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# MAPPINGS WITH SUBEXPONENTIALLY INTEGRABLE DISTORTION: MODULUS OF CONTINUITY, AND DISTORTION OF HAUSDORFF MEASURE AND MINKOWSKI CONTENT

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Abstract. We study mappings of finite distortion whose distortion functions are locally subexponentially integrable. We establish a local modulus of continuity estimate for the inverse of such a map. As applications, we describe the possible expansion and compression of certain Hausdorff measures and Minkowski contents under such mappings. We also exhibit examples that describe the extent to which our results are sharp.

## 1. Introduction

We call $\mathsf{R}^n \supset \Omega \xrightarrow{f} \mathsf{R}^n$ a *mapping of finite distortion* provided

- $f$ belongs to the Sobolev space $\mathsf{W}^{1,1}_{\mathrm{loc}}(\Omega;\mathsf{R}^n)$,
- $J(\cdot,f)$ belongs to the Lebesgue space $\mathsf{L}^1_{\mathrm{loc}}(\Omega;\mathsf{R})$,
- there exists a measurable function $K = K_f : \Omega \to [1,\infty]$ that is finite almost everywhere and is such that for almost every $x \in \Omega$,

$$|Df(x)|^n \le K(x) J(x,f). \tag{1.1}$$

Here $\Omega$ is a domain (open and connected) in Euclidean space $\mathsf{R}^n$ with $n \ge 2$, $|Df(x)|$ denotes the operator norm of the differential matrix of $f$ at the point $x$, and $J_f = J(\cdot,f)$ is the Jacobian determinant of $f$. Any (measurable) function $K$, with the *distortion inequality* (1.1) valid, is called a *distortion function* for $f$. When $K \in \mathsf{L}^\infty$ we recover the well known class of mappings of bounded distortion, also known as quasiregular mappings; see for instance [Res89].

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More generally, nonconstant mappings of finite distortion are continuous, discrete, and open, provided their distortion function satisfies certain conditions; see [IKO01], [KKM+03] and also the monograph [IM01]. For example, the finite distortion mappings of *exponentially integrable distortion*, that is, those for which $e^{pK} \in \mathsf{L}^1_{\mathrm{loc}}$ for some $p > 0$, have been extensively studied. See [IM01] and the references therein.

In this work, we study finite distortion homeomorphisms whose distortion functions $K$ are locally *subexponentially integrable*, which means that $\exp \mathcal{A}(pK) \in \mathsf{L}^1_{\mathrm{loc}}$ for some $p > 0$ and a given sublinear control function $\mathcal{A}$; see Section 2.2 for the precise hypotheses on $\mathcal{A}$. In this setting, we establish a sharp local modulus of continuity inequality for the inverse map. Then, using this inequality, we prove an estimate for the possible compression of certain Hausdorff measures induced by such maps, and similarly an estimate for the possible expansion of certain Minkowski contents. Finally, we exhibit examples that illustrate the sharpness of our results.

We start with our modulus of continuity result. It concerns homeomorphisms of finite distortion $K$ with $\exp \mathcal{A}(pK) \in \mathsf{L}^1_{\mathrm{loc}}$ and is a direct generalization of [HK03, Theorem B]. See Section 2.2 for the precise assumptions on $\mathcal{A}$. For brevity, we set

$$\omega(s) := s\mathcal{A}^{-1}(s)^{1/(n-1)}.$$

Theorem A. *Let $n \geq 2$ and $p > 0$. Assume $\mathcal{A}$ has the properties described in Section 2.2.1. There is a constant $C(\mathcal{A}, n)$, that depends only on the data associated with $\mathcal{A}$ and the dimension $n$, such that the following holds. Suppose $\Omega, \Omega' \subset \mathsf{R}^n$ are domains and $f : \Omega \to \Omega'$ is a finite distortion homeomorphism with $\exp \mathcal{A}(pK_f) \in \mathsf{L}^1_{\mathrm{loc}}(\Omega)$. Then for each ball $\mathsf{B}(z, R) \subset \Omega$ and all $x \in \mathsf{B}(z, R/6)$,*

$$\big|f(x) - f(z)\big| \geq D \exp\left( -\frac{C(\mathcal{A}, n)}{p^{1/(n-1)}} \omega\left( \log \frac{\Lambda R}{|x - z|} \right) \right), \tag{1.2}$$

*where*

$$D := \frac{1}{2} \operatorname{dist}\big(f(z), \partial f\big(\mathsf{B}(z; R/3)\big)\big)$$

*and*

$$\Lambda := \left( \frac{1}{|B(z, R)|} \int_{B(z,R)} \exp \mathcal{A}(pK_f) \right)^{1/n}.$$

Example 4.7 reveals the optimality of the above inequality. For future reference, we note that (1.2) is equivalent to the local modulus of continuity inequality

$$\big|g(y) - g(a)\big| \leq \Lambda R \exp\left( -\omega^{-1}\left( \frac{p^{1/(n-1)}}{C(\mathcal{A}, n)} \log \frac{D}{|y - a|} \right) \right), \tag{1.3}$$

where $g := f^{-1}$, $y := f(x)$, and $a := f(z)$.

Our first application of Theorem A describes the possible compression of Hausdorff measure under finite distortion homeomorphisms with subexponentially integrable distortion. An analogous result, for finite distortion homeomorphisms with exponentially integrable distortion, was established in [Zap11, Theorem 1.1]. She also constructed examples to illustrate the sharpness of her theorem; see [Zap11, Example 1.3] as well as our discussion in Section 4.2.

We assume the same conditions on $\mathcal{A}$ as above.

THEOREM B. *Let $n \geq 2$, $s \in (0,n]$, and $p > 0$. Let $C(\mathcal{A},n)$ be the constant from Theorem A and define the dimension gauge function*

$$h(t) = h_{s,p,\mathcal{A},n}(t) := \exp\left(-s\omega^{-1}\left(\frac{p^{1/(n-1)}}{C(\mathcal{A},n)}\log\frac{1}{t}\right)\right).$$

*Suppose $\Omega,\Omega' \subset \mathsf{R}^n$ are domains and $\Omega \xrightarrow{f} \Omega'$ is a finite distortion homeomorphism with $\exp\mathcal{A}(pK_f) \in \mathsf{L}^1_{\mathrm{loc}}(\Omega)$. Then for each $E \subset \Omega$ with $\mathcal{H}^s(E) > 0$, $\mathcal{H}^h(f(E)) > 0$.*

Examples 4.2, 4.3, 4.5, 4.6 illustrate the sharpness of the above compression result.

Our second application of Theorem A, Theorem C given below, describes the possible expansion of (upper) Minkowski content under a finite distortion homeomorphism with subexponentially integrable distortion. This is a direct generalization of [HK03, Theorem A]. Its proof utilizes certain volume growth estimates, and so now we consider control functions of the form $\mathcal{A}(t) = t/\mathcal{L}(t)$ for certain functions $\mathcal{L}$. Here our assumptions on $\mathcal{A}$ (see Section 2.2.2) are such that the self-improving integrability conditions (2.11) and (2.13) are in force.

We note that in this setting, with $\mathcal{A}(t) = t/\mathcal{L}(t)$, we have $\omega^{-1}(t) \simeq \mathcal{A}(t^{n-1})^{1/n}$ for all sufficiently large $t$ (see (2.14c)), and then

$$\omega^{-1}\left(\frac{p^{1/(n-1)}}{C}\log\frac{1}{t}\right) \simeq \mathcal{A}\left(\frac{p}{C}\log^{n-1}\frac{1}{t}\right)^{1/n} \simeq \frac{p^{1/n}}{C}\mathcal{A}\left(\log^{n-1}\frac{1}{t}\right)^{1/n},$$

where now the constant $C = C(\mathcal{L},n)$ depends only on the function $\mathcal{L}$ and dimension $n$. It follows that the modulus of continuity inequality (1.3) can be replaced with

$$\left|g(y) - g(a)\right| \leq \Lambda R\exp\left(-Cp^{1/n}\mathcal{A}\left(\log^{n-1}\frac{D}{|y-a|}\right)^{1/n}\right)$$

and the dimension gauge function in Theorem B can be replaced with

$$h(t) = h_{s,p,\mathcal{A},n}(t) := \exp\left(-Csp^{1/n}\mathcal{A}\left(\log^{n-1}\frac{1}{t}\right)^{1/n}\right). \tag{1.4}$$

THEOREM C. *Let $n \geq 2$, $k \in \mathsf{N}$, and $p > 0$. Assume $\mathcal{A}(t) = t/\mathcal{L}(t)$ where $\mathcal{L} = \mathcal{L}_k$ is as described in (2.5). There exists a constant $c = c(k, n)$ with the following property. Define the dimension gauge functions*

$$h_\beta(t) := t^n L_{k+1}(1/t)^\beta.$$

*Suppose $\Omega \xrightarrow{f} \Omega'$ is a finite distortion homeomorphism between domains $\Omega, \Omega' \subset \mathsf{R}^n$ with $\exp \mathcal{A}(pK_f) \in \mathsf{L}^1_{\mathrm{loc}}(\Omega)$. Then for every $\beta < cp$ and each compact set $E \subset \Omega$ with upper Minkowski dimension $\overline{\dim}_{\mathcal{M}}(E) < n$, $\bar{\mathcal{M}}^{h_\beta}(f(E)) = 0$.*

Results related to Theorem C can be found in the works [KZZ10], [KZZ09], [Raj11], [RZZ11a] and [RZZ11b]. The first three of these articles deal with planar Sobolev maps and provide sufficient conditions such that the images of certain sets have zero generalized Hausdorff measure when certain dimension gauges are used; the last has similar results for $\mathsf{R}^n$ with $n \geq 2$; the fourth paper deals with finite distortion homeomorphisms of spatial domains with subexponentially integrable distortion controlled by $\mathcal{A}(t) = t/\log(1+t)$.

We also mention the foundational work [AIKM00] that includes many modulus of continuity results. In addition, the idea behind our Cantor dust construction in Section 4.1 is based on the proof of [AIKM00, Theorem 7.2].

We prove Theorems A, B, C in Sections 3.1, 3.2, 3.3, respectively. Example 4.5 illustrates that, in a certain sense, as $s \to 0$, the gauge function in (1.4) gives an optimal result in Theorem B. In addition, it indicates that perhaps the factor $s$ should be replaced by $s/(n-s)$. We first discuss the related Examples 4.2 and 4.3 for finite distortion homeomorphisms with $\exp(pK) \in \mathsf{L}^1_{\mathrm{loc}}$; these slightly improve upon [Zap11, Example 1.3]. We end with Example 4.7 that is related to the modulus of continuity inequality (1.2).

## 2. Preliminaries

Our notation is relatively standard. We write $C = C(a, \ldots)$ to indicate a constant $C$ that depends only on the parameters $a, \ldots$; the notation $A \lesssim B$ means there exists a finite constant $c$ with $A \leq cB$, and $A \simeq B$ means that both $A \lesssim B$ and $B \lesssim A$ hold. Typically $a, b, c, C, K, \ldots$ are constants that depend on various parameters, and we try to make this as clear as possible often giving explicit values, however, at times $C$ will denote a generic constant whose value depends only on the data present but may differ even on the same line of inequalities.

We write $|x - y|$ for the Euclidean distance between points $x, y$ in Euclidean space $\mathsf{R}^n$; $\mathsf{B}(x; r) := \{y : |x - y| < r\}$ and $\mathsf{S}(x; r) := \{y : |x - y| = r\}$ are the open ball and the sphere of radius $r$ centered at the point $x$. We let $\mathsf{B}^n := \mathsf{B}(0; 1)$ and $\mathsf{S}^{n-1} = \partial \mathsf{B}^n$ denote the open unit ball and unit sphere, respectively; their natural measures are $\Omega_n$ and $\omega_{n-1}$. Given a ball $B$ and

$\sigma > 0$, we let $\sigma B$ denote the dilated ball with the same center; that is, $\sigma \mathsf{B}(x,r) := \mathsf{B}(x,\sigma r)$.

It is convenient to introduce the following convention. We say that a property holds "*as* $t \to \infty$" provided there is some $t_0$ such that the property holds for all $t \geq t_0$. For example, we write $\varphi(t) \lesssim \psi(t)$ as $t \to \infty$ to mean that there are $t_0$ (usually large) and $C \geq 1$ such that for all $t \geq t_0$,

$$\varphi(t) \leq C\psi(t).$$

Of course, $\varphi(t) \simeq \psi(t)$ as $t \to \infty$ provided both $\varphi(t) \lesssim \psi(t)$ and $\psi(t) \lesssim \varphi(t)$ as $t \to \infty$.

For example, for any $a > 0, \beta > 0, c > 0$, $\log(at^\beta + c) \simeq \log t$ as $t \to \infty$ where $t_0$ and $C$ depend only on $a, \beta, c$.

We require the following information; see [CK09, Proposition 5.1].

Fact 2.1. Let $(0,\infty) \xrightarrow{L} (0,\infty)$ be an increasing $\mathcal{C}^1$ function. Suppose that $L$ satisfies

$$\lim_{t\to\infty} L(t) = \infty \tag{2.1}$$

and there are constants $C_L \geq 0$ and $t_L \geq 1$ such that

$$\forall t \geq t_L, \quad t\frac{L'(t)}{L(t)} \leq \frac{C_L}{\log(1+t)}. \tag{2.2}$$

Then

$$\lim_{t\to\infty} \frac{\log L(t)}{\log t} = 0 \tag{2.3}$$

and for any $a > 0, \beta > 0, c > 0$,

$$L\big(at^\beta + c\big) \simeq L(t) \quad \text{as } t \to \infty, \tag{2.4}$$

where $t_0$ and $C$ depend only on $a, \beta, c, t_L, C_L$.

Condition (2.3) says that $L$ grows to infinity more slowly than any power and (2.4) says that $L$ does not see exponents.

Examples of such functions include both $L_k$ and $\mathcal{L}_k$ for any $k \in \mathsf{N}$. Here

$$\mathcal{L}_k(t) := L_1(t)L_2(t)\cdots L_k(t) \tag{2.5}$$

and $L_k$ is a $k$-times iterated logarithm defined by

$$L_k(t) := \log^{\circ k}(e_k + t) \quad \text{with } e_k := \exp^{\circ k}(0),$$

where $F^{\circ k}$ denotes $F$ composed with itself $k$ times, which is defined by

$$F^{\circ 1} := F \quad \text{and} \quad \text{for } k \geq 2, \quad F^{\circ k} := F \circ F^{\circ (k-1)}.$$

The constant $e_k$ is defined so that $L_k(0) = 0$. Notice that $L_k^{-1}(1) = e_{k+1} - e_k$. For example,

$$L_3(t) = \log\log\log\big(e^e + t\big) \quad \text{and} \quad L_3^{-1}(1) = \exp\big(e^e\big) - e^e.$$

We require the following technical facts. For example, we make use of item (c) below in our proof of Theorem C.

Lemma 2.2. *Let* $[1,\infty) \xrightarrow{L} [1,\infty)$ *be a* $\mathcal{C}^1$ *homeomorphism that satisfies* (2.1) *and* (2.2).

(a) *We always have* $\lim_{t\to\infty} \frac{L(t)}{L(tL(t))} = 1$.

(b) *Let* $[1,\infty) \xrightarrow{\varphi,\psi} [1,\infty)$ *be functions with* $\lim_{t\to\infty}\varphi(t) = \infty = \lim_{t\to\infty}\psi(t)$. *Then*

$$\lim_{t\to\infty} \frac{\varphi(t)}{\psi(t)} = 1 \quad \Longrightarrow \quad \lim_{t\to\infty} \frac{L(\varphi(t))}{L(\psi(t))} = 1.$$

(c) *Given* $\beta > 0$, *define* $Q(s) = Q_\beta(s) := sL^{-1}(s^{1/\beta})$. *Then* $Q^{-1}(t) \simeq L(t)^\beta$ *as* $t \to \infty$. *More precisely, for all sufficiently large* $t$,

$$(1+\beta)^{-\beta C} L(t)^\beta \le Q^{-1}(t) \le L(t)^\beta,$$

*where* $C = C_L$.

*Proof.* (a) Since $L$ is increasing, $L(t) \le L(tL(t))$ for all $t \ge 1$. Let $\varepsilon > 0$ be given. Using (2.3), we produce a $\tau = \tau(\varepsilon) > 1$ with the property that for all $t \ge \tau$, $L(t) \le t^\varepsilon$. Thus $L(tL(t)) \le L(t^{1+\varepsilon})$. Then from (2.2) we obtain

$$\log \frac{L(t^{1+\varepsilon})}{L(t)} = \int_t^{t^{1+\varepsilon}} \frac{L'(u)}{L(u)}\,du \le C \int_t^{t^{1+\varepsilon}} \frac{du}{u\log u} = C\log(1+\varepsilon)$$

and therefore for all $t \ge \tau$,

$$1 \le \frac{L(tL(t))}{L(t)} \le \frac{L(t^{1+\varepsilon})}{L(t)} \le (1+\varepsilon)^C,$$

where $C = C_L$.

(b) Assume that $\lim_{t\to\infty}(\varphi(t)/\psi(t)) = 1$. Then $\lim_{t\to\infty}(\log\varphi(t)/\log\psi(t)) = 1$. Thus,

$$\left|\log \frac{L(\varphi(t))}{L(\psi(t))}\right| = \left|\int_{\psi(t)}^{\varphi(t)} \frac{L'(u)}{L(u)}\,du\right| \le C\left|\int_{\psi(t)}^{\varphi(t)} \frac{du}{u\log u}\right| = C\left|\log\frac{\log\varphi(t)}{\log\psi(t)}\right|,$$

where again $C = C_L$.

(c) The change of variables $t = Q(L(u)^\beta) = L(u)^\beta L^{-1}(L(u)) = uL(u)^\beta$ gives

$$\frac{Q^{-1}(t)}{L(t)^\beta} = \left(\frac{L(u)}{L(uL(u)^\beta)}\right)^\beta.$$

Since $L$ is increasing with $L \ge 1$, we have

$$L(u) \le L\big(uL(u)^\beta\big) \le L\big(u^{1+\beta}\big) \le (1+\beta)^C L(u)$$

with the latter two inequalities holding for all sufficiently large $u$.

The first inequality implies that $Q^{-1}(t) \le L(t)^\beta$ and the latter two inequalities provide the lower estimate for $Q^{-1}(t)$. □

As a simple example, (b) above gives us that for any $a, b > 0$, $\lim_{t\to\infty} L(at)/L(at+b) = 1$. More importantly, from the proof of (b) we have that

$\lim_{t\to\infty}(\log\varphi(t)/\log\psi(t)) = 1$ and thus

$$\lim_{t\to\infty}\frac{L(\log\varphi(t))}{L(\log\psi(t))} = 1, \quad \text{and similarly} \quad \lim_{t\to\infty}\frac{L(L(\log\varphi(t)))}{L(L(\log\psi(t)))} = 1,$$

and so forth. Using similar ideas, it is straightforward to verify the following.

LEMMA 2.3. *Let $k\in\mathsf{N}$, $C>0$, and define the $k$-times iterated logarithm*

$$L(t) := \log^{\circ k}(C+t).$$

*Then for all $a>0, b>0, \alpha>0$,*

$$\lim_{t\to\infty}\frac{L(at)}{L(bt^\alpha)} = \begin{cases} 1/\alpha & \textit{when } k=1,\\ 1 & \textit{when } k\geq 2.\end{cases}$$

In particular, the above is valid for the functions $L_k$ defined just after (2.5).

**2.1. Orlicz spaces.** For our purposes, any homeomorphism $[0,\infty)\xrightarrow{P}[0,\infty)$ is an *Orlicz function* and the associated *Orlicz space* $\mathsf{L}^P(\Omega,\mathsf{R}^n)$ consists of all Lebesgue measurable functions $f:\Omega\to\mathsf{R}^n$ with the property that for some positive finite $\lambda$, $\int_\Omega P(\lambda|f|)<\infty$. Then the non-linear *Luxemburg functional* is defined, for $f\in\mathsf{L}^P(\Omega,\mathsf{R}^n)$, by

$$\|f\|_P = \|f\|_{\mathsf{L}^P(\Omega,\mathsf{R}^n)} := \inf\left\{\lambda>0 \,\middle|\, \int_\Omega P\bigl(\lambda^{-1}|f|\bigr) < P(1)\right\}.$$

For example, if $A$ is a measurable subset of $\mathsf{R}^n$ with positive measure $|A|$, then

$$\int_{\mathsf{R}^n} P\bigl(\lambda^{-1}\chi_A\bigr) = \int_A P\bigl(\lambda^{-1}\bigr) = P\bigl(\lambda^{-1}\bigr)|A|$$

and therefore

$$\|\chi_A\|_P = P^{-1}\left(\frac{P(1)}{|A|}\right)^{-1}. \tag{2.6}$$

A standard reference for Orlicz spaces and Orlicz functions is the text [RR91].

A pair of Orlicz functions $P$ and $Q$ satisfy *Young's inequality* provided for all $x,y\geq 0$,

$$xy \leq P(x)+Q(y). \tag{2.7}$$

When this holds, we have the *Orlicz–Hölder inequality*

$$\|gh\|_{\mathsf{L}^1} \leq C\|g\|_{\mathsf{L}^P}\|h\|_{\mathsf{L}^Q}, \tag{2.8}$$

where $C=P(1)+Q(1)$; see [RR91, Proposition 1, p. 58].

There is a useful way to produce such a pair of Orlicz functions. Let $[0,\infty)\xrightarrow{F}[0,\infty)$ be a homeomorphism and put $G:=F^{-1}$. Given any $\beta>0$, define

$$P(x) = P_\beta(x) := xF(x)^\beta \quad \text{and} \quad Q(y) = Q_\beta(y) := yG\bigl(y^{1/\beta}\bigr).$$

If $xy > P(x)$, then $y^{1/\beta} > F(x)$, so $G(y^{1/\beta}) > x$ and hence $Q(y) > xy$. Thus we see that $P$ and $Q$, as defined above, satisfy Young's inequality (2.7), so the Orlicz–Hölder inequality (2.8) holds with $C = F(1)^\beta + G(1)$.

We will apply the above construction to maps $[0,\infty) \xrightarrow{L} [0,\infty)$ that satisfy the hypotheses in Fact 2.1; see also Lemma 2.2(c).

**2.2. Subexponential integrability.** In this paper, we study homeomorphisms $f$ of finite distortion $K_f$ that are subexponentially integrable, meaning that there is a sublinear control function $\mathcal{A}$ such that for some $p > 0$,

$$\exp \mathcal{A}(pK_f) \in \mathsf{L}^1_{\mathrm{loc}}.$$

Everywhere here and below $[0,\infty) \xrightarrow{\mathcal{A}} [0,\infty)$ is a homeomorphism with the property that

$$\int_1^\infty \frac{\mathcal{A}(t)}{t^2}\, dt = \infty.$$

This assumption is critical in order for the mapping $f$ to be discrete, open, and to satisfy Lusin's $N$-condition; see [KKM$^+$03].

2.2.1. *Hypotheses for $\mathcal{A}$ in Theorems A and B.* For both Theorem A and Theorem B we also assume that $\mathcal{A}$ is a $\mathcal{C}^1$ diffeomorphism on $(0,\infty)$ with $t \mapsto t\mathcal{A}'(t)$ an increasing function,

$$\lim_{t\to\infty} t\frac{\mathcal{A}'(t)}{\mathcal{A}(t)} = 1, \tag{2.9}$$

and with both $\mathcal{A}$ and $\mathcal{A}^{-1}$ *doubling.*

The doubling condition for $\mathcal{A}$ asserts that $\mathcal{A}(2t) \simeq \mathcal{A}(t)$. The assumption that $t \mapsto t\mathcal{A}'(t)$ is increasing is a minor requirement that allows us to avoid many technicalities. The condition (2.9) implies that

$$\lim_{t\to\infty} \frac{\log \mathcal{A}(t)}{\log t} = 1, \quad \text{and hence that} \quad \lim_{t\to\infty} t\mathcal{A}'(t) = \infty. \tag{2.10}$$

2.2.2. *Hypotheses for $\mathcal{A}$ in Theorem C.* Our proof of Theorem C utilizes a volume growth estimate established in [CK09]. Given a sublinear control function $\mathcal{A}$, we define

$$\mathcal{E}(t) := 1 + \int_1^t \frac{\mathcal{A}(\xi)}{\xi^2}\, d\xi$$

and then for each $\beta > 0$ we set

$$\mathcal{P}_\beta(t) := t\mathcal{E}(t)^\beta.$$

In [CK09] it was shown that, with certain additional hypotheses on $\mathcal{A}$, there exists a constant $c(\mathcal{A}, n)$, that depends only on the data associated with $\mathcal{A}$ and the dimension $n$, such that

$$\exp \mathcal{A}(pK_f) \in \mathsf{L}^1_{\mathrm{loc}} \quad \Longrightarrow \quad \forall \beta < c(\mathcal{A}, n)p, \quad J_f \in \mathsf{L}^{\mathcal{P}_\beta}_{\mathrm{loc}}. \tag{2.11}$$

In [CK09] the authors work with a control function of the form $\mathcal{A}(t)=t/\mathcal{L}(t)$ where $\mathcal{L}$ satisfies (2.1) and (2.2) (and some other conditions too).

For Theorem C, we further assume that $\mathcal{A}(t)=t/\mathcal{L}_k(t)$ for some $k\in\mathsf{N}$, where $\mathcal{L}_k$ is defined in (2.5). We note that $\mathcal{L}_k$ satisfies the hypotheses in Fact 2.1 and that such an $\mathcal{A}$ satisfies all the assumptions listed above in Section 2.2.1 including those necessary for the work in [CK09]. In particular, it is straightforward to check that for such an $\mathcal{A}$ we have

$$\mathcal{E}(t)\simeq\frac{\mathcal{A}(t)}{t}\mathcal{A}^{-1}(\log t)\simeq\frac{(\log t)\mathcal{L}_k(\log t)}{\mathcal{L}_k(t)}\simeq L_{k+1}(t)\quad\text{as } t\to\infty. \tag{2.12}$$

Thus, in this setting, (2.11) reads as

$$\exp\mathcal{A}(pK_f)\in\mathsf{L}^1_{\mathrm{loc}}\quad\Longrightarrow\quad\forall\beta<cp,\quad J_f\in\mathsf{L}^{P_\beta}_{\mathrm{loc}}, \tag{2.13}$$

where $c=c(k,n)$ and $P_\beta(t):=tL_{k+1}(t)^\beta$. Gill [Gil10] established a more precise result in the plane setting.

It is worth mentioning that any requirements on $\mathcal{A}$ need only hold as $t\to\infty$: Any $\mathcal{A}$ with the needed properties valid for all $t\geq t_0$ can be modified for $0\leq t\leq t_0$ so that the desired conditions hold for all $t\geq 0$.

2.2.3. *Technical $\mathcal{A}$ facts.* In both parts of our proof of Theorem A, we would like to estimate certain integrals by using Jensen's Inequality with the auxiliary function $\varphi(t):=\exp\mathcal{A}(pt^\alpha)$ for some $\alpha>0$. However, such a function $\varphi$ may not be convex. To circumvent this problem, we employ a "Jensen's Inequality Replacement Trick" that makes use of the fact that $t\mapsto t^{-1}\varphi(t)$ is increasing on the interval $[\tau_p,\infty)$. To determine $\tau_p$, we note that

$$\left(\frac{\varphi(t)}{t}\right)'=\frac{\varphi(t)}{t^2}\big(\alpha pt^\alpha\mathcal{A}'\big(pt^\alpha\big)-1\big).$$

Thus,

$$\tau_p:=(t_\alpha/p)^{1/\alpha},\quad\text{where } t_\alpha:=\inf\big\{t\geq 0\,|\,t\mathcal{A}'(t)\geq\alpha^{-1}\big\}.$$

That such a $t_\alpha$ (which depends on both $\alpha$ and the data associated with $\mathcal{A}$) exists (i.e., that $t_\alpha<\infty$) follows from (2.10).

In both parts of our proof of Theorem A, the "Jensen's Inequality Replacement Trick" works provided a certain quantity exceeds $\tau_p$. As we cannot guarantee that this requirement is met, we need the following result. (We use this in two cases: first with $\alpha=1/n$ and $M=6^n$ and then with $\alpha=n-1$ and $M=4^n$.)

LEMMA 2.4. *Let $p>0$ and $\alpha>0$. Assume that $\mathcal{A}$ satisfies the conditions described in Section 2.2.1. Define $\varphi,t_\alpha,\tau_p$ as in the above paragraph. Suppose that $M\geq 1$ is such that $\alpha\log M>1$. Then there is a constant $C=C(M,\alpha,\mathcal{A})\geq 1$ (that does not depend on $p$) such that $\tau_p\leq C\varphi^{-1}(M)$.*

*Proof.* Since $\varphi^{-1}(s)=(p^{-1}\mathcal{A}^{-1}(\log s))^{1/\alpha}$, we see that

$$\tau_p\leq C\varphi^{-1}(M)\quad\Longleftrightarrow\quad\mathcal{A}\big(C^{-\alpha}t_\alpha\big)\leq\log M.$$

It is now easy to check that we can take $C=1$ if either $t_\alpha \le \mathcal{A}^{-1}(\log M)$ or $t_\alpha \ge \vartheta$, where $\vartheta = \vartheta(M,\alpha,\mathcal{A}) > 0$ is such that

$$\forall t \ge \vartheta, \quad t\frac{\mathcal{A}'(t)}{\mathcal{A}(t)} \ge \frac{1}{\alpha \log M}.$$

That such a $\vartheta$ exists follows from (2.9), since $\alpha \log M > 1$. Notice that when $t_\alpha \ge \vartheta$ we have

$$\frac{1}{\alpha \log M} \le t_\alpha \frac{\mathcal{A}'(t_\alpha)}{\mathcal{A}(t_\alpha)} = \frac{1/\alpha}{\mathcal{A}(t_\alpha)}, \quad \text{so } \mathcal{A}(t_\alpha) \le \log M.$$

For the case when $\mathcal{A}^{-1}(\log M) \le t_\alpha \le \vartheta$, we put $C := (\vartheta/\mathcal{A}^{-1}(\log M))^{1/\alpha}$. Then

$$\frac{t_\alpha}{C^\alpha} = \frac{t_\alpha}{\vartheta}\mathcal{A}^{-1}(\log M) \le \mathcal{A}^{-1}(\log M), \quad \text{so } \tau_p \le C\varphi^{-1}(M). \qquad \square$$

We also require the following technical information, especially in our later examples.

LEMMA 2.5. *Let* $[1,\infty) \xrightarrow{\mathcal{L}} [1,\infty)$ *be a* $\mathcal{C}^1$ *homeomorphism that satisfies* (2.1) *and* (2.2). *Define*

$$\mathcal{A}(t) := \frac{t}{\mathcal{L}(t)} \quad \textit{and} \quad \omega(s) := s\mathcal{A}^{-1}(s)^{1/(n-1)}.$$

*Then*

$$\forall C > 0, \quad \mathcal{A}(Ct) \le C_{\mathcal{L}} C \mathcal{A}(t) \quad \textit{for all sufficiently large } t > 1, \tag{2.14a}$$

*where* $C_{\mathcal{L}}$ *is a constant that depends only on* $\mathcal{L}$. *In addition*:

$$\lim_{s\to\infty} \frac{\mathcal{A}^{-1}(s)}{s\mathcal{L}(s)} = 1. \tag{2.14b}$$

$$\textit{As } s \to \infty, \quad \omega(s) \simeq \mathcal{A}^{-1}\big(s^n\big)^{1/(n-1)}. \tag{2.14c}$$

$$\forall C > 0, \quad \lim_{s\to\infty} \frac{\mathcal{A}^{-1}(C(s+1))}{\mathcal{A}^{-1}(Cs)} = 1. \tag{2.14d}$$

$$\lim_{s\to\infty} \frac{\mathcal{A}^{-1}(s)^{1/(n-1)}}{\omega'(s)} = \frac{n-1}{n}. \tag{2.14e}$$

$$\forall a > 0, \quad \lim_{N\to\infty} \frac{1}{\omega(aN)} \sum_{k=1}^{N} \mathcal{A}^{-1}(ak)^{1/(n-1)} = \frac{1}{a}\frac{n-1}{n}. \tag{2.14f}$$

$$\forall C > 0, \quad \lim_{u\to\infty} \frac{\mathcal{L}(u^n)}{\mathcal{L}(C\omega(u)^{n-1})} = 1. \tag{2.14g}$$

*Proof.* Here we refer to the above assertions as (a), ..., (g), respectively. To check (a), we note that when $C \ge 1$ we can take $C_{\mathcal{L}} = 1$. Assume $0 < C < 1$. An appeal to Fact 2.1 reveals that for all sufficiently large $t > 1$: $Ct \ge \sqrt{t}$, so $\mathcal{L}(Ct) \geq \mathcal{L}(\sqrt{t}) \geq C_{\mathcal{L}}\mathcal{L}(t)$ and therefore $\mathcal{A}(Ct) = Ct/\mathcal{L}(Ct) \leq C_{\mathcal{L}}Ct/\mathcal{L}(t) = C_{\mathcal{L}}C\mathcal{A}(t)$.

The limit in (b) is just [CK09, Lemma 2.2] from which it is easy to see that (c) and (d) hold (remembering Fact 2.1 for (c)). To establish (e), we compute

$$\begin{aligned}\omega'(s) &= \mathcal{A}^{-1}(s)^{1/(n-1)} + \frac{s}{n-1}\mathcal{A}^{-1}(s)^{\frac{1}{n-1}-1}\frac{d}{ds}\left[\mathcal{A}^{-1}(s)\right] \\ &= \mathcal{A}^{-1}(s)^{1/(n-1)}\left(1 + \frac{s}{n-1}\frac{(\mathcal{A}^{-1})'(s)}{\mathcal{A}^{-1}(s)}\right).\end{aligned}$$

Thus (e) will follow once we verify that

$$\lim_{s\to\infty}\frac{s(\mathcal{A}^{-1})'(s)}{\mathcal{A}^{-1}(s)} = 1.$$

Writing $t := \mathcal{A}^{-1}(s)$, and remembering that $\mathcal{A}(t) = t/\mathcal{L}(t)$, we find that

$$\frac{s(\mathcal{A}^{-1})'(s)}{\mathcal{A}^{-1}(s)} = \frac{\mathcal{A}(t)}{t\mathcal{A}'(t)} \quad \text{and then} \quad \frac{t\mathcal{A}'(t)}{\mathcal{A}(t)} = 1 - t\frac{\mathcal{L}'(t)}{\mathcal{L}(t)} \longrightarrow 1.$$

Now we turn our attention to (f). Since $\mathcal{A}^{-1}$ is increasing, we can use Riemann sums to obtain the estimates

$$\int_0^N \mathcal{A}^{-1}(\xi)^{1/(n-1)}\,d\xi \leq \sum_{k=1}^N \mathcal{A}^{-1}(ak)^{1/(n-1)} \leq \int_1^{N+1} \mathcal{A}^{-1}(\xi)^{1/(n-1)}\,d\xi.$$

Since each of the integrals above tends to infinity with $N$, we can use l'Hôpital's Rule to determine the limit of their quotients when we divide by $\omega(aN)$. Using (e) for the left-hand quotient, and (d) and (e) for the right-hand one, we see that both have the same limit thereby establishing (f).

To validate (g) we start with the change of variable $t = u^n$, so

$$\omega(u)^{n-1} = \left(u\mathcal{A}^{-1}(u)^{1/(n-1)}\right)^{n-1} = t^{(n-1)/n}\mathcal{A}^{-1}\left(t^{1/n}\right)$$

and the claim is that

$$\lim_{t\to\infty}\frac{\mathcal{L}(t)}{\mathcal{L}(Ct^{\frac{n-1}{n}}\mathcal{A}^{-1}(t^{\frac{1}{n}}))} = 1.$$

We write

$$\frac{\mathcal{L}(t)}{\mathcal{L}(Ct^{\frac{n-1}{n}}\mathcal{A}^{-1}(t^{\frac{1}{n}}))} = F(t)\cdot G(t)\cdot H(t),$$

where

$$F(t) := \frac{\mathcal{L}(t)}{\mathcal{L}(Ct)}, \qquad G(t) := \frac{\mathcal{L}(Ct)}{\mathcal{L}(Ct\mathcal{L}(t^{\frac{1}{n}}))}, \qquad H(t) := \frac{\mathcal{L}(Ct\mathcal{L}(t^{\frac{1}{n}}))}{\mathcal{L}(Ct^{\frac{n-1}{n}}\mathcal{A}^{-1}(t^{\frac{1}{n}}))}.$$

We demonstrate that

$$\lim_{t\to\infty}F(t) = \lim_{t\to\infty}G(t) = \lim_{t\to\infty}H(t) = 1.$$

If $C\geq 1$, then $\mathcal{L}(Ct)\geq\mathcal{L}(t)$ and so for all sufficiently large $t$ (so that $\mathcal{L}(t)\geq C$) we have

$$1\geq F(t)=\frac{\mathcal{L}(t)}{\mathcal{L}(Ct)}\geq\frac{\mathcal{L}(t)}{\mathcal{L}(t\mathcal{L}(t))}\longrightarrow 1\quad\text{as } t\to\infty$$

thanks to Lemma 2.2(a). Assume that $0<C<1$. The change of variable $\tau:=Ct$ gives us

$$\frac{\mathcal{L}(t)}{\mathcal{L}(Ct)}=\frac{\mathcal{L}(C^{-1}\tau)}{\mathcal{L}(\tau)}=\left(\frac{\mathcal{L}(\tau)}{\mathcal{L}(C^{-1}\tau)}\right)^{-1}\longrightarrow 1\quad\text{as } t\to\infty$$

thanks to the first case.

Next, we claim that for all sufficiently large $t>1$ (e.g., so that $\mathcal{L}(t^{1/n})\geq 1$),

$$1\geq G(t)\geq\frac{\mathcal{L}(Ct)}{\mathcal{L}(Ct\mathcal{L}(t))}=\frac{\mathcal{L}(Ct)}{\mathcal{L}(Ct\mathcal{L}(Ct))}\cdot\frac{\mathcal{L}(Ct\mathcal{L}(Ct))}{\mathcal{L}(Ct\mathcal{L}(t))}.$$

For the first fraction, we again use Lemma 2.2(a) to see that

$$\frac{\mathcal{L}(Ct)}{\mathcal{L}(Ct\mathcal{L}(Ct))}=\frac{\mathcal{L}(\tau)}{\mathcal{L}(\tau\mathcal{L}(\tau))}\longrightarrow 1\quad\text{as } t=C\tau\to\infty.$$

Similarly,

$$\frac{Ct\mathcal{L}(Ct)}{Ct\mathcal{L}(t)}=\frac{\mathcal{L}(Ct)}{\mathcal{L}(t)}\longrightarrow 1\quad\text{as } t\to\infty$$

and thus according to Lemma 2.2(b)

$$\frac{\mathcal{L}(Ct\mathcal{L}(Ct))}{\mathcal{L}(Ct\mathcal{L}(t))}\longrightarrow 1\quad\text{as } t\to\infty.$$

Finally, by (2.14b)

$$\frac{Ct\mathcal{L}(t^{\frac{1}{n}})}{Ct^{\frac{n-1}{n}}\mathcal{A}^{-1}(t^{\frac{1}{n}})}=\frac{Ct\mathcal{L}(t^{\frac{1}{n}})}{Ct^{\frac{n-1}{n}}t^{\frac{1}{n}}\mathcal{L}(t^{\frac{1}{n}})}\cdot\frac{t^{\frac{1}{n}}\mathcal{L}(t^{\frac{1}{n}})}{\mathcal{A}^{-1}(t^{\frac{1}{n}})}=\frac{t^{\frac{1}{n}}\mathcal{L}(t^{\frac{1}{n}})}{\mathcal{A}^{-1}(t^{\frac{1}{n}})}\longrightarrow 1$$

and thus another appeal to Lemma 2.2(b) tells us that $H(t)\to 1$ as $t\to\infty$. □

**2.3. Hausdorff and Minkowski dimensions.** A non-decreasing function $(0,\infty)\xrightarrow{h}(0,\infty)$ is called a *dimension gauge* provided $\lim_{t\to 0^+}h(t)=0$. We use a dimension gauge $h$ to define the (generalized) *Hausdorff measure* $\mathcal{H}^h$ via

$$\mathcal{H}^h(E):=\lim_{r\to 0^+}\left[\inf\left\{\sum_i h(\operatorname{diam}A_i):E\subset\bigcup_i A_i,\operatorname{diam}(A_i)\leq r\right\}\right]$$

for any set $E\subset\mathsf{R}^n$.

Typically we are only interested in knowing whether this quantity is zero, or positive and finite, or infinite. For this we can assume that the covering sets $A_i$ are balls $\mathsf{B}(a_i,r_i)$ with $r_i\leq r$, and then $h(\operatorname{diam}A_i)$ is replaced with $h(2r_i)$; doing this does not change the positivity or the finiteness of $\mathcal{H}^h(E)$.

When we consider covering sets that are balls all having the same radius, we are lead to the notion of Minkowski content; the (generalized) *upper Minkowski content* $\bar{\mathcal{M}}^h$ is defined by

$$\bar{\mathcal{M}}^h(E) := \limsup_{r\to 0^+} |E_r| h(r) r^{-n},$$

where $E \subset \mathsf{R}^n$ is any set and $|E_r|$ denotes the Lebesgue $n$-measure of the set

$$E_r := \{x \in \mathsf{R}^n \mid \operatorname{dist}(x,E) \le r\} = \bigcup_{x\in E} \bar{\mathsf{B}}(x,r).$$

When $h(t) = t^s$ for some $s > 0$, we use the standard notations $\mathcal{H}^s$ and $\bar{\mathcal{M}}^s$ instead of $\mathcal{H}^h$ and $\bar{\mathcal{M}}^h$, and then $\mathcal{H}^s(E)$ is called the *$s$-dimensional Hausdorff measure* of a set $E$ and $\bar{\mathcal{M}}^s(E)$ is the *$s$-dimensional outer Minkowski content* of $E$. The *Hausdorff dimension* of $E$ is determined by

$$\dim_{\mathcal{H}}(E) := \inf\{s > 0 \mid \mathcal{H}^s(E) = 0\}$$

and the *upper Minkowski dimension* of $E$ is

$$\overline{\dim}_{\mathcal{M}}(E) := \inf\{s > 0 \mid \bar{\mathcal{M}}^s(E) = 0\}.$$

However, in this paper we require a finer notion of "size"; for example, we will need to distinguish the "sizes" of certain zero-dimensional sets.

It is easy to check that for any two dimension gauge functions $g$ and $h$,

$$\mathcal{H}^h(E) \le \limsup_{t\to 0^+} \frac{h(t)}{g(t)} \mathcal{H}^g(E) \quad \text{and} \quad \bar{\mathcal{M}}^h(E) \le \limsup_{t\to 0^+} \frac{h(t)}{g(t)} \bar{\mathcal{M}}^g(E).$$

With this in mind, we impose an ordering on dimension gauges as follows: given two such functions $g$ and $h$ we write

$$g \preceq h \iff \limsup_{t\to 0^+} \frac{h(t)}{g(t)} < \infty \quad \text{and} \quad g \prec h \iff \lim_{t\to 0^+} \frac{h(t)}{g(t)} = 0.$$

Here are some simple examples.

(1) When $r > 0$ and $s > 0$: $r < s \iff t^r \prec t^s$.
(2) When $p > 0$ and $q > 0$: $t^p[\log(1/t)]^q \prec t^p \prec t^p/[\log(1/t)]^q$.
(3) For any $\alpha, \beta \in \mathsf{R}$: $\alpha < \beta \iff [\log(1/t)]^{-\alpha} \prec [\log(1/t)]^{-\beta}$.
(4) When $p > 0$: $\alpha < \beta \iff \exp(-\alpha[\log(1/t)]^p) \prec \exp(-\beta[\log(1/t)]^p)$.

Notice that when $g \preceq h$, $\mathcal{H}^h \ll \mathcal{H}^g$. Also, if $g \prec h$, then $\mathcal{H}^g(E) < \infty \implies \mathcal{H}^h(E) = 0$; that is, sets that are "small" with respect to $\mathcal{H}^g$ are $\mathcal{H}^h$-null sets.

We can use this order to see what are the "best" gauges. For example, in Theorem B, we verify that a certain set has positive measure. In this setting, the "best" gauge is the biggest: if $g \prec h$, then $h$ is a better gauge in the sense that $\mathcal{H}^h(E) > 0$ is a stronger statement than $\mathcal{H}^g(E) > 0$. On the other hand, in many of our examples we construct a certain set with zero measure, and in this setting the "best" gauge is the smallest: if $g \prec h$, then $g$ is a better gauge in the sense that $\mathcal{H}^g(E) = 0$ is a stronger statement than $\mathcal{H}^h(E) = 0$.

We remark that for any $\alpha \in \mathsf{R}$ and any $p<1$, both of the gauges $[\log(1/t)]^{-\alpha}$ and $\exp(-\alpha[\log(1/t)]^p)$ are zero-dimensional. That is, if the measure of a set $E$ (with respect to either of these gauges) is finite, then $\dim_{\mathcal{H}}(E)=0$.

**2.4. Capacity estimates.** Our proof of Theorem A depends on a capacity estimate that we provide here. The *(variational) p-capacity* of a compact set $E\subset\Omega$, relative to $\Omega$, is

$$\operatorname{cap}_p(E;\Omega) := \inf_{u\in\mathcal{W}} \int_\Omega |\nabla u|^p,$$

where $\mathcal{W} := \mathcal{C}(\Omega)\cap \mathsf{W}_0^{1,p}(\Omega)$ is the family of all functions $u$ that are continuous in $\Omega$, possess weak derivatives whose $p$th-powers are integrable, have zero 'boundary values', and satisfy $u\geq 1$ on $E$. Standard arguments permit us to assume that $u\in\mathcal{C}_0^\infty(\Omega)$ with $0\leq u\leq 1$, and we call these latter functions *admissible* for $\operatorname{cap}_p(E;\Omega)$; see [HKM93, pp. 27–28].

We write $\operatorname{cap}=\operatorname{cap}_n$ for the *conformal* $n$-capacity in $\mathsf{R}^n$.

The following is [HK03, Corollary 2.5].

FACT 2.6. Let $E$ be a continuum joining some point $a$ to the sphere $\mathsf{S}(a;r)$. Suppose that $v\in\mathsf{W}^{1,p}(\mathsf{B}(a;r),\mathsf{R})$ is continuous, satisfies $v\geq 1$ on $E$, and has integral average $v_{\mathsf{B}(a;r)}\leq 1/2$. Then for each $n-1<p<n$,

$$\fint_{\mathsf{B}(a;r)} |\nabla v|^p \geq C(p,n) r^{-p}.$$

**2.5. Quasiconformal compression.** We recall that for $\lambda\geq 1$, the map $x\mapsto |x|^{\lambda-1}x$ defines a $K$-quasiconformal self-homeomorphism of $\mathsf{R}^n$, with $K=\lambda^{n-1}$. Given $\lambda\geq 1$ and $\sigma\in(0,1)$, we define

$$\Psi(x) := \begin{cases} x & \text{for } x\in\mathsf{R}^n\setminus\mathsf{B}^n,\\ |x|^{\lambda-1}x & \text{for } x\in\mathsf{B}^n\setminus\sigma\mathsf{B}^n,\\ \sigma^{\lambda-1}x & \text{for } x\in\sigma\mathsf{B}^n.\end{cases}$$

We note that

$$\Psi(x)=x \quad \text{on } |x|=1 \quad \text{and} \quad \Psi(x)=\sigma^{\lambda-1}x \quad \text{on } |x|=\sigma.$$

In particular, $\Psi$ is a $\lambda^{n-1}$-quasiconformal self-homeomorphism of $\mathsf{R}^n$ that is the identity in $\mathsf{R}^n\setminus\mathsf{B}^n$, conformal in $\sigma\mathsf{B}^n$, and with

$$\Psi\big(\mathsf{B}^n\big)=\mathsf{B}^n \quad \text{and} \quad \Psi\big(\sigma\mathsf{B}^n\big)=\sigma^\lambda\mathsf{B}^n, \quad \text{so } \Psi\big(\mathsf{B}^n\setminus\sigma\mathsf{B}^n\big)=\mathsf{B}^n\setminus\sigma^\lambda\mathsf{B}^n.$$

Moreover, the distortion of $\Psi$ "lives" in $\mathsf{B}^n\setminus\sigma\mathsf{B}^n$; that is, $\Psi$ is conformal in $\sigma\mathsf{B}^n\cup(\mathsf{R}^n\setminus\bar{\mathsf{B}}^n)$.

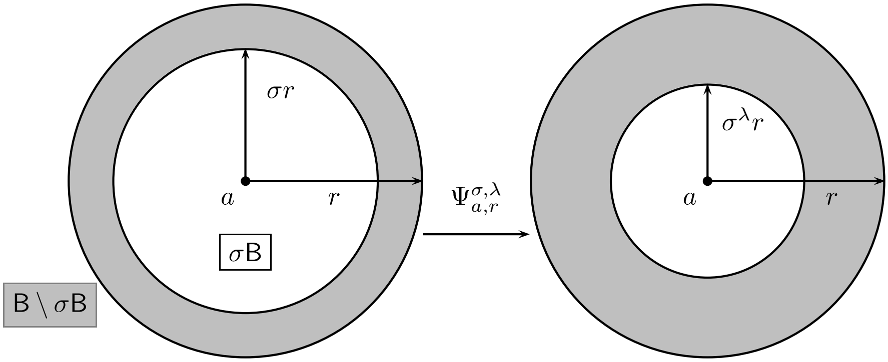


FIGURE 1. The radial squeeze-stretch map $\Psi_{a,r}^{\sigma,\lambda}$.

By employing auxiliary similarity maps, we can transport the action of $\Psi$ to any ball $\mathsf{B} := \mathsf{B}(a, r)$; see Figure 1. We define $\Psi_{a,r}^{\sigma,\lambda}$ via

$$\Psi_{a,r}^{\sigma,\lambda}(x) := \begin{cases} x & \text{for } x \in \mathsf{R}^n \setminus \mathsf{B}, \\ a + |\frac{x-a}{r}|^{\lambda-1}(x-a) & \text{for } x \in \mathsf{B} \setminus \sigma\mathsf{B}, \\ a + \sigma^{\lambda-1}(x-a) & \text{for } x \in \sigma\mathsf{B}. \end{cases}$$

Then $\Psi_{a,r}^{\sigma,\lambda}$ is a $\lambda^{n-1}$-quasiconformal self-homeomorphism of $\mathsf{R}^n$ that is the identity in $\mathsf{R}^n \setminus \mathsf{B}$, conformal in $\sigma\mathsf{B}$, and with

$$\begin{aligned} &\Psi_{a,r}^{\sigma,\lambda}(\mathsf{B}) = \mathsf{B} \quad \text{and} \quad \Psi_{a,r}^{\sigma,\lambda}(\sigma\mathsf{B}) = \sigma^\lambda\mathsf{B}, \quad \text{so } \Psi_{a,r}^{\sigma,\lambda}(\mathsf{B}\setminus\sigma\mathsf{B}) = \mathsf{B}\setminus\sigma^\lambda\mathsf{B}, \\ &\quad \text{and also for all points } x \in \mathsf{R}^n, \quad \left|\Psi_{a,r}^{\sigma,\lambda}(x) - x\right| \le r. \end{aligned} \tag{2.15}$$

We call $\Psi_{a,r}^{\sigma,\lambda}$ a *radial squeeze-stretch mapping*: it "squeezes" the ball $\sigma\mathsf{B}$ to $\sigma^\lambda\mathsf{B}$ via scaling by $\sigma^{\lambda-1}$ and "stretches" the spherical ring $\mathsf{B}\setminus\sigma\mathsf{B}$ to the ring $\mathsf{B}\setminus\sigma^\lambda\mathsf{B}$ via the radial map $x \mapsto |x|^{\lambda-1}x$. In addition, the distortion of $\Psi_{a,r}^{\sigma,\lambda}$ "lives" in the ring $\mathsf{B}\setminus\sigma\bar{\mathsf{B}}$; that is, $\Psi_{a,r}^{\sigma,\lambda}$ is conformal in $\sigma\mathsf{B}\cup(\mathsf{R}^n\setminus\bar{\mathsf{B}})$. Finally, we note that the radial squeeze-stretch map $\Psi_{a,r}^{\sigma,\lambda}$ is uniquely determined by the concentric ball triple

$$\big(\mathsf{B}, \sigma\mathsf{B}, \sigma^\lambda\mathsf{B}\big) := \big(\mathsf{B}(a,r), \mathsf{B}(a,\sigma r), \mathsf{B}\big(a, \sigma^\lambda r\big)\big).$$

## 3. Proofs of theorems

Here—in Sections 3.1, 3.2 and 3.3—we corroborate Theorems A, B and C.

**3.1. Proof of Theorem A.** We assume that $\Omega \xrightarrow{f} \Omega'$ is a finite distortion homeomorphism (between domains in $\mathsf{R}^n$) with $K = K_f$ satisfying $\exp\mathcal{A}(pK) \in \mathsf{L}^1_{\mathrm{loc}}(\Omega)$ for some $p > 0$; see Section 2.2.1 for the precise hypotheses on $\mathcal{A}$. We establish inequality (1.2).

An affine change of variables permits us to assume that $z=0$, $f(0)=0$, and $R=3/2$, in which case the asserted inequality (1.2) reduces to

$$\forall |x|<\frac{1}{4}, \quad \bigl|f(x)\bigr| \geq D\exp\biggl(-\frac{C(n)}{p^{\frac{1}{n-1}}}\omega\biggl(\log\frac{3\Lambda}{2|x|}\biggr)\biggr);$$

here $D=(1/2)\operatorname{dist}(0,\partial B')$ and $\Lambda^n = ⨍_{\mathsf{B}(0,3/2)}\exp\mathcal{A}(pK)$, where $B=\mathsf{B}(0,1/2)$ and $B'=f(B)$.

Fix a point $a\in\mathsf{B}(0,1/4)$ and let $a'=f(a)$. We can assume $|a'|<D$ (for otherwise we are done) and then $\operatorname{dist}(a',\partial B')>D$, so the line segment $E'=[0,a']$ lies inside of $B'$. We then have the standard capacity estimate [Väi71, 7.5, p. 22]

$$\operatorname{cap}\bigl(E',B'\bigr) \leq \omega_{n-1}\Big/\biggl(\log\frac{D}{|a'|}\biggr)^{n-1},$$

where $\omega_{n-1}$ denotes the measure of the unit sphere $\mathsf{S}^{n-1}$.

Having established this upper bound, we now seek a lower bound for this capacity. Let $u$ be an admissible test function for $\operatorname{cap}(E',B')$ and put $v=u\circ f$. The chain rule in conjunction with the distortion inequality (1.1) yield

$$\int_{B'}|\nabla u|^n \geq \int_{B'}|\nabla v\circ f|^n J_f \geq \int_B \frac{|\nabla v|^n}{K}.$$

Thus we search for lower bounds for the integral on the right-hand side. In fact, we show that

$$\int_B \frac{|\nabla v|^n}{K} \geq C(\mathcal{A},n)p\Big/\omega\biggl(\log\frac{3\Lambda}{2|a|}\biggr)^{n-1} \tag{3.1}$$

and this will finish the proof. Indeed, combining (3.1) with the above capacity estimate, and taking the infimum over all testing functions $u$, we obtain

$$\frac{\omega_{n-1}}{\log^{n-1}(D/|a'|)} \geq \frac{C(\mathcal{A},n)p}{\omega(\log\frac{3\Lambda}{2|a|})^{n-1}}$$

and therefore, as asserted,

$$\bigl|a'\bigr| \geq D\exp\biggl(-\frac{C(\mathcal{A},n)}{p^{1/(n-1)}}\omega\biggl(\log\frac{3\Lambda}{2|a|}\biggr)\biggr).$$

To establish (3.1), we examine two cases, depending on whether or not the average value $v_A$ of $v$ over the ball $A:=\mathsf{B}(0,|a|)\subset B$ exceeds $1/2$.

*The case* $v_A\leq 1/2$. Here we appeal to Fact 2.6, taking $p:=n^2/(n+1)$, and use Hölder's inequality to see that

$$\frac{C(n)}{|a|^p} \leq ⨍_A |\nabla v|^p \leq \biggl(⨍_A \frac{|\nabla v|^n}{K}\biggr)^{p/n}\biggl(⨍_A K^n\biggr)^{p/n^2},$$

so,

$$\fint_A \frac{|\nabla v|^n}{K} \geq \frac{C(n)}{|a|^n}\left(\fint_A K^n\right)^{-1/n}$$

and hence

$$\int_B \frac{|\nabla v|^n}{K} \geq |A| \fint_A \frac{|\nabla v|^n}{K} \geq C(n)\left(\fint_A K^n\right)^{-1/n}.$$

Our next goal is to obtain an upper bound for $\fint_A K^n$. Consider the auxiliary function

$$\varphi(t) := \exp \mathcal{A}\big(pt^{1/n}\big).$$

We would like to make use of Jensen's Inequality; see the discussion at the beginning of Section 2.2.3. If we knew that $\varphi$ were convex, then we would obtain

$$\varphi\left(\fint_A K^n\right) \leq \fint_A \varphi\big(K^n\big) \leq \frac{1}{|A|}\int_{\mathsf{B}(0,3/2)} \exp \mathcal{A}(pK) = \left(\frac{3}{2}\frac{\Lambda}{|a|}\right)^n \tag{3.2}$$

so that

$$\fint_A K^n \leq \varphi^{-1}\left(\left(\frac{3}{2}\frac{\Lambda}{|a|}\right)^n\right) = \left(\frac{1}{p}\mathcal{A}^{-1}\left(n \log\left(\frac{3}{2}\frac{\Lambda}{|a|}\right)\right)\right)^n \tag{3.3}$$

and thus

$$\int_B \frac{|\nabla v|^n}{K} \geq C(n)\left(\fint_A K^n\right)^{-1/n} \geq \frac{C(n)p}{\mathcal{A}^{-1}(n\log(\frac{3}{2}\frac{\Lambda}{|a|}))} \geq \frac{C(\mathcal{A},n)p}{\mathcal{A}^{-1}(\log(\frac{3}{2}\frac{\Lambda}{|a|}))}, \tag{3.4}$$

where the doubling property of $\mathcal{A}^{-1}$ was used to obtain the very last inequality. The above estimate is, in fact, stronger than inequality (3.1).

The problem with this approach is that we do not know that $\varphi$ is convex. To deal with this issue, we use the facts that $\varphi$ is increasing and that $t \mapsto \varphi(t)/t$ is increasing on $[\tau_p, \infty)$, where

$$\tau_p := (t_{1/n}/p)^n \quad \text{and} \quad t_{1/n} := \inf\big\{t \geq 0 \mid t\mathcal{A}'(t) \geq n\big\}.$$

Thus for any $\tau \geq \tau_p$ we have

$$\begin{aligned}
\int_A K^n &\leq \int_{A\cap\{K^n \geq \tau\}} K^n + \tau|A| \leq \frac{\tau}{\varphi(\tau)}\int_{A\cap\{K^n\geq\tau\}} \varphi\big(K^n\big) + \tau|A| \\
&\leq \frac{\tau}{\varphi(\tau)}\int_{B(0;3/2)} \exp \mathcal{A}(pK) + \tau|A| \\
&\leq \tau|A|\left(1 + \frac{1}{\varphi(\tau)}\left(\frac{3}{2}\frac{\Lambda}{|a|}\right)^n\right);
\end{aligned}$$

the last inequality just above is, again, a consequence of the fact

$$\frac{1}{|A|}\int_{B(0;3/2)} \exp \mathcal{A}(pK) = \frac{|\mathsf{B}(0,3/2)|}{|A|}\Lambda^n = \left(\frac{3}{2}\frac{\Lambda}{|a|}\right)^n.$$

So, for each $\tau \geq \tau_p$,

$$\fint_A K^n \leq \tau\left(1 + \frac{1}{\varphi(\tau)}\left(\frac{3}{2}\frac{\Lambda}{|a|}\right)^n\right).$$

If $\tau_p \leq \varphi^{-1}(6^n)$, then, as $\frac{3\Lambda}{2|a|} \geq 6$, we can apply the above with $\tau := \varphi^{-1}((\frac{3\Lambda}{2|a|})^n)$ to get

$$\fint_A K^n \leq 2\tau = 2\left(\frac{1}{p}\mathcal{A}^{-1}\left(n\log\left(\frac{3}{2}\frac{\Lambda}{|a|}\right)\right)\right)^n.$$

Except for the extra factor of 2, this is just inequality (3.3), and again we get (3.4).

For the general case, we appeal to Lemma 2.4 (with $\alpha = 1/n$ and $M = 6^n$) to get a constant $C = C(\mathcal{A}, n) \geq 1$ such that $\tau_p \leq C\varphi^{-1}(6^n)$. Then we apply the above with $\tau := C\varphi^{-1}((3\Lambda/2|a|)^n)$ to get

$$\fint_A K^n \leq \tau\left(1 + \frac{\varphi(\tau/C)}{\varphi(\tau)}\right) \leq 2\tau = 2C\left(\frac{1}{p}\mathcal{A}^{-1}\left(n\log\left(\frac{3}{2}\frac{\Lambda}{|a|}\right)\right)\right)^n.$$

Except for the extra factor of $2C$, this is just inequality (3.3), and once again we get (3.4).

*The case* $v_A \geq 1/2$. Here we utilize a chaining argument together with a Poincaré inequality. In order to facilitate a technical calculation below, we first rescale via the change of variable $g(x) := f(x/\sigma)$. Then $K_g(x) = K(x/\sigma)$ is a finite distortion function for $g$ with

$$L_g := \int_{B(0;3\sigma/2)} \exp\mathcal{A}(pK_g) = \sigma^n L_f, \quad \text{where } L_f := \int_{B(0;3/2)} \exp\mathcal{A}(pK_f);$$

so taking $\sigma = (\Omega_n/L_f)^{1/n}$ we obtain $L_g = \Omega_n$. Next, let $w(x) := v(x/\sigma)$ and note that $w_{\sigma A} = v_A$ and also $\int_{\sigma B} |\nabla w|^n/K_g = \int_B |\nabla v|^n/K$. Thus, we are still in the case $w_{\sigma A} \geq 1/2$ searching for a lower bound for the integral $\int_{\sigma B} |\nabla w|^n/K_g$.

Let $\nu \geq 2$ be the integer with $1/2^{\nu+1} < \sigma|a| \leq 1/2^\nu$; so $\nu \simeq \log(1/(\sigma|a|)$. Put $b := (1, 0, \ldots, 0)$ and consider the balls

$$A_i = \mathsf{B}(a_i; r_i/2) \quad \text{and} \quad B_i = \mathsf{B}(b_i; r_i),$$

where

$$r_i := 1/2^{\nu-i+1}, \qquad b_i := 2r_i b, \qquad a_i := b_i + (r_i/2)b, \quad \text{and } i \geq 1.$$

Also, put $B_0 := \mathsf{B}(0; 1/2^\nu)$ and $A_0 := \mathsf{B}(a_0; r_1/4)$ where $a_0 := (5/4)r_1 b$. Then for each $i \geq 1$: $A_{i-1}, A_i \subset B_i$ with each of $\partial A_{i-1}$, $\partial A_i$ being tangent to $\partial B_i$ and $2\operatorname{diam} A_{i-1} = \operatorname{diam} A_i = (1/2)\operatorname{diam} B_i$; also, $\sigma A, A_0 \subset B_0$.

Let $\ell$ be the smallest integer with $1/2^{\nu-\ell} \geq \sigma/2$; so $1 \leq \ell \leq \nu - 1$ as $\sigma < 1$. Then $A_\ell$ lies in the complement of $\sigma B = \mathsf{B}(0; \sigma/2)$, so $w_{A_\ell} = 0$ because the support of $w$ lies in $\sigma B$. Thus we can write

$$1/2 \le w_{\sigma A} = (w_{\sigma A} - w_{A_0}) + (w_{A_0} - w_{A_1}) + \cdots + (w_{A_{\ell-1}} - w_{A_\ell}).$$

Next, employing a Poincaré inequality, we can estimate the absolute value of each of these terms thereby obtaining

$$C(n) \le \sum_{i=0}^{\ell} \operatorname{diam}(B_i) \fint_{B_i} |\nabla w|.$$

Now we use Hölder's inequality twice, first on each of the integrals, and then on the sum itself, to get

$$C(n) \le \left( \sum_{i=0}^{\ell} \big(\operatorname{diam}(B_i)\big)^n \fint_{B_i} \frac{|\nabla w|^n}{K_g} \right)^{1/n} \left( \sum_{i=0}^{\ell} \fint_{B_i} K_g^{1/(n-1)} \right)^{(n-1)/n}.$$

The first factor on the right-hand side above can be estimated from above by (a constant times) $(\int_{\sigma B} |\nabla w|^n / K_g)^{1/n}$; this is because $B_i \cap \operatorname{supp}(w) \subset \sigma B$ and the balls $B_i$ have bounded overlap. Thus, raising to the power $n$ provides us with

$$C(n) \le \left( \int_{\sigma B} \frac{|\nabla w|^n}{K_g} \right) \left( \sum_{i=0}^{\ell} \fint_{B_i} K_g^{1/(n-1)} \right)^{n-1}. \tag{3.5}$$

It therefore remains to exhibit an upper bound for $(\sum_{i=0}^{\ell} \fint_{B_i} K_g^{1/(n-1)})^{n-1}$.

In fact we verify that

$$\left( \sum_{i=0}^{\ell} \fint_{B_i} K_g^{1/(n-1)} \right)^{n-1} \le \frac{C(\mathcal{A}, n)}{p} \omega \left( \log \frac{3\Lambda}{2|a|} \right)^{n-1}. \tag{3.6}$$

It is easy to see that (3.1) is an immediate consequence of (3.6) in conjunction with (3.5), once we recall that $\int_{\sigma B} |\nabla w|^n / K_g = \int_B |\nabla v|^n / K$.

Our next goal is to obtain an upper bound for each integral average $\fint_{B_i} K_g^{1/(n-1)}$. Notice that for each $0 \le i \le \ell$, $B_i \subset (3\sigma/2)\mathsf{B}^n = \sigma B$. Consider the auxiliary function

$$\varphi(t) := \exp \mathcal{A}\big(pt^{n-1}\big).$$

We would like to use Jensen's Inequality; see the discussion at the beginning of Section 2.2.3. If we knew that $\varphi$ were convex, then (recall the rescaling done above to ensure that $L_g = \Omega_n$) we would obtain

$$\begin{aligned} \varphi\left( \fint_{B_i} K_g^{1/(n-1)} \right) &\le \fint_{B_i} \exp \mathcal{A}(pK_g) \\ &\le \frac{1}{|B_i|} \int_{3\sigma B} \exp \mathcal{A}(pK_g) = \frac{L_g}{|B_i|} = r_i^{-n} \end{aligned} \tag{3.7}$$

so

$$\fint_{B_i} K_g^{1/(n-1)} \leq \varphi^{-1}\big(r_i^{-n}\big) = \left(\frac{1}{p}\mathcal{A}^{-1}\left(n\log\frac{1}{r_i}\right)\right)^{1/(n-1)}. \tag{3.8}$$

The problem with this approach is that we do not know that $\varphi$ is convex. To deal with this issue (see the discussion at the beginning of Section 2.2.3), we use the facts that $\varphi$ is increasing and that $t \mapsto \varphi(t)/t$ is also increasing on $[\tau_p, \infty)$, where

$$\tau_p := (t_{n-1}/p)^{1/(n-1)} \quad \text{and} \quad t_{n-1} := \inf\big\{t \geq 0 \mid t\mathcal{A}'(t) \geq 1/(n-1)\big\}.$$

Thus for any $\tau \geq \tau_p$ we have

$$\begin{aligned}
\int_{B_i} K_g^{1/(n-1)} &\leq \int_{B_i \cap \{K_g \geq \tau^{n-1}\}} K_g^{1/(n-1)} + \tau|B_i| \\
&\leq \frac{\tau}{\varphi(\tau)} \int_{B_i \cap \{K_g \geq \tau^{n-1}\}} \varphi\big(K_g^{1/(n-1)}\big) + \tau|B_i| \\
&\leq \frac{\tau}{\varphi(\tau)} \int_{3\sigma B} \exp\mathcal{A}(pK_g) + \tau|B_i| \\
&\leq \tau|B_i|\left(1 + \frac{1}{\varphi(\tau) r_i^n}\right);
\end{aligned}$$

the last inequality just above is, again, a consequence of the fact that $L_g = \Omega_n$.

So, for each $\tau \geq \tau_p$,

$$\fint_{B_i} K_g^{1/(n-1)} \leq \tau\left(1 + \frac{1}{\varphi(\tau) r_i^n}\right).$$

If $\tau_i = \varphi^{-1}(r_i^{-n}) \geq \tau_p$, then we can apply the above to get

$$\fint_{B_i} K_g^{1/(n-1)} \leq 2\tau_i = 2\left(\frac{1}{p}\mathcal{A}^{-1}\left(n\log\frac{1}{r_i}\right)\right)^{1/(n-1)}.$$

Except for the extra factor of 2, this is just inequality (3.8).

For the general case, we note that for each $0 \leq i \leq \ell$, $r_i^{-n} \geq 4^n$. We appeal to Lemma 2.4 (with $\alpha = n-1$ and $M = 4^n$) to get a constant $C = C(\mathcal{A}, n) \geq 1$ such that $\tau_p \leq C\varphi^{-1}(4^n)$. Then we apply the above with $\tau_i := C\varphi^{-1}(r_i^{-n}) \geq \tau_p$ to get

$$\fint_{B_i} K_g^{1/(n-1)} \leq 2\tau_i = 2C\varphi^{-1}\big(r_i^{-n}\big) = 2C\left(\frac{1}{p}\mathcal{A}^{-1}\left(n\log\frac{1}{r_i}\right)\right)^{1/(n-1)}. \tag{3.9}$$

Except for the extra factor of $2C$, (3.9) is once again inequality (3.8).

Finally, we demonstrate that inequality (3.9) implies (3.6). We have

$$r_i^{-1} = 2^{\nu-i+1} = 2^j, \quad \text{where } 2 \leq j := \nu - \ell + 1 \leq \nu + 1.$$

Using the facts that $\mathcal{A}^{-1}$ is both increasing and doubling we obtain

$$\begin{aligned}
&\sum_{i=0}^{\ell}\mathcal{A}^{-1}\left(n\log\frac{1}{r_i}\right)^{1/(n-1)}\\
&\quad=\sum_{j=\nu-\ell+1}^{\nu+1}\mathcal{A}^{-1}(jn\log 2)^{1/(n-1)}\\
&\quad\le\sum_{j=2}^{\nu+1}\mathcal{A}^{-1}(jn\log 2)^{1/(n-1)}\le\nu\mathcal{A}^{-1}\big((\nu+1)n\log 2\big)^{1/(n-1)}\\
&\quad\le C\nu\mathcal{A}^{-1}(\nu)^{1/(n-1)}\le C\log\frac{3\Lambda}{2|a|}\mathcal{A}^{-1}\left(\log\frac{3\Lambda}{2|a|}\right)^{1/(n-1)}\\
&\quad=C\omega\left(\log\frac{3\Lambda}{2|a|}\right);
\end{aligned}$$

here the last two inequalities hold because

$$\nu\simeq\log\frac{1}{\sigma|a|}=\log\frac{3\Lambda}{2|a|}.$$

Evidently, the above in conjunction with (3.9) gives (3.6).

**3.2. Proof of Theorem B.** We assume $\Omega\xrightarrow{f}\Omega'$ is a finite distortion homeomorphism with $\exp\mathcal{A}(pK_f)\in\mathsf{L}^1_{\mathrm{loc}}(\Omega)$ for some $p>0$. Also, we have the Hausdorff gauge function

$$h(t)=h_{s,p,\mathcal{A},n}(t):=\exp\left(-s\omega^{-1}\left(\frac{p^{1/(n-1)}}{C}\log\frac{1}{t}\right)\right).$$

Here $s\in(0,n]$ and $C=C(\mathcal{A},n)$ is the constant from Theorem A. We demonstrate that for each $E\subset\Omega$, $\mathcal{H}^h(f(E))=0$ implies that $\mathcal{H}^s(E)=0$.

Set $g:=f^{-1}$. According to (1.3), for each point $a\in\Omega'$, there are constants $L=L(a)$ and $D=D(a)\in(0,\mathrm{dist}(a,\partial\Omega'))$ such that for each $y\in\mathsf{B}(a,D)$,

$$\big|g(y)-g(a)\big|\le L\exp\left(-\omega^{-1}\left(\frac{p^{1/(n-1)}}{C}\log\frac{D}{|y-a|}\right)\right).$$

Thus for all $r\in(0,D]$

$$\mathrm{diam}\big(g\big[\mathsf{B}(a,r)\big]\big)\le 2L\exp\left(-\omega^{-1}\left(\frac{p^{1/(n-1)}}{C}\log\frac{D}{r}\right)\right).$$

For integers $j$, $k$ with $j\ge 2$, $k\ge 1$ we define

$$F_{jk}:=\big\{a\in\Omega'\mid D(a)\ge 1/j,2L(a)\le k\big\}.$$

Then $\Omega'=\bigcup_{j,k}F_{jk}$. Also, for each $a\in F_{jk}$ and all $r\in(0,1/j)$,

$$\mathrm{diam}\big(g\big[\mathsf{B}(a,r)\big]\big)\le k\exp\left(-\omega^{-1}\left(\frac{p^{1/(n-1)}}{C}\log\frac{1}{jr}\right)\right).$$

Suppose $E \subset \Omega$ with $\mathcal{H}^h(f(E)) = 0$. Fix integers $j$, $k$ with $j \geq 2$, $k \geq 1$. We show that $\mathcal{H}^s(E \cap g(F_{jk})) = 0$. Let $\varepsilon > 0$ be given. Select $\rho \in (0, 2/j^2)$. Note that

$$0 < r < \frac{2}{j^2} \quad \Longrightarrow \quad \frac{1}{2r} < \frac{1}{(jr)^2} \quad \text{and so} \quad \log\frac{1}{2r} < 2\log\frac{1}{jr}.$$

Since $\mathcal{H}^h(f(E) \cap F_{jk}) = 0$, there are balls $B_i := \mathsf{B}(a_i, r_i)$ with $a_i \in f(E) \cap F_{jk}$, $r_i \in (0, \rho)$, $f(E) \cap F_{jk} \subset \bigcup_i B_i$ and such that $\sum_i h(2r_i) < \varepsilon$. As $a_i \in F_{jk}$, $r_i < \rho < 2/j^2 \leq 1/j \leq D(a_i)$ and thus

$$\begin{aligned} \operatorname{diam}\bigl(g(B_i)\bigr) &\leq k \exp\left(-\omega^{-1}\left(\frac{p^{1/(n-1)}}{C}\log\frac{1}{jr_i}\right)\right) \\ &\leq k \exp\left(-\omega^{-1}\left(\frac{p^{1/(n-1)}}{C}\log\frac{1}{2r_i}\right)\right). \end{aligned}$$

Therefore

$$\begin{aligned} \sum_i \bigl(\operatorname{diam}\bigl(g(B_i)\bigr)\bigr)^s &\leq k^s \sum_i \exp\left(-s\omega^{-1}\left(\frac{p^{1/(n-1)}}{C}\log\frac{1}{2r_i}\right)\right) \\ &= k^s \sum_i h(2r_i) \leq k^s \varepsilon. \end{aligned}$$

Since $E \cap g(F_{jk}) \subset \bigcup_i g(B_i)$, by letting $\varepsilon \searrow 0$ in the above we conclude that

$$\mathcal{H}^s\bigl(E \cap g(F_{jk})\bigr) = 0. \qquad \square$$

**3.3. Proof of Theorem C.** Here we assume that

$$\mathcal{A}(t) = t/\mathcal{L}(t), \quad \text{where } \mathcal{L}(t) = \mathcal{L}_k(t) = L_1(t) \cdots L_k(t)$$

for some $k \in \mathsf{N}$. See (2.5). We work with the gauge functions

$$h_\beta(t) := t^n L_{k+1}(1/t)^\beta.$$

Suppose $\Omega \xrightarrow{f} \Omega'$ is a finite distortion homeomorphism between domains $\Omega, \Omega' \subset \mathsf{R}^n$ with $\exp \mathcal{A}(pK_f) \in \mathsf{L}^1_{\mathrm{loc}}(\Omega)$ for some $p > 0$. In this setting, (2.13) tells us that

$$\forall \beta < cp, \quad J_f \in \mathsf{L}^{P_\beta}_{\mathrm{loc}}(\Omega),$$

where $c = c(k, n)$ and $P_\beta(t) := tL_{k+1}(t)^\beta$. With this in mind, we define $Q_\beta(t) := tL^{-1}_{k+1}(t^{1/\beta})$. Then $P_\beta$ and $Q_\beta$ satisfy Young's inequality (2.7), so the Orlicz–Hölder inequality (2.8) is in force. Also, thanks to Lemma 2.2(c) we know that

$$Q_\beta^{-1}(t) \simeq L_{k+1}(t)^\beta \quad \text{as } t \to \infty.$$

Now fix $\beta \in (0, cp)$. Suppose a compact set $E \subset \Omega$ has upper Minkowski dimension $\overline{\dim}_{\mathcal{M}}(E) < n$. Let $F := f(E)$ and pick $\varepsilon > 0$ with $n - \varepsilon > \overline{\dim}_{\mathcal{M}}(E)$; so, $\bar{\mathcal{M}}^{n-\varepsilon}(E) = 0$. Noting that

$$\alpha > \beta \quad \text{and} \quad \bar{\mathcal{M}}^{h_\alpha}(F) < \infty \quad \Longrightarrow \quad \bar{\mathcal{M}}^{h_\beta}(F) = 0,$$

we deduce that it suffices to show that $\bar{\mathcal{M}}^{h_\beta}(F) < \infty$. To do this, we demonstrate that $|F_r| h_\beta(r) r^{-n}$ has a finite upper bound that is independent of $r$ and valid for all sufficiently small $r > 0$. Of course, $|F_r| h_\beta(r) r^{-n} = |F_r| L_{k+1}(1/r)^\beta$, and $|F_r|$ denotes the Lebesgue $n$-measure of the set $F_r := \{y \mid \operatorname{dist}(y, F) \le r\} = \bigcup_{y \in F} \bar{\mathsf{B}}(y, r)$.

To start, we choose $R > 0$ so that

$$E_R := \big\{x \mid \operatorname{dist}(x, E) \le R\big\} = \bigcup_{x \in E} \bar{\mathsf{B}}(x, R) \subset \Omega$$

and—by taking $R$ sufficiently small—so that

$$\forall \rho \in (0, R], \quad |E_\rho| \le \rho^\varepsilon,$$

where $|E_\rho|$ denotes the $n$-measure of $E_\rho := \bigcup_{x \in E} \bar{\mathsf{B}}(x, \rho)$. Next, since $f(E_R)$ is compact (so $f^{-1}$ is uniformly continuous on $f(E_R)$), there exists an $r_0 > 0$ such that for all points $a, y \in f(E_R)$ (say, $a = f(z), y = f(x)$) we have

$$|y - a| < r_0 \quad \Longrightarrow \quad |x - z| = \big|f^{-1}(y) - f^{-1}(a)\big| < R/6.$$

Suppose $r \in (0, r_0)$ and $y = f(x) \in F_r$. Then there is a point $a = f(z) \in F$ with $y \in \bar{\mathsf{B}}(a, r)$. Now $z \in E$, so $\mathsf{B}(z, R) \subset E_R \subset \Omega$, and $x \in \mathsf{B}(z, R/6)$. According to Theorem A, we thus have

$$r \ge |y - a| = \big|f(x) - f(z)\big| \ge D(z) \exp\biggl(-\frac{C}{p^{1/(n-1)}} \omega\biggl(\log \frac{\Lambda(z) R}{|x - z|}\biggr)\biggr).$$

Here $C = C(k, n)$ and $D(z) := (1/2) \operatorname{dist}(f(z), \partial f[\mathsf{B}(z, R/3)])$. As $f$ is a homeomorphism and $E$ is compact, there is a $\delta > 0$ with $D(\zeta) \ge \delta$ for all $\zeta \in E$. Also, for all $\zeta \in E$,

$$\Lambda(\zeta) := \biggl(\fint_{\mathsf{B}(\zeta, R)} \exp \mathcal{A}(pK)\biggr)^{1/n} \le \biggl(\frac{|E_R|}{\Omega_n R^n}\biggr)^{1/n} \biggl(\fint_{E_R} \exp \mathcal{A}(pK)\biggr)^{1/n} =: M,$$

where the constant $M$ depends only on the data. Therefore,

$$r \ge \delta \exp\biggl(-\frac{C}{p^{1/(n-1)}} \omega\biggl(\log \frac{MR}{|x - z|}\biggr)\biggr)$$

and so appealing to (2.14c) we obtain

$$p^{1/(n-1)} \log \frac{\delta}{r} \le C \omega\biggl(\log \frac{MR}{|x - z|}\biggr) \le C \mathcal{A}^{-1}\biggl(\log^n \frac{MR}{|x - z|}\biggr)^{1/(n-1)}$$

or equivalently,

$$\mathcal{A}\biggl(\frac{p}{C} \log^{n-1} \frac{\delta}{r}\biggr) \le \log^n \frac{MR}{|x - z|}.$$

Summarizing, for each $r \in (0, r_0)$ and all points $y = f(x) \in F_r$, there exists a $z \in E$ with

$$|x - z| \le \rho = \rho(r) := MR \exp\left(-\mathcal{A}\left(\frac{p}{C}\log^{n-1}\frac{\delta}{r}\right)^{1/n}\right).$$

In particular, we see that $x \in E_\rho$ and thus $F_r \subset f(E_\rho)$. By adjusting our choice of $r_0$, if necessary, we may ensure that for $r \in (0, r_0]$ we also have $\rho \in (0, R]$; for example, it suffices to pick $r_0$ with $\log^n M \le \mathcal{A}((p/C)\log^{n-1}(\delta/r_0))$.

It now follows that for all $r \in (0, r_0)$,

$$|F_r| \le \big|f(E_\rho)\big| \le \int_{E_\rho} J_f \le C\|J_f\|_{P_\beta}\|\chi_{E_\rho}\|_{Q_\beta};$$

here (2.8) provides the right-most inequality above. A glance back at (2.6) reveals that

$$\|\chi_{E_\rho}\|_{Q_\beta} = Q_\beta^{-1}\left(\frac{Q_\beta(1)}{|E_\rho|}\right)^{-1}, \quad \text{where } Q_\beta(1) = L_{k+1}^{-1}(1) = e_{k+2} - e_{k+1} =: C_k.$$

According to Lemma 2.2(c) and Fact 2.1,

$$Q_\beta^{-1}(t) \simeq L_{k+1}(t)^\beta \quad \text{and} \quad L_{k+1}\big(at^\varepsilon\big) \simeq L_{k+1}(t).$$

We claim that $L_{k+1}(\rho^{-1}) \simeq L_{k+1}(r^{-1})$ as $r \to 0^+$, and therefore as $r \to 0^+$

$$\|\chi_{E_\rho}\|_{Q_\beta} \simeq L_{k+1}\left(\frac{C_k}{|E_\rho|}\right)^{-\beta} \le L_{k+1}\left(\frac{C_k}{\rho^\varepsilon}\right)^{-\beta} \simeq L_{k+1}\big(\rho^{-1}\big)^{-\beta} \simeq L_{k+1}\big(r^{-1}\big)^{-\beta}.$$

The middle inequality above holds because $\rho \in (0, R]$ ensures that $|E_\rho| \le \rho^\varepsilon$.

Finally, by making use of the first and last estimates in the above paragraph, we see that for all $r \in (0, r_0)$ (again, adjusting $r_0$ as necessary),

$$|F_r| L_{k+1}\big(r^{-1}\big)^\beta \le C\|J_f\|_{P_\beta} < \infty;$$

this demonstrates that $\bar{\mathcal{M}}^{h_\beta}(F) < \infty$ as asserted.

It remains to check the claim that $L_{k+1}(\rho^{-1}) \simeq L_{k+1}(r^{-1})$ as $r \to 0^+$. This follows from the fact that for any positive constants $B, C, D$,

$$L_{k+1}\big(B\exp\big[\mathcal{A}\big(C\log^{n-1} Dt\big)^{1/n}\big]\big) \simeq L_{k+1}(t) \quad \text{as } t \to \infty.$$

To see this, we use the properties of $L_k$ explained in Fact 2.1 in conjunction with the two estimates that

$$\text{as } t \to \infty, \quad L_k\big(\mathcal{A}(t)\big) \simeq L_k(t) \quad \text{and} \quad L_k(\log t) \simeq L_{k+1}(t).$$

Thus for all sufficiently large $t$, we have

$$\begin{aligned} L_{k+1}\big(B\exp\big[\mathcal{A}\big(C\log^{n-1} Dt\big)^{1/n}\big]\big) &\simeq L_k\big(\log B + \mathcal{A}\big(C\log^{n-1} Dt\big)^{1/n}\big) \\ &\simeq L_k\big(\mathcal{A}\big(C\log^{n-1} t\big)^{1/n}\big) \\ &\simeq L_k\big(\mathcal{A}\big(C\log^{n-1} t\big)\big) \end{aligned}$$

$$\simeq L_k\big(C\log^{n-1} t\big) \simeq L_k\big(\log^{n-1} t\big)$$
$$\simeq L_k(\log t) \simeq L_{k+1}(t). \qquad \square$$

## 4. Compression examples

Here we present examples that illustrate to what extent Theorem A and Theorem B are optimal. See Example 4.7 for the former.

Our examples for Theorem B center on the gauge functions $h_{s,p,\mathcal{A},n}$ and are based on Cantor sets. A *generalized Cantor dust* is a compact set $E = \bigcap_1^\infty E_N$ where $E_1 \supset E_2 \supset \cdots \supset E_N \supset \cdots$ is a decreasing sequence of compact sets and each $E_N$ is a finite union of disjoint closed balls. In our examples, $E_N$ will be a union of certain closed subballs that are chosen from each of the balls that comprise $E_{N-1}$. We first give an overview, then describe our general construction, and then give specific examples.

**4.1. General construction.** We start with the closed unit ball $E_0 := \bar{\mathsf{B}} := \bar{\mathsf{B}}^n \subset \mathsf{R}^n$. We pick $m_1$ disjoint closed balls $E_i^1 \subset E_0$ $(1 \le i \le m_1)$ and put $E_1 := \bigcup_1^{m_1} E_i^1$. Next, for each $1 \le i \le m_1$, we pick $m_2$ disjoint closed balls $E_{ij}^2 \subset E_i^1$ $(1 \le j \le m_2)$ and put $E_2 := \bigcup_{i=1}^{m_1}\bigcup_{j=1}^{m_2} E_{ij}^2$. (In fact, we do this so that the sets $E_i^1 \setminus \bigcup_{j=1}^{m_2} E_{ij}^2$ are "isomorphic".) Continuing in this manner we get

$$E_N := \bigcup_J E_J^N, \quad \text{where } J = (j_1,\dots,j_N) \in \{1,\dots,m_1\} \times \cdots \times \{1,\dots,m_N\}.$$

Thus $E_N$ is a union of $m_1 \cdots m_N$ disjoint closed balls $E_J^N$. By appropriately specifying the radii of these balls, we obtain a finite upper bound for the Hausdorff measure of $E = \bigcap_1^\infty E_N$, and by choosing the balls "fairly uniformly distributed" we also get a positive lower bound for this measure; see [Mat95, pp. 63–64].

We follow this method for our general construction. We require the fact that for each positive integer $m \in \mathsf{N}$, there are $m$ disjoint closed balls in $\mathsf{B}^n$ each with the same radius $r$ and such that $mr^n = \kappa^n$ where $\kappa = \kappa(n)$ is a dimensional constant. By working with dyadic cubes, it is straightforward to confirm this with $\kappa(n) := 1/\sqrt{8n}$. We start with a given $s \in (0,n)$ (and later a given $p > 0$, $\mathcal{A}$, and a given Hausdorff gauge $h$). We construct generalized Cantor dusts $E, F \subset \mathsf{B}^n$ and a self-homeomorphism $f$ of $\mathsf{R}^n$ such that

$$f(E) = F, \qquad \mathcal{H}^s(E) \simeq 1, \quad \text{either} \quad \mathcal{H}^h(F) = 0 \quad \text{or} \quad \mathcal{H}^h(F) < \infty,$$

and so that $f$ has finite distortion $K_f$ with $\exp \mathcal{A}(pK_f) \in \mathsf{L}^1_{\mathrm{loc}}$. The precise details for these latter conditions will be provided in each example.

In each specific example, we will select integers $m_N \ge 2$ and distortion constants $\lambda_N \ge 1$. At each step $1, 2, \dots, N, \dots$ we choose $m_N$ disjoint closed balls $\bar{\mathsf{B}}(a_i^N, R_N) \subset \mathsf{B}^n$ (so here $1 \le i \le m_N$) each of radius $R_N$ where $R_N$

is chosen so that $m_N R_N^n = \kappa_N^n$; we choose these balls "fairly uniformly distributed" in $\mathsf{B}^n$. Here $0 < \kappa_N \le \kappa(n) = 1/\sqrt{8n}$. We also select $\sigma_N \in (0,1)$ so that $m_N(\sigma_N R_N)^s = 1$. (Such a $\sigma_N$ exists provided $m_N R_N^s > 1$, so provided we take $m_N > (1/\kappa_N)^{ns/(n-s)}$.)

Thus, starting with $0 < \kappa_N \le \kappa(n)$ and $m_N > (1/\kappa_N)^{ns/(n-s)}$, we take

$$R_N := \kappa_N m_N^{-1/n} \quad \text{and} \quad \sigma_N := \kappa_N^{-1} m_N^{(1/n)-(1/s)} = \kappa_N^{-1} m_N^{(s-n)/ns}.$$

Let $\varphi_i^N$ and $\vartheta_i^N$ be the similarities of $\mathsf{R}^n$ given by

$$\varphi_i^N(x) := a_i^N + \sigma_N R_N x \quad \text{and} \quad \vartheta_i^N(x) := a_i^N + \sigma_N^{\lambda_N} R_N x,$$

so that

$$\varphi_i^N\big(\mathsf{B}^n\big) = \mathsf{B}\big(a_i^N, \sigma_N R_N\big) \quad \text{and} \quad \vartheta_i^N\big(\mathsf{B}^n\big) = \mathsf{B}\big(a_i^N, \sigma_N^{\lambda_N} R_N\big);$$

here $\lambda_N \ge 1$ are auxiliary parameters that will be chosen later to determine the distortion of $f$. Notice that $a_i^N = \varphi_i^N(0)$. Next—see Figure 2 and recall that $\bar{\mathsf{B}} := \bar{\mathsf{B}}^n$—we define

$$\begin{aligned} E_i^1 &:= \varphi_i^1(\bar{\mathsf{B}}) \quad \text{for } 1 \le i \le m_1, \\ E_{ij}^2 &:= \varphi_i^1 \circ \varphi_j^2(\bar{\mathsf{B}}) \quad \text{for } 1 \le i \le m_1 \text{ and } 1 \le j \le m_2, \end{aligned}$$

and, in general, for $J = (j_1, \dots, j_N) \in \{1, \dots, m_1\} \times \cdots \times \{1, \dots, m_N\}$,

$$E_J^N := \Phi_J^N(\bar{\mathsf{B}}), \quad \text{where } \Phi_J^N := \varphi_{j_1}^1 \circ \varphi_{j_2}^2 \circ \cdots \circ \varphi_{j_N}^N.$$

Similarly, define

$$F_J^N := \Theta_J^N(\bar{\mathsf{B}}), \quad \text{where } \Theta_J^N := \vartheta_{j_1}^1 \circ \vartheta_{j_2}^2 \circ \cdots \circ \vartheta_{j_N}^N.$$

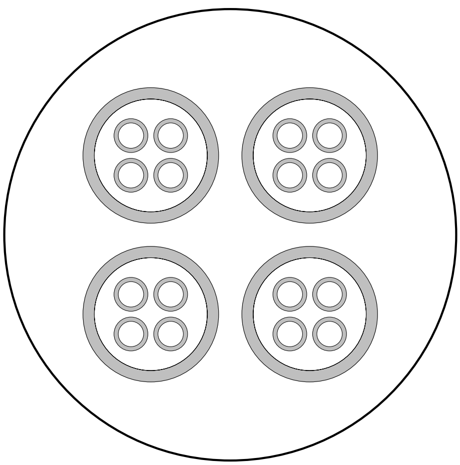

FIGURE 2. The 2nd generation set $E_2$ with $m_1 = m_2 = 4$.

We obtain generalized Cantor dusts

$$E := \bigcap_1^\infty E_N \quad \text{and} \quad F := \bigcap_1^\infty F_N,$$

where

$$E_N := \bigcup_{\text{all } J} E_J^N \quad \text{and} \quad F_N := \bigcup_{\text{all } J} E_J^N.$$

It is straightforward to calculate the centers and radii of the balls $E_J^N, F_J^N$. For example, the latter ball has center

$$\Theta_J^N(0) = a_{j_1}^1 + \sigma_1^{\lambda_1} R_1 \big( a_{j_2}^2 \\ + \sigma_2^{\lambda_2} R_2 \big[ \cdots + \sigma_{N-2}^{\lambda_{N-2}} R_{N-2} \big( a_{j_{N-1}}^{N-1} + \sigma_{N-1}^{\lambda_{N-1}} R_{N-1} a_{j_N}^N \big) \big] \big).$$

Also, the balls $E_J^N$ each have radius $\sigma_1 R_1 \cdots \sigma_N R_N$. Since these balls form a cover of $E$ with

$$\sum_{\text{all } J} (\sigma_1 R_1 \cdots \sigma_N R_N)^s = m_1 \cdots m_N (\sigma_1 R_1 \cdots \sigma_N R_N)^s = 1,$$

it is clear that $\mathcal{H}^s(E) \lesssim 1$. In fact, since these balls are chosen "fairly uniformly distributed" in their parent, it follows that $\mathcal{H}^s(E) \simeq 1$; see [Mat95, pp. 63–64]. In the examples that follow, we also determine the size of $F$. For this it is useful to know that each $F_J^N$ has radius $t_N := \sigma_1^{\lambda_1} R_1 \cdots \sigma_N^{\lambda_N} R_N$.

Finally, we construct a homeomorphism $\mathsf{R}^n \xrightarrow{f} \mathsf{R}^n$ with the property that $f(E) = F$ and such that $f$ has a given distortion; in fact, $f(E_N) = F_N$ for all $N$, and $f$ will have distortion $K_N := \lambda_N^{n-1}$ in the union of certain spherical rings and will be conformal elsewhere. This map $f$ is given as the limit of a sequence $(f_N)_{N=1}^\infty$ of quasiconformal self-homeomorphisms of $\mathsf{R}^n$; the maps $f_N$ are defined via a recursive relation. In order to accomplish this task, we introduce triples $(B_J, C_J, D_J)$ of concentric balls defined by

$$\begin{aligned} B_J &:= \mathsf{B}(c_J, r_N), \\ C_J &:= \sigma_N B_J = \mathsf{B}(c_J, \sigma_N r_N), \\ D_J &:= \sigma_N^{\lambda_N} B_J = \mathsf{B}\big(c_J, \sigma_N^{\lambda_N} r_N\big), \\ &\quad \text{where } r_N := \sigma_{N-1}^{\lambda_{N-1}} r_{N-1} R_N \ (\text{with } r_0 = \sigma_0 = \lambda_0 := 1) \end{aligned}$$

and it remains to specify the centers $c_J$. In fact, $c_J := \Theta_J^N(0)$, but this is more easily understood by starting at the beginning. Write $f_0$ to denote the identity map: $f_0(x) = x$.

*Step* 1. For $1 \le i \le m_1$, put

$$\begin{aligned} B_i &:= f_0 \circ \varphi_i^1\big(\sigma_1^{-1} \mathsf{B}\big), \\ C_i &:= \sigma_1 B_i = f_0 \circ \varphi_i^1(\mathsf{B}), \\ D_i &:= \sigma_1^{\lambda_1} B_i. \end{aligned}$$

One can readily check that

$$B_i = \mathsf{B}(c_i, r_1) \quad \text{and} \quad \bar{C}_i = f_0\big(E_i^1\big),$$

where $c_i := \vartheta_i^1(0) = a_i^1$ and $r_1 := R_1$. For each triple $(B_i, C_i, D_i)$, we have a radial squeeze-stretch map $\Psi_i^1 := \Psi_{c_i, r_1}^{\sigma_1, \lambda_1}$ (see Section 2.5) and we define

$$g_1(x) := \begin{cases} \Psi_i^1(x) & \text{for } x \in B_i,\ 1 \le i \le m_1, \\ x & \text{for } x \in \mathsf{R}^n \setminus \bigcup_{i=1}^{m_1} B_i. \end{cases}$$

Thus $\mathsf{R}^n \xrightarrow{g_1} \mathsf{R}^n$ is $K_1$-quasiconformal, $K_1 := \lambda_1^{n-1}$, and conformal in $\mathsf{R}^n \setminus \bigcup_1^{m_1} (\bar{B}_i \setminus C_i)$ with

$$g_1(B_i) = B_i \quad \text{and} \quad g_1(C_i) = D_i \quad \text{via a scaling by } \sigma_1^{\lambda_1 - 1} \quad \text{and}$$
$$g_1(B_i \setminus C_i) = B_i \setminus D_i \quad \text{via the radial stretch } x \mapsto |x|^{\lambda_1 - 1} x.$$

We set $f_1 := g_1 \circ f_0$. Note that $B_i \setminus C_i = \varphi_i^1(\sigma_1^{-1}\mathsf{B} \setminus \mathsf{B})$, so the distortion of $f_1$ is given via

$$K_{f_1} = \begin{cases} K_1 & \text{in } \bigcup_{i=1}^{m_1} \varphi_i^1(\sigma_1^{-1}\mathsf{B} \setminus \bar{\mathsf{B}}), \\ 1 & \text{in } \mathsf{R}^n \setminus \bigcup_{i=1}^{m_1} \varphi_i^1(\sigma_1^{-1}\bar{\mathsf{B}} \setminus \mathsf{B}). \end{cases}$$

Also, by comparing centers and radii, we see that

$$f_1\big(E_i^1\big) = f_1 \circ \varphi_i^1(\bar{\mathsf{B}}) = g_1(\bar{C}_i) = \bar{D}_i = F_i^1 \quad \text{and so} \quad f_1(E_1) = F_1.$$

*Step* 2. For $1 \le i \le m_1$ and $1 \le j \le m_2$, put

$$\begin{aligned} B_{ij} &:= f_1 \circ \Phi_{ij}^2\big(\sigma_2^{-1}\mathsf{B}\big), \\ C_{ij} &:= \sigma_2 B_{ij} = f_1 \circ \Phi_{ij}^2(\mathsf{B}), \\ D_{ij} &:= \sigma_2^{\lambda_2} B_{ij}^2. \end{aligned}$$

One can readily check that

$$B_{ij} = f_1 \circ \varphi_i^1\big[\mathsf{B}\big(a_j^2, R_2\big)\big] = \mathsf{B}(c_{ij}, r_2) \quad \text{and} \quad \bar{C}_{ij} = f_1\big(E_{ij}^2\big),$$

where $c_{ij} := \Theta_{ij}^2(0)$ and $r_2 := \sigma_1^{\lambda_1} r_1 R_1$. For each triple $(B_{ij}, C_{ij}, D_{ij})$ we have radial a squeeze-stretch map $\Psi_{ij}^2 := \Psi_{c_{ij}, r_2}^{\sigma_2, \lambda_2}$ (see Section 2.5) and we define

$$g_2(x) := \begin{cases} \Psi_{ij}^2(x) & \text{for } x \in B_{ij},\ 1 \le i \le m_1,\ 1 \le j \le m_2, \\ x & \text{for } x \in \mathsf{R}^n \setminus \bigcup_{i=1}^{m_1} \bigcup_{j=1}^{m_2} B_{ij}. \end{cases}$$

Thus $\mathsf{R}^n \xrightarrow{g_2} \mathsf{R}^n$ is $K_2$-quasiconformal, $K_2 := \lambda_2^{n-1}$, and conformal in $\mathsf{R}^n \setminus \bigcup_{i,j} (\bar{B}_{ij} \setminus C_{ij})$ with

$$g_2(B_{ij}) = B_{ij} \quad \text{and} \quad g_2(C_{ij}) = D_{ij} \quad \text{via a scaling by } \sigma_2^{\lambda_2 - 1} \quad \text{and}$$
$$g_2(B_{ij} \setminus C_{ij}) = B_{ij} \setminus D_{ij} \quad \text{via the radial squeeze-stretch } x \mapsto |x|^{\lambda_2 - 1} x.$$

We set $f_2 := g_2 \circ f_1$. Then, since

$$f_1^{-1}(B_{ij}) = \Phi_{ij}^2\big(\sigma_2^{-1}\mathsf{B}\big),$$

we see that

$$f_2(x) = \begin{cases} \Psi^2_{ij} \circ f_1(x) & \text{for } x \in \Phi^2_{ij}(\sigma_2^{-1}\mathsf{B}), \\ f_1(x) & \text{otherwise.} \end{cases}$$

Note that $\Phi^2_{ij}(\sigma_2^{-1}\mathsf{B}) = \varphi^1_i[\mathsf{B}(a^2_j, R_2)] \subset \varphi^1_i(\mathsf{B}) = C_i$. In $C_i$, $f_1 = g_1$ is conformal (being a linear scaling/squeeze by $\sigma_1^{\lambda_1 - 1}$). Therefore, the distortion of $f_2$ in $\Phi^2_{ij}(\sigma_2^{-1}\mathsf{B})$ comes only from $\Psi^2_{ij}$ (which has distortion $K_2$ in $B^2_{ij} \setminus \bar{C}_{ij}$ and is conformal elsewhere). In particular, we deduce that the distortion of $f_2$ is given via

$$K_{f_2} = \begin{cases} K_2 & \text{in } \bigcup_{i,j} \Phi^2_{ij}(\sigma_2^{-1}\mathsf{B} \setminus \bar{\mathsf{B}}), \\ K_1 & \text{in } \bigcup_{i=1}^{m_1} \varphi^1_i(\sigma_1^{-1}\mathsf{B} \setminus \bar{\mathsf{B}}), \\ 1 & \text{everywhere else.} \end{cases}$$

Also, by comparing centers and radii, we confirm that

$$f_2\big(E^2_{ij}\big) = g_2(\bar{C}_{ij}) = \bar{D}_{ij} = F^2_{ij} \quad \text{and so} \quad f_2(E_2) = F_2.$$

*Step N.* For each $J = I \times \{j\} = (j_1, \dots, j_N) \in \{1, \dots, m_1\} \times \cdots \times \{1, \dots, m_N\}$, put

$$\begin{aligned} B_J &:= f_{N-1} \circ \Phi^N_J\big(\sigma_N^{-1}\mathsf{B}\big), \\ C_J &:= \sigma_N B^N_J = f_{N-1} \circ \Phi^N_J(\mathsf{B}), \\ D_J &:= \sigma_N^{\lambda_N} B^N_J. \end{aligned}$$

One can check that

$$B_J = f_{N-1} \circ \Phi^{N-1}_I\big[\mathsf{B}\big(a^N_j, R_N\big)\big] = \mathsf{B}(c_J, r_N) \quad \text{and} \quad \bar{C}_J = f_{N-1}\big(E^N_J\big),$$

where $c_J := \Theta^N_J(0)$ and $r_N := \sigma_{N-1}^{\lambda_{N-1}} r_{N-1} R_N$. For each triple $(B_J, C_J, D_J)$ we have radial a squeeze-stretch map $\Psi^N_J := \Psi^{\sigma_N, \lambda_N}_{c_J, r_N}$ (see Section 2.5) and we define

$$g_N(x) := \begin{cases} \Psi^N_J(x) & \text{for } x \in B_J, \\ x & \text{for } x \in \mathsf{R}^n \setminus \bigcup_J B_J. \end{cases}$$

Thus $\mathsf{R}^n \xrightarrow{g_N} \mathsf{R}^n$ is $K_N$-quasiconformal, $K_N := \lambda_N^{n-1}$, and conformal in $\mathsf{R}^n \setminus \bigcup_J(\bar{B}_J \setminus C_J)$ with

$$g_N(B_J) = B^N_J \quad \text{and} \quad g_N(C_J) = D_J \quad \text{via a scaling by } \sigma_N^{\lambda_N - 1} \quad \text{and}$$
$$g_N(B_J \setminus C_J) = B_J \setminus D_J \quad \text{via the radial squeeze-stretch } x \mapsto |x|^{\lambda_N - 1} x.$$

We set $f_N := g_N \circ f_{N-1}$. Then, since

$$f_{N-1}^{-1}(B_J) = \Phi^N_J\big(\sigma_N^{-1}\mathsf{B}\big),$$

we see that

$$f_N(x) = \begin{cases} \Psi^N_J \circ f_{N-1}(x) & \text{for } x \in \Phi^N_J(\sigma_N^{-1}\mathsf{B}), \\ f_{N-1}(x) & \text{otherwise.} \end{cases}$$

Note that for $J=I\times\{j\}$, $\Phi_J^N(\sigma_N^{-1}\mathsf{B})=\Phi_I^{N-1}[\mathsf{B}(a_j^N,R_N)]\subset\Phi_I^{N-1}(\mathsf{B})$.

In $\Phi_I^{N-1}(\mathsf{B})$, $f_{N-2}$ is conformal (in fact, a linear scaling/squeeze) with $f_{N-2}[\Phi_I^{N-1}(\mathsf{B})]=C_I$. In $C_I$, $g_{N-1}$ is conformal (being a dilation by $\sigma_{N-1}^{\lambda_{N-1}-1}$). Therefore, in each ball $\Phi_I^{N-1}(\mathsf{B})$, $f_{N-1}=g_{N-1}\circ f_{N-2}$ is conformal.

It now follows that the distortion of $f_N$ in $\Phi_J^N(\sigma_N^{-1}\mathsf{B})$ comes only from $\Psi_J^N$ (which has distortion $K_N$ in $B_J^N\setminus\bar{C}_J$ and is conformal elsewhere). In particular, we deduce that the distortion of $f_N$ is given via

$$K_{f_N}=\begin{cases}K_N & \text{in } \bigcup_J \Phi_J^N(\sigma_N^{-1}\mathsf{B}\setminus\bar{\mathsf{B}}),\\ \vdots & \vdots\\ K_2 & \text{in } \bigcup_{i,j}\Phi_{ij}^2(\sigma_2^{-1}\mathsf{B}\setminus\bar{\mathsf{B}}),\\ K_1 & \text{in } \bigcup_{i=1}^{m_1}\varphi_i^1(\sigma_1^{-1}\mathsf{B}\setminus\bar{\mathsf{B}}),\\ 1 & \text{everywhere else.}\end{cases}$$

Also, by comparing centers and radii, we corroborate that

$$f_N\big(E_J^N\big)=g_N(\bar{C}_J)=\bar{D}_J=F_J^N \quad\text{and so}\quad f_N(E_N)=F_N.$$

*Final step.* We thus have a sequence $(f_N)_1^\infty$ of quasiconformal self-homeomorphisms of $\mathsf{R}^n$. In fact, using (2.16) we see that this sequence is uniformly Cauchy, so there is a limit map $f:=\lim_{N\to\infty}f_N$ that is evidently a homeomorphism. Since $f_N(E_N)=F_N$ for each $N$, $f(E)=F$. Also, since

$$f_N=f_{N-1}\quad\text{in } \mathsf{R}^n\setminus\bigcup_{\text{all }J}\Phi_J^N\big(\sigma_N^{-1}\mathsf{B}\big),$$

and, for $J=I\times\{j\}$,

$$\Phi_J^N\big(\sigma_N^{-1}\mathsf{B}\big)=\Phi_I^{N-1}\big[\mathsf{B}\big(a_j^N,R_N\big)\big]\subset E_I^{N-1},$$

we deduce that

$$\{f_N\neq f_{N-1}\}\subset\bigcup_{\text{all }J}\Phi_J^N\big(\sigma_N^{-1}\mathsf{B}\big)\subset E_{N-1}.$$

Recalling that $E=\bigcap E_N$ is a Lebesgue null set, we see that for almost every $x$ in $\mathsf{R}^n$, the tail of the sequence $(f_N(x))_1^\infty$ is constant.

Now, for each fixed $x\notin E$, there is an open ball $B:=\mathsf{B}(x;\varepsilon)$ and an $N\in\mathsf{N}$ such that $f|_B=f_N|_B$; here $\varepsilon$ and $N$ both depend on $x$ (or rather on $\operatorname{dist}(x,E)$). In particular, since each $f_N$ is a quasiconformal self-homeomorphism of $\mathsf{R}^n$, we see that $f$ is absolutely continuous on lines and differentiable almost everywhere. Below we address the question of whether or not $f$ is a Sobolev homeomorphism.

We see that $f$ has the distortion function

$$K_f=\begin{cases}K_N & \text{in } \bigcup_J\Phi_J^N(\sigma_N^{-1}\mathsf{B}\setminus\bar{\mathsf{B}}),\\ 1 & \text{everywhere else.}\end{cases}$$

We note that $K_f = K_N$ occurs in the union of $M_N := m_1 m_2 \cdots m_N$ spherical rings each with outer radius $\sigma_1 R_1 \cdots \sigma_{N-1} R_{N-1} R_N$ and inner radius $\sigma_1 R_1 \cdots \sigma_N R_N$; here we set $\sigma_0 = R_0 := 1$.

To determine the local integrability properties of the function $x \mapsto P(K_f(x))$—here $P(t)$ can be $t^p$ or $e^{pt}$ or $\exp \mathcal{A}(pt)$—it suffices to examine the convergence of the series

$$\sum_{N=1}^{\infty} M_N \big[(\sigma_1 R_1 \cdots \sigma_{N-1} R_{N-1} R_N)^n - (\sigma_1 R_1 \cdots \sigma_N R_N)^n\big] P(K_N).$$

Recalling that $\sigma_N R_N^n = m_N^{-n/s}$ and $\sigma_N = \kappa_N^{-1} m_N^{(s-n)/ns}$, we find that the above series equals

$$\begin{aligned}(4.1)\qquad \sum_{N=1}^{\infty} & M_N \big(\sigma_N^{-n} - 1\big)(\sigma_1 R_1 \cdots \sigma_N R_N)^n P(K_N) \\ &= \sum_{N=1}^{\infty} \big(\sigma_N^{-n} - 1\big) M_N^{1-n/s} P(K_N) \\ &= \sum_{N=1}^{\infty} \big(\kappa_N^n m_N^{(n-s)/s} - 1\big) M_N^{1-n/s} P(K_N).\end{aligned}$$

In certain of our specific examples we consider *regular Cantor dusts* by which we mean that $\kappa_N = \kappa$ and $m_N = m$ are some fixed constants. In this setting, the above convergence question simplifies to looking at convergence of the series

$$(4.2)\qquad \sum_{N=1}^{\infty} m^{N(1-n/s)} P(K_N).$$

We also need to estimate $\mathcal{H}^h(F)$, at least to show that this is zero or finite. For this it suffices to examine the behavior of $M_N h(\operatorname{diam}(F_N^J))$ as $N \to \infty$. Recall that $F = \bigcap F_N$ where $F_N$ is the union of $M_N$ disjoint closed balls each of radius $t_N := \sigma_1^{\lambda_1} R_1 \cdots \sigma_N^{\lambda_N} R_N$; this simplifies to $t_N = \sigma^{\lambda_1 + \cdots + \lambda_N} R^N$ when $F$ is a regular Cantor dust.

Finally, we discuss the question of whether or not $f$ is a mapping of finite distortion. We know that $f$ is absolutely continuous on lines and differentiable almost everywhere with a Jacobian that is positive almost everywhere. Once we know that the differential $Df$ of $f$ is locally integrable, then the absolute continuity on lines permits us to assert that $f$ belongs to $\mathsf{W}^{1,1}_{\mathrm{loc}}(\mathsf{R}^n, \mathsf{R}^n)$, and then being a Sobolev homeomorphism, we also know that the Jacobian of $f$ belongs to $\mathsf{L}^1_{\mathrm{loc}}(\mathsf{R}^n)$; see for example [IM01, p. 106 and Corollary 6.3.1, p. 108].

Thus we must determine whether or not $Df$ is locally integrable; in fact, we need only consider the integral over the doubled unit ball $A := 2\mathsf{B}^n$. From our construction, we can calculate $Df$, but its integral depends on the parameters in our construction in a non-trivial manner. One way to circumvent these difficulties is to employ a weak compactness argument using uniform $\mathsf{L}^p_{\mathrm{loc}}$ estimates for $Df_N$; here $1<p<n$. To this end, we apply Young's inequality to

$$|Df_N|^p = \big(|Df_N|^n\big)^{p/n} = K_{f_N}^{p/n} J_{f_N}^{p/n}$$

to obtain

$$|Df_N|^p \le \frac{n-p}{n} K_{f_N}^{p/(n-p)} + \frac{p}{n} J_{f_N}.$$

This yields

$$\int_A |Df_N|^p \le \frac{n-p}{n}\int_A K_{f_N}^{p/(n-p)} + \frac{p}{n}|A|.$$

Thus uniform $\mathsf{L}^p_{\mathrm{loc}}$ bounds for $Df_N$ exist provided

$$\sup_{N\in\mathsf{N}} \int_A K_{f_N}^{p/(n-p)} < \infty.$$

Such a condition will hold, for some $1<p<n$, if we know that for some $q>1$, the series in (4.1) converges with $P(t)=t^q$; we then get $p:=qn/(1+q)$.

Assuming this latter condition, the above discussion reveals that $(f_N|_A)_1^\infty$ is a bounded sequence in $\mathsf{W}^{1,p}(A,\mathsf{R}^n)$. Therefore, there is a subsequence $(f_M)$ of $(f_N|_A)$ that converges weakly to some $g$ in $\mathsf{W}^{1,p}(A,\mathsf{R}^n)$. In particular, this means that for all test functions $\varphi$ (i.e., $\varphi\in\mathcal{C}^\infty_{\mathrm{c}}(A,\mathsf{R}^n)$),

$$\int_A f_M\cdot\varphi \longrightarrow \int_A g\cdot\varphi \quad \text{as } M\to\infty \tag{4.3a}$$

and

$$\int_A (Df_M)^{\mathrm{t}}\varphi \longrightarrow \int_A (Dg)^{\mathrm{t}}\varphi \quad \text{as } M\to\infty. \tag{4.3b}$$

We know that $(f_N)_1^\infty$ converges to $f$ uniformly on all of $\mathsf{R}^n$, so it follows from (4.3a) that $f|_A$ and $g$ are equal as $\mathsf{L}^1_{\mathrm{loc}}(A)$ functions.

Now

$$Df_M = [D_1 f_M \cdots D_n f_M], \quad \text{where } D_j f_M = \frac{\partial f_M}{\partial x_j},$$

so

$$\begin{aligned}\int_A \frac{\partial g}{\partial x_j}\cdot\varphi &= \lim_{M\to\infty}\int_A \frac{\partial f_M}{\partial x_j}\cdot\varphi = -\lim_{M\to\infty}\int_A f_M\cdot\frac{\partial\varphi}{\partial x_j}\\ &= -\int_A g\cdot\frac{\partial\varphi}{\partial x_j} = -\int_A f\cdot\frac{\partial\varphi}{\partial x_j};\end{aligned}$$

here the two left-most equalities follow from (4.3b) and (4.3a), respectively, and the last equality holds because $f=g$. The equality of the first and last integrals above implies that $Dg$ is the distributional derivative of $f$.

Since $g \in \mathsf{W}^{1,p}(A, \mathsf{R}^n)$, we conclude that $f$ belongs to the Sobolev space $\mathsf{W}^{1,1}_{\mathrm{loc}}(\mathsf{R}^n; \mathsf{R}^n)$.

In summary, the above construction produces a homeomorphism $\mathsf{R}^n \xrightarrow{f} \mathsf{R}^n$ and Cantor dusts $E, F \subset \mathsf{B}^n$ with $f(E) = F$ and $\mathcal{H}^s(E) \simeq 1$. The map $f$ is differentiable almost everywhere, and the integrability of the distortion of $f$ can be determined by checking the convergence of the appropriate series in (4.1) or (4.2); when this series converges, $f$ is a mapping of finite distortion. Finally, we can provide upper estimates for the Hausdorff measure $\mathcal{H}^h(F)$ by controlling the radii $t_N$ of the balls used to construct $F$.

Here is a precise statement.

THEOREM 4.1. *Let $n \geq 2, s \in (0, n), p > 0$ be given. Let $(m_N)_{N=1}^{\infty}$ and $(\kappa_N)_{N=1}^{\infty}, (\lambda_N)_{N=1}^{\infty}$ be sequences of integers and real numbers, respectively, that satisfy $m_N > (1/\kappa_N)^{ns/(n-s)}$, $0 < \kappa_N \leq \kappa(n) := 1/\sqrt{8n}$, and $\lambda_N \geq 1$ for all $N$. Then there are generalized Cantor dusts $E, F \subset \mathsf{B}^n$ and a homeomorphism $f : \mathsf{R}^n \to \mathsf{R}^n$ with the properties that*

$$\mathcal{H}^s(E) \simeq 1 \quad \textit{and} \quad f(E) = F \quad \textit{and} \quad \forall x \in \mathsf{R}^n \setminus \mathsf{B}^n, \quad f(x) = x.$$

*The map $f$ is absolutely continuous on lines, differentiable almost everywhere, and if the series (4.1) converges with $P(t) = t^q$ for some $q > 1$, then $f$ is a mapping of finite distortion. Moreover, for $P(t)$ equal to $t^p$ or $\exp(pt)$ or $\exp \mathcal{A}(pt)$, we have $P(K_f) \in \mathsf{L}^1_{\mathrm{loc}}(\mathsf{R}^n)$ if and only if the series in (4.1) converges (or in (4.2) for the special case where $m_N = m$ and $\kappa_N = \kappa(n)$ for all $N$). Here $K_N := \lambda_N^{n-1}$ and $M_N := m_1 \cdots m_N$.*

**4.2. Compression examples with $\exp(pK) \in \mathsf{L}^1_{\mathrm{loc}}$.** Here we examine Theorem B in the special case where $\mathcal{A}(t) = t$. In part, we do this as it provides a simpler version of what we present below in Section 4.3, but our results here are also relevant for the case of exponentially integrable distortion.

In this setting, we have $\omega^{-1}(t) = t^{(n-1)/n}$ and so the gauge function $h = h_{s,p,\mathcal{A},n}$ (that appears in the statement of Theorem B) is of the form $h = h_\alpha$ where

$$h_\alpha(t) := \exp\left(-\alpha\left(\log\frac{1}{t}\right)^{(n-1)/n}\right).$$

The analog of Theorem B in this special case was established by Zapadinskaya (see [Zap11, Theorem 1.1]) and she proved that we can use the gauge function

$$h_{\gamma_0}, \quad \text{where } \gamma_0 := C(n)sp^{1/n};$$

here $C(n)$ is, essentially, the constant from [HK03, Theorem B]. She also constructed an example to illustrate the sharpness of her theorem; see [Zap11, Example 1.3]. Briefly, given $s \in (0, n)$, and $p > 0$, she constructs a finite distortion homeomorphism $f : \mathsf{R}^n \to \mathsf{R}^n$, with $\exp(qK_f) \in \mathsf{L}^1_{\mathrm{loc}}(\mathsf{R}^n)$ for all $q \in$

$(0,p)$, and a set $E\subset \mathsf{R}^n$ with $\mathcal{H}^s(E)>0$ but $\mathcal{H}^{h_\alpha}(f(E))=0$ for all $\alpha>\zeta_0:=Zsp^{1/n}$, where $Z=Z(s,n)$ is given by

$$Z(s,n):=\left(\frac{n}{n-1}\right)^{\frac{n-1}{n}}\frac{\zeta(s,n)}{(n-s)^{\frac{1}{n}}},$$
$$\zeta(s,n):=\begin{cases}\frac{1}{(1-s)^{\frac{n-1}{n}}} & \text{when } 0<s<1,\\ \frac{\log^{\frac{n-1}{n}} m(s)}{\log 2} & \text{when } 1\le s<n,\end{cases}$$

and $m(s):=(\lceil 2^{\frac{1}{n-s}}n^{\frac{s}{2(n-s)}}\rceil)^n$.

In our example, $\exp(pK_f)\in \mathsf{L}^1_{\text{loc}}(\mathsf{R}^n)$, and our range of allowable gauge functions is slightly better (because $A(s,n)<Z(s,n)$).

Example 4.2. Let $n\ge 2$, $s\in(0,n)$, and $p>0$ be given. Fix $\alpha>\alpha_0:=Asp^{1/n}$ where

$$A=A(s,n):=\left(\frac{n^2}{n-1}\right)^{(n-1)/n}\frac{1}{n-s}.$$

There exists a finite distortion homeomorphism $\mathsf{R}^n\xrightarrow{f}\mathsf{R}^n$ and a regular Cantor dust $E$ in $\mathsf{B}^n$ such that $f$ has $p$-exponentially integrable distortion, that is, $\exp(pK_f)\in \mathsf{L}^1_{\text{loc}}(\mathsf{R}^n)$, and $\mathcal{H}^s(E)\simeq 1$ but $\mathcal{H}^{h_\alpha}(f(E))=0$. Moreover, for all $x\in \mathsf{R}^n\setminus \mathsf{B}^n$, $f(x)=x$.

*Proof.* For each integer $m>(1/\kappa)^{ns/(n-s)}$ (recall that $\kappa=\kappa(n)=1/\sqrt{8n}$), set

$$\alpha_m:=\left(\frac{n}{n-1}\right)^{(n-1)/n}p^{1/n}\left(\frac{s}{n-s}\right)^{1/n}\Big/\left(\frac{n-s}{ns}-\frac{\log(1/\kappa)}{\log m}\right)^{(n-1)/n}.$$

Then as $m\to\infty$, $\alpha_m\searrow\alpha_0$. Thus we may select $m$ sufficiently large so that $\alpha>\alpha_m>\alpha_0$, and these inequalities will also hold if we take a larger $m$.

For this $m$, we pick $R$ and $\sigma$ so that $mR^n=\kappa^n$ and $m(\sigma R)^s=1$. Thus,

$$R:=\kappa m^{-1/n}\quad\text{and}\quad \sigma:=\kappa^{-1}m^{(1/n)-(1/s)}=\kappa^{-1}m^{(s-n)/ns}.$$

Using these values of $m,R,\sigma$—and taking $\lambda_N:=(aN)^{1/(n-1)}$, so that $K_N=aN$—we "do" the Cantor dust construction to obtain a finite distortion homeomorphism $f:\mathsf{R}^n\to\mathsf{R}^n$ and regular Cantor dusts $E,F\subset\mathsf{B}^n$ with $F=f(E)$ and $\mathcal{H}^s(E)\simeq 1$. We claim that the constant $a$ can be chosen so that both $\exp(pK_f)\in\mathsf{L}^1_{\text{loc}}(\mathsf{R}^n)$ and $\mathcal{H}^{h_\alpha}(F)=0$.

Recalling—see (4.2)—that the integrability condition $e^{pK_f}\in\mathsf{L}^1_{\text{loc}}(\mathsf{R}^n)$ is equivalent to convergence of the series

$$\sum_{N=1}^{\infty}m^{(1-n/s)N}e^{pK_N}=\sum_{N=1}^{\infty}m^{(1-n/s)N}e^{paN},$$

and writing

$$m^{(1-n/s)N}e^{paN} = \exp\left(\frac{s-n}{s}N\log m + paN\right) = \exp\left(N\left(\frac{s-n}{s}\log m + pa\right)\right),$$

we see that

$$\exp(pK_f) \in \mathsf{L}^1_{\mathrm{loc}}(\mathsf{R}^n) \quad \Longleftrightarrow \quad a < \frac{1}{p}\frac{n-s}{s}\log m.$$

Below we demonstrate that by choosing

$$a > \alpha^{-n}\left(\frac{n}{n-1}\right)^{n-1}\log m \Big/ \left(\frac{n-s}{ns} - \frac{\log(1/\kappa)}{\log m}\right)^{n-1}$$

we obtain $\mathcal{H}^{h_\alpha}(F) = 0$. Thus we must check that we can pick a constant $a$ that satisfies

$$\alpha^{-n}\left(\frac{n}{n-1}\right)^{n-1}\log m \Big/ \left(\frac{n-s}{ns} - \frac{\log(1/\kappa)}{\log m}\right)^{n-1} < a < \frac{1}{p}\frac{n-s}{s}\log m.$$

This is equivalent to requiring that

$$\alpha^n > \left(\frac{n}{n-1}\right)^{n-1} p\left(\frac{s}{n-s}\right) \Big/ \left(\frac{n-s}{ns} - \frac{\log(1/\kappa)}{\log m}\right)^{n-1}$$

and this holds because $\alpha > \alpha_m$.

It remains to confirm that the above lower bound on $a$ forces $\mathcal{H}^{h_\alpha}(F) = 0$. This holds provided $m^N h_\alpha(t_N) \to 0$ as $N \to \infty$, where

$$t_N = \sigma^{S_N}R^N$$

is the radius of the balls used to construct $F_N = f(E_N)$ and

$$S_N = \lambda_1 + \cdots + \lambda_N = K_1^{1/(n-1)} + \cdots + K_N^{1/(n-1)} = a^{1/(n-1)}\sum_{k=1}^{N} k^{1/(n-1)}.$$

Notice that

$$m^N h_\alpha(t_N) = \exp\left(N\log m - \alpha\log^{(n-1)/n}\frac{1}{t_N}\right) \longrightarrow 0$$

if and only if

$$\alpha\log^{(n-1)/n}\frac{1}{t_N} - N\log m \longrightarrow \infty.$$

We have

$$\begin{aligned}\log\frac{1}{t_N} &= S_N\log\frac{1}{\sigma} - N\log R \\ &= S_N\left(\left(\frac{n-s}{ns}\right)\log m - \log\frac{1}{\kappa}\right) + N\left(\frac{1}{n}\log m + \log\frac{1}{\kappa}\right) \\ &= NT_N\log m,\end{aligned}$$

where

$$T_N := \frac{S_N}{N}\left(\left(\frac{n-s}{ns}\right) - \frac{\log(1/\kappa)}{\log m}\right) + \left(\frac{1}{n} + \frac{\log(1/\kappa)}{\log m}\right).$$

Thus we must check that

$$\alpha(NT_N \log m)^{(n-1)/n} - N\log m \longrightarrow \infty \quad \text{as } N \to \infty.$$

To establish this limit, we first rewrite the above left-hand side as

$$N(\log m)^{(n-1)/n}\left[\alpha\left(\frac{T_N}{N^{1/(n-1)}}\right)^{(n-1)/n} - (\log m)^{1/n}\right].$$

Using the fact that

$$\begin{aligned}\frac{S_N}{N^{n/(n-1)}} &= a^{1/(n-1)}\sum_{k=1}^{N}\left(\frac{k}{N}\right)^{1/(n-1)}\frac{1}{N}\\ &> a^{1/(n-1)}\int_0^1 x^{1/(n-1)}\,dx = a^{1/(n-1)}\frac{n-1}{n}\end{aligned}$$

we see that

$$\begin{aligned}\frac{T_N}{N^{1/(n-1)}} &= \frac{1}{N^{1/(n-1)}}\left[\frac{S_N}{N}\left(\left(\frac{n-s}{ns}\right) - \frac{\log(1/\kappa)}{\log m}\right) + \left(\frac{1}{n} + \frac{\log(1/\kappa)}{\log m}\right)\right]\\ &> \frac{S_N}{N^{n/(n-1)}}\left(\frac{n-s}{ns} - \frac{\log(1/\kappa)}{\log m}\right)\\ &> a^{1/(n-1)}\frac{n-1}{n}\left(\frac{n-s}{ns} - \frac{\log(1/\kappa)}{\log m}\right)\end{aligned}$$

and therefore

$$\begin{aligned}&\alpha\left(\frac{T_N}{N^{1/(n-1)}}\right)^{(n-1)/n} - (\log m)^{1/n}\\ &\quad > \alpha\left[a^{1/(n-1)}\frac{n-1}{n}\left(\frac{n-s}{ns} - \frac{\log(1/\kappa)}{\log m}\right)\right]^{(n-1)/n} - (\log m)^{1/n}\\ &\quad = \alpha\left[a\left(\frac{n-1}{n}\right)^{n-1}\left(\frac{n-s}{ns} - \frac{\log(1/\kappa)}{\log m}\right)^{n-1}\right]^{1/n} - (\log m)^{1/n}.\end{aligned}$$

Finally, the right-hand side immediately above, which contains no $N$ terms, is strictly positive if and only if

$$a > \alpha^{-n}\left(\frac{n}{n-1}\right)^{(n-1)}\log m \Big/ \left(\frac{n-s}{ns} - \frac{\log(1/\kappa)}{\log m}\right)^{n-1},$$

and when this holds, the displayed quantity at the beginning of this paragraph does indeed tend to $\infty$ as $N \to \infty$. □

It is not difficult to use the above to construct an example where the map does not depend on either of the parameters $\alpha, s$. Let $(s_j)_1^\infty$ and $(\alpha_j)_1^\infty$ be monotone sequences in $(0,n)$ and $(\alpha_0,\infty)$, respectively with $s_j \nearrow n$ and $\alpha_j \searrow \alpha_0$ as $j\to\infty$. Let $f_j$ and $E_j$ be the maps and sets constructed in Example 4.2 using the parameters $s_j,\alpha_j$ (with some fixed $p>0$). By translating the set $E_j$, we may assume that $E_j \subset \mathsf{B}_j := \mathsf{B}(2je,1)$ where $e:=(1,0,\dots,0)\in\mathsf{R}^n$. In particular, for all $x\in\mathsf{R}^n\setminus\mathsf{B}_j$, $f_j(x)=x$. Thus we may define $f:\mathsf{R}^n\to\mathsf{R}^n$ by letting $f(x):=f_j(x)$ for $x\in\mathsf{B}_j$ and $f(x):=x$ for $x\in\mathsf{R}^n\setminus A$ where $A:=\bigcup\mathsf{B}_j$. We summarize this as follows.

Example 4.3. Let $n\ge 2$ and $p>0$ be given. There exists a finite distortion homeomorphism $f:\mathsf{R}^n\to\mathsf{R}^n$ with $\exp(pK_f)\in\mathsf{L}^1_{\mathrm{loc}}(\mathsf{R}^n)$ and a set $A\subset\mathsf{R}^n$ (a union of open balls each of radius one) with the following property. For each $s\in(0,n)$ and each $\alpha>\alpha_0$, there is a regular Cantor dust $E\subset A$ such that $\mathcal{H}^s(E)>0$ but $\mathcal{H}^{h_\alpha}(f(E))=0$.

We point out that the above provides a set $A$ with $\dim_{\mathcal{H}}(A)=n$ and $\dim_{\mathcal{H}}(f(A))=0$.

To verify the above conclusion, let $s\in(0,n)$ and $\alpha>\alpha_0$ be given. Pick $j$ so that $s\le s_j$ and $\alpha\ge\alpha_j$, and let $E=E_j$. Then $\mathcal{H}^{s_j}(E)\simeq 1$ implies that $\mathcal{H}^s(E)>0$ (quite likely, $\mathcal{H}^s(E)=\infty$). Similarly, since $\alpha_j\le\alpha$, $h_{\alpha_j}\preceq h_\alpha$, and therefore $\mathcal{H}^{h_\alpha}\ll\mathcal{H}^{h_{\alpha_j}}$.

The above example reveals several things regarding Theorem B (for the special case where $\mathcal{A}(t)=t$). A natural question is whether or not there is an improved version of this result that holds with a gauge that is better than the gauge function $h_{\gamma_0}$. Assume $h$ is a gauge function with the conclusion of Theorem B (with $\mathcal{A}(t)=t$) holding. Then it cannot be that $h_\alpha\preceq h$ for any $\alpha>\alpha_0$. This means that

$$\forall\alpha>\alpha_0,\quad \limsup_{t\to 0^+}\left[h(t)\exp\left(\alpha\log^{(n-1)/n}\frac{1}{t}\right)\right]=\infty.$$

In particular, for gauges of the form $h=h_\beta$, this implies that $\beta\le\alpha_0$.

The above discussion leads to the following questions. Here we take $\mathcal{A}(t)=t$.

Questions 4.4. (a) What is the largest constant $C(n)$ such that Theorem B holds for the gauge function $h_\gamma$ with $\gamma=C(n)sp^{1/n}$?

(b) Does Theorem B hold for some gauge function $h$ with $h_{\gamma_0}\prec h$?
(c) Does Theorem B hold for some gauge function $h_\beta$ with $\beta>\gamma_0$?
(d) Does Theorem B hold for the gauge function $h_{\alpha_0}$?
(e) Does Theorem B hold for any gauge function $h_\gamma$ with $\gamma=C(n)(s/(n-s))p^{1/n}$?
(f) Is there an example like Example 4.2 but with $\alpha=\alpha_0$?
(g) Is there an example like Example 4.2 but for some gauge function $h$ with $h\preceq h_{\alpha_0}$?

We note that the gauge functions in item (e) are better than those in item (a), at least for $s > n-1$, and so would give a stronger result. Also, Example 4.2 provides the following information about the constant $C(n)$ in item (a): any such constant must satisfy

$$C(n) \le n^{-1/n}\left(\frac{n}{n-1}\right)^{(n-1)/n} \qquad \text{so, for example, } C(2) \le 1.$$

We mention that this also provides information regarding [HK03, Problem B].

**4.3. Compression examples with** $\exp \mathcal{A}(pK) \in \mathsf{L}^1_{\mathrm{loc}}$**.** We continue our discussion of the optimality of the gauge function $h = h_{s,p,\mathcal{A},n}$ that appears in Theorem B. We assume that the control function has the form $\mathcal{A}(t) = t/\mathcal{L}(t)$ as in Lemma 2.5. As noted in (1.4), here the gauge $h$ is of the form $h = g_\beta$ where $\beta = Csp^{1/n}$ (with $C = C(\mathcal{L}, n)$) and

$$g_\beta(t) := \exp\left(-\beta \mathcal{A}\left(\log^{n-1}\frac{1}{t}\right)^{1/n}\right).$$

In addition, we further assume that $\mathcal{L} = \mathcal{L}_k$ for some $k \in \mathsf{N}$; see (2.5). This assumption is only used once, when we appeal to Lemma 2.3.

Example 4.5. Let $n \ge 2$, $s \in (0,n)$, and $p > 0$ be given. Fix $\beta > \beta_0 := Bsp^{1/n}$ where

$$B = B(s,n) := \left(\frac{n}{n-1}\right)^{(n-1)/n} \frac{n}{n-s}.$$

There exists a finite distortion homeomorphism $\mathsf{R}^n \xrightarrow{f} \mathsf{R}^n$ and a regular Cantor dust $E$ in $\mathsf{B}^n$ such that $f$ has $p$-subexponentially integrable distortion, that is, $\exp \mathcal{A}(pK_f) \in \mathsf{L}^1_{\mathrm{loc}}(\mathsf{R}^n)$, and $\mathcal{H}^s(E) \simeq 1$ but $\mathcal{H}^{g_\beta}(f(E)) = 0$. Moreover, for all $x \in \mathsf{R}^n \setminus \mathsf{B}^n$, $f(x) = x$.

*Proof.* We proceed as in Example 4.2, but here there are more technical details. For each integer $m > (1/\kappa)^{ns/(n-s)}$ (recall that $\kappa = \kappa(n) = 1/\sqrt{8n}$), set

$$\beta_m := \left(\frac{n}{n-1}\right)^{(n-1)/n} p^{1/n}\left(\frac{ns}{n-s}\right)^{1/n} \Big/ \left(\frac{n-s}{ns} - \frac{\log(1/\kappa)}{\log m}\right)^{(n-1)/n}.$$

Then as $m \to \infty$, $\beta_m \searrow \beta_0$. Thus we may select $m$ sufficiently large so that $\beta > \beta_m > \beta_0$, and these inequalities will also hold if we take a larger $m$.

For this $m$ we pick $R$ and $\sigma$ so that $mR^n = \kappa^n$ and $m(\sigma R)^s = 1$. Thus

$$R := \kappa m^{-1/n} \quad \text{and} \quad \sigma := \kappa^{-1} m^{(1/n)-(1/s)} = \kappa^{-1} m^{(s-n)/ns}.$$

Now we select $\lambda_N$ so that with $K_N := \lambda_N^{n-1}$ we have

$$\mathcal{A}(pK_N) = aN, \quad \text{where } a > 0 \text{ is a constant described below.}$$

We use these values of $m, R, \sigma, \lambda_N$ in the Cantor dust construction to obtain a finite distortion homeomorphism $f : \mathsf{R}^n \to \mathsf{R}^n$ and regular Cantor dusts $E, F \subset \mathsf{B}^n$ with $F = f(E)$ and $\mathcal{H}^s(E) \simeq 1$. We claim that the constant $a$ can be chosen so that both $\exp \mathcal{A}(pK_f) \in \mathsf{L}^1_{\mathrm{loc}}(\mathsf{R}^n)$ and $\mathcal{H}^{h_\beta}(F) = 0$.

Recalling—see (4.2)—that the integrability condition $e^{\mathcal{A}(pK_f)} \in \mathsf{L}^1_{\mathrm{loc}}(\mathsf{R}^n)$ is equivalent to convergence of the series

$$\sum_{N=1}^{\infty} m^{(1-n/s)N} e^{\mathcal{A}(pK_N)} = \sum_{N=1}^{\infty} m^{(1-n/s)N} e^{aN},$$

and writing

$$m^{(1-n/s)N} e^{aN} = \exp\left(\frac{s-n}{s} N \log m + aN\right) = \exp\left(N\left(\frac{s-n}{s}\log m + a\right)\right),$$

we see that

$$\exp(pK_f) \in \mathsf{L}^1_{\mathrm{loc}}\big(\mathsf{R}^n\big) \quad\Longleftrightarrow\quad a < \frac{n-s}{s}\log m.$$

Below we demonstrate that by choosing

$$a > \beta^{-n}\left(\frac{n}{n-1}\right)^{n-1} np\log m \Big/ \left(\frac{n-s}{ns} - \frac{\log(1/\kappa)}{\log m}\right)^{n-1} \tag{4.4}$$

we obtain $\mathcal{H}^{g_\beta}(F) = 0$. Thus we must check that we can pick a constant $a$ that satisfies

$$\beta^{-n}\left(\frac{n}{n-1}\right)^{n-1} np\log m \Big/ \left(\frac{n-s}{ns} - \frac{\log(1/\kappa)}{\log m}\right)^{n-1} < a < \frac{n-s}{s}\log m.$$

This is equivalent to requiring that

$$\beta^n > \left(\frac{n}{n-1}\right)^{n-1} p\left(\frac{ns}{n-s}\right) \Big/ \left(\frac{n-s}{ns} - \frac{\log(1/\kappa)}{\log m}\right)^{n-1}$$

and this holds because $\beta > \beta_m$.

It remains to confirm that the above lower bound on $a$, in (4.4), forces $\mathcal{H}^{g_\beta}(F) = 0$. This holds provided $m^N g_\beta(t_N) \to 0$ as $N \to \infty$, where

$$t_N = \sigma^{S_N} R^N$$

is the radius of the balls used to construct $F_N = f(E_N)$ and

$$S_N := \lambda_1 + \cdots + \lambda_N = K_1^{1/(n-1)} + \cdots + K_N^{1/(n-1)}.$$

Notice that

$$m^N g_\beta(t_N) = \exp\left(N\log m - \beta\mathcal{A}\left(\log^{(n-1)/n}\frac{1}{t_N}\right)\right) \longrightarrow 0$$

if and only if

$$\beta\mathcal{A}\left(\log^{(n-1)/n}\frac{1}{t_N}\right) - N\log m \longrightarrow \infty. \tag{4.5}$$

We have

$$\begin{aligned}
\log\frac{1}{t_N} &= S_N\log\frac{1}{\sigma} - N\log R\\
&= S_N\left(\left(\frac{n-s}{ns}\right)\log m - \log\frac{1}{\kappa}\right) + N\left(\frac{1}{n}\log m + \log\frac{1}{\kappa}\right)\\
&= \frac{\log m}{n}\left[S_N\left(\frac{n-s}{s} - n\frac{\log 1/\kappa}{\log m}\right) + N\left(1+n\frac{\log 1/\kappa}{\log m}\right)\right]\\
&= \frac{\log m}{n}S_N\left[\left(\frac{n-s}{s} - n\frac{\log 1/\kappa}{\log m}\right) + \frac{N}{S_N}\left(1+n\frac{\log 1/\kappa}{\log m}\right)\right]\\
&= \frac{\log m}{n}S_N\left[\frac{n-s}{s} + \frac{N}{S_N} + \left(\frac{N}{S_N}-1\right)n\frac{\log 1/\kappa}{\log m}\right]\\
&= \frac{\log m}{n}S_N T_N,
\end{aligned}$$

where

$$T_N := \frac{n-s}{s} + \frac{N}{S_N} + \left(\frac{N}{S_N}-1\right)n\frac{\log 1/\kappa}{\log m}.$$

Recalling that $\mathcal{A}(t)=t/\mathcal{L}(t)$ we obtain

$$\mathcal{A}\left(\log^{n-1}\frac{1}{t_N}\right)^{1/n} = \frac{((\log m/n)S_N T_N)^{(n-1)/n}}{\mathcal{L}(((\log m/n)S_N T_N)^{n-1})^{1/n}} = \frac{(Q_N S_N)^{(n-1)/n}}{\mathcal{L}((Q_N S_n)^{n-1})^{1/n}},$$

where $Q_N := (\log m/n)T_N$, and so

$$\beta\mathcal{A}\left(\log^{n-1}\frac{1}{t_N}\right)^{1/n} - N\log m = N\log m\left(\frac{\beta}{N\log m}\frac{(Q_N S_N)^{(n-1)/n}}{\mathcal{L}((Q_N S_n)^{n-1})^{1/n}} - 1\right).$$

We (eventually) show that

$$\lim_{N\to\infty}\frac{\beta}{N\log m}\frac{(Q_N S_N)^{(n-1)/n}}{\mathcal{L}((Q_N S_n)^{n-1})^{1/n}} > 1 \quad\iff\quad (4.4)\text{ holds;} \tag{4.6}$$

that is, the above limit exists and is strictly larger than one if and only if (4.4) holds. Thus by choosing the constant $a$ so that (4.4) holds, the limit inequality in (4.6) will hold, so (4.5) will be true, which in turn gives $m^N g_\beta(t_N)\to 0$ as $N\to\infty$ and therefore $\mathcal{H}^{g_\beta}(F)=0$. Thus it remains to establish (4.6).

To this end, we recall that $\omega(aN)=aN\mathcal{A}^{-1}(aN)^{1/(n-1)}$, and write

$$\begin{aligned}
&\frac{1}{N\log m}\frac{(Q_N S_N)^{(n-1)/n}}{\mathcal{L}((Q_N S_n)^{n-1})^{1/n}}\\
&\quad= \left(\frac{ap^{1/(n-1)}S_N}{\omega(aN)}\right)^{(n-1)/n}\cdot\frac{\omega(aN)^{(n-1)/n}}{a^{(n-1)/n}p^{1/n}N}\cdot\frac{Q_N^{(n-1)/n}/\log m}{\mathcal{L}((Q_N S_n)^{n-1})^{1/n}}\\
&\quad= \left(\frac{ap^{1/(n-1)}S_N}{\omega(aN)}\right)^{(n-1)/n}\cdot\frac{Q_N^{(n-1)/n}}{p^{1/n}\log m}\cdot\left(\frac{\mathcal{A}^{-1}(aN)}{N\mathcal{L}((Q_N S_n)^{n-1})}\right)^{1/n}.
\end{aligned}$$

Next, we claim that as $N \to \infty$,

$$\lim_{N\to\infty} \frac{ap^{1/(n-1)} S_N}{\omega(aN)} = \frac{n-1}{n}, \tag{4.7a}$$

$$\lim_{N\to\infty} Q_N = \frac{\log m}{n}\left(\frac{n-s}{s} - n\frac{\log 1/\kappa}{\log m}\right), \tag{4.7b}$$

$$\lim_{N\to\infty} \frac{\mathcal{A}^{-1}(aN)}{N\mathcal{L}((Q_N S_N)^{n-1})} = \frac{a}{n}. \tag{4.7c}$$

Armed with this information, we see that the limit on the left-hand side of (4.6) exists and is equal to $\beta/p^{1/n}\log m$ times the appropriate "product-power combination" of the above limits; that is,

$$\begin{aligned}
&\lim_{N\to\infty} \frac{\beta}{N\log m}\frac{(Q_N S_N)^{(n-1)/n}}{\mathcal{L}((Q_N S_N)^{n-1})^{1/n}} \\
&\quad = \frac{\beta}{p^{1/n}\log m}\left[\left(\frac{n-1}{n}\frac{\log m}{n}\left(\frac{n-s}{s} - n\frac{\log 1/\kappa}{\log m}\right)\right)^{n-1}\frac{a}{n}\right]^{1/n} \\
&\quad = \beta\left(\frac{n-1}{n}\right)^{(n-1)/n}\left(\frac{a}{pn\log m}\right)^{1/n}\left(\frac{n-s}{ns} - \frac{\log 1/\kappa}{\log m}\right)^{(n-1)/n}.
\end{aligned}$$

Evidently, the limit above is strictly larger than one if and only if (4.4) holds, and this establishes (4.6) (under the assumption that (4.7a), (4.7b), (4.7c) all hold).

Finally, it remains to establish the limits expressed in (4.7a), (4.7b), and (4.7c). The first of these, (4.7a), follows immediately from (2.14f) once we remember that

$$S_N := \sum_{j=1}^{n} \lambda_j = \sum_{j=1}^{n} K_j^{1/(n-1)} = p^{-1/(n-1)} \sum_{j=1}^{n} \mathcal{A}^{-1}(aj)^{1/(n-1)}.$$

Next, since $Q_N = (\log m/n)T_N$, we see that (4.7b) is equivalent to

$$\lim_{N\to\infty} T_N = \left(\frac{n-s}{s} - n\frac{\log 1/\kappa}{\log m}\right).$$

The above limit follows easily from the definition of $T_N$ and the fact that $\lim_{N\to\infty}(N/S_N) = 0$; this latter limit is found by writing

$$\frac{N}{S_N} = \frac{\omega(aN)}{S_N}\cdot\frac{N}{\omega(aN)} = \frac{\omega(aN)}{S_N}\cdot\frac{1}{\mathcal{A}^{-1}(aN)^{1/(n-1)}},$$

using (4.7a), and remembering that $\mathcal{A}^{-1}(s) \to \infty$ as $s \to \infty$.

To verify (4.7c), we first use (2.14b) to see that

$$\lim_{N\to\infty} \frac{\mathcal{A}^{-1}(aN)}{N\mathcal{L}((Q_N S_N)^{n-1})} = \lim_{N\to\infty} \frac{aN\mathcal{L}(aN)}{N\mathcal{L}((Q_N S_N)^{n-1})} = \lim_{N\to\infty} \frac{a\mathcal{L}(aN)}{\mathcal{L}((Q_N S_N)^{n-1})}.$$

We let $u := aN$ and write

$$\frac{\mathcal{L}(aN)}{\mathcal{L}((Q_N S_N)^{n-1})} = \frac{\mathcal{L}(u)}{\mathcal{L}(u^n)} \cdot \frac{\mathcal{L}(u^n)}{\mathcal{L}(\Lambda\omega(u)^{n-1})} \cdot \frac{\mathcal{L}(\Lambda\omega(u)^{n-1})}{\mathcal{L}((Q_N S_N)^{n-1})},$$

where $\Lambda$ is a constant that is described below. The three fractions on the above right-hand side have limits $1/n, 1, 1$, respectively, as $N \to \infty$, and thus (4.7c) holds. The first of these limits is an easy consequence of Lemma 2.3. The second is just (2.14g). For the third, we note that—by employing both (4.7a) and (4.7b)—we have

$$\Lambda := \lim_{N\to\infty}\left(\frac{Q_N S_N}{\omega(u)}\right)^{n-1} = \left(\frac{1}{ap^{1/(n-1)}}\frac{n-1}{n}\frac{\log m}{n}\left(\frac{n-s}{s} - n\frac{\log 1/\kappa}{\log m}\right)\right)^{n-1};$$

that is, the above limit exists and equals the right-hand quantity. Therefore

$$\lim_{N\to\infty}\frac{\Lambda\omega(u)^{n-1}}{(Q_N S_N)^{n-1}} = 1,$$

so by Lemma 2.2(b),

$$\lim_{N\to\infty}\frac{\mathcal{L}(\Lambda\omega(u)^{n-1})}{\mathcal{L}((Q_N S_N)^{n-1})} = 1. \qquad \square$$

It is not difficult to use the above to construct an example where the map does not depend on either of the parameters $\alpha, s$. Let $(s_j)_1^\infty$ and $(\alpha_j)_1^\infty$ be monotone sequences in $(0,n)$ and $(\alpha_0,\infty)$ respectively with $s_j \nearrow n$ and $\alpha_j \searrow \alpha_0$ as $j \to \infty$. Let $f_j$ and $E_j$ be the maps and sets constructed in Example 4.5 using the parameters $s_j, \alpha_j$ (and some given control function $\mathcal{A}$ and fixed $p > 0$). By translating the set $E_j$, we may assume that $E_j \subset \mathsf{B}_j := \mathsf{B}(2je, 1)$, where $e := (1,0,\dots,0) \in \mathsf{R}^n$. In particular, for all $x \in \mathsf{R}^n \setminus \mathsf{B}_j$, $f_j(x) = x$. Thus, we may define $f : \mathsf{R}^n \to \mathsf{R}^n$ by letting $f(x) := f_j(x)$ for $x \in \mathsf{B}_j$ and $f(x) := x$ for $x \in \mathsf{R}^n \setminus A$ where $A := \bigcup \mathsf{B}_j$. We summarize this as follows.

Example 4.6. Let $n \geq 2$, $p > 0$, and $\mathcal{A}$ be given. There exists a finite distortion homeomorphism $f : \mathsf{R}^n \to \mathsf{R}^n$ with $\exp\mathcal{A}(pK_f) \in \mathsf{L}^1_{\mathrm{loc}}(\mathsf{R}^n)$ and a set $A \subset \mathsf{R}^n$ with the following property. For each $s \in (0,n)$ and each $\beta > \beta_0$, there is a regular Cantor dust $E \subset A$ such that $\mathcal{H}^s(E) > 0$ but $\mathcal{H}^{g_\beta}(f(E)) = 0$.

**4.4. Modulus of continuity example.** We conclude with an example that illustrates to what extent Theorem A is best possible. We assume $\mathcal{L} : [0,\infty) \to [0,\infty)$ is a $\mathcal{C}^1$ homeomorphism that satisfies (2.1) and (2.2) and define

$$\mathcal{A}(t) := \frac{t}{\mathcal{L}(t)} \quad \text{and} \quad \omega(s) := s\mathcal{A}^{-1}(s)^{1/(n-1)}.$$

Example 4.7. Define $\mathsf{R}^n \xrightarrow{f} \mathsf{R}^n$ by $f(x) := \rho(|x|)\frac{x}{|x|}$ where

$$\rho(t) := \exp\left(-\frac{M}{p^{\frac{1}{n-1}}}\omega\left(\log\frac{1}{t}\right)\right);$$

here $p>0$ is given and $M>0$ will be chosen appropriately. Then $f$ is a finite distortion homeomorphism. When $M$ is sufficiently small (i.e., $M \le C(\mathcal{L},n)$), $\exp \mathcal{A}(pK_f)$ is locally integrable in a neighborhood of the origin.

*Proof.* Since $f$ is a radial map, it is not difficult to check that, with $r := |x|$,

$$\big|Df(x)\big| = \max\left\{\rho'(r), \frac{\rho(r)}{r}\right\} \quad \text{and} \quad J(x,f) = \rho'(r)\left(\frac{\rho(r)}{r}\right)^{n-1}.$$

A calculation reveals that for $r$ sufficiently small, $|Df(x)| = \rho'(r)$ and so

$$K_f(x) = p^{-1}M^{n-1}\omega'\big(\log(1/r)\big)^{n-1}.$$

Thanks to (2.14e), $\omega'(s) \le 2\mathcal{A}^{-1}(s)^{1/(n-1)}$ for all sufficiently large $s>0$, so for all sufficiently small $r=|x|$,

$$pK_f(x) \le (2M)^{n-1}\mathcal{A}^{-1}\left(\log\frac{1}{r}\right).$$

Appealing to (2.14a) we now deduce that for all sufficiently small $r=|x|$,

$$\exp \mathcal{A}\big(pK_f(x)\big) \le \exp\left(C(2M)^{n-1}\log\frac{1}{r}\right) = \frac{1}{r^\beta},$$

where $C = C(\mathcal{L})$ depends on $\mathcal{L}$ and $\beta = 2^{n-1}CM^{n-1}$.

Thus by choosing $M>0$ so that $\beta<n$, that is, with $M^{n-1} < n/(2^{n-1}C)$, we obtain $\exp\mathcal{A}(pK_f)$ locally integrable in a neighborhood of the origin. $\square$

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