Illinois Journal of Mathematics
Volume 61, Numbers 1–2, Spring and Summer 2017, Pages 171–186
S 0019-2082

EVALUATION OF TORNHEIM'S TYPE OF DOUBLE SERIES

SHIN-YA KADOTA, TAKUYA OKAMOTO AND KOJI TASAKA

ABSTRACT. We examine values of certain Tornheim's type of double series with odd weight. As a result, an affirmative answer to a conjecture about the parity theorem for the zeta function of the root system of the exceptional Lie algebra G_2, proposed by Komori, Matsumoto and Tsumura, is given.

1. Introduction and main theorem

For integers a, b, k_1, k_2, $k_3 \geq 1$, let

$$\zeta_{a,b}(k_1,k_2,k_3) := \sum_{m,n>0} \frac{1}{m^{k_1} n^{k_2} (am+bn)^{k_3}},$$

which converges absolutely and gives a real number. Since Tornheim [12] first studied the value $\zeta_{1,1}(k_1,k_2,k_3)$, we call the value $\zeta_{a,b}(k_1,k_2,k_3)$ Tornheim's type of double series (note that the function $\zeta_{a,b}(s_1,s_2,s_3)$ with $s_i \in \mathbb{C}$ can be viewed as a special case of the Shintani zeta function, but we will focus on its special values). In [8], the second author examined the values $\zeta_{a,b}(k_1,k_2,k_3)$ in the study of evaluations of special values of the zeta functions of root systems associated with A_2, B_2 and G_2. The goal was to express the special values of the zeta functions of root systems as $\mathbb{Q}$-linear combinations of two products of certain zeta values. As a prototype, we have in mind the analogous story for the parity theorem for multiple zeta values [3, Corollary 8] (see also [15]) and for Tornheim's series [2, Theorem 2] (see also [16]). For example, the identity

$$\zeta_{1,1}(1,1,3) = 4\zeta(5) - 2\zeta(2)\zeta(3)$$

is well known. Similar studies have been done in many articles [7], [11], [13], [14], [16], [17], [19] (see also [9]). In this paper, we will generalize the above

Received March 29, 2017; received in final form September 3, 2017.
This work was partially supported by JSPS KAKENHI Grant Numbers 15K17517 and 16H07115.
2010 *Mathematics Subject Classification.* Primary 11M32. Secondary 40B05.

expression to the value $\zeta_{a,b}(k_1,k_2,k_3)$ with $k_1+k_2+k_3$ odd. As a consequence, we give an affirmative answer to a conjecture about special values of the zeta function of the root system of G_2, which was proposed by Komori, Matsumoto and Tsumura [5, Eq. (7.1)].

We now state our main result. We use the Clausen-type functions defined for a positive integer $k\geq 2$ and $x\in\mathbb{R}$ by

$$\begin{aligned} C_k(x) &:= \operatorname{Re} Li_k\big(e^{2\pi ix}\big) = \sum_{m>0}\frac{\cos(2\pi mx)}{m^k},\\ S_k(x) &:= \operatorname{Im} Li_k\big(e^{2\pi ix}\big) = \sum_{m>0}\frac{\sin(2\pi mx)}{m^k}, \end{aligned} \tag{1}$$

where $Li_k(z)$ is the polylogarithm $\sum_{m>0}\frac{z^m}{m^k}$. Note that $C_k(x)$ equals the Riemann zeta value $\zeta(k):=\sum_{m>0}\frac{1}{m^k}$ when $x\in\mathbb{Z}$, and $S_k(x)$ is 0 when $x\in\frac{1}{2}\mathbb{Z}$.

THEOREM 1. *For positive integers N, a, b, k, k_1, k_2, k_3 with $N=\operatorname{lcm}(a,b)$ and $k=k_1+k_2+k_3$ odd, the value $\zeta_{a,b}(k_1,k_2,k_3)$ can be expressed as $\mathbb{Q}$-linear combinations of $\pi^{2n}C_{k-2n}(\frac{d}{N})$ and $\pi^{2n+1}S_{k-2n-1}(\frac{d}{N})$ for $0\leq n\leq\frac{k-3}{2}$ and $d\in\mathbb{Z}/N\mathbb{Z}$.*

Theorem 1 will be proved in Section 4 by using the generating functions. This leads to a recipe for giving a formula for the $\mathbb{Q}$-linear combination in Theorem 1. More precisely, one can deduce an explicit formula from Corollary 3 and Propositions 4, 7 and 8, but it might be much complicated (we do not develop the explicit formulas in this paper). As an example of a simple identity, we have

$$\zeta_{1,3}(1,1,3)=\frac{1}{81}\left(367\zeta(5)-19\pi^2\zeta(3)-27\pi S_4\left(\frac{1}{3}\right)-4\pi^3S_2\left(\frac{1}{3}\right)\right). \tag{2}$$

We apply Theorem 1 to proving the conjecture suggested by Komori, Matsumoto and Tsumura [5, Eq. (7.1)]. This will be described in Section 5.

It is worth mentioning that since the value $\zeta_{a,b}(k_1,k_2,k_3)$ can be expressed as $\mathbb{Q}$-linear combinations of double polylogarithms

$$Li_{k_1,k_2}(z_1,z_2)=\sum_{0<m<n}\frac{z_1^m z_2^n}{m^{k_1}n^{k_2}}, \tag{3}$$

Theorem 1 might be proved by the parity theorem for double polylogarithms obtained by Panzer [10] and Nakamura [7], which is illustrated in Remark 2. In this paper, we however do not use their result to prove Theorem 1, since we want to keep this paper self-contained.

The contents of this paper are as follows. In Section 2, we give an integral representation of the generating function of the values $\zeta_{a,b}(k_1,k_2,k_3)$ for any integers a, $b\geq 1$. In Section 3, the integral is computed. Section 4 gives a proof of Theorem 1.1. In Section 5, we recall the question [5, Eq. (7.1)] and give an affirmative answer to this.

2. Integral representation

In this section, we give an integral representation of the generating function of the values $\zeta_{a,b}(k_1,k_2,k_3)$ for any integers $a,\ b\geq 1$. The integral representation of the value $\zeta_{a,b}(k_1,k_2,k_3)$ was first given by the second author [8, Theorem 4.4], following the method used by Zagier (see also [6]). We recall it briefly.

For an integer $k\geq 0$, the Bernoulli polynomial $B_k(x)$ of order k is defined by

$$\sum_{k\geq 0} B_k(x)\frac{t^k}{k!}=\frac{te^{xt}}{e^t-1}.$$

The polynomial $B_k(x)$ admits the following expression (see [1, Theorem 4.11]): for $k\geq 1$ and $x\in\mathbb{R}$ ($x\in\mathbb{R}-\mathbb{Z}$, if $k=1$)

$$B_k\bigl(x-[x]\bigr)=\begin{cases}-2i\frac{k!}{(2\pi i)^k}\sum_{m>0}\frac{\sin(2\pi mx)}{m^k}, & k\geq 1:\text{odd},\\ -2\frac{k!}{(2\pi i)^k}\sum_{m>0}\frac{\cos(2\pi mx)}{m^k}, & k\geq 2:\text{even},\end{cases}$$

where $i=\sqrt{-1}$ and the summation $\sum_{m>0}$ is regarded as $\lim_{N\to\infty}\sum_{N>m>0}$ when $k=1$ (this ensures convergence). We define the modified (generalized) Clausen function for $k\geq 1$ and $x\in\mathbb{R}$ ($x\in\mathbb{R}-\mathbb{Z}$, if $k=1$) by

$$Cl_k\bigl(x-[x]\bigr)=\begin{cases}-\frac{k!}{(2\pi i)^{k-1}}\sum_{m>0}\frac{\cos(2\pi mx)}{m^k}, & k\geq 1:\text{odd},\\ -i\frac{k!}{(2\pi i)^{k-1}}\sum_{m>0}\frac{\sin(2\pi mx)}{m^k}, & k\geq 2:\text{even}.\end{cases}$$

With this, for $k\geq 1$ and $x\in\mathbb{R}$ ($x\in\mathbb{R}-\mathbb{Z}$ if $k=1$), the polylogarithm $Li_k(e^{2\pi ix})$ can be written in the form

$$Li_k\bigl(e^{2\pi ix}\bigr)=-\frac{(2\pi i)^{k-1}}{k!}\Bigl(Cl_k\bigl(x-[x]\bigr)+\pi iB_k\bigl(x-[x]\bigr)\Bigr). \tag{4}$$

We introduce formal generating functions. For $x\in\mathbb{R}-\mathbb{Z}$, let

$$\beta(x;t):=\sum_{k>0}\frac{B_k(x-[x])t^k}{k!}\quad\text{and}\quad\gamma(x;t):=\sum_{k>0}\frac{Cl_k(x-[x])t^k}{k!}.$$

Proposition 2. *For integers $a,\ b\geq 1$, we have*

$$\begin{aligned}&\sum_{k_1,k_2,k_3>0}\zeta_{a,b}(k_1,k_2,k_3)t_1^{k_1}t_2^{k_2}t_3^{k_3}\\&\quad=-\frac{1}{4\pi i}\int_0^1\bigl(\gamma(ax;2\pi it_1)\beta(bx;2\pi it_2)+\beta(ax;2\pi it_1)\gamma(bx;2\pi it_2)\bigr)\\&\qquad\times\beta(x;-2\pi it_3)\,dx\end{aligned}$$

$$+\frac{1}{4\pi^2}\int_0^1 \big(\gamma(ax;2\pi i t_1)\gamma(bx;2\pi i t_2)-\pi^2\beta(ax;2\pi i t_1)\beta(bx;2\pi i t_2)\big)$$
$$\times\,\beta(x;-2\pi i t_3)\,dx,$$

where the integrals on the right-hand side are defined formally by term-by-term integration.

Proof. When $k_1,\ k_2,\ k_3\geq 2$, it follows that

$$\int_0^1 Li_{k_1}\big(e^{2\pi i a x}\big)Li_{k_2}\big(e^{2\pi i b x}\big)\overline{Li_{k_3}\big(e^{2\pi i x}\big)}\,dx$$
$$=\int_0^1\sum_{m,n,l>0}\frac{e^{2\pi i m a x}e^{2\pi i n b x}e^{-2\pi i l x}}{m^{k_1}n^{k_2}l^{k_3}}\,dx$$
$$=\sum_{m,n,l>0}\frac{1}{m^{k_1}n^{k_2}l^{k_3}}\int_0^1 e^{2\pi i x(am+bn-l)}\,dx=\zeta_{a,b}(k_1,k_2,k_3),$$

where $\overline{Li_{k_3}(e^{2\pi i x})}$ stands for complex conjugate of $Li_{k_3}(e^{2\pi i x})$. For $k_1,\ k_2,\ k_3\geq 1$, the above equality is justified by replacing the integral $\int_0^1$ with

$$\lim_{\varepsilon\to 0}\sum_{j=1}^{\operatorname{lcm}(a,b)}\int_{\frac{j-1}{\operatorname{lcm}(a,b)}+\varepsilon}^{\frac{j}{\operatorname{lcm}(a,b)}-\varepsilon}, \tag{5}$$

where $\operatorname{lcm}(a,b)$ is the least common multiple of a and b (see [8, Theorem 4.4] for the details). Letting $Li(x;t):=\sum_{k>0}Li_k(e^{2\pi i x})t^k$, we therefore obtain

$$\sum_{k_1,k_2,k_3>0}\zeta_{a,b}(k_1,k_2,k_3)t_1^{k_1}t_2^{k_2}t_3^{k_3} =\int_0^1 Li(ax;t_1)Li(bx;t_2)\overline{Li(x;t_3)}\,dx. \tag{6}$$

Furthermore, the generating function of $Li_k(e^{2\pi i x})$ with $x\in\mathbb{R}-\mathbb{Z}$ can be written in the form

$$Li(x;t)=-\frac{1}{2\pi i}\big(\gamma(x;2\pi i t)+\pi i\beta(x;2\pi i t)\big), \tag{7}$$

and hence, the right-hand side of (6) is equal to

$$\frac{1}{(2\pi i)^3}\int_0^1\big(\gamma(ax;2\pi i t_1)+\pi i\beta(ax;2\pi i t_1)\big) \times\big(\gamma(bx;2\pi i t_2)+\pi i\beta(bx;2\pi i t_2)\big)\big(\gamma(x;-2\pi i t_3)-\pi i\beta(x;-2\pi i t_3)\big)\,dx. \tag{8}$$

We note that, similarly to (6), one obtains the relation

$$\int_0^1 Li(ax;t_1)Li(bx;t_2)Li(x;-t_3)\,dx=0,$$

and substituting (7) to the above identity, one has

$$\begin{aligned}&\int_0^1 \big(\gamma(ax;2\pi it_1)+\pi i\beta(ax;2\pi it_1)\big)\big(\gamma(bx;2\pi it_2)+\pi i\beta(bx;2\pi it_2)\big)\\&\qquad\times\gamma(x;-2\pi it_3)\,dx\\&\quad=-\pi i\int_0^1 \big(\gamma(ax;2\pi it_1)\\&\qquad+\pi i\beta(ax;2\pi it_1)\big)\big(\gamma(bx;2\pi it_2)+\pi i\beta(bx;2\pi it_2)\big)\beta(x;-2\pi it_3)\,dx.\end{aligned}$$

With this, (8) is reduced to

$$\begin{aligned}&-\frac{1}{(2\pi i)^2}\int_0^1 \big(\gamma(ax;2\pi it_1)+\pi i\beta(ax;2\pi it_1)\big)\big(\gamma(bx;2\pi it_2)+\pi i\beta(bx;2\pi it_2)\big)\\&\quad\times\beta(x;-2\pi it_3)\,dx,\end{aligned}$$

which completes the proof. □

The coefficient of t^k in $\gamma(x;2\pi it)$ (resp. $\beta(x;2\pi it)$) is a real-valued function, if k is even, and a real-valued function times $i=\sqrt{-1}$, if k is odd. Thus, comparing the coefficient of both sides, we have the following corollary. For simplicity, for integers a, $b\geq 1$ we let

$$F_{a,b}(t_1,t_2,t_3):=\int_0^1 \gamma(ax;t_1)\beta(bx;t_2)\beta(x;-t_3)\,dx, \tag{9}$$

where the integral is defined formally by term-by-term integration and by (5).

Corollary 3. *One has*

$$\begin{aligned}&\sum_{\substack{k_1,k_2,k_3>0\\k_1+k_2+k_3:\mathrm{odd}}} \zeta_{a,b}(k_1,k_2,k_3)t_1^{k_1}t_2^{k_2}t_3^{k_3}\\&\quad=-\frac{1}{4\pi i}F_{a,b}(2\pi it_1,2\pi it_2,2\pi it_3)-\frac{1}{4\pi i}F_{b,a}(2\pi it_2,2\pi it_1,2\pi it_3).\end{aligned}$$

Remark that, using the same method, one can give an integral expression of the generating function of the Riemann zeta values, which will be used later.

Proposition 4. *For integers a, $b\geq 1$, we have*

$$\begin{aligned}&\frac{1}{2\pi i}\int_0^1 \gamma(ax;2\pi it_1)\beta(bx;-2\pi it_2)\,dx\\&\quad=\sum_{\substack{r,s>0\\r+s:\mathrm{odd}}}\frac{\gcd(a,b)^{r+s}}{a^s b^r}\zeta(r+s)t_1^r t_2^s.\end{aligned} \tag{10}$$

Proof. Let $d=\gcd(a,b)$ and set $a=a'd, b=b'd$. It follows that

$$\begin{aligned}\int_0^1 & Li_r\left(e^{2\pi iax}\right)\overline{Li_s\left(e^{2\pi ibx}\right)}\,dx\\ &=\sum_{m,n>0}\frac{1}{m^rn^s}\int_0^1 e^{2\pi ix(am-bn)}\,dx\\ &=\sum_{\substack{m,n>0\\ m=\frac{b'}{a'}n}}\frac{1}{m^rn^s}=\left(\frac{a'}{b'}\right)^r\sum_{\substack{n>0\\ a'|n}}\frac{1}{n^{r+s}}\\ &=\frac{1}{a'^sb'^r}\zeta(r+s).\end{aligned}$$

Hence, we have

$$\int_0^1 Li(ax;t_1)\overline{Li(bx;t_2)}\,dx=\sum_{r,s>0}\frac{\gcd(a,b)^{r+s}}{a^sb^r}\zeta(r+s)t_1^rt_2^s.$$

By the relation $\int_0^1 Li(ax;t_1)Li(bx;-t_2)\,dx=0$ (a, $b\geq 1$) and (7), the left-hand side of the above equation can be reduced to

$$\frac{1}{2\pi i}\int_0^1\big(\gamma(ax;2\pi it_1)+\pi i\beta(ax;2\pi it_1)\big)\beta(bx;-2\pi it_2)\,dx.$$

Comparing the coefficients of $t_1^rt_2^s$, we complete the proof. □

3. Evaluation of integrals

In this section, we compute the integral $F_{a,b}(t_1,t_2,t_3)$.

We denote the generating function of the Bernoulli polynomials by $\beta_0(x;t)$:

$$\beta_0(x;t):=\frac{te^{xt}}{e^t-1}=\sum_{k\geq 0}B_k(x)\frac{t^k}{k!}.$$

For integers b, $c\geq 1$, we set

$$\begin{aligned}\alpha_b(t_1,t_2)&:=\beta_0(0;t_1)\beta_0(0;-t_2)\frac{e^{bt_1-t_2}-1}{bt_1-t_2},\\ \widetilde{\alpha}_{b,c}(t_1,t_2)&:=-t_1e^{-ct_1}\beta_0(0;-t_2)\frac{e^{bt_1-t_2}-1}{bt_1-t_2},\end{aligned}$$

which are elements in the formal power series ring $\mathbb{Q}[[t_1,t_2]]$.

Lemma 5. *For any integers b, $d\geq 1$, we have*

$$e^{-dt_1}\alpha_b(t_1,t_2)=\alpha_b(t_1,t_2)+\sum_{c=1}^{d}\widetilde{\alpha}_{b,c}(t_1,t_2).$$

Proof. By the relation $B_k(x) = B_k(x+1) - kx^{k-1}$ for $k \in \mathbb{Z}_{\geq 0}$ (see [1, Proposition 4.9 (2)]), we have $\beta_0(x;t) = \beta_0(x+1;t) - te^{xt}$. Using this formula with $x = -d, -d+1, \ldots, 1$ repeatedly, one gets

$$\beta_0(-d;t) = \beta_0(-d+1;t) - te^{-dt} = \cdots = \beta_0(0;t) - t\sum_{c=1}^{d} e^{-ct}.$$

Hence, we obtain

$$\begin{aligned} e^{-dt_1}\alpha_b(t_1,t_2) &= \beta_0(-d;t_1)\beta_0(0;-t_2)\frac{e^{bt_1-t_2}-1}{bt_1-t_2} \\ &= \alpha_b(t_1,t_2) - t_1\sum_{c=1}^{d} e^{-ct_1}\beta_0(0;-t_2)\frac{e^{bt_1-t_2}-1}{bt_1-t_2} \\ &= \alpha_b(t_1,t_2) + \sum_{c=1}^{d}\widetilde{\alpha}_{b,c}(t_1,t_2), \end{aligned}$$

which completes the proof. □

REMARK 1. Let us denote by $A_b(r,s)$ (resp. $\widetilde{A}_{b,c}(r,s)$) the coefficient of $t_1^r t_2^s$ in $\alpha_b(t_1,t_2)$ (resp. in $\widetilde{\alpha}_{b,c}(t_1,t_2)$). Then, we have

$$A_b(r,s) = \sum_{\substack{p_1+q_1=r \\ p_2+q_2=s \\ p_1,p_2,q_1,q_2\geq 0}} \frac{(-1)^{q_2+p_2} b^{p_1} B_{q_1} B_{q_2}}{p_1!p_2!q_1!q_2!(p_1+p_2+1)}$$

and

$$\widetilde{A}_{b,c}(r,s) = \sum_{\substack{p_1+q_1=r \\ p_2+q_2=s \\ p_1,p_2,q_2\geq 0 \\ q_1\geq 1}} \frac{(-1)^{q_1+q_2+p_2} c^{q_1-1} b^{p_1} B_{q_2}}{p_1!(q_1-1)!p_2!q_2!(p_1+p_2+1)},$$

where $B_k = B_k(1) = (-1)^k B_k(0)$ is the kth Bernoulli number. We note that since $\widetilde{\alpha}_{b,c}(t_1,t_2) \in t_1\mathbb{Q}[[t_1,t_2]]$, we have $\widetilde{A}_{b,c}(0,s) = 0$ for any $s \in \mathbb{Z}_{\geq 0}$.

LEMMA 6. *Let b, d be positive integers with $d \in \{0,1,\ldots,b-1\}$. Then, for $x \in (\frac{d}{b}, \frac{d+1}{b})$, we have*

$$\beta(bx;t_1)\beta(x;-t_2) = e^{-dt_1}\alpha_b(t_1,t_2)\beta_0(x;bt_1-t_2) - \beta(bx;t_1) - \beta(x;-t_2) - 1,$$

where we recall $\beta(x;t) = \sum_{k>0} \frac{B_k(x-[x])}{k!}t^k$.

Proof. Since $bx - [bx] = bx - d$ when $x \in (\frac{d}{b}, \frac{d+1}{b})$, one has

$$\begin{aligned} &\big(\beta(bx;t_1)+1\big)\big(\beta(x;-t_2)+1\big) \\ &\quad = \frac{t_1 e^{(bx-d)t_1}}{e^{t_1}-1}\,\frac{-t_2 e^{-xt_2}}{e^{-t_2}-1} \end{aligned}$$

$$
\begin{aligned}
&= e^{-dt_1} \frac{t_1}{e^{t_1}-1} \frac{-t_2}{e^{-t_2}-1} e^{(bt_1-t_2)x} \\
&= e^{-dt_1} \beta_0(0;t_1)\beta_0(0;-t_2) \frac{e^{bt_1-t_2}-1}{bt_1-t_2} \frac{(bt_1-t_2)e^{(bt_1-t_2)x}}{e^{bt_1-t_2}-1} \\
&= e^{-dt_1} \alpha_b(t_1,t_2)\beta_0(x;bt_1-t_2),
\end{aligned}
$$

from which the statement follows. □

PROPOSITION 7. *For any integers a, $b \geq 1$, we have*

$$
\begin{aligned}
F_{a,b}(t_1,t_2,t_3) = \alpha_b(t_2,t_3) &\int_0^1 \gamma(ax;t_1)\beta_0(x;bt_2-t_3)\,dx \\
&+ \sum_{c=1}^{b-1} \widetilde{\alpha}_{b,c}(t_2,t_3) \int_{\frac{c}{b}}^1 \gamma(ax;t_1)\beta_0(x;bt_2-t_3)\,dx \\
&- \int_0^1 \gamma(ax;t_1)\big(\beta(bx;t_2)+\beta(x;-t_3)\big)\,dx.
\end{aligned} \tag{11}
$$

Proof. Splitting the integral $\int_0^1 = \sum_{d=0}^{b-1} \int_{\frac{d}{b}}^{\frac{d+1}{b}}$ in the definition of $F_{a,b}$ (see (9)) and then using Lemma 6, we have

$$
\begin{aligned}
&F_{a,b}(t_1,t_2,t_3) \\
&\quad = \sum_{d=0}^{b-1} \int_{\frac{d}{b}}^{\frac{d+1}{b}} \gamma(ax;t_1)\beta(bx;t_2)\beta(x;-t_3)\,dx \\
&\quad = \sum_{d=0}^{b-1} e^{-dt_2}\alpha_b(t_2,t_3) \int_{\frac{d}{b}}^{\frac{d+1}{b}} \gamma(ax;t_1)\beta_0(x;bt_2-t_3)\,dx \\
&\qquad - \sum_{d=0}^{b-1} \int_{\frac{d}{b}}^{\frac{d+1}{b}} \gamma(ax;t_1)\big(\beta(bx;t_2)+\beta(x;-t_3)+1\big)\,dx \\
&\quad = \sum_{d=0}^{b-1} \left(\alpha_b(t_2,t_3) + \sum_{c=1}^{d} \widetilde{\alpha}_{b,c}(t_2,t_3)\right) \int_{\frac{d}{b}}^{\frac{d+1}{b}} \gamma(ax;t_1)\beta_0(x;bt_2-t_3)\,dx \\
&\qquad - \int_0^1 \gamma(ax;t_1)\big(\beta(bx;t_2)+\beta(x;-t_3)+1\big)\,dx,
\end{aligned}
$$

where for the last equality we have used Lemma 5. Since $\int_0^1 Li(ax;t)\,dx = 0$ holds, we have

$$
\int_0^1 \gamma(ax;t_1)\,dx = 0. \tag{12}
$$

Hence, the statement follows from and the interchange of order of summation $\sum_{d=1}^{b-1}\sum_{c=1}^{d} = \sum_{c=1}^{b-1}\sum_{d=c}^{b-1}$. □

We now deal with the integral of the second term of the right-hand side of (11).

PROPOSITION 8. *For any integers* a, $b \geq 1$ *and* $c \in \{0, 1, \ldots, b-1\}$, *we have*

$$\frac{1}{2\pi i}\int_{\frac{c}{b}}^{1} \gamma(ax; 2\pi i t_1)\beta_0\big(x; 2\pi i(bt_2 - t_3)\big)\, dx$$

$$= -i \sum_{\substack{s \geq 1 \\ p,q \geq 0 \\ p+s:\text{odd}}} \frac{(-1)^s (2\pi i)^{q-1}}{q! a^s} S_{p+s+1}\left(\frac{ac}{b}\right) B_q\left(\frac{c}{b}\right) t_1^{p+1} (bt_2 - t_3)^{q+s-1}$$

$$+ \sum_{\substack{s \geq 1 \\ p,q \geq 0 \\ p+s:\text{even}}} \frac{(-1)^s (2\pi i)^{q-1}}{q! a^s} \left(\zeta(p+s+1) B_q - C_{p+s+1}\left(\frac{ac}{b}\right) B_q\left(\frac{c}{b}\right)\right)$$

$$\times t_1^{p+1} (bt_2 - t_3)^{q+s-1},$$

where $S_n(x)$ *and* $C_n(x)$ *are defined in* (1).

Proof. For an integer $s \geq 1$, we let

$$\gamma_s(x; t) = \sum_{k \geq s} \frac{Cl_k(x - [x])}{k!} t^k.$$

It is easily seen that for any integer $s \geq 2$ we have

$$\frac{d}{dx}\gamma_s(ax; t) = at\gamma_{s-1}(ax; t) \quad \text{and} \quad \frac{d}{dx}\beta_0(x; t) = t\beta_0(x; t).$$

By repeated use of the integration by parts and noting that $\gamma_1(x; t) = \gamma(x; t)$, we have

$$\int_{\frac{c}{b}}^{1} \gamma(ax; 2\pi i t_1)\beta_0\big(x; 2\pi i(bt_2 - t_3)\big)\, dx$$

$$= \sum_{s \geq 2} \frac{(-2\pi i(bt_2 - t_3))^{s-2}}{(2\pi i a t_1)^{s-1}} \Big[\gamma_s(ax; 2\pi i t_1)\beta_0\big(x; 2\pi i(bt_2 - t_3)\big)\Big]_{\frac{c}{b}}^{1}$$

$$= \sum_{\substack{s \geq 2 \\ p \geq s \\ q \geq 0}} \frac{(-1)^s (2\pi i)^{p+q-1}}{p! q! a^{s-1}}$$

$$\times \Big[Cl_p\big(ax - [ax]\big) B_q(x)\Big]_{\frac{c}{b}}^{1} t_1^{p-s+1} (bt_2 - t_3)^{q+s-2}$$

$$= \sum_{\substack{s \geq 1 \\ p,q \geq 0}} \frac{(-1)^{s+1} (2\pi i)^{p+q+s}}{(p+s+1)! q! a^s}$$

$$\times \Big[Cl_{p+s+1}\big(ax - [ax]\big) B_q(x)\Big]_{\frac{c}{b}}^{1} t_1^{p+1} (bt_2 - t_3)^{q+s-1}.$$

By definition, for any $x \in \mathbb{Q}$ and $k \geq 2$ we have

$$Cl_k\big(x-[x]\big) = \begin{cases} -\frac{k!}{(2\pi i)^{k-1}} C_k(x), & k:\text{odd}, \\ -i\frac{k!}{(2\pi i)^{k-1}} S_k(x), & k:\text{even}, \end{cases}$$

and hence, the above last line is computed as follows:

$$\begin{aligned} i \sum_{\substack{s\geq 1\\ p,q\geq 0\\ p+s:\text{odd}}} & \frac{(-1)^s(2\pi i)^q}{q!a^s}\left(S_{p+s+1}(a)B_q(1) - S_{p+s+1}\left(\frac{ac}{b}\right)B_q\left(\frac{c}{b}\right)\right) \\ & \times t_1^{p+1}(bt_2-t_3)^{q+s-1} \\ & + \sum_{\substack{s\geq 1\\ p,q\geq 0\\ p+s:\text{even}}} \frac{(-1)^s(2\pi i)^q}{q!a^s}\left(C_{p+s+1}(a)B_q(1) - C_{p+s+1}\left(\frac{ac}{b}\right)B_q\left(\frac{c}{b}\right)\right) \\ & \times t_1^{p+1}(bt_2-t_3)^{q+s-1}, \end{aligned}$$

which completes the proof. □

4. Proof of Theorem 1

We can now complete the proof of Theorem 1 as follows.

Proof of Theorem 1. We compute the real part of the coefficient of $t_1^{k_1}t_2^{k_2}t_3^{k_3}$ in the generating function $\frac{1}{2\pi i}F_{a,b}(2\pi i t_1, 2\pi i t_2, 2\pi i t_3)$ for positive integers k, k_1, k_2, k_3 with $k = k_1+k_2+k_3$ odd. By (11) with $t_j \to 2\pi i t_j$, we have

$$\begin{aligned} & \frac{1}{2\pi i}F_{a,b}(2\pi i t_1, 2\pi i t_2, 2\pi i t_3) \\ & \quad = \alpha_b(2\pi i t_2, 2\pi i t_3) \\ & \qquad \times \frac{1}{2\pi i}\int_0^1 \gamma(ax; 2\pi i t_1)\beta_0\big(x; -2\pi i(t_3-bt_2)\big)\,dx \qquad (13) \\ & \qquad + \sum_{c=1}^{b-1} \widetilde{\alpha}_{b,c}(2\pi i t_2, 2\pi i t_3) \qquad (14) \\ & \qquad \times \frac{1}{2\pi i}\int_{\frac{c}{b}}^1 \gamma(ax; 2\pi i t_1)\beta_0\big(x; 2\pi i(bt_2-t_3)\big)\,dx \\ & \qquad - \frac{1}{2\pi i}\int_0^1 \gamma(ax; 2\pi i t_1)\Big(\beta\big(bx; -2\pi i(-t_2)\big) + \beta(x; -2\pi i t_3)\Big)\,dx. \qquad (15) \end{aligned}$$

By (10), the coefficient of $t_1^{k_1}t_2^{k_2}t_3^{k_3}$ in the last term (15) is a rational multiple of $\zeta(k)$. For the first term (13), using (10) and (12), we have

$$\frac{1}{2\pi i}\int_0^1 \gamma(ax;2\pi i t_1)\beta_0\bigl(x;-2\pi i(t_3-bt_2)\bigr)\,dx \in \sum_{\substack{k_1,k_2,k_3>0\\ k_1+k_2+k_3:\mathrm{odd}}} \mathbb{Q}\zeta(k_1+k_2+k_3)t_1^{k_1}t_2^{k_2}t_3^{k_3},$$

where $\sum a_r t^r \in \sum V_r t^r$ means $a_r \in V_r$ for all r. We also have

$$\alpha_b(2\pi i t_1, 2\pi i t_2) \in \sum_{r,s\geq 0} \mathbb{Q}(2\pi i)^{r+s}t_1^r t_2^s.$$

Hence the real part of the coefficient of $t_1^{k_1}t_2^{k_2}t_3^{k_3}$ in (13) can be expressed as $\mathbb{Q}$-linear combinations of $\pi^{2n}\zeta(k-2n)$ with $0\leq n\leq \frac{k-3}{2}$. For the second term (14), using Proposition 8 (see also Remark 1), we have

$$\begin{aligned}
(16)\qquad &\widetilde{\alpha}_{b,c}(2\pi i t_2, 2\pi i t_3) \times \frac{1}{2\pi i}\int_{\frac{c}{b}}^1 \gamma(ax;2\pi i t_1)\beta_0\bigl(x;2\pi i(bt_2-t_3)\bigr)\,dx \\
&= -i\sum_{\substack{n_2\geq 1\\ n_3\geq 0}}\sum_{\substack{s\geq 1\\ p,q\geq 0\\ p+s:\mathrm{odd}}} \frac{(-1)^s\widetilde{A}_{b,c}(n_2,n_3)}{q!a^s} \times (2\pi i)^{n_2+n_3+q-1}S_{p+s+1}\left(\frac{ac}{b}\right)B_q\left(\frac{c}{b}\right) \times t_1^{p+1}(bt_2-t_3)^{q+s-1}t_2^{n_2}t_3^{n_3} \\
&\quad + \sum_{\substack{n_2\geq 1\\ n_3\geq 0}}\sum_{\substack{s\geq 1\\ p,q\geq 0\\ p+s:\mathrm{even}}} \frac{(-1)^s\widetilde{A}_{b,c}(n_2,n_3)}{q!a^s}(2\pi i)^{n_2+n_3+q-1} \times \Biggl(\zeta(p+s+1)B_q - C_{p+s+1}\left(\frac{ac}{b}\right)B_q\left(\frac{c}{b}\right)\Biggr)t_1^{p+1}(bt_2-t_3)^{q+s-1}t_2^{n_2}t_3^{n_3},
\end{aligned}$$

where we note that in the above both summations, $p+s+1$ runs over integers greater than 1. Since for any $x\in\mathbb{Q}$ and $k\geq 0$ we have $B_k(x)\in\mathbb{Q}$, the real part of the coefficient of $t_1^{k_1}t_2^{k_2}t_3^{k_3}$ in the first term (resp. the second term) of the right-hand side of (16) is a $\mathbb{Q}$-linear combination of $\pi^{2n+1}S_{k-2n-1}(\frac{ac}{b})$ with $0\leq n\leq\frac{k-3}{2}$ (resp. $\pi^{2n}C_{k-2n}(\frac{ac}{b})$ and $\pi^{2n}\zeta(k-2n)$ with $0\leq n\leq \frac{k-3}{2}$). We therefore find that the real part of the coefficient of $t_1^{k_1}t_2^{k_2}t_3^{k_3}$ in the generating function $\frac{1}{2\pi i}F_{a,b}(2\pi i t_1, 2\pi i t_2, 2\pi i t_3)$ can be expressed as $\mathbb{Q}$-linear combinations of $\pi^{2n+1}S_{k-2n-1}(\frac{ac}{b})$ and $\pi^{2n}C_{k-2n}(\frac{ac}{b})$ with $0\le n\le \frac{k-3}{2}$ and $c\in\mathbb{Z}/b\mathbb{Z}$. Thus by Corollary 3, we complete the proof. $\square$

REMARK 2. As mentioned in the introduction, the value $\zeta_{a,b}(k_1,k_2,k_3)$ is expressible as $\mathbb{Q}$-linear combinations of double polylogarithms $Li_{r,s}(z_1,z_2)$ defined in (3), where the expression is obtained from the partial fractional decomposition

$$\frac{1}{x^r y^s}=\sum_{\substack{p+q=r+s\\ p,q\ge 1}}\frac{1}{(x+y)^p}\left(\binom{p-1}{s-1}\frac{1}{x^q}+\binom{p-1}{r-1}\frac{1}{y^q}\right)\qquad (r,s\in\mathbb{Z}_{\ge 1})$$

and the orthogonality relation

$$\frac{1}{N}\sum_{n\in\mathbb{Z}/N\mathbb{Z}}\mu_N^{dn}=\begin{cases}1, & N\mid d,\\ 0, & N\nmid d,\end{cases}$$

where $\mu_N=e^{2\pi i/N}$ and $d\in\mathbb{Z}$. For example, one can check

$$\zeta_{1,3}(1,1,3)=\sum_{u\in\mathbb{Z}/3\mathbb{Z}}Li_{1,4}\big(\mu_3^{-u},\mu_3^{u}\big)+\sum_{u\in\mathbb{Z}/3\mathbb{Z}}Li_{1,4}\big(\mu_3^{u},1\big). \tag{17}$$

From this, Theorem 1 might be proved by the parity theorem for double polylogarithms examined in [10, Eq. (3.2)]. Although we do not proceed with this in general, let us illustrate an example. As a special case of [10, Eq. (3.2)], one obtains

$$\begin{aligned}&Li_{1,4}(z_1,z_2)+Li_{1,4}\big(z_1^{-1},z_2^{-1}\big)\\ &\quad=\sum_{n=1}^{5}(-1)^{n+1}Li_n(z_1)\mathcal{B}_{5-n}(z_1z_2)-Li_1(z_1)\mathcal{B}_4(z_2)\\ &\qquad+\sum_{n=4}^{5}\binom{n-1}{3}Li_n\big(z_2^{-1}\big)\mathcal{B}_{5-n}(z_1z_2)-Li_5(z_1z_2),\end{aligned}$$

where for each integer $k\ge 0$ we set $\mathcal{B}_k(z)=\frac{(2\pi i)^k}{k!}B_k(\frac12+\frac{\log(-z)}{2\pi i})$. We note that $Li_k(\mu_3^u)=C_k(\frac{u}{3})+iS_k(\frac{u}{3})$ and $\mathcal{B}_k(\mu_3)=\frac{(2\pi i)^k}{k!}B_k(\frac13)$ since $\log(-\mu_3)=-\frac{\pi i}{3}$. With this, the above formula gives

$$\begin{aligned}&\mathrm{Re}\big(Li_{1,4}\big(\mu_3^{-1},\mu_3\big)+Li_{1,4}\big(\mu_3^{-2},\mu_3^{2}\big)\big)\\ &\quad=\frac{1}{243}\big(-843\zeta(5)+36\pi^2\zeta(3)+4\pi^4\log 3\big),\\ &\mathrm{Re}\big(Li_{1,4}(\mu_3,1)+Li_{1,4}\big(\mu_3^{2},1\big)\big)\\ &\quad=\frac{1}{243}\left(972\zeta(5)-12\pi^2\zeta(3)-4\pi^4\log 3-81\pi S_4\left(\frac13\right)-12\pi^3S_2\left(\frac13\right)\right),\\ &2Li_{1,4}(1,1)=4\zeta(5)-\frac13\pi^2\zeta(3),\end{aligned}$$

where we have used $C_k(\frac{1}{3}) = C_k(\frac{2}{3}) = \frac{1-3^{k-1}}{2\cdot 3^{k-1}}\zeta(k)$ for $k \geq 2$ and $C_1(\frac{1}{3}) = C_1(\frac{2}{3}) = -\frac{1}{2}\log 3$. Substituting the above formulas to (17), one gets (2). We have checked Theorem 1 for $(a,b)=(1,3)$ and $(2,3)$ in this direction.

5. The zeta function of the root system G_2

In this section, we give an affirmative answer to the question posed by Komori, Matsumoto and Tsumura [5, Eq. (7.1)].

The zeta-function associated with the exceptional Lie algebra G_2 is defined for complex variables $\mathbf{s}=(s_1,s_2,\dots,s_6)\in\mathbb{C}^6$ by

$$\zeta(\mathbf{s};G_2) := \sum_{m,n>0} \frac{1}{m^{s_1}n^{s_2}(m+n)^{s_3}(m+2n)^{s_4}(m+3n)^{s_5}(2m+3n)^{s_6}}.$$

The function $\zeta(\mathbf{s};G_2)$ was first introduced by Komori, Matsumoto and Tsumura (see [4], [5]), where they developed its analytic properties and functional relations. They also examined explicit evaluations of the special values of $\zeta(\mathbf{k};G_2)$ at $\mathbf{k}\in\mathbb{Z}^6_{>0}$ (see [18] for $\mathbf{k}\in\mathbb{Z}^6_{\geq 0}$), where we note that the series $\zeta(\mathbf{k};G_2)$ converges absolutely for $\mathbf{k}\in\mathbb{Z}^6_{>0}$. For example, they showed

$$\zeta(2,1,1,1,1,1;G_2) = -\frac{109}{1296}\zeta(7) + \frac{1}{18}\zeta(2)\zeta(5).$$

Komori, Matsumoto and Tsumura [5, Eq. (7.1)] suggested a conjecture that the value $\zeta(k_1,\dots,k_6;G_2)$ with $k_1+\cdots+k_6$ odd lies in the polynomial ring over $\mathbb{Q}$ generated by $\zeta(k)$ $(k\in\mathbb{Z}_{\geq 2})$ and $L(k,\chi_3)$ $(k\in\mathbb{Z}_{\geq 1})$, where $L(s,\chi_3)$ is the Dirichlet L-function associated with the character χ_3 defined by

$$L(s,\chi_3) = \sum_{m>0}\frac{\chi_3(m)}{m^s}$$

and the character χ_3 is determined by $\chi_3(n)=1$ if $n\equiv 1 \mod 3$, $\chi_3(n)=-1$ if $n\equiv 2 \mod 3$ and $\chi_3(n)=0$ if $n\equiv 0 \mod 3$. We remark that the second author [8] showed that the value $\zeta(k_1,\dots,k_6;G_2)$ with $k_1+\cdots+k_6$ odd can be written in terms of $\zeta(s), L(s,\chi_3), S_r(\frac{d}{N}), C_r(\frac{d}{N})$ for $N=4,12$ and $0<d<N, (d,N)=1$ (see also [5, §7]). The following theorem gives an affirmative answer to the question.

THEOREM 9. *For any integers $k,k_1,\dots,k_6\geq 1$ with $k=k_1+\cdots+k_6$ odd, the value $\zeta(k_1,\dots,k_6;G_2)$ can be expressed as $\mathbb{Q}$-linear combinations of $\zeta(2n)\zeta(k-2n)$ $(0\leq n\leq\frac{k-3}{2})$ and $L(2n+1,\chi_3)L(k-2n-1,\chi_3)$ $(0\leq n\leq\frac{k-3}{2})$, where $\zeta(0)=-\frac{1}{2}$.*

Proof. In [8, Theorem 2.3], the second author proved that for any integers $l_1,\dots,l_6\geq 1$, the value $\zeta(l_1,\dots,l_6;G_2)$ can be expressed as $\mathbb{Q}$-linear combinations of $\zeta_{a,b}(n_1,n_2,n_3)$ with $(a,b)=(1,1),(1,2),(1,3),(2,3)$, $n_1+n_2+n_3=$

$l_1+\cdots+l_6$ and $n_1,n_2,n_3\in\mathbb{Z}_{>0}$. As a consequence, it follows from Theorem 1 that the value $\zeta(k_1,\ldots,k_6;G_2)$ can be written as $\mathbb{Q}$-linear combinations of $\pi^{2n}C_{k-2n}(\frac{d}{6})$ and $\pi^{2n+1}S_{k-2n-1}(\frac{d}{6})$ with $0\le n\le \frac{k-3}{2}$ and $d\in\mathbb{Z}/6\mathbb{Z}$. Now consider the values $C_k(\frac{d}{6})$ and $S_k(\frac{d}{6})$. They are expressible as $\mathbb{Q}$-linear combinations of

$$\zeta_l^{(d)}(k)=\sum_{\substack{m>0\\ m\equiv d \mod l}}\frac{1}{m^k}\quad (d\in\mathbb{Z}/l\mathbb{Z}).$$

For $k\ge 2$, using the identities $\zeta(k)=\sum_{d\in\mathbb{Z}/l\mathbb{Z}}\zeta_l^{(d)}(k)$ and $\zeta_l^{(0)}(k)\in\mathbb{Q}\zeta(k)$, we have $C_k(\frac{1}{2})=\zeta_2^{(0)}(k)-\zeta_2^{(1)}(k)\in\mathbb{Q}\zeta(k)$ and $C_k(\frac{1}{3})=C_k(\frac{2}{3})=\zeta_3^{(0)}(k)-\frac{1}{2}(\zeta_3^{(1)}(k)+\zeta_3^{(2)}(k))\in\mathbb{Q}\zeta(k)$. Furthermore, using the identity $\zeta_{al}^{(ad)}(k)=a^{-k}\zeta_l^{(d)}(k)$, we have

$$\begin{aligned}C_k\left(\frac{1}{6}\right)&=C_k\left(\frac{5}{6}\right)\\&=\zeta_6^{(0)}(k)-\zeta_6^{(3)}(k)+\frac{1}{2}\big(\zeta_6^{(1)}(k)+\zeta_6^{(5)}(k)\big)-\frac{1}{2}\big(\zeta_6^{(2)}(k)+\zeta_6^{(4)}(k)\big)\\&\in\mathbb{Q}\zeta(k).\end{aligned}$$

Thus, $C_k(\frac{d}{6})\in\mathbb{Q}\zeta(k)$ holds for any $d\in\mathbb{Z}/6\mathbb{Z}$ and $k\ge 2$. Likewise, it is easily seen that $S_k(\frac{d}{6})\in\mathbb{Q}\sqrt{3}L(k,\chi_3)$ holds. Then the result follows from the well-known formula: $\zeta(2n)\in\mathbb{Q}\pi^{2n}, L(2n+1,\chi_3)\in\mathbb{Q}\sqrt{3}\pi^{2n+1}$ for any $n\in\mathbb{Z}_{\ge 0}$ (see [1, Theorem 9.6]). $\square$

Let us illustrate an example of the formula for $\zeta(k_1,\ldots,k_6;G_2)$. Applying the partial fractional decomposition repeatedly to the form $(m+n)^{-k_3}(m+2n)^{-k_4}(m+3n)^{-k_5}(2m+3n)^{-k_6}$, we get

$$\begin{aligned}&\zeta(1,1,1,1,1,2;G_2)\\&\quad=\frac{1}{2}\zeta_{1,1}(5,1,1)-16\zeta_{1,2}(5,1,1)+\frac{9}{2}\zeta_{1,3}(5,1,1)+9\zeta_{2,3}(4,1,2)+18\zeta_{2,3}(5,1,1).\end{aligned}$$

Then, by Theorem 1 (actually we use Corollary 3 together with Propositions 4, 7 and 8), we have

$$\begin{aligned}\zeta(1,1,1,1,1,2;G_2)&=\frac{2507}{1296}\zeta(7)-\frac{505}{648}\pi^2\zeta(5)+\frac{9}{4}\pi S_6\left(\frac{1}{3}\right)\\&=\frac{2507}{1296}\zeta(7)-\frac{505}{108}\zeta(2)\zeta(5)+\frac{3}{8}L(1,\chi_3)L(6,\chi_3),\end{aligned}$$

where $L(1,\chi_3)=\frac{\pi}{3\sqrt{3}}$.

Acknowledgments. The authors are grateful to Professors Kohji Matsumoto, Takashi Nakamura and Hirofumi Tsumura for initial advice and many useful comments.

References

[1] T. Arakawa, T. Ibukiyama and M. Kaneko, *Bernoulli numbers and zeta functions*, Springer Monographs in Mathematics, Springer, Tokyo, 2014. MR 3307736

[2] J. G. Huard, K. S. Williams and N. Y. Zhang, *On Tornheim's double series*, Acta Arith. **75** (1996), no. 2, 105–117.

[3] K. Ihara, M. Kaneko and D. Zagier, *Derivation and double shuffle relations for multiple zeta values*, Compos. Math. **142** (2006), no. 2, 307–338.

[4] Y. Komori, K. Matsumoto and H. Tsumura, *On Witten multiple zeta-functions associated with semi-simple Lie algebras IV*, Glasg. Math. J. **53** (2011), no. 1, 185–206. MR 2747143

[5] Y. Komori, K. Matsumoto and H. Tsumura, *On Witten multiple zeta-function associated with semi-simple Lie algebras V*, Glasg. Math. J. **57** (2015), no. 1, 107–130. MR 3292681

[6] T. Nakamura, *A functional relation for the Tornheim double zeta function*, Acta Arith. **125** (2006), no. 3, 257–263.

[7] T. Nakamura, *A simple proof of the functional relation for the Lerch type Tornheim double zeta function*, Tokyo J. Math. **35** (2012), no. 2, 333–337.

[8] T. Okamoto, *Multiple zeta values related with the zeta-function of the root system of type A_2, B_2 and G_2*, Comment. Math. Univ. St. Pauli **61** (2012), no. 1, 9–27.

[9] T. Okamoto, *On alternating analogues of the Mordell–Tornheim triple zeta values*, J. Ramanujan Math. Soc. **28** (2013), no. 2, 247–269.

[10] E. Panzer, *The parity theorem for multiple polylogarithms*, J. Number Theory **172** (2017), 93–113. MR 3573145

[11] M. V. Subbarao and R. Sitaramachandra, *On some infinite series of L. J. Mordell and their analogues*, Pacific J. Math. **119** (1985), no. 1, 245–255.

[12] L. Tornheim, *Harmonic double series*, Amer. J. Math. **72** (1950), 303–314. MR 0034860

[13] H. Tsumura, *On alternating analogues of Tornheim's double series*, Proc. Amer. Math. Soc. **131** (2003), no. 12, 3633–3641.

[14] H. Tsumura, *Evaluation formulas for Tornheim's type of alternating double series*, Math. Comp. **73** (2004), no. 245, 251–258.

[15] H. Tsumura, *Combinatorial relations for Euler–Zagier sums*, Acta Arith. **111** (2004), no. 1, 27–42.

[16] H. Tsumura, *On Mordell-Tornheim zeta values*, Proc. Amer. Math. Soc. **133** (2005), 2387–2393.

[17] H. Tsumura, *On alternating analogues of Tornheim's double series. II*, Ramanujan J. **18** (2009), no. 1, 81–90.

[18] J. Zhao, *Multi-polylogs at twelfth roots of unity and special values of Witten multiple zeta function attached to the exceptional Lie algebra $\mathfrak{g}_2$*, J. Algebra Appl. **9** (2010), no. 2, 327–337.

[19] X. Zhou, T. Cai and D. M. Bradley, *Signed q-analogs of Tornheim's double series*, Proc. Amer. Math. Soc. **136** (2008), no. 8, 2689–2698.

Shin-ya Kadota, Graduate School of Mathematics, Nagoya University, Furo-cho, Chikusa-ku, Nagoya-shi, Aichi 464-8602, Japan

E-mail address: m13018c@math.nagoya-u.ac.jp

Takuya Okamoto, Department of Mathematics, College of Liberal Arts and Sciences, Kitasato University, Kitasato 1-15-1, Minami-ku, Sagamihara, Kanagawa 252-0373, Japan

E-mail address: takuyaok@kitasato-u.ac.jp

Koji Tasaka, Department of Information Science and Technology, Aichi Prefectural University, 1522-3 Ibaragabasama, Nagakute, Aichi Prefecture 480-1198, Japan

E-mail address: tasaka@ist.aichi-pu.ac.jp