))$ for all l. We can easily prove that b is a miniwall, which contradicts the choice of a_- and a. $\square$

Lemma 2.2. *The structural morphism $i = i^{\mathbf{f}} : Q^{\mathbf{f}} \to Q^{ss}_{-}$ is a closed immersion.*

Proof. For $s \in Q^{\mathbf{f}}$ we put $t = i(s)$. First we claim that their residue fields satisfy $k(s) = k(t)$. Indeed, any member $\lambda \in \mathrm{Gal}(k(s)/k(t))$ induces another $k(s)$-valued point

$$\mathrm{Spec}(k(s)) \xrightarrow{\lambda} \mathrm{Spec}(k(s)) \to Q^{\mathbf{f}}$$

of $Q^{\mathbf{f}}$. We denote this $k(s)$-valued point by s'. s and s' respectively give exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & (\mathcal{U}_{-} \otimes k(t)) \otimes k(s) & \longrightarrow & \mathcal{G} \otimes k(s) & \longrightarrow & 0 \\ & & & & \| & & & & \\ 0 & \longrightarrow & K' & \longrightarrow & (\mathcal{U}_{-} \otimes k(t)) \otimes k(s') & \longrightarrow & \mathcal{G} \otimes k(s') & \longrightarrow & 0. \end{array} \tag{2.4}$$

Because of the definition of $\mathbf{f}$ and $Q^{\mathbf{f}}$, it holds that

$$0 < \{c_1(K) - c_1(\mathcal{G} \otimes k(s))\} \cdot H_{+} \quad \text{and that} \quad 0 < \{c_1(K') - c_1(\mathcal{G}' \otimes k(s))\} \cdot H_{+}.$$

Besides, the lemma above tells us that both $\mathcal{G}\otimes k(s)$ and $\mathcal{G}\otimes k(s')$ are torsion-free and rank-one. Thus two horizontal rows in (2.4) respectively give the Harder-Narasimhan filtration of $\mathcal{U}_{-} \otimes k(s)$ with respect to H_{+}-stability. Because of the uniqueness of the Harder-Narasimhan filtration, two quotient sheaves in (2.4) are isomorphic, that is, $s = s'$. Accordingly $\mathrm{Gal}(k(s)/k(t)) = \{1\}$, and hence $k(s) = k(t)$ since $\mathrm{ch}(k(t)) = 0$.

Next, i is injective and hence finite. Indeed, suppose that two points s and s' in $Q^{\mathbf{f}}$ satisfy that $i(s) = i(s') = t$. Then $k(s) = k(s') = k(t)$ as mentioned above, and we have two exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & \mathcal{U}_{-} \otimes k(t) & \longrightarrow & \mathcal{G} \otimes k(s) & \longrightarrow & 0 \\ & & & & \| & & & & \\ 0 & \longrightarrow & K' & \longrightarrow & \mathcal{U}_{-} \otimes k(t) \otimes k(s') & \longrightarrow & \mathcal{G} \otimes k(s') & \longrightarrow & 0. \end{array}$$

Then one can prove that $s = s'$ in $Q^{\mathbf{f}}$, in the same way as the preceding paragraph.

Next, i is unramified. To prove this, we only need to show that the tangent map $T_t\, i : T_t\, Q^{\mathbf{f}} \to T_s\, Q^{ss}_{-}$ is injective. $t \in Q^{\mathbf{f}}$ gives an exact sequence

$$0 \longrightarrow K \longrightarrow \mathcal{U}_{-} \otimes k(s) \longrightarrow \mathcal{G} \otimes k(t) \longrightarrow 0 \tag{2.5}$$

on $X_{k(s)} = X_{k(t)}$. By [HL, p. 43] $\mathrm{Ker}(T_t\, i)$ equals $\mathrm{Hom}_{X_{k(t)}}(K, \mathcal{G} \otimes k(t))$, which is equal to zero because (2.5) gives the Harder-Narasimhan filtration of $\mathcal{U}_{-} \otimes k(s)$.

Last, i is a closed immersion. Since i is injective and unramified, the fiber $i^{-1}(t)$ is naturally isomorphic to $\mathrm{Spec}(k(s))$ for $s \in Q^{\mathbf{f}}$. Since i is finite,

$i^{-1}(t)$ is isomorphic to $\mathrm{Spec}(i_*\mathcal{O}_{Q^{\mathbf{f}}} \otimes k(t))$. These facts tell us that the natural homomorphism $\mathcal{O}_{Q^{ss}_-} \otimes k(t) \to i_*\mathcal{O}_{Q^{\mathbf{f}}} \otimes k(t)$ is surjective since $k(t) = k(s)$. So $\mathcal{O}_{Q^{\mathbf{f}}} \to i_*\mathcal{O}_{Q^{\mathbf{f}}}$ itself should be surjective. This means that a finite morphism i is a closed immersion. □

We therefore obtain a closed subscheme $Q^{\mathbf{f}}$ of Q^{ss}_-, which is contained in Q^s_- by virtue of Lemma 1.6. Remembering the way to define the natural action $\bar{\sigma} : \bar{G} \times Q^s_- \to Q^s_-$, one can verify the following:

Lemma 2.3. *Denote by $\overline{\sigma}_- : \overline{G} \times Q^s_- \to Q^s_-$ the natural action of $\overline{G}$ on Q^s_-. Then the morphism* $\mathrm{id} \times \overline{\sigma}_- : \overline{G} \times Q^s_- \to \overline{G} \times Q^s_-$ *satisfies that* $(\mathrm{id} \times \overline{\sigma}_-)(\overline{G} \times Q^{\mathbf{f}}) = \overline{G} \times Q^{\mathbf{f}}$ *as subschemes of* $\overline{G} \times Q^{\mathbf{f}}$.

This lemma means that

$$\mathrm{pr}_2^* \mathcal{O}_{Q^{\mathbf{f}}} = \mathcal{O}_{\overline{G} \times Q^{\mathbf{f}}} = (\mathrm{id}_{\overline{G}} \times \overline{\sigma}_-)^* \mathcal{O}_{\overline{G} \times Q^{\mathbf{f}}} = \overline{\sigma}_-^* \mathcal{O}_{Q^{\mathbf{f}}} \tag{2.6}$$

as quotient sheaves of $\mathcal{O}_{\overline{G} \times Q^s_-}$. Since $\pi_- : Q^s_- \to M^s_-$ is a principal fiber bundle with group $\overline{G}$, $(\overline{\sigma}_-, \mathrm{pr}_2) : \overline{G} \times Q^s_- \to Q^s_- \times_{M^s_-} Q^s_-$ is isomorphic. Thus, the identification (2.6) corresponds to an isomorphism

$$\alpha_2 : \mathrm{pr}_2^* \mathcal{O}_{Q^{\mathbf{f}}} \to \mathrm{pr}_1^* \mathcal{O}_{Q^{\mathbf{f}}} \tag{2.7}$$

of quotient sheaves of $\mathcal{O}_{Q^s_- \times_M Q^s_-}$, where $\mathrm{pr}_i : Q^s_- \times_{M^s_-} Q^s_- \to Q^s_-$ is the i-th projection for $i = 1, 2$. Since (2.6) results from Lemma 2.3, one can check that the isomorphism (2.7) satisfies that $\mathrm{pr}_{12}^*(\alpha_2) \circ \mathrm{pr}_{23}^*(\alpha_2) = \mathrm{pr}_{13}^*(\alpha_2)$, where $\mathrm{pr}_{ij} : Q^s_- \times_{M^s_-} Q^s_- \times_{M^s_-} Q^s_- \to Q^s_- \times_{M^s_-} Q^s_-$ is the (i, j)-th projection.

By faithfully-flat quasi-compact descent theory, we get a coherent sheaf $\mathcal{F}$ on M_- and a homomorphism $p' : \mathcal{O}_{M^s_-} \to \mathcal{F}$ such that $\pi_-^* \mathcal{F} = \mathcal{O}_{Q^{\mathbf{f}}}$ and that $\pi_-^*(p') = p^{\mathbf{f}}$. This $p' : \mathcal{O}_{M^s_-} \to \mathcal{F}$ should be surjective since $\pi_- : Q^s_- \to M^s_-$ is faithfully-flat, and hence p' gives a closed subscheme $P^{\mathbf{f}}$ of M^s_- such that $\pi_-^{-1}(P^{\mathbf{f}}) = Q^{\mathbf{f}}$. On the other hand $Q^{\mathbf{f}}$ is a closed subscheme of Q^{ss}_- fixed by $\overline{G}$, and so $P^{\mathbf{f}} = \pi_-(Q^{\mathbf{f}})$ is closed not only in M^s_- but also in M_- by the property of good quotient. Summarizing:

Lemma 2.4. *The closed subscheme $Q^{\mathbf{f}}$ of Q^{ss}_- obtained in Lemma* 2.2 *descends to a closed subscheme $P^{\mathbf{f}}$ of M_- such that $\pi_-^{-1}(P^{\mathbf{f}}) = Q^{\mathbf{f}}$, where $\pi_- : Q^{ss}_- \to M_-$ is the quotient map. $P^{\mathbf{f}}$ is contained in M^s_-. Set-theoretically, $\coprod_{\mathbf{f} \in A^+(a)} P^{\mathbf{f}}$ coincides with the subset* (2.1). *Both $Q^{\mathbf{f}} \cap Q^{\mathbf{f}'}$ and $P^{\mathbf{f}} \cap P^{\mathbf{f}'}$ are empty if $\mathbf{f}$ and $\mathbf{f}'$ are mutually different member of $A^+(a)$.*

At the end of this section, we define a closed subset

$$M_+ \supset \{[E] \mid E \text{ is not } a_-\text{-semistable}\} \tag{2.8}$$

similarly to the above M_-. First we define $-\mathbf{f}$.

Definition 2.5. For $\mathbf{f} = (f, m, n) \in A^+(a)$, we define $-\mathbf{f} \in \mathrm{Num}(X) \times \mathbb{Z}_{\geq 0}^{\times 2}$ by $-\mathbf{f} = (-f, n, m)$.

In the same way as the case of $i^{\mathbf{f}} : Q^{\mathbf{f}} \to Q_-^{ss}$ and $P^{\mathbf{f}} \subset Q_-^{ss}$, we can show that a projective Q_+^{ss}-scheme $Q^{-\mathbf{f}}$ can be defined; the structural morphism $i^{-\mathbf{f}} : Q^{-\mathbf{f}} \to Q_+^{ss}$ is a closed immersion which factors through Q_+^s; by using faithfully-flat quasi-compact descent theory, we can obtain a closed subscheme $P^{-\mathbf{f}} \subset M_+^s$ such that $\pi_+^{-1}(P^{-\mathbf{f}}) = Q^{-\mathbf{f}}$; for different members $\mathbf{f}$ and $\mathbf{f}'$ of $A^+(a)$, we see that $P^{-\mathbf{f}} \cap P^{-\mathbf{f}'}$ is empty; set-theoretically, $\coprod_{\mathbf{f}\in A^+(a)} P^{-\mathbf{f}}$ coincides with the subset (2.8) of M_+.

3. A sequence of morphisms connecting $M_-(c_1, c_2)$ with $M_+(c_1, c_2)$

Let V_- be a closed subscheme $\coprod_{\mathbf{f}\in A^+(a)} Q^{\mathbf{f}}$ of Q_-^{ss}, and $\varphi_- : \tilde{Q}_-^{ss} \to Q_-^{ss}$ the blowing-up of Q_-^{ss} along V_-, with exceptional divisor D_-. Similarly, let P_- be a closed subscheme $\coprod_{\mathbf{f}\in A^+(a)} P^{\mathbf{f}}$ of M_-, and $\phi_- : \tilde{M}_- \to M_-$ the blowing-up of M_- along P_-, with exceptional divisor E_-.

$$\begin{tikzcd}
V_- = \coprod_{\mathbf{f}} Q^{\mathbf{f}} \arrow[r, hook] & Q_-^{ss} \arrow[d, "\pi_-"] & \tilde{Q}_-^{ss} \arrow[l, "\varphi_-"'] \arrow[d, "\tilde{\pi}_-"] & D_- \arrow[l, hook'] \\
P_- = \coprod_{\mathbf{f}} P^{\mathbf{f}} \arrow[r, hook] & M_- & \tilde{M}_- \arrow[l, "\phi_-"'] & E_- \arrow[l, hook']
\end{tikzcd}$$

Because $\varphi_-^{-1}\pi_-^{-1}(P_-) = \varphi_-(V_-) = D_-$ is an effective Cartier divisor on $\tilde{Q}_-^{ss}$, a morphism $\tilde{\pi}_-$ is induced. In this section, we begin with constructing a morphism $\tilde{\varphi}_+ : \tilde{Q}_-^{ss} \to M_+$ using the method of elementary transformation. Joining the universal quotient sheaf $\mathcal{U}_-|_{X_{Q^{\mathbf{f}}}} \twoheadrightarrow \mathcal{G} = \mathcal{G}^{\mathbf{f}}$ of $Q^{\mathbf{f}}$, we have a quotient sheaf $\mathcal{U}_-|_{X_{V_-}} \twoheadrightarrow \mathcal{G}$ on $X_{V_-} = \coprod_{\mathbf{f}} X_{Q^{\mathbf{f}}}$. This results in an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{U}_-|_{V_-} \longrightarrow \mathcal{G} \longrightarrow 0 \tag{3.1}$$

of V_--flat X_{V_-}-modules. Pulling back this by $\mathrm{id}_X \times \varphi_- : X_{D_-} \to X_{V_-}$, we get an exact sequence of D_--flat sheaves

$$0 \longrightarrow \tilde{\mathcal{F}} \longrightarrow \tilde{\mathcal{U}}_-|_{D_-} \longrightarrow \tilde{\mathcal{G}} \longrightarrow 0 \tag{3.2}$$

on X_{D_-}. Now let $\mathcal{W}_+$ denote $\mathrm{Ker}(\tilde{\mathcal{U}}_- \twoheadrightarrow \tilde{\mathcal{U}}_-|_{D_-} \to \tilde{\mathcal{G}})$, that is,

$$0 \longrightarrow \mathcal{W}_+ \longrightarrow \tilde{\mathcal{U}}_- \longrightarrow \tilde{\mathcal{G}} \longrightarrow 0 \tag{3.3}$$

is exact. From [Fr, Lemma A.3] $\mathcal{W}_+$ is flat over $\tilde{Q}_-^{ss}$. (3.2) and (3.3) induce a commutative diagram on $X_{\tilde{Q}_-^{ss}}$

(3.4)
$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \tilde{\mathcal{U}}_-(-D_-) \arrow[r] \arrow[d, equal] & \mathcal{W}_+ \arrow[r, "f"] \arrow[d] & \tilde{\mathcal{F}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tilde{\mathcal{U}}_-(-D_-) \arrow[r] & \tilde{\mathcal{U}}_- \arrow[r] \arrow[d, "h"] & \tilde{\mathcal{U}}_-|_{D_-} \arrow[r] \arrow[d] & 0 \\
 & & \tilde{\mathcal{G}} \arrow[r, equal] \arrow[d] & \tilde{\mathcal{G}} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

whose rows and columns are exact. The second column of (3.4) gives rise to an exact sequence

$$0 \longrightarrow Tor_1^{X_{\tilde{Q}_-^{ss}}}(\tilde{\mathcal{G}}, \mathcal{O}_{X_{D_-}}) = \tilde{\mathcal{G}}(-D_-) \longrightarrow \mathcal{W}_+|_{X_{D_-}} \longrightarrow \tilde{\mathcal{U}}_-|_{X_{D_-}} \longrightarrow \tilde{\mathcal{G}} \longrightarrow 0.$$

From (3.4), this results in an exact sequence

$$0 \longrightarrow \tilde{\mathcal{G}}(-D_-) \longrightarrow \mathcal{W}_+|_{X_{D_-}} \overset{f|_{D_-}}{\longrightarrow} \tilde{\mathcal{F}} \longrightarrow 0. \tag{3.5}$$

(3.5) and the first row of (3.4) induce the following commutative diagram on $X_{\tilde{Q}_-^{ss}}$:

(3.6)
$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{W}_+(-D_-) \arrow[r] \arrow[d, equal] & \tilde{\mathcal{U}}_-(-D_-) \arrow[r, "\bar{h}"] \arrow[d] & \tilde{\mathcal{G}}(-D_-) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{W}_+(-D_-) \arrow[r] & \mathcal{W}_+ \arrow[r] \arrow[d, "f"] & \mathcal{W}_+|_{D_-} \arrow[r] \arrow[d, "f|_{D_-}"] & 0 \\
 & & \tilde{\mathcal{F}} \arrow[r, equal] \arrow[d] & \tilde{\mathcal{F}} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

such that its second column is equal to the first row of (3.4), and that all rows and columns are exact. For homomorphisms h in (3.4) and $\bar{h}$ in (3.6), one can find an isomorphism $j_g : \tilde{\mathcal{G}}(-D_-) \to \tilde{\mathcal{G}}(-D_-)$ such that

(3.7)
$$\begin{CD}
\tilde{\mathcal{U}}_-(-D_-) @>{\bar{h}}>> \tilde{\mathcal{G}}(-D_-) \\
@| @VV{j_g}V \\
\tilde{\mathcal{U}}_-(-D_-) @>{h(-D_-)}>> \tilde{\mathcal{G}}(-D_-)
\end{CD}$$

is commutative, in view of the uniqueness of the Harder-Narasimhan filtration and the simplicity of torsion-free rank-one sheaf.

Now we recall some obstruction theory. By the exact sequence

(3.8)
$$0 \longrightarrow \mathcal{O}_{D_-}(-D_-) \longrightarrow \mathcal{O}_{2D_-} \longrightarrow \mathcal{O}_{D_-} \longrightarrow 0.$$

and (3.2), we have the following commutative diagram on X_{2D_-} whose rows and columns are exact:

(3.9)
$$\begin{CD}
@. @. 0 @. @. \\
@. @. @VVV @. @. \\
0 @>>> \tilde{\mathcal{F}}(-D_-) @>>> \tilde{\mathcal{U}}_- \otimes \mathcal{O}_{D_-}(-D_-) @>>> \tilde{\mathcal{G}}(-D_-) @>>> 0 \\
@. @. @VVV @. @. \\
@. @. \tilde{\mathcal{U}}_-|_{X_{2D_-}} @. @. \\
@. @. @VVV @. @. \\
0 @>>> \tilde{\mathcal{F}} @>>> \tilde{\mathcal{U}}_-|_{X_{D_-}} @>>> \tilde{\mathcal{G}} @>>> 0 \\
@. @. @VVV @. @. \\
@. @. 0 @. @. .
\end{CD}$$

From this we can get a complex $\tilde{\mathcal{F}}(-D_-) \xrightarrow{F} \tilde{\mathcal{U}}_-|_{X_{2D_-}} \xrightarrow{G} \tilde{\mathcal{G}}$, and check that its middle cohomology $B = \operatorname{Ker} G / \operatorname{Im} F$ is a $\mathcal{O}_{X_{D_-}}$-module. Then, again from (3.9) we can deduce an exact sequence

(3.10)
$$0 \longrightarrow \tilde{\mathcal{G}}(-D_-) \xrightarrow{p} B \xrightarrow{q} \tilde{\mathcal{F}} \longrightarrow 0$$

of D_--flat $\mathcal{O}_{X_{D_-}}$-modules.

Lemma 3.1. *The following conditions are equivalent for a closed point t of D_-:*

(1) *The exact sequence*

(3.11)
$$0 \longrightarrow \tilde{\mathcal{G}}(-D_-) \otimes k(t) \longrightarrow B \otimes k(t) \longrightarrow \tilde{\mathcal{F}} \otimes k(t) \longrightarrow 0$$

induced from (3.10) *is trivial*;

(2) *Let* $\tilde{m}_t \subset \mathcal{O}_{\tilde{Q}^{ss}_-}$ *be the maximal ideal defining* t *and* l *the integer such that* $I_{D_-,t} \subset \tilde{m}_t^l$ *and that* $I_{D_-,t} \not\subset \tilde{m}_t^{l+1}$. *Then there is a morphism* p_{l+1} : $\operatorname{Spec}(\mathcal{O}_{\tilde{Q}^{ss}_-}/\tilde{m}_t^{l+1}) \to V_- = \coprod_{\mathbf{f}} Q^{\mathbf{f}}$ *such that*

$$\begin{tikzcd}
\operatorname{Spec}(\mathcal{O}_{\tilde{Q}^{ss}_-}/\mathcal{O}(-D_-)+\tilde{m}_t^{l+1}) \arrow[r, hook] \arrow[d, hook] & D_- \arrow[r, "\varphi_-"] & V_- \arrow[d, hook] \\
\operatorname{Spec}(\mathcal{O}_{\tilde{Q}^{ss}_-}/\tilde{m}_t^{l+1}) \arrow[r, hook] \arrow[urr, "p_{l+1}"'] & \tilde{Q}^{ss}_- \arrow[r, "\varphi_-"'] & Q^{ss}_-
\end{tikzcd} \tag{3.12}$$

is commutative.

Proof. We put $A = \mathcal{O}_{\tilde{Q}^{ss}_-}/\tilde{m}_t^{l+1}$ and $A' = \mathcal{O}_{\tilde{Q}^{ss}_-}/(\mathcal{O}(-D_-)+\tilde{m}_t^{l+1})$, which are Artinian local rings. Tensoring A to (3.8), we have the following commutative diagram whose rows are exact.

(3.13)

$$\begin{tikzcd}
\mathcal{O}_{D_-}(-D_-) \otimes_{\tilde{Q}^{ss}_-} A \arrow[r] \arrow[d, "q"] & \mathcal{O}_{2D_-} \otimes_Q A \arrow[r] \arrow[d, equal] & \mathcal{O}_{D_-} \otimes_Q A \arrow[r] \arrow[d, equal] & 0 \\
I = \mathcal{O}(-D_-)+\tilde{m}_t^{l+1}/\tilde{m}_t^{l+1} \arrow[r, hook] & A \arrow[r] & A' \arrow[r] & 0
\end{tikzcd}$$

Remark that I is a $k(t)$-module because of the choice of l. From its bottom row and (3.2) we get the following commutative diagram on X_A whose rows and columns are exact, similarly to (3.9):

(3.14)

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & \tilde{\mathcal{F}}_{k(t)} \otimes I \arrow[r] & \tilde{\mathcal{U}}_{k(t)} \otimes I \arrow[r] \arrow[d] & \tilde{\mathcal{G}}_{k(t)} \otimes I \arrow[r] & 0 \\
 & & \tilde{\mathcal{U}}_-|_{X_A} \arrow[d] & & \\
0 \arrow[r] & \tilde{\mathcal{F}} \otimes_{D_-} A' \arrow[r] & \tilde{\mathcal{U}}_-|_{X_{A'}} \arrow[r] \arrow[d] & \tilde{\mathcal{G}} \otimes_{D_-} A' \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}.$$

Then one can deduce a complex $\tilde{\mathcal{F}}_{k(t)} \otimes I \xrightarrow{F'} \tilde{\mathcal{U}}_-|_{X_{A'}} \xrightarrow{G'} \tilde{\mathcal{G}} \otimes_{D_-} A'$ and an exact sequence of $X_{A'}$-modules

$$0 \longrightarrow \tilde{\mathcal{G}}_{k(t)} \otimes I \longrightarrow B' = \operatorname{Ker} G'/\operatorname{Im} F' \longrightarrow \tilde{\mathcal{F}} \otimes_{D_-} A' \longrightarrow 0. \tag{3.15}$$

Now recall that obstruction theory shows the following fact ([HL, p. 43]).

Fact 3.2. *The exact sequence* (3.15) *is trivial if and only if the condition* (ii) *in Lemma* 3.1 *is satisfied.*

From the commutativity of (3.13), we can make a homomorphism $B \otimes_{D_-} A' \to B'$ such that

(3.16)
$$\begin{tikzcd}
\tilde{\mathcal{G}}(-D_-) \otimes_{D_-} A' = \tilde{\mathcal{G}}(-D_-) \otimes_Q A \arrow[r, hook] \arrow[d, "\mathrm{id} \otimes q"] & B \otimes_{D_-} A' \arrow[r] \arrow[d] & \tilde{\mathcal{F}} \otimes_{D_-} A' \arrow[r] \arrow[d, equal] & 0 \\
\tilde{\mathcal{G}}_{k(t)} \otimes I = \tilde{\mathcal{G}} \otimes_{\tilde{Q}^{ss}_-} I \arrow[r, hook] & B' \arrow[r] & \tilde{\mathcal{F}} \otimes_{D_-} A' \arrow[r] & 0
\end{tikzcd}$$

is commutative, where the first row is obtained by tensoring A to (3.10), and the second row is (3.15). Further, the homomorphism q in (3.13) gives a surjective homomorphism $q \otimes k(t) : \mathcal{O}_{D_-} \otimes_{D_-} k(t) \twoheadrightarrow I$, which should be isomorphic because $\mathrm{rk}_{k(t)} \mathcal{O}_{D_-}(-D_-) \otimes k(t) = 1$ and $I \neq 0$. Accordingly we obtain a commutative diagram

(3.17)
$$\begin{tikzcd}
\mathrm{Ext}^1_{X_{A'}}(\tilde{\mathcal{F}} \otimes A', \tilde{\mathcal{G}}(-D_-) \otimes A') \arrow[r, "q_*"] \arrow[d, "(\pi_t)_*"] & \mathrm{Ext}^1_{X_{A'}}(\tilde{\mathcal{F}} \otimes A', \tilde{\mathcal{G}}_{k(t)} \otimes I) \\
\mathrm{Ext}^1_{X_{A'}}(\tilde{\mathcal{F}} \otimes A', \tilde{\mathcal{G}}(-D_-) \otimes k(t)) & \mathrm{Ext}^1_{X_{k(t)}}(\tilde{\mathcal{F}}_{k(t)}, \tilde{\mathcal{G}}(-D_-)_{k(t)}), \arrow[l, "\pi_t^*"] \arrow[u, "(q \otimes k(t))_*"']
\end{tikzcd}$$

where π_t is a natural homomorphism $A' \to k(t)$. Remark that π_t^* is isomorphic since $\tilde{\mathcal{F}}$ is D_--flat. Let $\lambda \in \mathrm{Ext}^1_{X_{A'}}(\tilde{\mathcal{F}} \otimes A', \tilde{\mathcal{G}}(-D_-) \otimes A')$ be the extension class of the first row of (3.16). Then one can prove that $(\pi_t^*)^{-1}(\pi_{t*}(\lambda))$ is the extension class of (3.11) and that $q_*(\lambda)$ is the extension class of (3.15) by using the commutativity of (3.16). Because $(q \otimes k(t))_*$ is isomorphic, $(\pi_t^*)^{-1}(\pi_{t*}(\lambda)) = 0$ if and only if $q_*(\lambda) = 0$. This and Fact 3.2 complete the proof of this lemma. □

Lemma 3.3. *There is an isomorphism* $r_0 : \mathcal{W}_+|_{X_{D_-}} \to B$ *such that the following diagram is commutative*:

(3.18)
$$\begin{tikzcd}
0 \arrow[r] & \tilde{\mathcal{G}}(-D_-) \arrow[r] \arrow[d, "j_g"] & \mathcal{W}_+|_{X_{D_-}} \arrow[r] \arrow[d, "r_0"] & \tilde{\mathcal{F}} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \tilde{\mathcal{G}}(-D_-) \arrow[r, "p"] & B \arrow[r, "q"] & \tilde{\mathcal{F}} \arrow[r] & 0.
\end{tikzcd}$$

Here the first row is the third column of (3.6), *the second row is* (3.10), *and* j_g *is the isomorphism in* (3.7).

Proof. Tensoring $\mathcal{O}_{2D_-}$ to (3.4), we have a commutative diagram on

X_{2D_-}

(3.19)
$$\begin{tikzcd}
 & \tilde{\mathcal{G}}(-2D_-) \arrow[r] \arrow[d, hook] & 0 \arrow[d] & \\
\tilde{\mathcal{U}}_-(-D_-)|_{2D_-} \arrow[r, "k'"] \arrow[d, equal] & \mathcal{W}_+|_{2D_-} \arrow[r, "f|_{2D_-}"] \arrow[d, "r"] & \tilde{\mathcal{F}} \arrow[r] \arrow[d] & 0 \\
\tilde{\mathcal{U}}_-(-D_-)|_{2D_-} \arrow[r] & \tilde{\mathcal{U}}_-|_{2D_-} \arrow[r] \arrow[d, "h|_{2D_-}=G"'] & \tilde{\mathcal{U}}_-|_{D_-} \arrow[r] \arrow[d] & 0 \\
 & \tilde{\mathcal{G}} \arrow[r, equal] \arrow[d] & \tilde{\mathcal{G}} \arrow[d] & \\
 & 0 & 0 &
\end{tikzcd}$$

whose rows and columns are exact. In this diagram $h|_{2D_-}$ clearly is equal to the homomorphism G defined just below (3.9), and so r factors into $\mathcal{W}_+|_{2D_-} \overset{r_1}{\twoheadrightarrow} \operatorname{Im} r = \operatorname{Ker} G \to \tilde{\mathcal{U}}_-|_{2D_-}$. One can readily check that

(3.20)
$$\begin{tikzcd}
\mathcal{W}_+|_{2D_-} \arrow[r, two heads] \arrow[d, "r_1"] & \mathcal{W}_+|_{D_-} \arrow[d, dotted, "r_0"] \arrow[dr, "f|_{D_-}"] & \\
\operatorname{Ker} G \arrow[r, two heads] & B = \operatorname{Ker} G/\operatorname{Im} F \arrow[r, "q"'] & \tilde{\mathcal{F}}
\end{tikzcd}$$

is commutative by the definition of q in (3.10). Since B is naturally regarded as an $\mathcal{O}_{X_{D_-}}$-module, we can induce a homomorphism $r_0 : \mathcal{W}_+|_{D_-} \to B$ such that the left side of (3.20) becomes commutative. Then one can also check the right side of (3.20) is commutative, since $\mathcal{W}_+|_{2D_-} \to \mathcal{W}_+|_{D_-}$ is surjective. Therefore the right side of (3.18) is surely commutative for this r_0.

Next, by the definition of p in (3.10) one can readily check that

$$\begin{tikzcd}
\tilde{\mathcal{U}}_-(-D_-)|_{2D_-} \arrow[rr, two heads, "h(-D_-)|_{2D_-}"] \arrow[d, "k'"] & & \tilde{\mathcal{G}}(-D_-) \arrow[d, "p"] \\
\mathcal{W}_+|_{2D_-} \arrow[r, two heads, "r_1"] & \operatorname{Ker} G \arrow[r, two heads] & B
\end{tikzcd}$$

is commutative, where $h|_{2D_-}$ and k' are those of (3.19). We have also the following commutative diagram:

$$\begin{tikzcd}
\tilde{\mathcal{U}}_-(-D_-)|_{2D_-} \arrow[r, "k'"] \arrow[d, "\bar{h}|_{2D_-}"] & \mathcal{W}_+|_{2D_-} \arrow[r, "r_1"] \arrow[d] & \operatorname{Ker} G \arrow[d] \\
\tilde{\mathcal{G}}(-D_-) \arrow[r] & \mathcal{W}_+|_{D_-} \arrow[r, two heads, "r_0"] & B,
\end{tikzcd}$$

where the left side is the upper-right side of (3.6), and the right side is the left side of (3.20). These two commutative diagrams gives rise to a commutative diagram

$$
\begin{tikzcd}
\tilde{\mathcal{U}}_-(-D_-)|_{2D_-} \arrow[r, two heads, "\bar{h}|_{2D_-}"] \arrow[d, equal] & \mathcal{G}(-D_-) \arrow[r] & \mathcal{W}_+|_{D_-} \arrow[d, "r_0"] \\
\tilde{\mathcal{U}}_-(-D_-)|_{2D_-} \arrow[r, two heads, "h(-D_-)"'] & \tilde{\mathcal{G}}(-D_-) \arrow[r, "p"] & B.
\end{tikzcd}
\tag{3.21}
$$

Then we can prove the right side of (3.18) is commutative from (3.7) and the surjectivity of $\bar{h}|_{2D_-}$. □

Corollary 3.4. *Let $t \in D_-$ be a closed point. Then the exact sequence*

$$0 \longrightarrow \tilde{\mathcal{G}}(-D_-) \otimes k(t) \longrightarrow \mathcal{W}_+ \otimes k(t) \longrightarrow \tilde{\mathcal{F}} \otimes k(t) \longrightarrow 0$$

induced from the third column of (3.6) *is nontrivial.*

Proof. Suppose not. Then Lemmas 3.1 and 3.3 lead to a morphism p_{l+1} : $\mathrm{Spec}(\mathcal{O}_{\tilde{Q}^{ss}_-}/\tilde{m}_t^{l+1}) \to V_-$ such that (3.12) becomes commutative. This p_{l+1} induces a $\tilde{Q}^{ss}_-$-morphism $q_{l+1} : \mathrm{Spec}(\mathcal{O}_{\tilde{Q}^{ss}_-}/\tilde{m}_t^{l+1}) \to \tilde{Q}^{ss}_- \times_{Q^{ss}_-} V_- = D_-$. Thus I_{D_-} is contained in $\tilde{m}_t^{l+1}$, which contradicts the choice of l in Lemma 3.1. □

From the corollary above one can show that $\mathcal{W}_+ \otimes k(t) \in \mathrm{Coh}(X_{k(t)})$ is a_+-semistable for every point $t \in \tilde{Q}^{ss}_-$ in a similar fashion to the proof of Lemma 1.6 (ii). This sheaf $\mathcal{W}_+$ accordingly gives a morphism $\tilde{\varphi}_+ : \tilde{Q}^{ss}_- \to M_+$. Now we intend to construct a morphism $\bar{\phi}_+ : \tilde{M}_- \to M_+$ such that $\bar{\phi}_+ \circ \tilde{\pi}_- : \tilde{Q}^{ss}_- \to M_+$ is equal to $\tilde{\varphi}_+$.

Lemma 3.5. *The natural morphism $\tilde{Q}^{ss}_- \to Q^{ss}_- \times_{M_-} \tilde{M}_-$ is isomorphic.*

Proof. $\tilde{Q}^s_-$ denotes the open subset $(\phi_- \circ \tilde{\pi}_-)^{-1}(M^s_-)$ of $\tilde{Q}^{ss}_-$, and $\tilde{M}^s_-$ denotes $\phi_-^{-1}(M^s_-)$. Because $E_- \subset \tilde{M}_-$ is contained in $\tilde{M}^s_-$ it suffices to show that $\tilde{Q}^s_- \to Q^s_- \otimes_{M^s_-} \tilde{M}^s_-$ is isomorphic. Since $\pi_- : Q^s_- \to M^s_-$ is flat one can show that $\pi_-^*(I_{P_-,M^s_-}) = I_{V_-,Q^s_-}$, and hence that $\pi_-^*(I^n_{P_-,M^s_-}) = I^n_{V_-,Q^s_-}$ for any n. □

Using this lemma one can induce an action $\bar{\Sigma}_- : \bar{G} \times \tilde{Q}^{ss}_- = (\bar{G} \times Q^{ss}_-) \times_{M_-} \tilde{M}_- \to \tilde{Q}^{ss}_- = Q^{ss}_- \times_{M_-} \tilde{M}_-$ from the action $\bar{\sigma}_- : \bar{G} \times Q^{ss}_- \to Q^{ss}_-$.

Lemma 3.6. *As to the morphism $\tilde{\varphi}_+$, the following is commutative*:

$$
\begin{tikzcd}
\bar{G} \times \tilde{Q}^{ss}_- \arrow[r, "\bar{\Sigma}_-"'] \arrow[d, "\mathrm{pr}_2"] & \tilde{Q}^{ss}_- \arrow[d, "\tilde{\varphi}_+"] \\
\tilde{Q}^{ss}_- \arrow[r, "\tilde{\varphi}_+"] & M_+.
\end{tikzcd}
$$

One can prove this lemma easily. $\pi_- : Q_-^{ss} \to M_-$ is a good quotient by $\bar{\sigma}_-$, so [M2, p. 8, Remark 5] and [M2, p. 27, Theorem 1] imply that $\tilde{\pi}_- : \tilde{Q}_-^{ss} = Q_-^{ss} \times_{M_-} \tilde{M}_- \to \tilde{M}_-$ is a categorical quotient by $\bar{\Sigma}_-$. Therefore there is a unique morphism $\bar{\phi}_+ : \tilde{M}_- \to M_+$ such that $\bar{\phi}_+ \circ \tilde{\pi}_- : \tilde{Q}_-^{ss} \to M_+$ is equal to $\tilde{\phi}_+$ because of the lemma above.

Consequently we can connect $M_- = M_-(c_1, c_2)$ with $M_+ = M_+(c_1, c_2)$ by

$$\begin{tikzcd} V_- \arrow[r, hook] & Q_-^{ss} \arrow[d, "\pi_-"'] & \tilde{Q}_-^{ss} \arrow[l, "\varphi_-"'] \arrow[d, "\tilde{\pi}_-"'] \arrow[dr, "\tilde{\varphi}_+"] & \\ P_- \arrow[r, hook] & M_- & \tilde{M}_- \arrow[l, "\phi_-"'] \arrow[r, "\bar{\phi}_+"'] & M_+ \end{tikzcd} \tag{3.22}$$

when $P_- \subset M_-$ is nowhere dense. (Without this hypothesis $\tilde{M}_-$ may be empty.)

We shall reverse M_- and M_+ and follow a similar argument. Let V_+ be a closed subscheme $\coprod_{\mathbf{f} \in A^+(a)} Q^{-\mathbf{f}}$, and P_+ a closed subscheme $\coprod_{\mathbf{f} \in A^+(a)} P^{-\mathbf{f}}$ of M_+, mentioned right after Definition 2.5. Let $\varphi_+ : \tilde{Q}_+^{ss} \to Q_+^{ss}$ be the blowing-up along V_+, and $\phi_+ : \tilde{M}_+ \to M_+$ the blowing-up along P_+. Denote their exceptional divisors by $D_+ \subset \tilde{Q}_+^{ss}$ and $E_+ \subset \tilde{M}_+$ respectively. Then we can construct a morphism $\tilde{\varphi}_- : \tilde{Q}_+^{ss} \to M_-$ and make it descend to a morphism $\bar{\phi}_- : \tilde{M}_+ \to M_-$. Thereby we get another sequence of morphisms connecting M_- and M_+ as follows:

$$\begin{tikzcd} & \tilde{Q}_+^{ss} \arrow[r, "\varphi_+"'] \arrow[dl, "\tilde{\varphi}_-"'] \arrow[d, "\tilde{\pi}_+"] & Q_+^{ss} \arrow[d, "\pi_+"] & V_+ \arrow[l, hook'] \\ M_- & \tilde{M}_+ \arrow[l, "\bar{\phi}_-"] \arrow[r, "\phi_+"'] & M_+ & P_+ . \arrow[l, hook'] \end{tikzcd} \tag{3.23}$$

4. $\bar{\phi}_+ : \tilde{M}_- \to M_+$ is blowing-up

In this section we would like to compare (3.22) with (3.23) assuming that P_- is nowhere dense in M_-. The following lemma shall be needed later.

Lemma 4.1. *Let $\mathcal{U}_+$ be a universal quotient sheaf of Q_+^{ss} on $X_{Q_+^{ss}}$, and $\mathcal{W}_+$ the $X_{\tilde{Q}_-^{ss}}$-module defined at (3.3). There are an open covering $\bigcup_\alpha U_\alpha$ of $\tilde{Q}_-^{ss}$, a morphism $\bar{\varphi}_+^\alpha : U_\alpha \to Q_+^{ss}$ such that*

$$\begin{tikzcd} U_\alpha \arrow[r, "\bar{\varphi}_+^\alpha"] \arrow[d, hook] & Q_+^{ss} \arrow[d, "\pi_+"] \\ \tilde{Q}_-^{ss} \arrow[r, "\tilde{\varphi}_+"] & M_+ \end{tikzcd}$$

is commutative, and an isomorphism $\Phi_+^\alpha : \mathcal{W}_+|_{U_\alpha} \to (\bar{\varphi}_+^\alpha)^*\mathcal{U}_+$ *of* X_{U_α}*-modules. Furthermore, we can assume that* $U_\alpha \cap U_\beta \subset \tilde{Q}_-^s$ *if* $\alpha \neq \beta$.

Proof. The proof of the first part is easy, so may be left to the reader. Recall that both Q_+^{ss} and Q_-^{ss} are open subsets of a Quot-scheme Q, and that $\mathcal{U}_+|_{Q_-^{ss}\cap Q_+^{ss}} = \mathcal{U}_-|_{Q_-^{ss}\cap Q_+^{ss}}$. $U_0 = \tilde{Q}_-^{ss} \setminus D_- = Q_-^{ss} \setminus V_-$ is an open neighborhood of $\tilde{Q}_-^{ss} \setminus \tilde{Q}_-^s$, and is contained in $Q_-^{ss} \cap Q_+^{ss}$. Let $\bar{\varphi}_+^0 : U_0 = Q_-^{ss} \setminus V_- \to Q_+^{ss}$ be a natural open immersion, and $\Phi_+^0 : \mathcal{W}_+|_{U_0} \to \mathcal{U}_+|_{U_0}$ an isomorphism $\mathcal{W}_+|_{\tilde{Q}_-^{ss}\setminus D_-} \to \tilde{\mathcal{U}}_-|_{\tilde{Q}_-^{ss}\setminus D_-} = \mathcal{U}_-|_{Q_-^{ss}\setminus V_-} = \mathcal{U}_+|_{Q_-^{ss}\setminus V_-}$ induced from (3.3). Then $\bar{\varphi}_+^0$ and Φ_+^0 satisfy the conditions in this lemma. Thus we can assume that $U_\alpha \cap U_\beta \subset \tilde{Q}_-^s$ if $\alpha \neq \beta$. □

Lemma 4.2. $\tilde{\varphi}_+^{-1}(P_+)$ *is equal to* $D_- = \varphi_-^{-1}(V_-)$ *as closed subschemes in* $\tilde{Q}_-^{ss}$.

Proof. Clearly $D_- \subset \tilde{\varphi}_+^{-1}(P_+)$ from the construction of $\tilde{\varphi}_+$. We first consider the case where $D'_- := \tilde{\varphi}_+^{-1}(P_+)$ is a Cartier divisor of $\tilde{Q}_-^{ss}$. By virtue of the definition of V_+, there is an exact sequence

$$0 \longrightarrow \mathcal{G}' \longrightarrow \mathcal{U}_+|_{X_{V_+}} \longrightarrow \mathcal{F}' \longrightarrow 0 \tag{4.1}$$

of V_+-flat X_{V_+}-modules such that, for every closed point t of $Q^{-\mathbf{f}}$, $(2c_1(\mathcal{G}'_{k(t)}) - c_1, c_2(\mathcal{G}'_{k(t)}), c_2(\mathcal{F}'_{k(t)}))$ is equal to $-\mathbf{f}$. Similarly to Lemma 2.1, $\mathcal{F}'$ and $\mathcal{G}'$ are flat family of torsion-free sheaves. Pulling back this by $\bar{\varphi}_+^\alpha$ of Lemma 4.1, we have an exact sequence

$$\begin{aligned} 0 \longrightarrow (\bar{\varphi}_+^\alpha)^*\,\mathcal{G}' = \mathcal{G}'_\alpha &\longrightarrow (\bar{\varphi}_+^\alpha)^*\,\mathcal{U}_+|_{D'_-\cap U_\alpha} \\ &= \bar{\mathcal{U}}_+^\alpha|_{D'_-\cap U_\alpha} \longrightarrow (\bar{\varphi}_+^\alpha)^*\,\mathcal{F}' = \mathcal{F}'_\alpha \longrightarrow 0 \end{aligned} \tag{4.2}$$

on $X \times (\bar{\varphi}_+^\alpha)^{-1}(V_+) = X_{D'_-\cap U_\alpha}$, where we put $(\bar{\varphi}_+^\alpha)^*\mathcal{U}_+ = \bar{\mathcal{U}}_+^\alpha$. Let $\mathcal{V}_-$ denote $\mathrm{Ker}(\bar{\mathcal{U}}_+^\alpha \twoheadrightarrow \bar{\mathcal{U}}_+^\alpha|_{D'_-\cap U_\alpha} \twoheadrightarrow \mathcal{F}'_\alpha)$, that is,

$$0 \longrightarrow \mathcal{V}_- \longrightarrow \bar{\mathcal{U}}_+^\alpha \longrightarrow \mathcal{F}'_\alpha \longrightarrow 0 \tag{4.3}$$

is exact. $\mathcal{V}_-$ is flat over U_α since D'_- is a Cartier divisor of $\tilde{Q}_-^{ss}$.

Because $D'_- \supset D_-$, the isomorphism Φ_+^α in Lemma 4.1 induces a surjection $\bar{\mathcal{U}}_+^\alpha|_{D'_-\cap U_\alpha} \twoheadrightarrow \mathcal{W}_+|_{D_-\cap U_\alpha}$. Hence we have a diagram on $X_{D'_-\cap U_\alpha}$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G}'_\alpha \arrow[r] & \bar{\mathcal{U}}_+^\alpha|_{D'_-\cap U_\alpha} \arrow[r] \arrow[d] & \mathcal{F}'_\alpha \arrow[r] \arrow[d, dotted, "r"] & 0 \\ 0 \arrow[r] & \tilde{\mathcal{G}}(-D_-)|_{U_\alpha} \arrow[r] & \mathcal{W}_+|_{D_-\cap U_\alpha} \arrow[r] & \tilde{\mathcal{F}}|_{U_\alpha} \arrow[r] & 0, \end{tikzcd} \tag{4.4}$$

where the first row is (4.2) and the second row is the restriction of the third column in (3.6) to $X_{D_-\cap U_\alpha}$. One can check that $Hom_{X_{D'_-\cap U_\alpha}/D'_-\cap U_\alpha}(\mathcal{G}'_\alpha, \tilde{\mathcal{F}}|_{U_\alpha}) =$

$Hom_{X_{D_-\cap U_\alpha}/D_-\cap U_\alpha}(\mathcal{G}'_\alpha|_{D_-\cap U_\alpha}, \tilde{\mathcal{F}}|_{U_\alpha}) = 0$ by base change theorem on relative Ext sheaves, and so one can find $r : \mathcal{F}'_\alpha \to \tilde{\mathcal{F}}|_{U_\alpha}$ such that (4.4) is commutative. Then the following also is commutative:

(4.5)
$$\begin{tikzcd}
0 \arrow[r] & \mathcal{V}_- \arrow[r] \arrow[d, dotted, "s"] & \bar{\mathcal{U}}^\alpha_+ \arrow[r] \arrow[d, "(\Phi^\alpha_+)^{-1}"] & \mathcal{F}'_\alpha \arrow[r] \arrow[d, "r"] & 0 \\
0 \arrow[r] & \tilde{\mathcal{U}}_-(-D_-)|_{U_\alpha} \arrow[r] & \mathcal{W}_+|_{U_\alpha} \arrow[r] & \tilde{\mathcal{F}}|_{U_\alpha} \arrow[r] & 0,
\end{tikzcd}$$

where the first row is (4.3) and the second row is the restriction of the second column in (3.6) to X_{U_α}.

Claim 4.3. Set-theoretically, $D_- \cap U_\alpha$ coincides with $D'_- \cap U_\alpha$.

Proof. Suppose not. Then one can find a closed point $t \in D'_-$ that is not contained in D_-. Since $t \in D'_-$, (4.2) implies that $\bar{\mathcal{U}}^\alpha_+ \otimes k(t)$ is not a_--semistable. Since $t \notin D_-$, (3.3) implies that $\mathcal{W}_+ \otimes k(t)$ is a_--semistable. This is a contradiction because $\bar{\mathcal{U}}^\alpha_+ \otimes k(t)$ is isomorphic to $\mathcal{W}_+ \otimes k(t)$. $\square$

One can obtain the following commutative diagram by tensoring $\mathcal{O}_{D'_-\cap U_\alpha}$ to the first row in (4.5) and $\mathcal{O}_{D_-\cap U_\alpha}$ to the second row in (4.5) since $D'_- \supset D_-$:

(4.6)
$$\begin{tikzcd}
\mathcal{F}'_\alpha(-D'_-) \arrow[r, hook] \arrow[d, "u"] & \mathcal{V}_-|_{D'_-\cap U_\alpha} \arrow[r] \arrow[d, "s'"] & \bar{\mathcal{U}}^\alpha_+|_{D'_-\cap U_\alpha} \arrow[r] \arrow[d, "(\Phi^{-1})'"] & \mathcal{F}'_\alpha \arrow[r] \arrow[d, "r"] & 0 \\
\tilde{\mathcal{F}}(-D_-)|_{U_\alpha} \arrow[r, hook] & \tilde{\mathcal{U}}(-D_-)|_{D_-\cap U_\alpha} \arrow[r] & \mathcal{W}_+|_{D_-\cap U_\alpha} \arrow[r] & \tilde{\mathcal{F}}|_{U_\alpha} \arrow[r] & 0.
\end{tikzcd}$$

Claim 4.4. $s\otimes k(t) : \mathcal{V}_-\otimes k(t) \to \tilde{\mathcal{U}}_-(-D_-)\otimes k(t)$ in (4.5) is isomorphic for every closed point $t \in U_\alpha$.

Proof. We have to verify this only in case where t is contained in D'_-. By Claim 4.3 t is also contained in D_-. Tensoring $k(t)$ to (4.6), we obtain a commutative diagram

(4.7)
$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}'_\alpha(-D'_-)_{k(t)} \arrow[r] \arrow[d, "u_t"] & \mathcal{V}_{-\,k(t)} \arrow[r] \arrow[d, "s'_t"] & \bar{\mathcal{U}}^\alpha_{+\,k(t)} \arrow[r] \arrow[d, "(\Phi^{-1})'_t"] & \mathcal{F}'_{\alpha\,k(t)} \arrow[r] \arrow[d, "r_t"] & 0 \\
0 \arrow[r] & \tilde{\mathcal{F}}(-D_-)_{k(t)} \arrow[r] & \tilde{\mathcal{U}}_-(-D_-)_{k(t)} \arrow[r] & \mathcal{W}_{+\,k(t)} \arrow[r] & \tilde{\mathcal{F}}_{k(t)} \arrow[r] & 0
\end{tikzcd}$$

whose rows are exact. $(\Phi^{-1})'_t$ is isomorphic by its definition. One can see that also r_t is isomorphic by the uniqueness of the Harder-Narasimhan filtration with respect to a_--stability. Thus s'_t is nonzero map. If u_t is zero map, then s'_t induces a nonzero homomorphism

$$\bar{s}'_t : \mathcal{G}'_\alpha \otimes k(t) = \mathrm{Cok}(\mathcal{F}'_\alpha(-D'_-)_{k(t)} \to \mathcal{V}_{-\,k(t)}) \to \tilde{\mathcal{U}}_-(-D_-)_{k(t)}$$

by (4.2). This $\bar{s}'_t$ should be injective because $\mathcal{G}'_\alpha \otimes k(t)$ is torsion-free and rank-one. This contradicts the a_--semistability of $\tilde{\mathcal{U}}_-(-D_-)_{k(t)}$, and so u_t should be nonzero, and hence injective. Then one can see s'_t is injective by diagram-chasing. (4.7) implies the Chern classes of $\mathcal{V}_{-\,k(t)}$ are equal to those of $\tilde{\mathcal{U}}_-(-D_-)_{k(t)}$, we see that $s'_t = s \otimes k(t)$ is isomorphic. □

Both $\mathcal{V}_-$ and $\tilde{\mathcal{U}}_-(-D_-)|_{U_\alpha}$ are U_α-flat, and hence the claim above implies that s in (4.5) is isomorphic. Then also r in (4.5) is isomorphic. Because $\mathcal{F}'_\alpha$ is $D'_- \cap U_\alpha$-flat and $\tilde{\mathcal{F}}|_{U_\alpha}$ is $D_- \cap U_\alpha$-flat, one can verify that $D_- \cap U_\alpha$ is equal to $D'_- \cap U_\alpha$. Since this holds good for every U_α, we conclude the proof of this lemma in case where $\tilde{\varphi}_+^{-1}(P_+)$ is a Cartier divisor.

Next, we consider the case where $\tilde{\varphi}_+^{-1}(P_+) = D'_-$ is not necessarily a Cartier divisor of $\tilde{Q}^{ss}_-$. Let $\varphi^{(2)}_- : \tilde{Q}^{(2)}_- \to \tilde{Q}^{ss}_-$ be the blowing up along D'_-. Let $\tilde{D}_-$ and $\tilde{D}'_-$ denote closed subschemes $(\varphi^{(2)}_-)^{-1}(D_-)$ and $(\varphi^{(2)}_-)^{-1}(D'_-)$ of $\tilde{Q}^{(2)}_-$, respectively. For a natural exact sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{Q}^{ss}_-}(-D_-) \longrightarrow \mathcal{O}_{\tilde{Q}^{ss}_-} \to \mathcal{O}_{D_-} \longrightarrow 0$$

on $\tilde{Q}^{ss}_-$, one can verify that also its pull-back by $\varphi^{(2)}_-$

$$0 \longrightarrow (\varphi^{(2)}_-)^*\mathcal{O}_{\tilde{Q}^{ss}_-}(-D_-) \longrightarrow \mathcal{O}_{\tilde{Q}^{(2)}_-} \longrightarrow \mathcal{O}_{\tilde{D}_-} \longrightarrow 0$$

is exact. In view of this, one can check that the pull-back of (3.6) by $\mathrm{id}_X \times \varphi^{(2)}_-$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \tilde{\mathcal{W}}^{(2)}_+(-\tilde{D}_-) \arrow[r] \arrow[d, equal] & \tilde{\mathcal{U}}^{(2)}_-(-\tilde{D}_-) \arrow[r] \arrow[d] & \tilde{\mathcal{G}}^{(2)}(-\tilde{D}_-) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tilde{\mathcal{W}}^{(2)}_+(-\tilde{D}_-) \arrow[r] & \tilde{\mathcal{W}}^{(2)}_+ \arrow[r] \arrow[d] & \tilde{\mathcal{W}}^{(2)}_+|_{\tilde{D}_-} \arrow[r] \arrow[d] & 0 \\
 & & \tilde{\mathcal{F}}^{(2)} \arrow[r, equal] \arrow[d] & \tilde{\mathcal{F}}^{(2)} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

satisfies that its rows and columns are exact, where $\tilde{\mathcal{W}}^{(2)}_+$ denotes $(\mathrm{id}_X \times \tilde{\varphi}^{(2)}_-)^*\mathcal{W}_+$, and so on. Now both $\tilde{D}'_-$ and $\tilde{D}_-$ are Cartier divisors, and we can show that $\tilde{D}'_- = \tilde{D}_-$ as subschemes of $\tilde{Q}^{(2)}_-$ in the same way as the proof in the preceding case.

Claim 4.5. Let R be a Noetherian ring, t an element of R which is not a zero-divisor, and $tR \supset I$ an ideal of R. Suppose that $\mathrm{Proj}_R(\oplus I^n/I^{n+1}) = \mathrm{Proj}_R(\oplus I^n/tI^n)$ as subschemes in $\mathrm{Proj}_R(\oplus_{n\geq 0} I^n)$. Then $tR = I$ if $\mathrm{Spec}(R/I)$ is nowhere dense in $\mathrm{Spec}(R)$.

Its proof is left to the reader. Now Lemma 4.2 is immediate from Claims 4.3 and 4.5. $\square$

Corollary 4.6. *$(\bar{\phi}_+)^{-1}(P_+)$ coincides with $E_- = \phi_-^{-1}(P_-)$ as subschemes of $\tilde{M}_-$.*

Proof. By Claim 4.3, closed subschemes E_- and $(\bar{\phi}_+)^{-1}(P_+)$ of $\tilde{M}_-$ are contained in $\tilde{M}_-^s$. Thus $\tilde{\pi}_- : \tilde{\pi}_-^{-1}(E_-) = D_- \to E_-$ and $\tilde{\pi}_- : \tilde{\pi}_-^{-1}\bar{\phi}_+^{-1}(P_+) = \tilde{\varphi}_+^{-1}(P_+) \to \bar{\phi}_+^{-1}(P_+)$ are faithfully-flat. Hence this corollary is immediate from Lemma 4.2. $\square$

By the corollary above, there is a morphism $\Delta_+ : \tilde{M}_- \to \tilde{M}_+$ such that $\phi_+ \circ \Delta_+ : \tilde{M}_- \to \tilde{M}_+ \to M_+$ is equal to $\bar{\phi}_+$. Likewise, for $U_\alpha \subset \tilde{Q}_-^{ss}$ and $\bar{\varphi}_+^\alpha$ in Lemma 4.1, there is a morphism $\Delta_+^\alpha : U_\alpha \to \tilde{Q}_+^{ss}$ such that $\varphi_+ \circ \Delta_+^\alpha : U_\alpha \to \tilde{Q}_+^{ss} \to Q_+^{ss}$ is equal to $\bar{\varphi}_+^\alpha$ since $(\bar{\varphi}_+^\alpha)^{-1}(V_+) = (\tilde{\varphi}_+)^{-1}(P_+) \cap U_\alpha$ is a Cartier divisor of U_α by Lemma 4.2.

Lemma 4.7.

$$\begin{tikzcd} U_\alpha \arrow[r, hook, "i_\alpha"'] \arrow[d, "\Delta_+^\alpha"] & \tilde{Q}_-^{ss} \arrow[r, "\tilde{\pi}_-"'] & \tilde{M}_- \arrow[d, "\Delta_+"'] \\ \tilde{Q}_+^{ss} \arrow[rr, "\tilde{\pi}_+"] & & \tilde{M}_+ \end{tikzcd}$$

is commutative.

Proof. One can check that both $\phi_+ \circ (\Delta_+ \circ \tilde{\pi}_- \circ i_\alpha) : U_\alpha \to \tilde{M}_+ \to M_+$ and $\phi_+ \circ (\tilde{\pi}_+ \circ \Delta_+^\alpha)$ coincide with $\pi_+ \circ \bar{\varphi}_+^\alpha : U_\alpha \to Q_+^{ss} \to M_+$. Then this lemma follows by the universal property of the blowing-up $\phi_+ : \tilde{M}_+ \to M_+$. $\square$

Proposition 4.8. *The morphism $\bar{\phi}_- \circ \Delta_+ : \tilde{M}_- \to \tilde{M}_+ \to M_-$ is equal to $\phi_- : \tilde{M}_- \to M_-$.*

Proof. First, let us verify the commutativity of

(4.8)
$$\begin{tikzcd} U_\alpha \arrow[r, hook, "i_\alpha"'] \arrow[d, "\Delta_+^\alpha"'] & \tilde{Q}_-^{ss} \arrow[r, "\varphi_-"'] & Q_-^{ss} \arrow[d, "\pi_-"] \\ \tilde{Q}_+^{ss} \arrow[rr, "\tilde{\varphi}_-"] & & M_-. \end{tikzcd}$$

Pulling back an exact sequence (4.1) on X_{V_+} by $\mathrm{id}_X \times \varphi_+ : X_{\tilde{Q}_+^{ss}} \to X_{Q_+^{ss}}$, we obtain a commutative diagram on $X_{\tilde{Q}^{ss}_+}$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{W}_- \arrow[r] \arrow[d] & \varphi_+^*\mathcal{U}_+ = \tilde{\mathcal{U}}_+ \arrow[r] \arrow[d] & \varphi_+^*\mathcal{F}' = \tilde{\mathcal{F}}' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \varphi_+^*\mathcal{G}' = \tilde{\mathcal{G}}' \arrow[r] & \tilde{\mathcal{U}}_+|_{D_+} \arrow[r] & \tilde{\mathcal{F}}' \arrow[r] & 0
\end{tikzcd}$$

whose rows are exact. Remark that $\mathcal{W}_-$ is $\tilde{Q}^{ss}_+$-flat. Pulling back this diagram by $\mathrm{id}_X \times \Delta^\alpha_+$, we obtain a commutative diagram on X_{U_α}

$$\begin{tikzcd}
 & (\Delta^\alpha_+)^*\mathcal{W}_- \arrow[r] \arrow[d] & (\Delta^\alpha_+)^*\tilde{\mathcal{U}}_+ \arrow[r] \arrow[d] & (\Delta^\alpha_+)^*\tilde{\mathcal{F}}' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (\Delta^\alpha_+)^*\tilde{\mathcal{G}}' \arrow[r] & (\Delta^\alpha_+)^*\tilde{\mathcal{U}}_+|_{D_-\cap U_\alpha} \arrow[r] & (\Delta^\alpha_+)^*\tilde{\mathcal{F}}' \arrow[r] & 0
\end{tikzcd} \tag{4.9}$$

whose rows are exact, because $(\Delta^\alpha_+)^{-1}(D_+) = D_- \cap U_\alpha$ by Lemma 4.2. Compare this with a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \tilde{\mathcal{U}}_-(-D_-) \arrow[r] \arrow[d] & \mathcal{W}_+ \arrow[r] \arrow[d] & \tilde{\mathcal{F}} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \tilde{\mathcal{G}}(-D_-) \arrow[r] & \mathcal{W}_+|_{D_-} \arrow[r] & \tilde{\mathcal{F}} \arrow[r] & 0
\end{tikzcd} \tag{4.10}$$

on $X_{\tilde{Q}^{ss}_-}$ in (3.6). Since $(\Delta^\alpha_+)^*\tilde{\mathcal{U}}_+ = (\bar{\varphi}^\alpha_+)^*\mathcal{U}_+$, an isomorphism Φ^α_+ in Lemma 4.1 connects the second row of (4.9) with that of (4.10):

$$\begin{tikzcd}
0 \arrow[r] & (\Delta^\alpha_+)^*\tilde{\mathcal{G}}' \arrow[r] & (\Delta^\alpha_+)^*\tilde{\mathcal{U}}_+|_{D_-\cap U_\alpha} \arrow[r] \arrow[d, "(\Phi^\alpha_+)^{-1}"'] & (\Delta^\alpha_+)^*\tilde{\mathcal{F}}' \arrow[r] \arrow[d, dotted, "\gamma"] & 0 \\
0 \arrow[r] & \tilde{\mathcal{G}}(-D_-)|_{U_\alpha} \arrow[r] & \mathcal{W}_+|_{D_-\cap U_\alpha} \arrow[r] & \tilde{\mathcal{F}}|_{U_\alpha} \arrow[r] & 0.
\end{tikzcd} \tag{4.11}$$

Remark that all sheaves in this diagram are flat over $D_- \cap U_\alpha$. One can check that two exact sequences in this diagram are relative Harder-Narasimhan filtrations of $(\Delta^\alpha_+)^*\tilde{\mathcal{U}}_+|_{D_-\cap U_\alpha} \simeq \mathcal{W}_+|_{D_-\cap U_\alpha}$ with respect to a_--stability, and hence we get a homomorphism $\gamma : (\Delta^\alpha_+)^*\tilde{\mathcal{F}}' \to \tilde{\mathcal{F}}|_{U_\alpha}$ which makes (4.11) commutative. $\gamma \otimes k(t)$ is isomorphic for any $t \in D_- \cap U_\alpha$ because of the uniqueness of HNF, and so γ should be isomorphic. (4.9), (4.10), Φ^α_+ and γ induce a surjective homomorphism

$$s : (\Delta^\alpha_+)^*\mathcal{W}_- \to \tilde{\mathcal{U}}_-(-D_-)|_{U_\alpha}. \tag{4.12}$$

In fact $s \otimes k(t)$ should be isomorphic for any closed point $t \in U_\alpha$, since $(\Delta^\alpha_+)^*\mathcal{W}_- \otimes k(t)$ and $\tilde{\mathcal{U}}_-(-D_-) \otimes k(t)$ has the same Chern classes. Thereby

(4.12) is isomorphic, and hence (4.8) is commutative. From Lemma 4.7 and (4.8), one can verify $\bar{\phi}_- \circ \Delta_+ \circ \tilde{\pi}_- \circ i_\alpha : U_\alpha \hookrightarrow \tilde{Q}_-^{ss} \to M_-$ equals $\phi_- \circ \bar{\pi}_- \circ i_\alpha : U_\alpha \to \tilde{Q}_-^{ss} \to M_-$ by diagram-chasing. Hence $(\bar{\phi}_- \circ \Delta_+) \circ \tilde{\pi}_- : \tilde{Q}_-^{ss} \to \tilde{M}_- \to M_-$ equals $\phi_- \circ \tilde{\pi}_- : \tilde{Q}_-^{ss} \to \tilde{M}_- \to M_-$. As mentioned in the preceding section, $\tilde{\pi}_- : \tilde{Q}_-^{ss} \to \tilde{M}_-$ is a categorical quotient by $\bar{G}$. Therefore we conclude that $\bar{\phi}_- \circ \Delta_+ : \tilde{M}_- \to \tilde{M}_+ \to M_-$ coincides with ϕ_-, thanks to the property of categorical quotients. □

From the proposition above we get a morphism Δ_+ such that

(4.13)
$$\begin{tikzcd} \tilde{M}_- \arrow[r, "\bar{\phi}_+"] \arrow[d, "\phi_-"'] \arrow[dr, "\Delta_+"] & M_+ \\ M_- & \tilde{M}_+ \arrow[l, "\bar{\phi}_-"] \arrow[u, "\phi_+"'] \end{tikzcd}$$

is commutative. Quite similarly, there is a morphism $\Delta_- : M_+ \to M_-$ such that

(4.14)
$$\begin{tikzcd} \tilde{M}_- \arrow[r, "\bar{\phi}_+"] \arrow[d, "\phi_-"'] & M_+ \\ M_- & \tilde{M}_+ \arrow[l, "\bar{\phi}_-"] \arrow[u, "\phi_+"'] \arrow[ul, "\Delta_-"'] \end{tikzcd}$$

is commutative. Thus $\phi_- \circ (\Delta_- \circ \Delta_+) : \tilde{M}_- \to \tilde{M}_- \to M_-$ is equal to $\phi_- : \tilde{M}_- \to M_-$, and so $\Delta_- \circ \Delta_+ : \tilde{M}_- \to \tilde{M}_+ \to \tilde{M}_-$ should be $\mathrm{id}_{\tilde{M}_-}$ because of the universal property of blowing-up ϕ_-. Likewise $\Delta_+ \circ \Delta_- : \tilde{M}_+ \to \tilde{M}_+$ equals $\mathrm{id}_{\tilde{M}_+}$, and hence both Δ_+ and Δ_- are isomorphic. Summarizing:

Proposition 4.9. *As to* (3.22) *and* (3.23)*, there are isomorphisms* $\Delta_+ : \tilde{M}_- \to \tilde{M}_+$ *and* $\Delta_- : \tilde{M}_+ \to \tilde{M}_-$ *such that* (4.13) *and* (4.14) *are commutative. In particular, the morphism* $\bar{\phi}_+ : \tilde{M}_- \to M_+$*, which is constructed by the method of elementary transform and descent theory, is the blowing-up of* M_+ *along* W_+*.*

5. Some structure of $P^{\mathbf{f}}$ over $\mathrm{Pic}(X) \times \mathrm{Hilb}(X) \times \mathrm{Hilb}(X)$

Let $\mathbf{f} = (f, m, n)$ be a member of $A^+(a)$. (3.1) gives an exact sequence

(5.1)
$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{U}_-|_{Q^{\mathbf{f}}} \longrightarrow \mathcal{G} \longrightarrow 0$$

of $Q^{\mathbf{f}}$-flat $\mathcal{O}_{X_{Q^{\mathbf{f}}}}$-modules. By Lemma 2.1 both $\mathcal{F} \otimes k(t)$ and $\mathcal{G} \otimes k(t)$ are torsion-free, rank-one, and hence H-stable for any $t \in Q^{\mathbf{f}}$. Denote by $M_H(1, F, m)$ the coarse moduli scheme of H-stable rank-one sheaves on X with Chern classes

$(F, m) \in \mathrm{Num}(X) \times \mathbb{Z}$. Then $\mathcal{F}$ and $\mathcal{G}$ in (5.1) induce morphisms $\tau_{\mathcal{F}} : Q^{\mathbf{f}} \to M_H(1, (c_1 + f)/2, m)$ and $\tau_{\mathcal{G}} : Q^{\mathbf{f}} \to M_H(1, (c_1 - f)/2, n)$. On the other hand $M_H(1, F, m)$ is isomorphic to $\mathrm{Pic}^F(X) \times \mathrm{Hilb}^m(X)$, where $\mathrm{Pic}^F(X)$ is an open closed subscheme $\{L \in \mathrm{Pic}(X) \,|\, [L] = F \text{ in } \mathrm{Num}(X)\}$ of $\mathrm{Pic}(X)$. Thereby, using $\tau_{\mathcal{F}}$ and $\tau_{\mathcal{G}}$ we obtain a morphism $\tau^Q : Q^{\mathbf{f}} \to \mathrm{Pic}^{(c_1+f)/2}(X) \times \mathrm{Hilb}^m(X) \times \mathrm{Hilb}^n(X)$ which has the following properties: Let $\mathcal{P} \in \mathrm{Coh}(X_{\mathrm{Pic}})$ be a universal line bundle of $\mathrm{Pic}(X)$, and let $I_{Z_1} \in \mathrm{Coh}(X_{\mathrm{Hilb}^m})$ (resp. $I_{Z_2} \in \mathrm{Coh}(X_{\mathrm{Hilb}^n})$) be the ideal sheaf of a universal sheaf of $\mathrm{Hilb}^m(X)$ (resp. $\mathrm{Hilb}^n(X)$). Define $\mathcal{F}_0$ and $\mathcal{G}_0 \in \mathrm{Coh}(X_{\mathrm{Pic} \times \mathrm{Hilb}^m \times \mathrm{Hilb}^n})$ by

$$\mathcal{F}_0 := \mathrm{pr}_{12}^*(\mathcal{P}) \otimes \mathrm{pr}_{13}^*(I_{Z_1}) \text{ and } \mathcal{G}_0 := c_1 \otimes \mathrm{pr}_{12}^*(\mathcal{P}^\vee) \otimes \mathrm{pr}_{14}^*(I_{Z_2}). \tag{5.2}$$

Then one can find line bundles L_1 and L_2 on $Q^{\mathbf{f}}$ such that

$$\mathcal{F} \simeq (\tau^Q)^* \mathcal{F}_0 \otimes L_1 \text{ and } \mathcal{G} \simeq (\tau^Q)^* \mathcal{G}_0 \otimes L_2. \tag{5.3}$$

From now on, we shorten $\mathrm{Pic}^{(c_1+f)/2}(X) \times \mathrm{Hilb}^m(X) \times \mathrm{Hilb}^n(X)$ to $T = T^{\mathbf{f}}$.

One can show that $\tau^Q : Q^{\mathbf{f}} \to T$ is $\bar{G}$-invariant in a similar fashion to the proof of Lemma 3.6, and hence τ^Q descends to a morphism $\tau_- : P^{\mathbf{f}} \to T$, since $\pi_- : \pi_-^{-1}(P^{\mathbf{f}}) = Q^{\mathbf{f}} \to P^{\mathbf{f}}$ is a categorical quotient by $\bar{G}$. In this section we would like to study some structure of $P^{\mathbf{f}}$ as a T-scheme.

One can find bounded complexes $F^\bullet$ and $G^\bullet$ of locally-free $\mathcal{O}_T$-modules of finite rank which allow quasi-isomorphisms $\tau_F : F^\bullet \to \mathcal{F}_0$ and $\tau_G : G^\bullet \to \mathcal{G}_0$ of complexes. Let $q : X_T \to T$ be the projection. The Serre duality [H1] asserts a natural homomorphism

$$\begin{aligned} \Theta_q : \mathbf{R}q_* \, \mathbf{R}Hom_{X_T} (Hom_{X_T}(F^\bullet, G^\bullet), \mathcal{O}_{X_T}[2]) \\ \to \mathbf{R}Hom_T (\mathbf{R}q_*(Hom_{X_T}(F^\bullet, G^\bullet(K_X)), \mathcal{O}_T) \end{aligned} \tag{5.4}$$

in the derived category $D(T)$ is isomorphism. Now we shall deduce the following from this.

Proposition 5.1. *For any T-scheme $f : S \to T$, there is an isomorphism*

$$\Theta_{f^*q} : Ext^1_{X_S/S}(f^*\mathcal{G}_0, f^*\mathcal{F}_0) \to Ext^1_{X_S/S}(f^*\mathcal{F}_0, f^*\mathcal{G}_0(K_X))^\vee$$

of relative Ext sheaves.

Proof. We prove this lemma only in case where $S = T$. It's easy to extend the proof to general case. As to the left side of (5.4), one can check that

$$[\mathbf{R}q_* \, \mathbf{R}Hom_{X_T} (Hom_{X_T}(F^\bullet, G^\bullet), \mathcal{O}_{X_T}[2])]_{-l} \simeq Ext^{2-l}_{X_T/T}(\mathcal{G}_0, \mathcal{F}_0) \tag{5.5}$$

for any integer l. Now consider the right side of (5.4). If we fix an affine open covering $\mathbf{U} = \{U_i\}_i$ of X_T such that $q : U_i \hookrightarrow X_T \to T$ is affine, then we can construct a quasi-isomorphism

$$\begin{aligned} Hom_{X_T}(F^\bullet, G^\bullet(K_X)) \longrightarrow Hom_{X_T}(F^\bullet, \mathcal{G}_0(K_X)) \\ \longrightarrow \mathcal{C}^\bullet(Hom_{X_T}(F^\bullet, \mathcal{G}_0(K_X)), \mathbf{U}) \end{aligned}$$

to the Cěch complex similarly to [H2, Lemma III.4.2].

$$q_*\left(\mathcal{C}^\bullet(Hom_{X_T}(F^\bullet,\mathcal{G}_0(K_X)),\mathbf{U})\right)$$

represents $\mathbf{R}q_*(Hom_{X_T}(F^\bullet,G^\bullet(K_X)))$ since $\mathcal{C}^p\left(Hom_{X_T}(F^q,\mathcal{G}_0(K_X)),\mathbf{U}\right)$ is q_*-acyclic. Therefore, for an injective resolution $\iota_T:\mathcal{O}_T\to K^\bullet$, a complex

$$Hom_T(q_*(\mathcal{C}^\bullet(Hom_{X_T}(F^\bullet,\mathcal{G}_0(K_X)),\mathbf{U})),K^\bullet)$$

represents $\mathbf{R}Hom_T\left(\mathbf{R}q_*(Hom_{X_T}(F^\bullet,G^\bullet(K_X))),\mathcal{O}_T\right)$. Furthermore, for any affine open subset T_α of T, there is a bounded complex $H^\bullet_\alpha$ of free $\mathcal{O}_{T_\alpha}$-modules of finite rank and with a quasi-isomorphism

$$h_\alpha: H^\bullet_\alpha\to q_*\mathcal{C}^\bullet(Hom_{X_T}(F^\bullet,\mathcal{G}_0(K_X)),\mathbf{U})|_{T_\alpha}. \tag{5.6}$$

by [M1, p. 47, Lemma 1.1]. This h_α and $\iota_T:\mathcal{O}_T\to K^\bullet$ give rise to an isomorphism

$$\begin{aligned}[][Hom_T(q_*\mathcal{C}^\bullet(Hom_{X_T}(F^\bullet,\mathcal{G}_0(K_X)),\mathbf{U}),K^\bullet)|_{T_\alpha}]_{-1}&\\ \simeq [Hom_{T_\alpha}(H^\bullet_\alpha,K^\bullet)]_{-1}\simeq [Hom_{T_\alpha}(H^\bullet_\alpha,\mathcal{O}_{T_\alpha})]_{-1}&.\end{aligned} \tag{5.7}$$

Claim 5.2. This complex $H^\bullet_\alpha$ induces an isomorphism

$$i_\alpha: Hom_{T_\alpha}([H^\bullet_\alpha]_1,\mathcal{O}_{T_\alpha})\to [Hom_{T_\alpha}(H^\bullet_\alpha,\mathcal{O}_{T_\alpha})]_{-1}.$$

Proof. As a result of the base change theorem for relative Ext sheaves [La, Theorem 1.4], $Ext^2_{X_T/T}(\mathcal{F}_0,\mathcal{G}_0(K_X))$ is equal to zero. Thus one can assume that $H^l_\alpha=0$ if $l\geq 2$. The remaining part of the proof is easy and left to the reader. □

From (5.6), (5.7) and the claim above, we obtain an isomorphism

$$\begin{aligned}j_\alpha:&[Hom_T(q_*\mathcal{C}^\bullet(Hom_{X_T}(F^\bullet,\mathcal{G}_0(K_X)),\mathbf{U}),K^\bullet)]_{-1}|_{T_\alpha}\\ &\to Hom_T\left([q_*\mathcal{C}^\bullet(Hom_{X_T}(F^\bullet,\mathcal{G}_0(K_X)),\mathbf{U})]_1,\mathcal{O}_T\right)|_{T_\alpha}\end{aligned}$$

Claim 5.3. Let T_α and T_β be affine open subsets in T. Then $j_\alpha|_{T_{\alpha\beta}}=j_\beta|_{T_{\alpha\beta}}$.

Proof. For h_α and h_β at (5.6), there are a bounded complex $K^\bullet_{\alpha\beta}$ of locally free $\mathcal{O}_{T_{\alpha\beta}}$-modules of finite rank, and quasi-isomorphisms k_α and k_β such that

$$\begin{tikzcd} K^\bullet_{\alpha\beta} \arrow[r, "k_\beta"'] \arrow[d, "k_\alpha"'] & H^\bullet_\beta \arrow[d, "h_\beta|_{T_{\alpha\beta}}"'] \\ H^\bullet_\alpha \arrow[r, "h_\alpha|_{T_{\alpha\beta}}"] & q_*\mathcal{C}^\bullet(Hom_{X_T}(F^\bullet,\mathcal{G}_0(K_X)),\mathbf{U})|_{T_{\alpha\beta}} \end{tikzcd}$$

is commutative up to homotopy. This $(K^\bullet_{\alpha\beta},k_\alpha,k_\beta)$ can be found by using [M1, p. 47, Lemma 1.1] and the mapping cone complex $Z^\bullet(f)$ ([H1, p. 26]). Then a quasi-isomorphism

$$k_\alpha|_{T_{\alpha\beta}}\circ k_\alpha: K^\bullet_{\alpha\beta}\to q_*\mathcal{C}^\bullet(Hom_{X_T}(F^\bullet,\mathcal{G}_0(K_X)),\mathbf{U})|_{T_{\alpha\beta}}$$

induces an isomorphism $j_{\alpha\beta}$ similarly to j_α. One can verify that both $j_\alpha|_{T_{\alpha\beta}}$ and $j_\beta|_{T_{\alpha\beta}}$ coincide with $j_{\alpha\beta}$. $\square$

By this claim we can glue $\{j_\alpha\}_\alpha$ to obtain an isomorphism

$$\begin{aligned} j : [\mathcal{H}om_T(q_*\mathcal{C}^\bullet(\mathcal{H}om_{X_T}(F^\bullet, \mathcal{G}_0(K_X)), \mathbf{U}),\, K^\bullet)]_{-1} \\ = [\mathbf{R}\mathcal{H}om_T(\mathbf{R}q_*(\mathcal{H}om_{X_T}(F^\bullet, G^\bullet(K_X))), \mathcal{O}_T)]_{-1} \\ \to \mathcal{H}om_T\left([q_*\mathcal{C}^\bullet(\mathcal{H}om_{X_T}(F^\bullet, \mathcal{G}_0(K_X)), \mathbf{U})]_1, \mathcal{O}_T\right) \\ = \mathcal{E}xt^1_{X_T/T}(\mathcal{F}_0, \mathcal{G}_0(K_X))^\vee. \end{aligned}$$

Now this j, (5.4) and (5.5) complete the proof of this lemma. $\square$

Remark that $\mathcal{E}xt^1_{X_T/T}(\mathcal{F}_0, \mathcal{G}_0(K_X))$ is not isomorphic to $\mathcal{E}xt^1_{X_T/T}(\mathcal{G}_0, \mathcal{F}_0)^\vee$ in general.

Lemma 5.4. *A natural homomorphism*

$$f^*\mathcal{E}xt^1_{X_T/T}(\mathcal{F}_0, \mathcal{G}_0(K_X)) \to \mathcal{E}xt^1_{X_S/S}(f^*\mathcal{F}_0, f^*\mathcal{G}_0(K_X))$$

is isomorphic for any T-scheme $f : S \to T$.

Proof. This lemma is immediate from base change theorem [La, p. 104]. $\square$

Now let us study a T-scheme $P^{\mathbf{f}}$.

Lemma 5.5. *There is a T-morphism $i_- : \mathbb{P}(\mathcal{E}xt^1_{X_T/T}(\mathcal{F}_0, \mathcal{G}_0(K_X))) \to P^{\mathbf{f}}$.*

Proof. We shorten $\mathbb{P}(\mathcal{E}xt^1_{X_T/T}(\mathcal{F}_0, \mathcal{G}_0(K_X)))$ to $\mathbb{P}_-$, and denote by $p_- : \mathbb{P}_- \to T$ its structural morphism. Proposition 5.1 and Lemma 5.4 lead to a natural isomorphism

$$\begin{aligned} &\mathrm{Hom}_{\mathbb{P}_-}(p_-^*\mathcal{E}xt^1_{X_T/T}(\mathcal{F}_0, \mathcal{G}_0(K_X)), \mathcal{O}(1)) \\ &\quad\simeq \Gamma(\mathbb{P}_-, \mathcal{E}xt^1_{X_{\mathbb{P}_-}/\mathbb{P}_-}(\mathcal{G}_0, \mathcal{F}_0 \otimes \mathcal{O}_-(1))) \\ &\quad\simeq \mathrm{Ext}^1_{X_{\mathbb{P}_-}}(\mathcal{G}_0, \mathcal{F}_0 \otimes \mathcal{O}_-(1)) \end{aligned}$$

since $\mathcal{H}om_{X_{\mathbb{P}_-}/\mathbb{P}_-}(\mathcal{G}_0, \mathcal{F}_0 \otimes \mathcal{O}_-(1)) = 0$ by base change theorem. A tautological quotient line bundle

$$p_-^*\mathcal{E}xt^1_{X_T/T}(\mathcal{F}_0, \mathcal{G}_0(K_X)) \twoheadrightarrow \mathcal{O}_-(1) \tag{5.8}$$

on $\mathbb{P}_-$ gives $\sigma \in \mathrm{Ext}^1_{X_{\mathbb{P}_-}}(\mathcal{G}_0, \mathcal{F}_0 \otimes \mathcal{O}_-(1))$ or an extension

$$0 \longrightarrow \mathcal{F}_0 \otimes \mathcal{O}_-(1) \longrightarrow \mathcal{V}_- \longrightarrow \mathcal{G}_0 \longrightarrow 0. \tag{5.9}$$

This $\mathcal{O}_{X_{\mathbb{P}_-}}$-module $\mathcal{V}_-$ is $\mathbb{P}_-$-flat. For any point t of $\mathbb{P}_-$, Proposition 5.1 and Lemma 5.4 provide us with homomorphisms

$$\begin{aligned}\kappa_1 \circ (k(t) \otimes \Theta_q) :\ & k(t) \otimes \Gamma(\mathbb{P}_-, Ext^1_{X_{\mathbb{P}_-}/\mathbb{P}_-}(\mathcal{G}_0, \mathcal{F}_0 \otimes \mathcal{O} - (1))) \\ &\longrightarrow k(t) \otimes \Gamma(\mathbb{P}_-, Ext^1_{X_{\mathbb{P}_-}/\mathbb{P}_-}(\mathcal{F}_0 \otimes \mathcal{O}_-(1), \mathcal{G}_0(K_X))^\vee) \\ &\longrightarrow \mathrm{Ext}^1_{X_{k(t)}}(\mathcal{F}_{0\,k(t)}, \mathcal{G}_{0\,k(t)}(K_X))^\vee\end{aligned}$$

and

$$\begin{aligned}\Theta_{q\otimes k(t)} \circ \kappa_2 :\ & k(t) \otimes \Gamma(\mathbb{P}_-, Ext^1_{X_{\mathbb{P}_-}/\mathbb{P}_-}(\mathcal{G}_0, \mathcal{F}_0 \otimes \mathcal{O}_-(1))) \\ &\longrightarrow \mathrm{Ext}^1_{X_{k(t)}}(\mathcal{G}_{0\,k(t)}, \mathcal{F}_{0\,k(t)}) \\ &\longrightarrow \mathrm{Ext}^1_{X_{k(t)}}(\mathcal{F}_{0\,k(t)}, \mathcal{G}_{0\,k(t)}(K_X))^\vee,\end{aligned}$$

where κ_i are natural maps. In fact these homomorphisms are equal to each other because a trace map $\mathrm{Tr}_q : R^2 q_*(K_X) \to \mathcal{O}_T$ is compatible with base change by [Co, p. 172, Theorem 3.6.5]. The extension class of the exact sequence

$$0 \longrightarrow \mathcal{F}_{0\,k(t)} \longrightarrow \mathcal{V}_{-\,k(t)} \longrightarrow \mathcal{G}_{0\,k(t)} \longrightarrow 0 \tag{5.10}$$

induced from (5.9) is equal to $\kappa_2(\sigma) \in \mathrm{Ext}^1_{X_{k(t)}}(\mathcal{G}_{0\,k(t)}, \mathcal{F}_{0\,k(t)})$. On the other hand $\kappa_1 \circ (k(t) \otimes \Theta_q)(\sigma) \in \mathrm{Hom}_{k(t)}(\mathrm{Ext}^1_{X_{k(t)}}(\mathcal{F}_{0\,k(t)}, \mathcal{G}_{0\,k(t)}), k(t))$ is nonzero since (5.8) is surjective. Therefore we see that (5.10) is not trivial, which means that $\mathcal{V}_-$ is a flat family of a_--stable sheaves by Lemma 1.6. $\mathcal{V}_-$ gives a morphism $i_- : \mathbb{P}_- \to M_-$. It's easy to see that i_- factors through $\mathbb{P}_- \to P^{\mathbf{f}} \hookrightarrow M_-$ and that $i_- : \mathbb{P}_- \to P^{\mathbf{f}}$ is a T-morphism. □

By (5.1) and (5.3), we have a natural exact sequence

$$0 \longrightarrow (\tau^Q)^* \mathcal{F}_0 \otimes L_1 \longrightarrow \mathcal{U}_-|_{Q^{\mathbf{f}}} \longrightarrow (\tau^Q)^* \mathcal{G}_0 \otimes L_2 \to 0 \tag{5.11}$$

on $X_{Q^{\mathbf{f}}}$. Similarly to the proof of the lemma above, one can show that

$$\begin{aligned}&\mathrm{Ext}^1_{X_{Q^{\mathbf{f}}}}((\tau^Q)^* \mathcal{G}_0 \otimes L_2, (\tau^Q)^* \mathcal{F}_0 \otimes L_1) \\ &\qquad \simeq \mathrm{Hom}_{Q^{\mathbf{f}}}((\tau^Q)^* Ext^1_{X_T/T}(\mathcal{F}_0, \mathcal{G}_0(K_X)),\ L_1 \otimes L_2^\vee),\end{aligned}$$

and that the homomorphism $(\tau^Q)^* Ext^1_{X_T/T}(\mathcal{F}_0, \mathcal{G}_0(K_X)) \twoheadrightarrow L_1 \otimes L_2^\vee$ induced by (5.11) is surjective. Thus $j^Q : Q^{\mathbf{f}} \to \mathbb{P}(Ext^1_{X_T/T}(\mathcal{F}_0, \mathcal{G}_0(K_X)))$ is derived. One can check that j^Q is $\bar{G}$-invariant. As a result, j^Q descends to a morphism

$$j_- : P^{\mathbf{f}} \to \mathbb{P}(Ext_{X_T/T}(\mathcal{F}_0, \mathcal{G}_0(K_X))). \tag{5.12}$$

Lemma 5.6. *For morphisms i_- in Lemma* 5.5 *and j_- at* (5.12), *it holds that $i_- \circ j_- = \mathrm{id}_{P^{\mathbf{f}}}$ and that $j_- \circ i_- = \mathrm{id}_{\mathbb{P}_-}$.*

Proof. Since $\pi_- : Q^{\mathbf{f}} \to P^{\mathbf{f}}$ is a categorical quotient by $\bar{G}$, $i_- \circ j_-$ is equal to $\mathrm{id}_{P^{\mathbf{f}}}$ if and only if $(i_- \circ j_-) \circ \pi_- = i_- \circ j^Q$ is equal to π_-. One can readily verify this, and hence its proof is omitted. T-morphism $i_- : \mathbb{P}(Ext^1_{X_T/T}(\mathcal{F}_0, \mathcal{G}_0(K_X))) = \mathbb{P}_- \to P^{\mathbf{f}} \hookrightarrow M_-$ is induced from an $\mathcal{O}_{X_T}$-module $\mathcal{V}_-$ in (5.9), and hence one can find an affine open covering $\{P_\alpha\}_\alpha$ of $\mathbb{P}_-$ and a morphism $i_\alpha : P_\alpha \to Q^{\mathbf{f}}$ such that $\pi_- \circ i_\alpha = i_-|_{U_\alpha}$. It's easy to show that $j^Q \circ i_\alpha = j_- \circ i_-|_{P_\alpha} : P_\alpha \to \mathbb{P}_-$ is equal to id_{P_α}, and hence its proof is left to the reader. $\square$

Summing up, we get the following:

Proposition 5.7. *Fix an element* $\mathbf{f}$ *of* $A^+(a)$. *We define a scheme* T, $\mathcal{O}_{X_T}$*-modules* $\mathcal{F}_0$ *and* $\mathcal{G}_0$, *and line bundles* L_1 *and* L_2 *over* $Q^{\mathbf{f}}$ *as in* (5.2) *and in* (5.3).

(1) $P^{\mathbf{f}}$ *can be regarded as a* T*-scheme.*

(2) *There is an isomorphism* $j_- : P^{\mathbf{f}} \to \mathbb{P}(Ext^1_{X_T/T}(\mathcal{F}_0, \mathcal{G}_0(K_X)))$ *over* T *such that* $L_1 \otimes L_2^\vee \in \mathrm{Pic}(Q^{\mathbf{f}})$ *in* (5.3) *is equal to* $(j_- \circ \pi_-)^* \mathcal{O}_-(1)$, *where* $\mathcal{O}_-(1)$ *is the tautological line bundle of* $\mathbb{P}(Ext^1_{X_T/T}(\mathcal{F}_0, \mathcal{G}_0(K_X)))$.

6. Algebro-geometric analogy of μ-map and the Donaldson polynomial

From now on we shall consider the case of $c_1 = 0$. Hence M_- stands for $M_{a_-}(0, c_2)$, and so on. We begin with reviewing the algebro-geometric analogy $\mu_- : \mathrm{NS}(X) \to \mathrm{NS}(M_-)$ of μ-map, which was introduced in [Li]. Let $C \subset X$ be a nonsingular curve, and θ_C a line bundle on C with $\deg(\theta_C) = g(C) - 1$. For a universal sheaf $\mathcal{U}_-$ of Q_-^{ss} on $X_{Q_-^{ss}}$, one can show that a complex $\mathbf{R}\,\mathrm{pr}_{2*}(\mathcal{U}_-|_C \otimes \theta_C)$ on Q_-^{ss} locally is quasi-isomorphic to a bounded complex of free modules of finite rank. Thus its determinantal line bundle $\det \mathbf{R}\,\mathrm{pr}_{2*}(\mathcal{U}_-|_C \otimes \theta_C)$ on Q_-^{ss} exists. In fact this line bundle descends to a line bundle $\mathcal{L}_{M_-}(C, \theta_C)^\vee$ on M_-, and its algebraic equivalence class $[\mathcal{L}_{M_-}(C, \theta_c)^\vee] \in \mathrm{NS}(M_-)$ is independent of the choice of θ_C. It can be checked that the correspondence $C \mapsto [\mathcal{L}_{M_-}(C, \theta_C)]$ induces a homomorphism $\mu_- : \mathrm{NS}(X) \to \mathrm{NS}(M_-)$. One can also construct a homomorphism $\mu_+ : \mathrm{NS}(X) \to \mathrm{NS}(M_+)$ likewise. Let $M_- \xleftarrow{\phi_-} \tilde{M}_- \xrightarrow{\phi_+} M_+$ be the sequence of morphisms (3.22). For $\mathbf{f} \in A^+(a)$, we denote by $E^{\mathbf{f}}$ the Cartier divisor $\phi_-^{-1}(P^{\mathbf{f}})$ on $\tilde{M}_-$, which is equal to $\tilde{\phi}_+^{-1}(P^{\mathbf{f}})$ by Corollary 4.6.

Lemma 6.1. *For* $\alpha \in \mathrm{NS}(X)$, *it holds that*

$$\phi_-^* \mu_-(\alpha) - \bar{\phi}_+^* \mu_+(\alpha) = \sum_{\mathbf{f} \in A^+(a)} \mathcal{O}_{\tilde{M}_-} \left(\langle f \cdot \alpha/2 \rangle E^{\mathbf{f}} \right)$$

in $\mathrm{NS}(\tilde{M}_-)$.

Proof. For the simplicity of notation, we prove this lemma in case of $\sharp A^+(a) = 1$. It's easy to extend the proof to general case. Let C and θ_C be as explained above, and

$$0 \longrightarrow \mathcal{W}_+ \longrightarrow \tilde{\mathcal{U}}_- \longrightarrow \tilde{\mathcal{G}} \longrightarrow 0 \tag{6.1}$$

the exact sequence (3.3) on $X_{\tilde{Q}^{ss}_-}$. Since $\mathcal{U}_-$ is a flat family of torsion free sheaves, one can show that a $\mathcal{O}_{C_{\tilde{Q}^{ss}_-}}$-module $\tilde{\mathcal{U}}_-|_C$ is $\tilde{Q}^{ss}_-$-flat. By using the method of Cěch complex, one get a quasi-isomorphism

(6.2)
$$\mathbf{L}f^*\mathbf{R}\,\mathrm{pr}_{2\,*}(\tilde{\mathcal{U}}_-|_C \otimes \theta_C) \to \mathbf{R}\,\mathrm{pr}'_{2\,*}\,\mathbf{L}f'^*(\tilde{\mathcal{U}}_-|_C \otimes \theta_C) = \mathbf{R}\,\mathrm{pr}'_{2\,*}\,f'^*(\tilde{\mathcal{U}}_-|_C \otimes \theta_C),$$

where

$$\begin{array}{ccc} C_S & \xrightarrow{f'} & C_{\tilde{Q}^{ss}_-} \\ \downarrow{\scriptstyle \mathrm{pr}'_2} & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ S & \xrightarrow{f} & \tilde{Q}^{ss}_- \end{array}$$

is a fiber product. The analogy to these result about $(\tilde{\mathcal{U}}_-, \tilde{Q}^{ss}_-)$ also holds to $(\mathcal{W}_+, \tilde{Q}^{ss}_-)$ and $(\tilde{\mathcal{G}}, D_-)$. (6.1) gives a triangle

$$\mathbf{R}\,\mathrm{pr}_{2\,*}(\mathcal{W}_+|_C \otimes \theta_C) \longrightarrow \mathbf{R}\,\mathrm{pr}_{2\,*}(\tilde{\mathcal{U}}_-|_C \otimes \theta_C) \longrightarrow \mathbf{R}\,\mathrm{pr}_{2\,*}(\tilde{\mathcal{G}}_-|_C \otimes \theta_C)$$

in $D(\tilde{Q}^{ss}_-)$, and hence an isomorphism

$$\det \mathbf{R}\,\mathrm{pr}_{2\,*}(\tilde{\mathcal{U}}_-|_C \otimes \theta_C) \simeq \det \mathbf{R}\,\mathrm{pr}_{2\,*}(\mathcal{W}_+|_C \otimes \theta_C) \cdot \det \mathbf{R}\,\mathrm{pr}_{2\,*}(\tilde{\mathcal{G}}|_C \otimes \theta_C) \tag{6.3}$$

in $\mathrm{Pic}(\tilde{Q}^{ss}_-)$ is induced.

$\det \mathbf{R}\,\mathrm{pr}_{2\,*}(\mathcal{W}_+|_C \otimes \theta_C)$ naturally is isomorphic to $\tilde{\pi}^*_+\bar{\phi}^*_+\mathcal{L}_{M_+}(C,\theta_C)^\vee$. Indeed, Lemma 4.1 gives an open covering $\bigcup_\alpha U_\alpha$ of $\tilde{Q}^{ss}_-$, a morphism $\bar{\varphi}^\alpha_+ : U_\alpha \to Q^{ss}_+$, and an isomorphism of X_{U_α}-modules $\Phi^\alpha_+ : \mathcal{W}_+|_{U_\alpha} \to (\bar{\varphi}^\alpha_+)^*\mathcal{U}_+$. By (6.2) Φ^α_+ naturally induces an isomorphism

$$\begin{aligned} &\det(\Phi^\alpha_+) : \det \mathbf{R}\,\mathrm{pr}_{2\,*}(\mathcal{W}_+|_C \otimes \theta_C)|_{U_\alpha} \to \det \mathbf{R}\,\mathrm{pr}_{2\,*}(\bar{\varphi}^\alpha_+)^*(\mathcal{U}_+|_C \otimes \theta_C) \\ &\quad \to (\bar{\varphi}^\alpha_+)^* \det \mathbf{R}\,\mathrm{pr}_{2\,*}(\mathcal{U}_+|_C \otimes \theta_C) \\ &\qquad = (\bar{\varphi}^\alpha_+)^*\pi^*_+\mathcal{L}_{M_+}(C,\theta_C)^\vee = \tilde{\varphi}^*_+\mathcal{L}_{M_+}(C,\theta_C)^\vee|_{U_\alpha}. \end{aligned}$$

In addition, if $\alpha \neq \beta$ then the isomorphism

$$(\Phi^\beta_+)^{-1} \circ \Phi^\alpha_+ : \mathcal{W}_+|_{U_{\alpha\beta}} \to \mathcal{W}_+|_{U_{\alpha\beta}} \tag{6.4}$$

on $X_{U_{\alpha\beta}} = X_{U_\alpha \cap U_\beta}$ is given by $\lambda_{\alpha\beta} \in \Gamma(\mathcal{O}^\times_{U_{\alpha\beta}})$ since $\mathcal{W}_+|_{U_{\alpha\beta}}$ is a flat family of simple sheaves as mentioned right after Corollary 3.4. One can define the rank R of a perfect complex $\mathbf{R}\,\mathrm{pr}_{2\,*}(\mathcal{W}_+|_C \otimes \theta_C)$, and then

$$\begin{aligned} \det(\Phi^\beta_+)^{-1} \circ \det(\Phi^\alpha_+) : \det \mathbf{R}\,\mathrm{pr}_{2\,*}(\mathcal{W}_+|_C \otimes \theta_C)|_{U_{\alpha\beta}} \\ \longrightarrow \det \mathbf{R}\,\mathrm{pr}_{2\,*}(\mathcal{W}_+|_C \otimes \theta_C)|_{U_{\alpha\beta}} \end{aligned}$$

is given by $\lambda_{\alpha\beta}^{\times R}$. This R turns out to be zero because the Riemann-Roch theorem implies that $\chi(C_{k(t)}, \mathcal{W}_+|_C \otimes \theta_C \otimes k(t)) = 0$ for every $t \in U_{\alpha\beta}$. Hence we can glue $\det(\Phi_+^\alpha)$ to obtain an isomorphism

$$\det \mathbf{R}\,\mathrm{pr}_{2\,*}(\mathcal{W}_+|_C \otimes \theta_C) \simeq \tilde{\varphi}_+^* \mathcal{L}_{M_+}(C, \theta_C)^\vee = \tilde{\pi}_-^* \bar{\phi}_+^* \mathcal{L}_{M_+}(C, \theta_C)^\vee. \tag{6.5}$$

From (6.2), we can get a natural isomorphisms

$$\det \mathbf{R}\,\mathrm{pr}_{2\,*}(\tilde{\mathcal{U}}_-|_C \otimes \theta_C) \to \varphi_-^* \det \mathbf{R}\,\mathrm{pr}_{2\,*}(\mathcal{U}_-|_C \otimes \theta_C) = \tilde{\pi}_-^* \phi_-^* \mathcal{L}_{M_-}(C, \theta_C)^\vee.$$

Hence by (6.3) and (6.5)

$$\tilde{\pi}_-^*(\tilde{\phi}_+^* \mathcal{L}_{M_+}(C, \theta_C) - \phi_-^* \mathcal{L}_{M_-}(C, \theta_C)) \simeq \det \mathbf{R}\,\mathrm{pr}_{2\,*}(\tilde{\mathcal{G}}|_C \otimes \theta_C).$$

$\tilde{\mathcal{G}}|_C \otimes \theta_C$ is a sheaf on $C_{D_-} \subset C_{\tilde{Q}_-^{ss}}$, and so $\det \mathbf{R}\,\mathrm{pr}_{2\,*}(\tilde{\mathcal{G}}|_C \otimes \theta_C)$ can be regarded as $\det \mathbf{R}\,\mathrm{pr}_{2\,*}\, j_*^C(\tilde{\mathcal{G}}|_C \otimes \theta_C) = \det j_* \mathbf{R}\,\mathrm{pr}'_{2\,*}(\tilde{\mathcal{G}}|_C \otimes \theta_C)$, where

$$\begin{array}{ccc} C_{D_-} & \overset{j^C}{\hookrightarrow} & C_{\tilde{Q}_-^{ss}} \\ \downarrow{\scriptstyle \mathrm{pr}'_2} & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ D_- & \overset{j}{\hookrightarrow} & \tilde{Q}_-^{ss} \end{array}$$

is a fiber product. By the Riemann-Roch theorem $\chi(C_{k(t)}, \tilde{\mathcal{G}}|_C \otimes \theta_C \otimes k(t)) = -f \cdot C/2$ for every $t \in D_-$. Thus the rank of a complex $\mathbf{R}\,\mathrm{pr}'_{2\,*}(\tilde{\mathcal{G}}|_C \otimes \theta_C)$ on D_- is equal to $-f \cdot C/2$. In view of this we can prove that $\det \mathbf{R}\,\mathrm{pr}_{2,*}(\tilde{\mathcal{G}}|_C \otimes \theta_C) = -\langle f \cdot C/2 \rangle D_-$ in $\mathrm{Pic}(\tilde{Q}_-^{ss})$; its proof is omitted. Summing up, we obtain an isomorphism

$$\begin{aligned} &\tilde{\pi}_-^*(\tilde{\phi}_+^* \mathcal{L}_{M_+}(C, \theta_C) - \phi_-^* \mathcal{L}_{M_-}(C, \theta_C)) \\ &\qquad \simeq -\sum_{\mathbf{f} \in A^+(a)} \mathcal{O}_{\tilde{Q}_-^{ss}}(\langle f \cdot C/2 \rangle D^{\mathbf{f}}) = -\sum_{\mathbf{f} \in A^+(a)} \tilde{\pi}_-^* \mathcal{O}_{\tilde{M}_-}(\langle f \cdot C/2 \rangle E^{\mathbf{f}}) \end{aligned} \tag{6.6}$$

in $\mathrm{Pic}(\tilde{Q}_-^{ss})$. Moreover, both sides in (6.6) respectively have a natural $\bar{G}$-linearized structure. One can check that (6.6) is an isomorphism of $\bar{G}$-linearized line bundles. By [Se, Theorem 4] and [HL, p. 87, Theorem 4.2.16] the natural homomorphism

$$\tilde{\pi}_-^* : \mathrm{Pic}(\tilde{M}_-) \to \mathrm{Pic}_{\bar{G}}(\tilde{Q}_-^{ss}) \tag{6.7}$$

is injective, where $\mathrm{Pic}_{\bar{G}}(\tilde{Q}_-^{ss})$ is the group of $\bar{G}$-linearized line bundles on $\tilde{Q}_-^{ss}$. Thereby (6.6) and (6.7) complete the proof of this lemma. □

Now we assume that

$$\dim M_{H_\pm}(0, c_2) = 4c_2 - 3\chi(\mathcal{O}_X) = d(c_2) \tag{6.8}$$

and that

$$\operatorname{codim}(M_\pm, P_\pm) \geq 2. \tag{6.9}$$

These assumptions can be considered to be reasonably weak by the following lemma.

Lemma 6.2. *Let* $\operatorname{Amp}(X)$ *be the ample cone of* X*, and* $S \subset \operatorname{Amp}(X)$ *a compact subset containing* $H_\pm$*. If* c_2 *is sufficiently large with respect to* S*, then assumptions* (6.8) *and* (6.9) *hold good.*

Proof. Refer to [Zu, Theorem 2], [GL], and the proof of [Q1, Theorem 2.3.]. □

By (6.8) we can define a multilinear map

$$\gamma_\pm = \gamma_\pm(c_2) : \operatorname{Sym}^{d(c_2)} \operatorname{NS}(X) \to \mathbb{Z}$$

by $\gamma_\pm(\alpha_1, \ldots, \alpha_{d(c_2)}) = \mu_\pm(\alpha_1) \cdot \cdots \cdot \mu_\pm(\alpha_{d(c_2)})$ using the intersection number of line bundles on $M_\pm = M_{a_\pm}(0, c_2)$. Similarly, a multilinear map

$$\gamma_{H_\pm} = \gamma_{H_\pm}(c_2) : \operatorname{Sym}^{d(c_2)} \operatorname{NS}(X) \to \mathbb{Z}$$

can be defined by the intersection number of line bundles on $M_{H_\pm}(0, c_2)$.

Concerning $\gamma_{H_\pm}(c_2)$ let us recall its relation to the Donaldson polynomials, which was stated in Proposition 0.1. As explained in Introduction, Proposition 0.1 originally results from differential geometry. We would like to observe this proposition from an algebro-geometric point of view. For this reason we shall study $\mu_-(C)^{d(c_2)} - \mu_+(C)^{d(c_2)}$ for a nonsingular curve C in X. We often shorten $d(c_2)$ to d.

Since (6.9) both $\phi_- : \tilde{M}_- \to M_-$ and $\bar{\phi}_+ : \tilde{M}_- \to M_+$ are birational, $\mu_+(C)^d - \mu_-(C)^d$ is equal to $\bar{\phi}_+^* \mu_+(C)^d - \phi_-^* \mu_-(C)^d$. For $f \in \operatorname{NS}(X)$, we set $\lambda_f^C := \langle f \cdot C/2 \rangle$. Then Lemma 6.1 implies that

$$\begin{aligned} &\bar{\phi}_+^* \mu_+(C)^d - \phi_-^* \mu_-(C)^d \\ &\quad = \{\bar{\phi}_+^* \mu_+(C) - \phi_-^* \mu_-(C)\} \cdot \sum_{k=0}^{d-1} \phi_-^* \mu_-(C)^k \cdot \bar{\phi}_+^* \mu_+(C)^{d-1-k} \\ &\quad = \sum_{k=0}^{d-1} \left[\phi_-^* \mu_-(C)^k \cdot \bar{\phi}_+^* \mu_+(C)^{d-1-k} \cdot - \sum_{\mathbf{f}} \lambda_f^C E^{\mathbf{f}} \right]_{\tilde{M}_-} , \end{aligned} \tag{6.10}$$

where $[\quad]_{\tilde{M}_-}$ stands for the multiplication of line bundles is calculated on $\tilde{M}_-$. By [KL, p. 297, Proposition 4], (6.10) is equal to

$$\begin{aligned} &\sum_{\mathbf{f} \in A^+(a)} -\lambda_f^C \sum_{k=0}^{d-1} \left[\phi_-^* \mu_-(C)|_{E^{\mathbf{f}}}^k \cdot \left\{ \phi_-^* \mu_-(C) - \sum_{\mathbf{f}} \lambda_f^C E^{\mathbf{f}} \right\} \Bigg|_{E^{\mathbf{f}}}^{d-1-k} \right]_{E^{\mathbf{f}}} \\ &\quad = \sum_{\mathbf{f}} -\lambda_f^C \sum_{k=0}^{d-1} [\phi_-^* \mu_-(C)|_{E^{\mathbf{f}}}^k \cdot \{\phi_-^* \mu_-(C) - \lambda_f^C E^{\mathbf{f}}\}|_{E^{\mathbf{f}}}^{d-1-k}]_{E^{\mathbf{f}}}, \end{aligned} \tag{6.11}$$

since $E^{\mathbf{f}} \cap E^{\mathbf{f}'} = \emptyset$ if $\mathbf{f}$ and $\mathbf{f}'$ are different by 2.4. ϕ_- and $\bar{\phi}_+$ induce a commutative diagram

$$\begin{tikzcd} E^{\mathbf{f}} \arrow[r, "\bar{\phi}_+"'] \arrow[d, "\phi_-"'] & P^{-\mathbf{f}} \arrow[d, "\tau_+"] \\ P^{\mathbf{f}} \arrow[r, "\tau_-"] & T. \end{tikzcd}$$

Let $\mathcal{F}_0$ and $\mathcal{G}_0$ be $\mathcal{O}_{X_T}$-modules defined in (5.2). By Proposition 5.7 there are tautological line bundles $\mathcal{O}_-^{\mathbf{f}}(1)$ and $\mathcal{O}_+^{-\mathbf{f}}(1)$ on, respectively, $P^{\mathbf{f}}$ and $P^{-\mathbf{f}}$.

Lemma 6.3. *A line bundle $\mathcal{O}_{E^{\mathbf{f}}}(-E^{\mathbf{f}})$ on $E^{\mathbf{f}}$ is naturally isomorphic to $\phi_-^*\mathcal{O}_-^{\mathbf{f}}(1) + \bar{\phi}_+^*\mathcal{O}_+^{-\mathbf{f}}(1)$.*

Proof. We shorten $\mathcal{O}_-^{\mathbf{f}}(1)$ and $\mathcal{O}_+^{-\mathbf{f}}(1)$ to, respectively, $\mathcal{O}_-(1)$ and $\mathcal{O}_+(1)$. Let $D^{\mathbf{f}}$ denote a closed subscheme $(\tilde{\pi}_-)^{-1}(E^{\mathbf{f}})$ of $\tilde{Q}_-^{ss}$. Then

$$\begin{tikzcd} Q^{\mathbf{f}} \arrow[d, "\pi_-"'] & D^{\mathbf{f}} \arrow[l, "\varphi_-"'] \arrow[d, "\tilde{\pi}_-"'] & D^{\mathbf{f}} \cap U_\alpha \arrow[l, hook'] \arrow[r, "\bar{\varphi}_+^\alpha"'] & Q^{-\mathbf{f}} \arrow[d, "\pi_+"] \\ P^{\mathbf{f}} \arrow[r, "\phi_-"'] & E^{\mathbf{f}} \arrow[rr, "\bar{\phi}_+"] & & P^{-\mathbf{f}} \end{tikzcd}$$

is commutative for $U_\alpha \subset \tilde{Q}_-^{ss}$ and $\bar{\varphi}_+^\alpha$ in Lemma 4.1. By (5.3), we can rewrite the exact sequence (3.5) to obtain

$$0 \longrightarrow \mathcal{G}_0 \otimes L_2(-D^{\mathbf{f}}) \longrightarrow \mathcal{W}_+|_{X_{D^{\mathbf{f}}}} \longrightarrow \mathcal{F}_0 \otimes L_1 \longrightarrow 0 \tag{6.12}$$

on $X_{D^{\mathbf{f}}}$. Next, let

$$0 \longrightarrow \mathcal{G}' \longrightarrow \mathcal{U}_+|_{Q^{-\mathbf{f}}} \longrightarrow \mathcal{F}' \longrightarrow 0 \tag{6.13}$$

be the exact sequence (4.1) on $X_{Q^{-\mathbf{f}}}$. Similarly to (5.3), there are isomorphisms

$$\mathcal{F}' \simeq \mathcal{F}_0 \otimes M_1 \quad \text{and} \quad \mathcal{G}' \simeq \mathcal{G}_0 \otimes M_2$$

with some line bundles M_1 and M_2 on $Q^{-\mathbf{f}}$. Analogously to Proposition 5.7, $M_2 \otimes M_1^\vee$ is isomorphic to $\pi_+^*\mathcal{O}_+(1)$. Thus we obtain an exact sequence

$$0 \longrightarrow \mathcal{G}_0 \otimes (\bar{\varphi}_+^\alpha)^* M_2 \longrightarrow (\bar{\varphi}_+^\alpha)^*\mathcal{U}_+|_{D^{\mathbf{f}} \cap U_\alpha} \longrightarrow \mathcal{F}_0 \otimes (\bar{\varphi}_+^\alpha)^* M_1 \longrightarrow 0 \tag{6.14}$$

on $X_{D^{\mathbf{f}}} \cap U_\alpha$, pulling (6.13) back by $\mathrm{id}_X \times \bar{\varphi}_+^\alpha : X_{D^{\mathbf{f}} \cap U_\alpha} \to X_{Q^{-\mathbf{f}}}$. Connecting (6.12) and (6.14) by the isomorphism Φ_+^α in Lemma 4.1, we get the following:

(6.15)

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G}_0 \otimes L_2(-D^{\mathbf{f}}) \arrow[r] \arrow[d, dotted, "r'_\alpha"] & \mathcal{W}_+|_{D^{\mathbf{f}} \cap U_\alpha} \arrow[r] \arrow[d, "\Phi_+^\alpha"] & \mathcal{F}_0 \otimes L_1|_{U_\alpha} \arrow[r] \arrow[d, dotted, "r_\alpha"] & 0 \\ 0 \arrow[r] & \mathcal{G}_0 \otimes (\bar{\varphi}_+^\alpha)^* M_2 \arrow[r] & (\bar{\varphi}_+^\alpha)^*\mathcal{U}_+|_{D^{\mathbf{f}} \cap U_\alpha} \arrow[r] & \mathcal{F}_0 \otimes (\bar{\varphi}_+^\alpha)^* M_1 \arrow[r] & 0. \end{tikzcd}$$

As observed in the proof of Lemma 4.2, there is an isomorphism $r_\alpha : \mathcal{F}_0 \otimes L_1|_{U_\alpha} \to \mathcal{F}_0 \otimes (\bar{\varphi}_+^\alpha)^* M_1$ which makes (6.15) commutative. This r_α induces an isomorphism r'_α in (6.15). Because both $\mathcal{F}_0$ and $\mathcal{G}_0$ are flat families of simple sheaves, r_α and r'_α induce isomorphisms

$$\Gamma(r_\alpha) : L_1|_{U_\alpha} \to (\bar{\varphi}_+^\alpha)^* M_1 \quad \text{and} \quad \Gamma(r'_\alpha) : L_2 \otimes \mathcal{O}_{D^{\mathbf{f}}}(-D^{\mathbf{f}})|_{U_\alpha} \to (\bar{\varphi}_+^\alpha)^* M_2$$

of line bundles on $D^{\mathbf{f}} \cap U_\alpha$. $\Gamma(r_\alpha)$ and $\Gamma(r'_\alpha)$ induce an isomorphism

$$\begin{aligned}\Gamma(r_\alpha)^{-1} \cdot \Gamma(r'_\alpha) &: \mathcal{O}_{D^{\mathbf{f}}}(-D^{\mathbf{f}})|_{U_\alpha} = \tilde{\pi}_-^* \mathcal{O}_{E^{\mathbf{f}}}(-E^{\mathbf{f}})|_{U_\alpha} \\ &\to (L_1 \otimes L_2^\vee)|_{U_\alpha} \otimes (\bar{\varphi}_+^\alpha)^*(M_2 \otimes M_1^\vee) \\ &\simeq \tilde{\pi}_-^* \phi_-^* \mathcal{O}_-(1)|_{U_\alpha} \otimes (\bar{\varphi}_+^\alpha)^* \pi_+^* \mathcal{O}_+(1) \\ &= \tilde{\pi}^*(\phi_-^* \mathcal{O}_-(1) + \bar{\phi}_+^* \mathcal{O}_+(1))|_{D^{\mathbf{f}} \cap U_\alpha}\end{aligned}$$

By (6.4) one can check that $\Gamma(r_\alpha)^{-1} \cdot \Gamma(r'_\alpha) = \Gamma(r_\beta)^{-1} \cdot \Gamma(r'_\alpha)$, and hence glue $\Gamma(r_\alpha)^{-1} \cdot \Gamma(r'_\alpha)$ to obtain an isomorphism

$$\tilde{\pi}_-^* \mathcal{O}_{E^{\mathbf{f}}}(-E^{\mathbf{f}}) \simeq \tilde{\pi}_-^*(\phi_-^* \mathcal{O}_-(1) + \bar{\phi}_+^* \mathcal{O}_+(1))$$

of line bundles on $D^{\mathbf{f}}$. One can also check this is an isomorphism of $\bar{G}$-linearized line bundles. Then we complete the proof of this lemma in similar fashion to the proof of Lemma 6.1. ☐

From (6.11) and Lemma 6.3 we obtain that

$$(6.16) \quad \mu_-(C)^{d(c_2)} - \mu_+(C)^{d(c_2)} = \sum_{\mathbf{f} \in A^+(a)} -\lambda_f^C \cdot \sum_{k=0}^{d(c_2)-1} \left[\phi_-^* \mu_-(C)|_{E^{\mathbf{f}}}^k \cdot \{ \phi_-^* \mu_-(C)|_{E^{\mathbf{f}}} + \lambda_f^C (\mathcal{O}_-^{\mathbf{f}}(1) + \mathcal{O}_+^{-\mathbf{f}}(1)) \}^{d(c_2)-1-k} \right]_{E^{\mathbf{f}}}.$$

In the following section, we shall in detail examine the right side of this equation in some special case.

7. The relation to the intersection theory of $\mathbb{P}(\mathcal{A}_-) \times \mathbb{P}(\mathcal{A}_-^\vee)$

From now on, adding to (6.8) and (6.9) we assume that the irregularity $q(X) = 0$ and that

(7.1) some section $\kappa \in \Gamma(K_X)$ gives a nonsingular curve $\mathcal{K} \subset X$

in view of Proposition 0.1. (We can expect this will be weakened to the condition $p_g(X) > 0$; to do so, we have to adjust the assumption in Proposition 7.1.) Moreover we assume the following about $\mathbf{f} = (f, m, n) \in A^+(a)$:

$$(7.2) \quad \begin{aligned} &\dim \operatorname{Ext}^1_{X_t}(\mathcal{F}_0 \otimes k(t), \mathcal{G}_0 \otimes k(t)) = L_+ \quad \text{and} \\ &\dim \operatorname{Ext}^1_{X_t}(\mathcal{G}_0 \otimes k(t), \mathcal{F}_0 \otimes k(t)) = L_- \end{aligned}$$

are independent of $t \in T = \mathrm{Pic}^{[f/2]}(X) \times \mathrm{Hilb}^m(X) \times \mathrm{Hilb}^n(X)$, where $\mathcal{F}_0$ and $\mathcal{G}_0$ are $\mathcal{O}_{X_T}$-modules defined in (5.2). This assumption (7.2) holds good if, for example, K_X is numerically equivalent to zero, but is not weak at all in general. Assuming (7.2) we see that both

$$(7.3) \qquad \mathcal{A}_- = Ext^1_{X_T/T}(\mathcal{F}_0, \mathcal{G}_0(K_X)) \quad \text{and} \quad \mathcal{A}_+ = Ext^1_{X_T/T}(\mathcal{G}_0, \mathcal{F}_0(K_X))$$

are locally free $\mathcal{O}_T$-modules, and hence $P^{\pm\mathbf{f}} = \mathbb{P}(\mathcal{A}_\mp)$ are projective bundles over a nonsingular scheme T. Under these assumptions we would like to examine

$$(7.4) \qquad \sum_{k=0}^{d(c_2)-1} \left[\phi_-^* \mu_-(C)|_{E^{\mathbf{f}}}^k \cdot \bar{\phi}_+^* \mu_+(C)|_{E^{\mathbf{f}}}^{d-1-k}\right]_{E^{\mathbf{f}}}$$
$$= \sum_{k=0}^{d(c_2)-1} \left[\phi_-^* \mu_-(C)|_{E^{\mathbf{f}}}^k \cdot \{\phi_-^* \mu_-(C) + \lambda_f^C(\mathcal{O}_-^{\mathbf{f}}(1) + \mathcal{O}_+^{-\mathbf{f}}(1))\}|_{E^{\mathbf{f}}}^{d-1-k}\right],$$

which appeared in (6.16). We shorten $\mathcal{O}_-^{\mathbf{f}}(1)$ and $\mathcal{O}_+^{-\mathbf{f}}$ to, respectively, $\mathcal{O}_-(1)$ and $\mathcal{O}_+(1)$ for the time being. Since $P^{\pm\mathbf{f}}$ are projective bundles over T there are line bundles $\beta_\pm$ on T and integers $N_\pm$ such that

$$\mu_-(C)|_{P^{\mathbf{f}}} = \tau_-^*(\beta_-) + \mathcal{O}_-(N_-) \text{ and } \mu_+(C)|_{P^{-\mathbf{f}}} = \tau_+^*(\beta_+) + \mathcal{O}_+(N_+)$$

in $\mathrm{Pic}(P^{\pm\mathbf{f}})$. By Lemmas 6.1 and 6.3 we have

$$(7.5) \quad \phi_-^* \tau_-^*(\beta_- - \beta_+) + \phi_-^* \mathcal{O}_-(N_-) - \bar{\phi}_+^* \mathcal{O}_+(N_+) = -\lambda_f^C(\phi_-^* \mathcal{O}_-(1) + \bar{\phi}_+^* \mathcal{O}_+(1))$$

in $\mathrm{Pic}(E^{\mathbf{f}})$. Suppose that $N_+ \neq \lambda_f^C = \langle f \cdot C/2 \rangle$. Then (7.5) implies that $\mathcal{O}_{E^{\mathbf{f}}}$ is ϕ_--ample since $\mathcal{O}_{E^{\mathbf{f}}}(-E^{\mathbf{f}}) = \phi_-^* \mathcal{O}_-(1) + \bar{\phi}_+^* \mathcal{O}_+(1)$ is ϕ_--ample. By [EGA, II.5.1], the proper morphism $\phi_- : E^{\mathbf{f}} \to P^{\mathbf{f}}$ should be finite if $\mathcal{O}_{E^{\mathbf{f}}}$ is ϕ_--ample. This contradicts (6.9). Therefore as divisors on $E^{\mathbf{f}}$ we have

$$\phi_-^* \mu_-(C)|_{E^{\mathbf{f}}} = \phi_-^*\{\tau_-^* \beta + \mathcal{O}_-(-\lambda_f^C)\} \qquad \text{and}$$
$$\bar{\phi}_+^* \mu_+(C)|_{E^{\mathbf{f}}} = \bar{\phi}_+^*\{\tau_+ \beta + \mathcal{O}_+(\lambda_f^C)\}$$

with $\beta = \beta_- \in \mathrm{Pic}(T)$. Hence one can check that (7.4) is equal to

$$(7.6) \quad \sum_{k=0}^{d-1} \left[\phi_-^*(\beta + \mathcal{O}_-(-\lambda_f^C))^k \cdot \bar{\phi}_+^*(\beta + \mathcal{O}_+(\lambda_f^C))^{d-1-k}\right]_{E^{\mathbf{f}}}$$
$$= \sum_{t=0}^{d-1} {}_{d-1}C_t \cdot (\lambda_f^C)^{d-1-t} \sum_{s=0}^{d-1-t} \left[\beta^t \cdot \mathcal{O}_+(1)^s \cdot \mathcal{O}_-(-1)^{d-1-t-s}\right]_{E^{\mathbf{f}}}$$

by using the equation $\sum_{l=0}^t {}_{s+l}C_l \cdot {}_{d-1-s-l}C_{s-l} = {}_{d-1}C_t$.

Let $E_0^{\mathbf{f}}, \ldots, E_n^{\mathbf{f}}$ be the reductions of all irreducible components of $E^{\mathbf{f}}$, and let $F_0^{\mathbf{f}}, \ldots, F_n^{\mathbf{f}} \subset P^{\mathbf{f}} \times_T P^{-\mathbf{f}}$ be their image schemes by $\phi_- \times_T \bar{\phi}_+ : E^{\mathbf{f}} \to P^{\mathbf{f}} \times_T$

$P^{-\mathbf{f}}$. [KL, Section 1] implies that $\sum_{s=0}^{d-1-t}\left[\beta^t\cdot\mathcal{O}_+(1)^s\cdot\mathcal{O}_-(-1)^{d-1-t-s}\right]_{E^{\mathbf{f}}}$ in (7.6) is equal to

$$\sum_{i=0}^{n}\deg_i\sum_{s=0}^{d-1-t}\left[\beta^t\cdot\mathcal{O}_+(1)^s\cdot\mathcal{O}_-(-1)^{d-1-t-s}\right]_{F_i^{\mathbf{f}}} \tag{7.7}$$

with some rational number $\deg_i$. We shorten $F_i^{\mathbf{f}}$ to $F^{\mathbf{f}}$ for the time being. We fix some integer M, and divide (the right side of) (7.7) into

$$\begin{aligned}\sum_{s=0}^{M}+\sum_{s=M+1}^{d-1-t}&=\left[\beta^t\cdot\mathcal{O}_-(-1)^{d-1-t-M}\cdot\sum_{s=0}^{M}(\mathcal{O}_+(1)^s\cdot\mathcal{O}_-(-1)^{M-s})\right]_{F^{\mathbf{f}}}\\&+\left[\beta^t\cdot\mathcal{O}_+(1)^{M+1}\cdot\sum_{s=0}^{d-2-t-M}(\mathcal{O}_+(1)^s\cdot\mathcal{O}_-(-1)^{d-2-t-s-M})\right]_{F^{\mathbf{f}}}.\end{aligned} \tag{7.8}$$

(7.8) is related to the intersection theory on $P^{\mathbf{f}}\times_T P^{-\mathbf{f}}=\mathbb{P}(\mathcal{A}_-)\times_T\mathbb{P}(\mathcal{A}_+)$ since $F^{\mathbf{f}}$ is its closed subscheme. In this section we would like to reduce the problem of computing (7.8) to the intersection theory on $\mathbb{P}(\mathcal{A}_-)\times_T\mathbb{P}(\mathcal{A}_-^{\vee})$ and $\mathbb{P}(\mathcal{A}_+^{\vee})\times_T\mathbb{P}(\mathcal{A}_+)$ by choosing M suitably. The reason why we would like to do so will be explained in the next section. It is possible to connect $P^{\mathbf{f}}\times_T P^{-\mathbf{f}}$ with $\mathbb{P}(\mathcal{A}_-)\times_T\mathbb{P}(\mathcal{A}_-^{\vee})$ because $p_g(X)>0$.

Since T is projective over $\mathbb{C}$, there is a line bundle β_0 on T such that coherent $\mathcal{O}_T$-modules $\mathcal{A}_-\otimes\beta_0$, $\mathcal{A}_-\otimes 2\beta_0$ and $\mathcal{A}_-\otimes(\beta+\beta_0)$ are generated by their global sections. Because $\beta=\{\mathcal{O}_-(1)+\beta+\beta_0\}-\{\mathcal{O}_-(1)+\beta_0\}$ and $\mathcal{O}_-(1)=2\{\mathcal{O}_-(1)+\beta_0\}-\{\mathcal{O}_-(1)+2\beta_0\}$, one can express $\beta^s\cdot\mathcal{O}_-(-1)^{d-1-t-M}$ in (7.8) as

$$\sum_{i=1}^{I}N_i\prod_{j=1}^{d-1-M}(\mathcal{O}_-(1)+L_j^i)$$

with integers N_i and line bundles L_j^i on T such that

$$\tau_{-*}(\mathcal{O}_-(1)+L_j^i)=\mathcal{A}_-\otimes L_j^i \text{ is generated by its global sections.} \tag{7.9}$$

Hence, in order to understand the first half of (7.8), let us examine

$$\left[\prod_{j=1}^{d-1-M}(\mathcal{O}_-(1)+L_j)\cdot\sum_{s=0}^{M}(\mathcal{O}_+(1)^s\cdot\mathcal{O}_-(-1)^{M-s})\right]_{F^{\mathbf{f}}}, \tag{7.10}$$

where $L_j\in\operatorname{Pic}(T)$ satisfies (7.9). We shall denote the natural projections by $p_\mp: F^{\mathbf{f}}\hookrightarrow P^{\mathbf{f}}\times_T P^{-\mathbf{f}}\to P^{\pm\mathbf{f}}$. (7.10) clearly is zero if $d-1-M>\dim p_-(F^{\mathbf{f}})$, and so we can assume that $d-1-M\le\dim p_-(F^{\mathbf{f}})$. Then one can find nonzero global sections $\lambda_j\in\Gamma(P^{\mathbf{f}},\mathcal{O}_-(1)\otimes L_j)=\Gamma(T,\mathcal{A}_-\otimes L_j)$ such that $\dim(F^{\mathbf{f}}\cap p_-^{-1}(\Lambda_1\cap\cdots\cap\Lambda_j))=\dim F^{\mathbf{f}}-j$, where $\Lambda_j\subset P^{\mathbf{f}}$ is the effective Cartier divisor of $P^{\mathbf{f}}$ corresponding to λ_j. These λ_j induce a homomorphism

$$\bigoplus_j\otimes\lambda_j: L_1^{-1}\oplus\cdots\oplus L_{d-1-M}^{-1}\to\mathcal{A}_-. \tag{7.11}$$

$\Lambda_1 \cap \cdots \cap \Lambda_{d-1-M} \subset P^{\mathbf{f}}$ is just a closed subscheme $\mathbb{P}(\mathrm{Cok}(\bigoplus_j \otimes\lambda_j)) \subset \mathbb{P}(A_-)$. By a general property of intersection number [KL, p. 297, Proposition 4], (7.10) is equal to

$$\sum_{s=0}^{M} \left[\mathcal{O}_+(1)^s \cdot \mathcal{O}_-(-1)^{M-s}\right]_{p_-^{-1}(\Lambda_1\cap\cdots\cap\Lambda_{d-1-M})}. \tag{7.12}$$

On the other hand, $\kappa \in \Gamma(K_X)$ in (7.1) induces a homomorphism

$$\otimes\kappa_- : \mathcal{A}_+^\vee = Ext^1_{X_T/T}(\mathcal{F}_0, \mathcal{G}_0) \to \mathcal{A}_- = Ext^1_{X_T/T}(\mathcal{F}_0, \mathcal{G}_0(K_X)) \tag{7.13}$$

by virtue of Proposition 5.1. We define l_- by $l_- = \mathrm{rk}(\mathrm{Cok}(\otimes\kappa_-))$ and prove the following proposition.

Proposition 7.1. *If $d-1-M \geq l_- + \dim T$, then we can choose λ_j so that*

(7.14)
$$\mathcal{A}_+^\vee = Ext^1_{X_T/T}(\mathcal{F}_0, \mathcal{G}_0) \stackrel{\otimes\kappa}{\to} \mathcal{A}_- = Ext^1_{X_T/T}(\mathcal{F}_0 \otimes \mathcal{G}_0(K_X)) \twoheadrightarrow \mathrm{Cok}(\bigoplus_j \otimes\lambda_j)$$

is surjective. In particular, $p_-^{-1}(\Lambda_1 \cap \cdots \Lambda_{d-1-M})$ can be regarded as a closed subscheme of $\mathbb{P}(\mathcal{A}_+^\vee) \times_T P^{-\mathbf{f}} = \mathbb{P}(\mathcal{A}_+^\vee) \times_T \mathbb{P}(\mathcal{A}_+)$.

Proof. Suppose that the following lemma is valid:

Lemma 7.2. *Define a closed subscheme T_i of T by*

$$T_i = \left\{ t \in T \;\middle|\; \begin{array}{l} \mathrm{rk}\,\mathrm{Cok}\{\otimes\kappa : \mathrm{Ext}^1_{X_t}(\mathcal{F}_0 \otimes k(t), \mathcal{G}_0 \otimes k(t)) \to \\ \quad \mathrm{Ext}^1_{X_t}(\mathcal{F}_0 \otimes k(t), \mathcal{G}_0(K_X) \otimes k(t))\} \geq l_- + i \end{array} \right\}.$$

Then $\mathrm{codim}(T_i, T) \geq i$ *for all* $i \geq 0$.

Then the dimension of a closed subscheme $\mathbb{P}(\mathrm{Cok}\,(\otimes\kappa))$ of $\mathbb{P}(\mathcal{A}_-) = P^{\mathbf{f}}$ is less than $l_- + \dim T$ since relative Ext sheaves $\mathcal{A}_-$ and $\mathcal{A}_+^\vee$ are compatible with base change by the assumption (7.2). Thus if $d-1-M \geq l_- + \dim T$, then one can choose λ_j suitably so that $\Lambda_1 \cap \cdots \cap \Lambda_{d-1-M} \cap \mathbb{P}(\mathrm{Cok}\,(\otimes\kappa)) = \emptyset$ in $P^{\mathbf{f}}$, or $L_1^{-1} \oplus \cdots \oplus L_{d-1-M}^{-1} \stackrel{\oplus\otimes\lambda_j}{\longrightarrow} \mathcal{A}_- = Ext^1_{X_T/T}(\mathcal{F}_0, \mathcal{G}_0(K_X)) \twoheadrightarrow \mathrm{Cok}(\otimes\kappa)$ is surjective. Hence also $\mathcal{A}_+^\vee \stackrel{\otimes\kappa}{\longrightarrow} \mathcal{A}_- \twoheadrightarrow \mathrm{Cok}(\oplus \otimes \lambda_j)$ is surjective, and so the proof of Proposition 7.1 is completed.

To prove Lemma 7.2 let us observe good properties of $\mathrm{Hilb}(X)$. $\mathcal{F}_0 \otimes k(t)$ and $\mathcal{G}_0 \otimes k(t)$ are isomorphic to, respectively, $\mathcal{O}(L) \otimes I_{Z_1}$ and $\mathcal{O}(c_1 - L) \otimes I_{Z_2}$ for some divisor L on X_t and codimension-two closed subschemes Z_1 and Z_2 in X_t. The long exact sequence of Ext sheaves associated with a short exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{O}(c_1 - L) = \mathcal{G}_0 \otimes k(t) &\stackrel{\otimes\kappa}{\longrightarrow} \mathcal{O}(c_1 - L + K_X) \otimes I_{Z_2} \\ &= \mathcal{G}_0(K_X) \otimes k(t) \longrightarrow \mathcal{O}(c_1 - L + K_X)|_{\mathcal{K}} \otimes I_{Z_2} \longrightarrow 0 \end{aligned}$$

tells us that

$$\begin{aligned}&\operatorname{rk}\operatorname{Cok}\{\kappa : \operatorname{Ext}^1_{X_t}(\mathcal{F}_0\otimes k(t), \mathcal{G}_0\otimes k(t)) \to \operatorname{Ext}^1_{X_t}(\mathcal{F}_0\otimes k(t), \mathcal{G}_0(K_X)\otimes k(t))\}\\ &\quad = L_- - L_+ - \hom_X(I_{Z_1}, \mathcal{O}(c_1 - 2L + K_X)\otimes I_{Z_2})\\ &\qquad + \hom_X(I_{Z_1}, \mathcal{O}(c_1 - 2L + K_X)|_{\mathcal{K}}\otimes I_{Z_2}),\end{aligned}$$

where $L_\pm$ are those of (7.2). Since

$$\dim \operatorname{Ext}^1_{X_t}(\mathcal{G}_0\otimes k(t), \mathcal{F}_0\otimes k(t)) = \dim \operatorname{Ext}^1_{X_t}(I_{Z_1}, \mathcal{O}(c_1 - 2L + K_X)\otimes I_{Z_2})$$

is independent of $t\in T$, $\hom_X(I_{Z_1}, \mathcal{O}(c_1 - 2L + K_X)\otimes I_{Z_2})$ is independent of $t\in T$. Moreover, if $t\in T$ is so general that $Z_1\cap\mathcal{K} = Z_2\cap\mathcal{K} = \emptyset$, then $\hom_X(I_{Z_1}, \mathcal{O}(c_1 - 2L + K_X)|_{\mathcal{K}}\otimes I_{Z_2})$ is equal to $h^0(\mathcal{O}(c_1 - 2L + K_X)|_{\mathcal{K}})$, which is independent of $t\in T$ since $q(X) = 0$. Therefore one can show that

$$\begin{aligned}&\operatorname{rk}\operatorname{Cok}\{\otimes\kappa : \operatorname{Ext}^1_{X_t}(\mathcal{F}_0\otimes k(t), \mathcal{G}_0\otimes k(t)) \to \operatorname{Ext}^1_{X_t}(\mathcal{F}_0\otimes k(t), \mathcal{G}_0(K_X)\otimes k(t))\}\\ &\quad - l_- = \hom_X(I_{Z_1}, \mathcal{O}(c_1 - 2L + K_X)|_{\mathcal{K}}\otimes I_{Z_2}) - h^0(\mathcal{O}(c_1 - 2L + K_X)|_{\mathcal{K}}).\end{aligned}$$

Now we divide Artinian schemes Z_1 and Z_2 into $Z_1 = W_1\coprod T_1$ and $Z_2 = W_2\coprod T_2$ so that, set-theoretically, $W_1 = W_2 = Z_1\cap Z_2\cap\mathcal{K}$.

Claim 7.3.

$$\begin{aligned}&\hom_X(I_{Z_1}, \mathcal{O}(c_1 - 2L + K_X)|_{\mathcal{K}}\otimes I_{Z_2}) - h^0(\mathcal{O}(c_1 - 2L + K_X)|_{\mathcal{K}})\\ &\quad \le l(Z_1\cap\mathcal{K}) + l(Z_2\cap\mathcal{K}) + \hom_X(\mathcal{O}_{W_2}, \mathcal{O}_{W_1})\\ &\qquad + \hom_X(\mathcal{O}_{W_1}, \operatorname{Im}(\otimes\kappa : \mathcal{O}_{W_2}\to\mathcal{O}_{W_2})).\end{aligned}$$

Proof. From the long exact sequence of Tor sheaves, one derives two exact sequences

$$0 \longrightarrow F_2 \longrightarrow \mathcal{O}_{Z_2} \stackrel{\otimes\kappa}{\longrightarrow} \mathcal{O}_{Z_2} \longrightarrow \mathcal{O}_{Z_2\cap\mathcal{K}} \longrightarrow 0 \tag{7.15}$$

and

$$0 \longrightarrow F_2 \longrightarrow I_{Z_2}|_{\mathcal{K}} \longrightarrow L_2 = \operatorname{Ker}(\mathcal{O}_{\mathcal{K}} \twoheadrightarrow \mathcal{O}_{\mathcal{K}\cap Z_2}) \longrightarrow 0.$$

Hence one can show that

$$\begin{aligned}&\hom_X(I_{Z_1}, \mathcal{O}(c_1 - 2L + K_X)|_{\mathcal{K}}\otimes I_{Z_2}) - h^0(\mathcal{O}(c_1 - 2L + K_X)|_{\mathcal{K}})\\ &\quad \le \hom_X(I_{Z_1}, \mathcal{O}(c_1 - 2L + K_X)|_{\mathcal{K}}) + \hom_X(I_{Z_1}, F_2) - h^0(\mathcal{O}(c_1 - 2L + K_X)|_{\mathcal{K}})\\ &\quad = [\chi(I_{Z_1}, \mathcal{O}(c_1 - 2L + K_X)) + \operatorname{ext}^1_X(I_{Z_1}, \mathcal{O}(c_1 - 2L + K_X)|_{\mathcal{K}})\\ &\qquad - \chi(I_{Z_1}, \mathcal{O}(c_1 - 2L))] - [\chi(\mathcal{O}(c_1 - 2L + K_X)) - \chi(\mathcal{O}(c_1 - 2L))\\ &\qquad + h^1(\mathcal{O}(c_1 - 2L + K_X)|_{\mathcal{K}})] + \chi(I_{Z_1}, F_2) + \operatorname{ext}^1_X(I_{Z_1}, F_2)\\ &\quad = \operatorname{ext}^1_X(I_{Z_1}, \mathcal{O}(c_1 - 2L + K_X)|_{\mathcal{K}}) + l(F_2) + \operatorname{ext}^1_X(I_{Z_1}, F_2)\\ &\qquad - h^1(\mathcal{O}(c_1 - 2L + K_X)|_{\mathcal{K}})\\ &\quad = \operatorname{ext}^1_X(I_{Z_1}, \mathcal{O}(c_1 - 2L + K_X)|_{\mathcal{K}}) - h^1(\mathcal{O}(c_1 - 2L + K_X)|_{\mathcal{K}})\end{aligned}$$

$$+\,l(Z_2\cap\mathcal{K})+\mathrm{ext}^1_X(I_{Z_1},F_2)$$

by the Riemann-Roch theorem and (7.15). If we define F_1 by an exact sequence

$$0\longrightarrow F_1\longrightarrow\mathcal{O}_{Z_1}\xrightarrow{\otimes\kappa}\mathcal{O}_{Z_1}\longrightarrow\mathcal{O}_{Z_1\cap\mathcal{K}}\longrightarrow 0,$$

then we have that

$$\begin{aligned}&\mathrm{ext}^1_X(I_{Z_1},\mathcal{O}(c_1-2L+K_X)|_{\mathcal{K}})-h^1(\mathcal{O}(c_1-2L+K_X)|_{\mathcal{K}})\\&\quad\le\mathrm{ext}^2_X(\mathcal{O}_{Z_1},\mathcal{O}(c_1-2L+K_X)|_{\mathcal{K}})=\mathrm{hom}_X(\mathcal{O}(c_1-2L)|_{\mathcal{K}},\mathcal{O}_{Z_1})\\&\quad=\mathrm{hom}_X(\mathcal{O}_{\mathcal{K}},\mathcal{O}_{Z_1})=\mathrm{hom}_X(\mathcal{O}_{\mathcal{K}},F_1)\le l(F_1)=l(Z_1\cap\mathcal{K})\end{aligned}$$

For $W_2\subset Z_2$ mentioned above, there is an exact sequence

$$0\longrightarrow G_2\longrightarrow\mathcal{O}_{W_2}\xrightarrow{\otimes\kappa}\mathcal{O}_{W_2}\longrightarrow\mathcal{O}_{W_2\cap\mathcal{K}}\longrightarrow 0.$$

$\mathrm{ext}^1_X(I_{Z_1},F_2)=\mathrm{ext}^2_X(\mathcal{O}_{Z_1},F_2)=\mathrm{hom}_X(F_2,\mathcal{O}_{Z_1})$ equals $\mathrm{hom}_X(G_2,\mathcal{O}_{W_1})$ naturally. G_2 induces an exact sequence

$$0\longrightarrow G_2\longrightarrow\mathcal{O}_{W_2}\longrightarrow\mathrm{Im}(\otimes\kappa)\longrightarrow 0.$$

This sequence implies that

$$\begin{aligned}&\mathrm{hom}_X(G_2,\mathcal{O}_{W_1})\\&\quad\le\mathrm{hom}_X(\mathcal{O}_{W_2},\mathcal{O}_{W_1})-\mathrm{hom}_X(\mathrm{Im}(\otimes\kappa),\mathcal{O}_{W_1})+\mathrm{ext}^1_X(\mathrm{Im}(\otimes\kappa),\mathcal{O}_{W_1})\\&\quad=-\chi(\mathrm{Im}(\otimes\kappa),\mathcal{O}_{W_1})+\mathrm{ext}^2_X(\mathrm{Im}(\otimes\kappa),\mathcal{O}_{W_1})+\mathrm{hom}_X(\mathcal{O}_{W_2},\mathcal{O}_{W_1})\\&\quad=\mathrm{hom}_X(\mathcal{O}_{W_1},\mathrm{Im}(\otimes\kappa))+\mathrm{hom}_X(\mathcal{O}_{W_2},\mathcal{O}_{W_1}).\end{aligned}$$

Hence we conclude the proof of this claim. □

For nonnegative integers p, q and r,

$$W^{mn}_{pqr}=W_{pqr}=\left\{\begin{array}{l|l}(Z_1,Z_2)\in & l(Z_1\cap\mathcal{K})=p,\ l(Z_2\cap\mathcal{K})=q,\\ \mathrm{Hilb}^m(X)\times & \mathrm{hom}(\mathcal{O}_{W_2},\mathcal{O}_{W_1})+\mathrm{hom}(\mathcal{O}_{W_1},\mathrm{Im}(\otimes\kappa:\\ \mathrm{Hilb}^n(X) & \mathcal{O}_{W_2}\to\mathcal{O}_{W_2}))=r\end{array}\right\}$$

is a locally-closed subscheme of $\mathrm{Hilb}^m(X)\times\mathrm{Hilb}^n(X)$. By the claim above, the proof of Lemma 7.2 is completed if we prove that

$$\mathrm{codim}(W^{mn}_{pqr},\mathrm{Hilb}^m(X)\times\mathrm{Hilb}^n(X))\ge p+q+r. \tag{7.16}$$

Let $\mathrm{Hilb}^m(X,x)$ denote $\mathrm{Hilb}^m(\mathrm{Spec}(\mathcal{O}_{X,x}))$ for a closed point $x\in X$, and let $Z^m_p\subset\mathrm{Hilb}^m(X)$ be a locally closed subscheme $\{z\in\mathrm{Hilb}^m(X)\mid l(Z\cap\mathcal{K})=p\}$ for $p\in\mathbb{N}$.

Claim 7.4. If we prove that

(7.17)
$$\mathrm{codim}(W^{mn}_{pqr}\cap[\mathrm{Hilb}^m(X,x)\times\mathrm{Hilb}^n(X,x)],\ \mathrm{Hilb}^m(X,x)\times\mathrm{Hilb}^n(X,x))\ge p+q+r+1$$

and that

$$\operatorname{codim}(Z_p^m, \operatorname{Hilb}^m(X)) \geq p, \tag{7.18}$$

then (7.16) follows.

Proof. The proof is by induction on (m, n). Fix (m, n) and suppose that (7.16) holds good for $(m', n') \neq (m, n)$ such that $m' \leq m$ and $n' \leq n$. If either m or n is zero, then (7.16) for (m, n) is immediate from (7.18). Hence we assume that both m and n are positive. We divide the proof into several cases. Let (Z_1, Z_2) be a member of $W^{mn}_{pqr} \subset \operatorname{Hilb}^m(X) \times \operatorname{Hilb}^n(X)$.

First, suppose that $\sharp \operatorname{supp}(Z_1) \geq 2$ and $\sharp \operatorname{supp}(Z_2) \geq 2$. Let m_1, m_2, n_1 and n_2 be positive integers such that $m_1 + m_2 = m$ and $n_1 + n_2 = n$. If we define an open subset U^{m_1} of $\operatorname{Hilb}^{m_1}(X) \times \operatorname{Hilb}^{m_2}(X)$ by

$$U^{m_1} = \{(Z_1^{(1)}, Z_1^{(2)}) \mid Z_1^{(1)} \cap Z_1^{(2)} = \emptyset\},$$

then we can define a natural map $\varphi^{m_1} : U^{m_1} \to \operatorname{Hilb}^m(X)$. Similarly we can define $\varphi^{n_1} : U^{n_1} \to \operatorname{Hilb}^n(X)$. Let V^{m_1,n_1} be an open subset of $U^{m_1} \times U^{n_1}$

$$\{(Z_1^{(1)}, Z_1^{(2)}, Z_2^{(1)}, Z_2^{(2)}) \mid Z_1^{(2)} \cap Z_2^{(1)} = Z_1^{(1)} \cap Z_2^{(2)} = \emptyset\}.$$

(Z_1, Z_2) is contained in $\varphi^{m_1} \times \varphi^{n_1}(V^{m_1,n_1})$ for some m_1, n_1. It's easy to prove that, in $\operatorname{Hilb}^{m_1} \times \operatorname{Hilb}^{n_1} \times \operatorname{Hilb}^{m_2} \times \operatorname{Hilb}^{n_2}$,

$$(\varphi^{m_1} \times \varphi^{n_1})^{-1}(W^{mn}_{pqr}) \cap V^{m_1,n_1} \subset \bigcup_{(p_i, q_i, r_i)} W^{m_1 n_1}_{p_1 q_1 r_1} \times W^{m_2 n_2}_{p_2 q_2 r_2}, \tag{7.19}$$

where (p_i, q_i, r_i) runs over the set of all triples such that $p_1 + p_2 = p$, $q_1 + q_2 = q$ and $r_1 + r_2 = r$. The inductive hypothesis tells us that the dimension of the right side of (7.19) is not exceeding $2(m + n) - (p + q + r)$, since $\dim \operatorname{Hilb}^n(X) = 2n$ by [Fo]. Hence $\dim(W^{mn}_{pqr} \cap (\varphi^{m_i} \times \varphi^{n_i})(V^{m_i,n_i}) \leq 2(m + n) - (p + q + r)$.

Unless $\sharp \operatorname{supp}(Z_1) \geq 2$ and $\sharp \operatorname{supp}(Z_2) \geq 2$, it holds either $\sharp \operatorname{supp}(Z_1) = 1$ and $\sharp \operatorname{supp}(Z_2) \geq 2$, $\sharp \operatorname{supp}(Z_1) \geq 2$ and $\sharp \operatorname{supp}(Z_2) = 1$, $\operatorname{supp}(Z_1) = \operatorname{supp}(Z_2) = \{x\}$ or $\operatorname{supp}(Z_1) \cap \operatorname{supp}(Z_2) = \emptyset$. In all cases one can verify that (Z_1, Z_2) is contained in a subscheme whose dimension does not exceed $2(m + n) - (p + q + r)$, similarly to the case where $\sharp \operatorname{supp}(Z_1) \geq 2$ and $\sharp \operatorname{supp}(Z_2) \geq 2$. □

Claim 7.5. Let us denote $Z_p^m \cap \operatorname{Hilb}^m(X, x)$ by $Z_p^m(x)$. If we prove that

$$\operatorname{codim}(Z_p^m(x), \operatorname{Hilb}^m(X, x)) \geq p - 1, \tag{7.20}$$

then (7.17) and (7.18) follow.

Proof. We can prove (7.18) by using (7.20) in a similar fashion to the proof of Claim 7.4. Shorten $W^{mn}_{pqr} \cap [\operatorname{Hilb}^m(X, x) \times \operatorname{Hilb}^n(X, x)]$ to $W^{mn}_{pqr}(x)$. If $(Z_1, Z_2) \in W^{mn}_{pqr}(x)$, then

$$\begin{aligned} r &= \hom_X(\mathcal{O}_{Z_2}, \mathcal{O}_{Z_1}) + \hom_X(\mathcal{O}_{Z_1}, \operatorname{Im}(\otimes\kappa : \mathcal{O}_{Z_2} \to \mathcal{O}_{Z_2})) \\ &\leq l(Z_1) + l(\operatorname{Im}(\otimes\kappa)) = l(Z_1) + l(Z_2) - l(Z_2 \cap \mathcal{K}) = m + n - q. \end{aligned}$$

Hence if $Z^{mn}_{pqr}(x) \neq \emptyset$, then (7.20) means that

$$\begin{aligned}2(m+n)-(p+q+r) &\geq 2(m+n)-(p+q+m+n-q)\\ &= m+n-p = m-1-(p-1)+(n-1)+1\\ &\geq \dim(Z^m_p(x)\times \mathrm{Hilb}^n(X,x))+1 \geq \dim(W^{mn}_{pqr}(x))+1\end{aligned}$$

since $\dim \mathrm{Hilb}^m(X,x) = m+1$ by [Br]. Thus (7.17) follows. □

Now we prove the following claim, which completes the proof of Proposition 7.1 because of the claim above.

Claim 7.6. For an integer $i \geq 2$ and a closed point $x \in X$, we define a locally closed subscheme $W^m_{qi}(x)$ of $Z^m_q(x) \subset \mathrm{Hilb}^m(X)$ by

$$W^m_{qi}(x) = \{z \in Z^m_q(x) \mid \dim_{\mathbb{C}}(I_Z \otimes k(t)) = i\}.$$

Then it holds that

$$\dim Z^m_q(x) \leq m-q \qquad (1 \leq q \leq m), \tag{7.21}$$

and that

$$\dim W^m_{qi}(x) \leq m-q+2-i \qquad (2 \leq i,\ 1 \leq q \leq m). \tag{7.22}$$

Proof. It suffices to prove this in case where $x \in \mathcal{K}$. The proof is by induction on m. It's easy to prove this claim for $m=1$. Fix m and suppose that this claim is valid for all $m' \leq m$. Referring to [ES], we here recall the incidence subvariety $H_{m,m+1}$ of $\mathrm{Hilb}^m(X)\times\mathrm{Hilb}^{m+1}(X)$:

$$H_{m,m+1} = \{(Z_1,Z_2) \in \mathrm{Hilb}^m(X)\times\mathrm{Hilb}^{m+1}(X) \mid Z_1 \subset Z_2\}.$$

Let $f : H_{m,m+1} \to \mathrm{Hilb}^m(X)$ and $g : H_{m,m+1} \to \mathrm{Hilb}^{m+1}(X)$ be the projections. There is a natural morphism $q : H_{m,m+1} \to X$ sending (Z_1,Z_2) to the unique point where Z_1 and Z_2 differ. They give a (birational) morphism $\phi = (f,q) : H_{m,m+1} \to \mathrm{Hilb}^m(X)\times X$. By [ES, Section 3] it holds that

$$\dim \phi^{-1}(Z_1,y) = \dim_{\mathbb{C}}(I_{Z_1}\otimes k(y)) - 1 \tag{7.23}$$

for $(Z_1,y) \in \mathrm{Hilb}^m(X)\times X$, and that if $(g,q)^{-1}(Z_2,y) \neq \emptyset$ then

$$\dim (g,q)^{-1}(Z_2,y) = \dim_{\mathbb{C}}(I_{Z_2}\otimes k(y)) - 2 \tag{7.24}$$

for $(Z_2,y) \in \mathrm{Hilb}^{m+1}(X)\times X$.

First let us show (7.21) for $m+1$. Suppose that $q \leq m$. Then for any $Z_2 \in Z^{m+1}_q(x)$ one can find $Z_1 \in Z^m_q(x)$ such that $(Z_1,Z_2) \in H_{m,m+1}$. Thus $Z^{m+1}_q(x) \subset g(\phi^{-1}(Z^m_q(x)\times\{x\}))$. $Z^m_q(x)$ clearly is equal to $\bigcup_{i\geq 2} W^m_{qi}(x)$, and so $\dim Z^{m+1}_q(x) \leq \max_{i\geq 2}\dim\phi^{-1}(W^m_{qi}(x)\times\{x\})$. The inductive hypothesis (7.22) and (7.23) imply that

$$\begin{aligned}&\dim \phi^{-1}(W^m_{qi}(x)\times\{x\})\\ &\qquad \leq \dim W^m_{qi}(x)+i-1 \leq m-q+2-i+i-1 = m-q+1.\end{aligned} \tag{7.25}$$

Now we claim that $\dim Z^{m+1}_{m+1}(x) = 0$. Indeed, if $Z_2 \in Z^{m+1}_{m+1}(x)$, then $\mathcal{O}_{Z_2}$ is isomorphic to $\mathcal{O}_{Z_2\cap\mathcal{K}}$, which is equal to $\mathcal{O}_{\mathcal{K}}/m^{m+1}_{\mathcal{K},x}$ since $\mathcal{K}$ is a nonsingular curve. Therefore (7.21) is valid for $m+1$.

Next let us show (7.22) for $m+1$. If $q = m+1$ or $i = 2$, then (7.22) results form (7.21). So suppose that $q \le m$ and $i \ge 3$. If $(Z_1, Z_2) \in H_{m,m+1}$ satisfies $Z_2 \in W^{m+1}_{qi}(x)$, then $Z_1 \in \mathrm{Hilb}^m(X, x)$, $l(Z_1 \cap \mathcal{K}) = q-1$ or q, and $\dim_{\mathbb{C}}(I_{Z_1} \otimes k(x)) = i-1$, i, or $i+1$. Hence

(7.26)

$$g^{-1}(W^{m+1}_{qi}(x)) \subset \bigcup_{j=i-1}^{i+1} \phi^{-1}\left(W^m_{q-1,j}(x) \times \{x\}\right) \cup \bigcup_{j=i-1}^{i+1} \phi^{-1}\left(W^m_{q,j}(x) \times \{x\}\right).$$

If $Z_1 \in Z^m_{q-1}(x)$ and $Z_2 \in Z^{m+1}_q(x)$ satisfy $Z_1 \subset Z_2$, then I_{Z_2} is equal to

$$\mathrm{Ker}(I_{Z_1} \twoheadrightarrow I_{Z_1}|_{\mathcal{K}} \twoheadrightarrow I_{Z_1\cap\mathcal{K}} = m^{q-1}_{\mathcal{K},x} \twoheadrightarrow m^{q-1}_{\mathcal{K},x}/m^q_{\mathcal{K},x} \simeq \mathbb{C})$$

since $\mathcal{K}$ is nonsingular, where $m_{\mathcal{K},x}$ is the ideal sheaf of $x \in \mathcal{K}$. Consequently the inductive hypothesis (7.22) implies that

$$\begin{aligned}(7.27)\quad \dim \phi^{-1}(W^m_{q-1,j}(x) \times \{x\}) \cap g^{-1}(W^{m+1}_{q,i}(x))&\\ \le \dim W^m_{q-1,j}(x) \times \{x\} \le m-q+3-j &\le m-q+1\end{aligned}$$

since $j \ge i-1 \ge 2$. (7.25), (7.26) and (7.27) mean that $\dim(W^{m+1}_{qi}(x)) \le m-q+1$. By (7.24) we have

$$\dim W^{m+1}_{qi}(x) \le m-q+1-(i-2) = m+1-q+2-i.$$

Therefore we have proved (7.22). □

Claim 7.6 concludes the proof of Proposition 7.1. □

Therefore (7.12), which is the first half of (7.8), is related to the intersection theory on $\mathbb{P}(\mathcal{A}^\vee_+) \times_T \mathbb{P}(\mathcal{A}_+)$ if $d-1-M \ge l_- + \dim T$.

8. The relation to incidence varieties

To understand (7.12) still more, let us examine subschemes $F^{\mathbf{f}}$ and $p_-^{-1}(\Lambda_1 \cap \cdots \cap \Lambda_{d-1-M})$ of $P^{\mathbf{f}} \times_T P^{-\mathbf{f}}$. We denote the reduction of $\varphi_-^{-1}(Q^{\mathbf{f}}) = \tilde{\varphi}_+^{-1}(P^{-\mathbf{f}})$ by $D^{\mathbf{f}}$.

Lemma 8.1. *Let $r : Hom_{X_{D^{\mathbf{f}}}/D^{\mathbf{f}}}(\tilde{\mathcal{G}}, \tilde{\mathcal{G}}(K_X)) \to Ext^1_{X_{D^{\mathbf{f}}}/D^{\mathbf{f}}}(\tilde{\mathcal{G}}, \tilde{\mathcal{F}}(K_X))$ be a homomorphism induced by the restriction of (3.2) to $X_{D^{\mathbf{f}}}$. (We here shorten $\tilde{\mathcal{F}}|_{X_{D^{\mathbf{f}}}}$ to $\tilde{\mathcal{F}}$, etc.) The extension class of the third column of (3.6) gives an element s of*

$$\begin{aligned}\mathrm{Ext}^1_{X_{D^{\mathbf{f}}}}(\tilde{\mathcal{F}}, \tilde{\mathcal{G}}(-D_-)) &= \Gamma(D^{\mathbf{f}}, Ext^1_{X_{D^{\mathbf{f}}}/D^{\mathbf{f}}}(\tilde{\mathcal{F}}, \tilde{\mathcal{G}}(-D_-)))\\ &\simeq \Gamma(D^{\mathbf{f}}, Ext^1_{X_{D^{\mathbf{f}}}/D^{\mathbf{f}}}(\tilde{\mathcal{G}}(-D_-), \tilde{\mathcal{F}}(K_X))^\vee)\\ &= \mathrm{Hom}_{D^{\mathbf{f}}}(Ext^1_{X_{D^{\mathbf{f}}}/D^{\mathbf{f}}}(\tilde{\mathcal{G}}, \tilde{\mathcal{F}}(K_X)), \mathcal{O}_{D^{\mathbf{f}}}(-D_-))\end{aligned}$$

by virtue of Proposition 5.1*. Then* $s\circ r: Hom_{X_{D^{\mathbf{f}}}/D^{\mathbf{f}}}(\tilde{\mathcal{G}}, \tilde{\mathcal{G}}(K_X)) \to \mathcal{O}_{D^{\mathbf{f}}}(-D_-)$ *is zero.*

Proof. We shall appeal to some obstruction theory. For a closed point t of $D^{\mathbf{f}}$ the third column of (3.6) induces an exact sequence

$$0 \longrightarrow \tilde{\mathcal{G}}_{k(t)}(-D_-) \longrightarrow \mathcal{W}_+|_{X_{k(t)}} \longrightarrow \tilde{\mathcal{F}}_{k(t)} \longrightarrow 0 \tag{8.1}$$

on $X_{k(t)}$. As observed in the proof of Lemma 3.1 and lemma 3.3, the extension class σ of (8.1) in $\mathrm{Ext}^1_{X_{k(t)}}(\tilde{\mathcal{F}}_{k(t)}, \tilde{\mathcal{G}}_{k(t)}(-D_-))$ is the obstruction to extend a morphism

$$\mathrm{Spec}(A') = \mathrm{Spec}(\mathcal{O}_{\tilde{Q}^{ss}_-}/\mathcal{O}(-D_-) + \tilde{m}_t^{l+1}) \longrightarrow D_- \xrightarrow{\varphi_-} V_-$$

to a morphism $\mathrm{Spec}(A) = \mathrm{Spec}(\mathcal{O}_{\tilde{Q}^{ss}_-}/\tilde{m}_t^{l+1}) \to V_-$, where l is the integer in Lemma 3.1. Next, let

$$r'_t : \mathrm{Ext}^1_X(\tilde{\mathcal{F}}_{k(t)}, \tilde{\mathcal{G}}_{k(t)}(-D_-)) \longrightarrow \mathrm{Ext}^2_X(\tilde{\mathcal{G}}_{k(t)}, \tilde{\mathcal{G}}_{k(t)}(-D_-))$$

be an homomorphism induced by (3.2). Then $r'_t(\sigma)$ is the obstruction to extend $\tilde{\mathcal{G}} \otimes_{D_-} \mathcal{O}_{A'} \in \mathrm{Coh}(X_{A'})$ to an A-flat family of simple sheaves on X_A by [HL, Section 2.A]. Moreover, the trace map

$$\mathrm{tr} : \mathrm{Ext}^2_X(\tilde{\mathcal{G}}_{k(t)}, \tilde{\mathcal{G}}_{k(t)}(-D_-)) \longrightarrow H^2(\mathcal{O}_X(-D_-)) \tag{8.2}$$

sends $r'_t(\sigma)$ to the obstruction to extend a line bundle $\det(\tilde{\mathcal{G}} \otimes_{D_-} \mathcal{O}_{A'})$ on $X_{A'}$ to a line bundle on X_A by [HL, Theorem 4.5.3]. Now $\mathrm{Pic}(X)$ is smooth over $\mathbb{C}$, and the trace map (8.2) is isomorphic since $\mathrm{rk}(\tilde{\mathcal{G}}_{k(t)}) = 1$. Therefore $r'_t(\sigma) = 0$. Remark that

$$\begin{tikzcd}
\mathrm{Ext}^1_X(\tilde{\mathcal{F}}_{k(t)}, \tilde{\mathcal{G}}_{k(t)}(-D_-)) \arrow[r, "r'_t"'] \arrow[d, "\Theta"] & \mathrm{Ext}^2_X(\tilde{\mathcal{G}}_{k(t)}, \tilde{\mathcal{G}}_{k(t)}(-D_-)) \arrow[d, "\Theta"'] \\
\mathrm{Ext}^1_X(\tilde{\mathcal{G}}_{k(t)}(-D_-), \tilde{\mathcal{F}}_{k(t)}(K_X))^\vee \arrow[r, "r_t^\vee"] & \mathrm{Hom}_X(\tilde{\mathcal{G}}_{k(t)}(-D_-), \tilde{\mathcal{G}}_{k(t)}(K_X))^\vee
\end{tikzcd}$$

is commutative, where Θ is an isomorphism induced by the Serre duality (5.4), and

$$r_t : \mathrm{Hom}(\tilde{\mathcal{G}}_{k(t)}(-D_-), \tilde{\mathcal{G}}_{k(t)}(K_X)) \longrightarrow \mathrm{Ext}^1_X(\tilde{\mathcal{G}}_{k(t)}(-D_-), \tilde{\mathcal{F}}_{k(t)}(K_X))$$

is defined similarly to r in this lemma. One can verify that

$$Hom_{X_{D^{\mathbf{f}}}/D^{\mathbf{f}}}(\tilde{\mathcal{G}}, \tilde{\mathcal{G}}(K_X)) \otimes k(t) \xrightarrow{\mathrm{can}} \mathrm{Hom}_{X_{k(t)}}(\tilde{\mathcal{G}}_{k(t)}, \tilde{\mathcal{G}}_{k(t)}(K_X)) \xrightarrow{r_t^\vee \circ \Theta(\sigma)} \mathcal{O}_{\tilde{Q}^{ss}_-}(-D_-) \otimes k(t)$$

is equal to $(s \circ r) \otimes k(t)$. Hence $(s \circ r) \otimes k(t) = 0$ for every closed point $t \in D^{\mathbf{f}}$, which implies $s \circ r = 0$ since $D^{\mathbf{f}}$ is reduced. □

An exact sequence (5.9) on $X_{P^{\mathbf{f}}}$ induces a homomorphism

$$r_P : Hom_{X_{P^{\mathbf{f}}}/P^{\mathbf{f}}}(\mathcal{G}_0, \mathcal{G}_0(K_X)) \to Ext^1_{X_{P^{\mathbf{f}}}/P^{\mathbf{f}}}(\mathcal{G}_0, \mathcal{F}_0 \otimes \mathcal{O}_-(1)(K_X)). \tag{8.3}$$

Lemma 8.2. *The image scheme of* $\phi_- \times_T \bar{\phi}_+ : (E^{\mathbf{f}})_{\mathrm{red}} \to P^{\mathbf{f}} \times_T P^{-\mathbf{f}}$ *is contained in a subscheme*

$$\mathbb{P}(\mathrm{Cok}(r_P)) \subset \mathbb{P}(Ext^1_{X_{P^{\mathbf{f}}}/P^{\mathbf{f}}}(\mathcal{G}_0, \mathcal{F}_0 \otimes \mathcal{O}_-(1)(K_X))) = P^{\mathbf{f}} \times_T P^{-\mathbf{f}}$$

defined in (8.3).

Proof. We shorten $(E^{\mathbf{f}})_{\mathrm{red}}$ to E_r in this proof. There is a natural exact sequence

$$0 \longrightarrow \mathcal{G}_0 \otimes \mathcal{O}_+(1) \longrightarrow \mathcal{V}_+ \longrightarrow \mathcal{F}_0 \longrightarrow 0 \tag{8.4}$$

on $X^{-\mathbf{f}}$ similarly to (5.9). Pulling back (5.9) and (8.4) by, respectively, ϕ_- and $\bar{\phi}_+$, we have two exact sequences

$$0 \longrightarrow \mathcal{F}_0 \otimes \mathcal{O}_-(1) \longrightarrow \tilde{\mathcal{V}}_- \longrightarrow \mathcal{G}_0 \longrightarrow 0, \tag{8.5}$$

$$0 \longrightarrow \mathcal{G}_0 \otimes \mathcal{O}_+(1) \longrightarrow \tilde{\mathcal{V}}_+ \longrightarrow \mathcal{F}_0 \longrightarrow 0 \tag{8.6}$$

on X_{E_r}. They induce two homomorphisms

$$r_E : Hom_{X_{E_r}/E_r}(\mathcal{G}_0, \mathcal{G}_0(K_X)) \longrightarrow Ext^1_{X_{E_r}/E_r}(\mathcal{G}_0, \mathcal{F}_0 \otimes \mathcal{O}_-(1)(K_X)) \quad \text{and}$$
$$s_E : Ext^1_{X_{E_r}/E_r}(\mathcal{G}_0, \mathcal{F}_0 \otimes \mathcal{O}_-(1)(K_X)) \longrightarrow \mathcal{O}_-(1) \otimes \mathcal{O}_+(1).$$

We pull them back by $\tilde{\pi}_- : D^{\mathbf{f}} \to E_r$. Then

$$\begin{tikzcd} \tilde{\pi}_-^* Hom_{X_{E_r}/E_r}(\mathcal{G}_0, \mathcal{G}_0(K_X)) \arrow[r, "\tilde{\pi}_-^*(s_E \circ r_E)"'] \arrow[d, "f_1"] & \tilde{\pi}_-^*(\mathcal{O}_-(1) \otimes \mathcal{O}_+(1)) \arrow[d, "f_2"'] \\ Hom_{X_{D^{\mathbf{f}}}/D^{\mathbf{f}}}(\tilde{\mathcal{G}}, \tilde{\mathcal{G}}(K_X)) \arrow[r, "s \circ r"] & \mathcal{O}_{D^{\mathbf{f}}}(-D_-) \end{tikzcd}$$

is commutative, where f_1 is a natural homomorphism and f_2 is the isomorphism in Lemma 6.3. One can prove this by recollecting the way to construct $\bar{\phi}_+$ and the proof of Proposition 5.7. Therefore $\tilde{\pi}_-^*(s_E \circ r_E) = 0$ by Lemma 8.1.

On the other hand, $\tilde{\pi}_- : (\tilde{\pi}_-)^{-1}(E^{\mathbf{f}}_{\mathrm{red}}) \to E^{\mathbf{f}}_{\mathrm{red}} = E_r$ is a principal $\bar{G}$-bundle since $E^{\mathbf{f}} \subset \tilde{M}^s_-$. Thereby $(\tilde{\pi}_-)^{-1}(E^{\mathbf{f}}_{\mathrm{red}})$ is reduced, and hence $(\tilde{\pi}_-)^{-1}(E^{\mathbf{f}})_{\mathrm{red}} = D^{\mathbf{f}}$. Accordingly $\tilde{\pi}_- : D^{\mathbf{f}} \to E_r$ is faithfully-flat, and so $\tilde{\pi}_-^*(s_E \circ r_E) = 0$ implies $s_E \circ r_E = 0$. In fact, s_E gives a morphism $\mathbb{P}(s_E) : E_r \to E_r \times_T P^{-\mathbf{f}} = \mathbb{P}(Ext^1_{X_{E_r}/E_r}(\mathcal{G}_0, \mathcal{F}_0(K_X))$ and $\mathbb{P}(s_E)$ is equal to $\mathrm{id} \times_T \bar{\phi}_+$ because of its definition. Thus $s_E \circ r_E = 0$ implies this lemma. $\square$

Here we remark that $F^{\mathbf{f}}$ also is contained in $\mathbb{P}(\mathrm{Cok}(r_P)) \subset P^{\mathbf{f}} \times_T P^{-\mathbf{f}}$ by virtue of its definition and the lemma above.

Let us proceed to study a closed subscheme $p_-^{-1}(\Lambda_1 \cap \cdots \cap \Lambda_{d-1-M})$ of $F^{\mathbf{f}}$ in (7.12). We assume that $q(X) > 0$, (6.8), (6.9), (7.1) and (7.2).

Definition 8.3. Dualizing a canonical quotient $\mathcal{A}_+ \otimes_T \mathcal{O}_{P^{-\mathbf{f}}} \twoheadrightarrow \mathcal{O}_+(1)$, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_+(-1) \longrightarrow \mathcal{A}_+^\vee \otimes_T \mathcal{O}_{P^{-\mathbf{f}}} \longrightarrow \mathrm{Cok}_+ \longrightarrow 0 \tag{8.7}$$

on $P^{-\mathbf{f}}$. The *incidence subvariety* $\mathbf{D}^{-\mathbf{f}}$ of $\mathbb{P}(\mathcal{A}_+^\vee \otimes_T \mathcal{O}_{P^{-\mathbf{f}}}) = \mathbb{P}(\mathcal{A}_+^\vee) \times_T \mathbb{P}(\mathcal{A}_+)$ is a closed subscheme $\mathbb{P}(\mathrm{Cok}_+)$.

Lemma 8.4. *Suppose that the homomorphism* (7.14) *is surjective. Then a closed subscheme* $p_-^{-1}(\Lambda_1 \cap \cdots \Lambda_{d-1-M})$ *of* $\mathbb{P}(Ext^1_{X_T/T}(\mathcal{F}_0, \mathcal{G}_0)) \times_T P^{-\mathbf{f}}$ *is contained in the incidence variety* $\mathbf{D}^{-\mathbf{f}}$.

Proof. In the proof we shorten $\Lambda_1 \cap \cdots \cap \Lambda_{d-1-M}$ to Λ for simplicity. From the assumption we have the following commutative diagram of $\mathcal{O}_T$-modules:

$$\begin{tikzcd}
\mathcal{A}_+^\vee = Ext^1_{X_T/T}(\mathcal{F}_0, \mathcal{G}_0) \arrow[r, two heads] \arrow[d, "\otimes\kappa"'] & \mathrm{Cok}(\oplus \otimes \lambda_j) \\
\mathcal{A}_- = Ext^1_{X_T/T}(\mathcal{F}_0, \mathcal{G}_0(K_X)). \arrow[ur, two heads] &
\end{tikzcd} \tag{8.8}$$

This induces two closed immersions:

$$\begin{tikzcd}
\mathbb{P}(\mathcal{A}_+^\vee) & \mathbb{P}(\mathrm{Cok}(\oplus \otimes \lambda_j)) = \Lambda \arrow[l, hook', "i^\vee"'] \arrow[dl, hook', "i"] \\
\mathbb{P}(\mathcal{A}_-) = P^{\mathbf{f}}. &
\end{tikzcd} \tag{8.9}$$

Now we can find isomorphisms

$$j^\vee : i^{\vee *}\mathcal{O}_+^\vee(1) \longrightarrow i^*\mathcal{O}_-(1) \quad \text{and} \quad j : i^*\mathcal{O}_-(1) \longrightarrow \mathcal{O}_\Lambda(1)$$

such that

$$\begin{tikzcd}
\mathcal{A}_+^\vee \otimes_T \mathcal{O}_\Lambda \arrow[r, "\otimes\kappa"'] \arrow[d, two heads, "(i^\vee)^*\lambda_+^\vee"'] & \mathcal{A}_- \otimes_T \mathcal{O}_\Lambda \arrow[r, two heads] \arrow[d, two heads, "i^*(\lambda_-)"'] & \mathrm{Cok}(\oplus \otimes \lambda_j) \otimes_T \mathcal{O}_\Lambda \arrow[d, two heads, "\lambda_\Lambda"'] \\
(i^\vee)^*\mathcal{O}_+^\vee(1) \arrow[r, "j^\vee"'] & i^*\mathcal{O}_-(1) \arrow[r, "j"'] & \mathcal{O}_\Lambda(1)
\end{tikzcd} \tag{8.10}$$

is commutative, where λ_-, $\lambda_+^\vee$ and λ_Λ are the natural surjections on $P^{\mathbf{f}}$, $\mathbb{P}(\mathcal{A}_+^\vee)$ and Λ, respectively. Pull back this diagram by $\phi_- : E_\Lambda := \phi_-^{-1}(\Lambda) \to \Lambda$, which is a restriction of $\phi_- : E^{\mathbf{f}}_{\mathrm{red}} \to P^{\mathbf{f}}$. The following diagram on E_Λ is commutative:

$$
\begin{tikzcd}
\mathcal{O}_+(-1)|_{E_\Lambda} \arrow[r, "\mathrm{can}"'] \arrow[dd, "l"'] & Hom_{X_{E_\Lambda}/E_\Lambda}(\mathcal{F}_0 \otimes \mathcal{O}_+(1), \mathcal{F}_0) \arrow[d, "\otimes\kappa"'] \\
 & Hom_{X_{E_\Lambda}/E_\Lambda}(\mathcal{F}_0 \otimes \mathcal{O}_+(1), \mathcal{F}_0(K_X)) \arrow[d, "r'_E"'] \\
\mathcal{A}_+^\vee \otimes_T \mathcal{O}_{E_\Lambda} \arrow[r, "\otimes\kappa"] \arrow[d, "(i^\vee\circ\phi_-)^*\lambda_+^\vee"'] & \mathcal{A}_- \otimes_T \mathcal{O}_{E_\Lambda} = Ext^1_{X_{E_\Lambda}/E_\Lambda}(\mathcal{F}_0, \mathcal{G}_0(K_x)) \arrow[d, "(i\circ\phi_-)^*\lambda_-"'] \\
\mathcal{O}_+^\vee(1) \arrow[r, "\phi_-^*(j^\vee)"] & \mathcal{O}_-(1).
\end{tikzcd}
\tag{8.11}
$$

Here, l is the pull back of $\mathcal{O}_+(-1) \to \mathcal{A}_+^\vee \otimes_T \mathcal{O}_{P^{-\mathbf{f}}}$ in (8.7) by $E_\Lambda \hookrightarrow E^{\mathbf{f}} \to P^{-\mathbf{f}}$, r'_E is defined by using (8.6) similarly to r_P (8.3), and the lower diagram is obtained from the left side of (8.11). Then, $(i \circ \phi_-)^*\lambda_-$ in (8.11) coincides with the homomorphism $s'_E : Ext^1_{X_{E_\Lambda}/E_\Lambda}(\mathcal{F}_0, \mathcal{G}_0(K_X)) \twoheadrightarrow \mathcal{O}_-(1)$ defined by using (8.5) similarly to s_E in the proof of Lemma 8.4. Hence one can verify that $(i \circ \phi_-)^*\lambda_- \circ r'_E = 0$ in the same way as the proof of Lemma 8.4, which implies $(i^\vee \circ \phi_-)^*\lambda_+^\vee \circ l = 0$ in (8.11). This and (8.10) imply that

$$
\mathcal{O}_+(-1)|_{E_\Lambda} \xrightarrow{l} \mathcal{A}_+^\vee \otimes_T \mathcal{O}_{E_\Lambda} \twoheadrightarrow \mathrm{Cok}(\oplus \otimes \lambda_j) \otimes_T \mathcal{O}_{E_\Lambda} \xrightarrow{\phi_-^*(\lambda_\Lambda)} \mathcal{O}_\Lambda(1)
$$

is the zero map. By this we can conclude the proof. □

For the time being we suppose a homomorphism (7.14) is surjective. Moreover, we assume that $\dim F^{\mathbf{f}} = \dim E^{\mathbf{f}} = d-1$ since (7.7) is zero unless this holds good. Then, by the lemma above a subvariety $p_-^{-1}(\Lambda_1 \cap \cdots \cap \Lambda_{d-1-M})$ of $\mathbb{P}(\mathcal{A}_+^\vee) \times_T \mathbb{P}(\mathcal{A}_+)$ gives an algebraic cycle $\omega \in A^r(\mathbf{D}^{-\mathbf{f}})$ of the incidence variety $\mathbf{D}^{-\mathbf{f}}$ with $r = \mathrm{codim}(p_-^{-1}(\Lambda_1 \cap \cdots \cap \Lambda_{d-1-M}), \mathbf{D}^{-\mathbf{f}})$. $\mathbf{D}^{-\mathbf{f}}$ is nonsingular, so we can use the intersection theory of $\mathbf{D}^{\mathbf{f}}$. Because $\mathcal{O}_-(1)|_{\Lambda_1\cap\cdots\cap\Lambda_{d-1-M}} = \mathcal{O}_+^\vee(1)|_{\Lambda_1\cap\cdots\cap\Lambda_{d-1-M}}$ as mentioned in (8.10), one can verify that (7.12) is equal to

$$
\sum_{t=0}^{M} \deg\left(c_1(\mathcal{O}_+(1))^t \cdot c_1(\mathcal{O}_+^\vee(-1))^{M-t} \cdot \omega\right)_{\mathbf{D}^{-\mathbf{f}}}, \tag{8.12}
$$

where $(\ )_{\mathbf{D}^{\mathbf{f}}}$ designates the multiplication in the Chow ring $A(\mathbf{D}^{-\mathbf{f}})$. We shall omit $c_1(\ )$ from now on. Moreover, one can write $\omega \in A^r(\mathbf{D}^{-\mathbf{f}})$ as $\omega = \sum_{j=0}^{L_+-2} b_j\, \mathcal{O}_+^\vee(1)^j$ with some $b_j \in A^{r-j}(P^{-\mathbf{f}}) = A^{r-j}(\mathbb{P}(\mathcal{A}_+))$ because the sheaf Cok_+ in (8.7) is a vector bundle on $P^{-\mathbf{f}}$ whose rank is $L_+ - 1 = \mathrm{rk}\mathcal{A}_+^\vee - 1$. By (8.7), $(-1)^M$ times (8.12) is equal to

(8.13)

$$\begin{aligned}
&\deg \sum_{j=0}^{L_+-2} \left(b_j \cdot \mathcal{O}_+^\vee(1)^j \cdot \sum_{t=0}^{M} \mathcal{O}_+^\vee(1)^t \cdot \mathcal{O}_+(-1)^{M-t} \right)_{\mathbf{D}^{-\mathbf{f}}} \\
&\quad = \deg \sum_{j=0}^{L_+-2} \left(b_j \cdot \sum_{t=0}^{M+j} \mathcal{O}_+^\vee(1)^t \cdot \mathcal{O}_+(-1)^{M+j-t} \right)_{\mathbf{D}^{-\mathbf{f}}} \\
&\qquad - \deg \sum_{j=0}^{L_+-2} \left(b_j \cdot \sum_{t=0}^{j-1} \mathcal{O}_+^\vee(1)^t \cdot \mathcal{O}_+(-1)^{M+j-t} \right)_{\mathbf{D}^{-\mathbf{f}}} \\
&\quad = \deg \sum_{j=0}^{L_+-2} \left(b_j \cdot \sum_{t=0}^{M+j} \mathcal{O}_{\mathbb{P}(\mathrm{Cok}_+)}(1)^t \cdot \mathcal{O}_+(-1)^{M+j-t} \right)_{\mathbb{P}(\mathrm{Cok}_+)} \\
&\qquad - \deg \sum_{j=0}^{L_+-2} \left(b_j \cdot \mathcal{O}_+(-1)^{M-1} \cdot \sum_{t=0}^{j-1} \mathcal{O}_{\mathbb{P}(\mathrm{Cok}_+)}(1)^t \cdot \mathcal{O}_+(-1)^{j-1-t} \right)_{\mathbb{P}(\mathrm{Cok}_+)} \\
&\quad = \deg \sum_{j=0}^{L_+-2} \left(b_j \cdot \sum_{t=0}^{M-j} s_{t-(L_+-2)}((\mathrm{Cok}_+)^\vee) \cdot \mathcal{O}_+(-1)^{M+j-t} \right)_{P^{-\mathbf{f}}} \\
&\qquad - \deg \sum_{j=0}^{L_+-2} \left(b_j \cdot \mathcal{O}_+(-1)^{M-1} \cdot \sum_{t=0}^{j-1} s_{t-(L_+-2)}((\mathrm{Cok}_+)^\vee) \right. \\
&\qquad\qquad \left. \cdot\, \mathcal{O}_+(-1)^{j-1-t} \right)_{P^{-\mathbf{f}}} .
\end{aligned}$$

Here $s_l((\mathrm{Cok}_+)^\vee) \in A^l(P^{-\mathbf{f}})$ is the Segre class of a vector bundle $(\mathrm{Cok}_+)^\vee$ on $P^{-\mathbf{f}}$, which is explained in [Fu, Section 3.1].

In general, the Chern polynomial $c_t(\mathcal{V}) = \sum_{j=0}^{\infty} c_j(\mathcal{V})\, t^j$ of a vector bundle $\mathcal{V}$ satisfies that $c_t(\mathcal{V})^{-1} = \sum_{j=0}^{\infty} s_j(\mathcal{V})\, t^j$ as power serieses. Thus the dual of (8.7) tells us that

$$s_t((\mathrm{Cok}_+)^\vee) \cdot (1 + \sum_{j>0} \mathcal{O}_+(-1)^j\, t^j) = s_t(\mathcal{A}_+ \otimes_T \mathcal{O}_{P^{-\mathbf{f}}}).$$

In addition, $s_j(\mathcal{V}) = 0$ if $j < 0$. We see that (8.13) is equal to the degree of

$$\begin{aligned}
&\sum_{j=0}^{L_+-2} \big(b_j \cdot s_{M+j-(L_+-2)}(\mathcal{A}_+ \otimes_T \mathcal{O}_{P^{-\mathbf{f}}})\big)_{P^{-\mathbf{f}}} \\
&\quad - \sum_{j=0}^{L_+-2} \big(b_j \cdot \mathcal{O}_+(-1)^{M+1} \cdot s_{j-1-(L_+-2)}(\mathcal{A}_+ \otimes_T \mathcal{O}_{P^{-\mathbf{f}}})\big)_{P^{-\mathbf{f}}} \qquad (8.14)\\
&\quad = \sum_{j=0}^{L_+-2} \big(b_j \cdot s_{M+j-(L_+-2)}(\mathcal{A}_+ \otimes_T \mathcal{O}_{P^{-\mathbf{f}}})\big)_{P^{-\mathbf{f}}} ,
\end{aligned}$$

taking into account that $j-1-(L_+-2)<0$. Since $\mathcal{A}_+$ is a vector bundle on T, $s_{M+j-(L_+-2)}(\mathcal{A}_+\otimes_T \mathcal{O}_{P^{-\mathbf{f}}})=0$ provided that $M+j-(L_+-2)\geq M-(L_+-2)>\dim T$. Therefore we obtain the following proposition as a result of (7.8), Proposition 7.1, (8.12), (8.14), etc.

Proposition 8.5. *If $d-1-M\geq l_-+\dim T$ and $M-(L_+-2)>\dim T$, then the first term $\sum_{s=0}^{M}$ in* (7.8) *is zero.*

Furthermore, the second term $\sum_{s=M+1}^{d-1-t}$ of (7.8),

$$\left[\beta^t\cdot\mathcal{O}_+(1)^{M+1}\cdot\sum_{s=0}^{d-2-t-M}\mathcal{O}_+(1)^s\cdot\mathcal{O}_-(-1)^{d-2-t-s-m}\right]_{F^{\mathbf{f}}},\tag{8.15}$$

clearly is zero if $M+1>\dim P^{-\mathbf{f}}=L_+-1+\dim T$.

Proposition 8.6. *The contribution of $E^{\mathbf{f}}$ to $\mu_-(C)^{d(c_2)}-\mu_+(C)^{d(c_2)}$, that is* (7.4), *is equal to zero if $d\geq L_+-l_-+2\dim T$.*

Proof. By Proposition 8.5 and (8.15), (7.8) is equal to zero if $M\leq d-1-l_--\dim T$ and $M>L_+-2+\dim T$. One can find such an integer M if $d-1-l_--\dim T>L_+-2+\dim T$. □

As observed before Claim 7.3,

$$\begin{aligned}l_-&=L_--L_+-\hom_X(I_{Z_1},\mathcal{O}(c_1-2L+K_X)\otimes I_{Z_2})\\&\quad+h^0(\mathcal{O}(c_1-2L+K_X)|_{\mathcal{K}})\\&=-\chi(I_{Z_1},\mathcal{O}(c_1-2L+K_X)\otimes I_{Z_2})-L_++h^0(\mathcal{O}(c_1-2L+K_X)|_{\mathcal{K}}),\end{aligned}$$

where L and Z_i satisfies that $([2L-c_1],l(Z_1),l(Z_2))=\mathbf{f}=(f,m,n)$. From the Riemann-Roch theorem and Clifford's theorem [H2, Theorem IV.5.4], we deduce that

$$\begin{aligned}l_-+L_+&=-f\cdot(f-K_X)/2+(m+n)-\chi(\mathcal{O}_X)+h^0(\mathcal{O}(c_1-2L+K_X)|_{\mathcal{K}})\\&\leq-f\cdot(f-K_X)/2+(m+n)-\chi(\mathcal{O}_X)\\&\quad+\max(-K_X\cdot f,(K_X-f)\cdot K_X/2+1,0).\end{aligned}$$

On the other hand, $\dim T=\dim(\mathrm{Pic}(X)\times\mathrm{Hilb}^m(X)\times\mathrm{Hilb}^n(X))=2(m+n)$ and $m+n=c_2+(f^2-c_1^2)/4$ since $(f,m,n)\in A^+(a)$. Therefore one can verify

$$\begin{aligned}&d-(L_++l_-+2\dim T)\\&\qquad\geq-c_2-(3/4)f^2-2\chi(\mathcal{O}_X)+\min(\pm K_X\cdot f/2,-K_X^2/2-1).\end{aligned}$$

Now fix a compact subset $\mathcal{S}$ in $\mathrm{Amp}(X)$. Then one can find a constant $d_0(\mathcal{S})$ depending on $\mathcal{S}$ such that $|f\cdot K_X|\leq d_0(\mathcal{S})\cdot\sqrt{-f^2}$ if $W^f\cap\mathcal{S}\neq\emptyset$, as shown in the proof of [Q1, Lemma 2.1]. Hence one can find constants $d_1(\mathcal{S})$ and $d_2(\mathcal{S})$ depending on $\mathcal{S}$ such that if $-f^2>(4/3)c_2+d_1(\mathcal{S})\sqrt{c_2}+d_2(\mathcal{S})$, then (8) is greater than zero.

Therefore we arrive at Theorem 0.2 in Introduction, which is the observation of Proposition 0.1 in algebro-geometric view.

Remark 8.7. Suppose that X is $K3$ surface and that assumptions (6.8) and (6.9) hold good for $(0, c_2)$. Then (7.1) and (7.2) are always valid, and furthermore, the homomorphism (7.14) is always surjective. (It is not necessary to assume that $d-1-M \geq l_- + \dim T$.) Thus one can prove $\gamma_{H_-}(c_2) = \gamma_{H_+}(c_2)$.

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References

[Br] J. Briançon, *Description de* $H\mathrm{ilb}^n C\{x, y\}$, Invent. Math. **41**-1 (1977), 45–89.

[Co] B. Conrad, *Grothendieck duality and base change*, Lecture Notes in Math. **1750**, Springer-Verlag, Berlin, 2000.

[EG] G. Ellingsrud and L. Göttsche, *Variation of moduli spaces and Donaldson invariants under change of polarization*, J. Reine Angew. Math. **467** (1995), 1–49.

[ES] G. Ellingsrud and S. A. Strømme, *An intersection number for the punctual hilbert scheme of a surface*, Trans. Amer. Math. Soc. **350** (1998), 2547–2552.

[Fo] J. Fogarty, *Algebraic families on an algebraic surface*, Amer. J. Math. **90** (1968), 511–521.

[Fr] R. Friedman, *Vector bundles and* so(3)*-invariants for elliptic surfaces*, J. Amer. Math. Soc. **8**.

[FM] R. Friedman and J. W. Morgan, *Smooth four-manifolds and complex surfaces*, Springer-Verlag, Berlin, 1994.

[FQ] R. Friedman and Z. Qin, *Flips of moduli spaces and transition formulas for Donaldson polynomial invariants of rational surfaces*, Comm. Anal. Geom. **3**-1-2 (1995), 11–83.

[Fu] W. Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics **2**, Springer-Verlag, Berlin, 1998.

[GL] D. Gieseker and J. Li, *Irreducibility of moduli of rank-*2 *vector bundles on algebraic surfaces*, J. Differential Geom. **40**-1 (1994), 23–104.

[EGA] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique II*, Inst. Hautes Études Sci. Publ. Math. **8** (1961).

[H1] R. Hartshorne, *Residues and duality*, Lecture Notes in Math. **20**, Springer-Verlag, Berlin, 1966.

[H2] ———, *Algebraic geometry*, Grad. Texts in Math. Springer-Verlag, New York, **52**, 1977.

[HL] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, Friedr, Vieweg & Sohn, 1997.

[KL] S. L. Kleiman, *Toward a numerical theory of ampleness*, Ann. of Math. (2) **84** (1966), 293–344.

[La] H. Lange, *Universal families of extensions*, J. Algebra **83**-1 (1983), 101–112.

[Li] J. Li, *Algebraic geometric interpretation of Donaldson's polynomial invariants*, J. Differential Geom. **37**-2 (1993), 417–466.

[Ma] M. Maruyama, *Moduli of stable sheaves. II*, J. Math. Kyoto Univ. **18**-3 (1978), 557–614.

[MW] K. Matsuki and R. Wentworth, *Mumford-Thaddeus principle on the moduli space of vector bundles on an algebraic surface*, Internat. J. Math. **8**-1 (1997), 97–148.

[Mo] T. Mochizuki, *The theory of the invariants obtained from the moduli stack of stable objects on a smooth polarized surface*, presented in http://www.math.ias.edu/~takuro/list.html.

[M1] D. Mumford, *Abelian varieties*, Published for the Tata Institute of Fundamental Research, Bombay, 1970.

[M2] D. Mumford, J. Fogarty and F. Kirwan, *Geometric invariant theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (2) **34**, Springer-Verlag, 1994.

[Q1] Z. Qin, *Birational properties of moduli spaces of stable locally free rank-2 sheaves on algebraic surfaces*, Manuscripta Math. **72**-2 (1991), 163–180.

[Q2] ———, *Equivalence classes of polarizations and moduli spaces of sheaves*, J. Differential Geom. **37**-2 (1993), 397–415.

[Se] C. S. Seshadri, *Geometric reductivity over arbitrary base*, Adv. in Math. **26** (1977), 225–274.

[Yk] K. Yokogawa, *Compactification of moduli of parabolic sheaves and moduli of parabolic Higgs sheaves*, J. Math. Kyoto Univ. **33**-2 (1993), 451–504.

[Zu] K. Zuo, *Generic smoothness of the moduli spaces of rank two stable vector bundles over algebraic surfaces*, Math. Z. **207**-4 (1991), 629–643.