Illinois Journal of Mathematics
Volume 57, Number 3, Fall 2013, Pages 629–664
S 0019-2082

DIVISION OF HOLOMORPHIC FUNCTIONS AND GROWTH CONDITIONS

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ABSTRACT. Let D be a strictly convex domain of $\mathbb{C}^n$, f_1 and f_2 be two holomorphic functions defined on a neighbourhood of $\overline{D}$ and set $X_l = \{z, f_l(z) = 0\}$, $l = 1, 2$. Suppose that $X_l \cap bD$ is transverse for $l = 1$ and $l = 2$, and that $X_1 \cap X_2$ is a complete intersection. We give necessary conditions when $n \geq 2$ and sufficient conditions when $n = 2$ under which a function g can be written as $g = g_1 f_1 + g_2 f_2$ with g_1 and g_2 in $L^q(D)$, $q \in [1, +\infty)$, or g_1 and g_2 in $\mathrm{BMO}(D)$. In order to prove the sufficient condition, we explicitly write down the functions g_1 and g_2 using integral representation formulae and new residue currents.

1. Introduction

In this article, we are interested in ideals of holomorphic functions and corona type problems. More precisely, being given a domain D of $\mathbb{C}^n$ and k functions $f_1, \dots, f_k$ holomorphic in a neighbourhood of $\overline{D}$, we are looking for condition(s), as close as possible to being necessary and sufficient, under which a function g, holomorphic on D, can be written as

$$g = f_1 g_1 + \cdots + f_k g_k, \tag{1}$$

with $g_1, \dots, g_k$ holomorphic on D and satisfying growth conditions at the boundary of D. We restrict ourselves to a strictly convex domain D of $\mathbb{C}^n$ and we consider the case of two generators f_1 and f_2, holomorphic in a neighbourhood of $\overline{D}$. We write D as $D = \{z \in \mathbb{C}^n, \rho(z) < 0\}$ where ρ is a smooth strictly convex function defined on $\mathbb{C}^n$ such that the gradient of ρ does not vanish in a neighbourhood $\mathcal{U}$ of the boundary of D. We denote by D_r, $r \in \mathbb{R}$, the set $D_r = \{z \in \mathbb{C}^n, \rho(z) < r\}$, by bD_r its boundary, by η_ζ the outer unit normal to $bD_{\rho(\zeta)}$ at a point $\zeta \in \mathcal{U}$ and by v_ζ a smooth unitary complex vector

Received February 7, 2013; received in final form September 2, 2013.
2010 *Mathematics Subject Classification.* 32A22, 32A26, 32A27, 32A37, 32A40, 32A55.

field tangent at ζ to $bD_{\rho(\zeta)}$. We denote by X_1 the set $X_1=\{z,f_1(z)=0\}$, and by X_2 the set $X_2=\{z,f_2(z)=0\}$. We assume that the intersections $X_1\cap bD$ and $X_2\cap bD$ are transverse in the sense of tangent cones and that $X_1\cap X_2$ is a complete intersection. Let us recall that an analytic subset A of pure co-dimension m in $\mathbb{C}^n$ is said to be a complete intersection if there are m holomorphic functions $h_1,\dots,h_m$ such that $A=\bigcap_{i=1}^m\{z,h_i(z)=0\}$; and that the intersection $X_l\cap D$, $l=1$ or $l=2$, is said to be transverse if for every $p\in X_l\cap bD$, the complex tangent space to bD at p and the tangent cone to X_l at p span $T_p\mathbb{C}^n$.

Our goal here is to find assumptions on g, holomorphic in D, as close as possible to being necessary and sufficient, under which we can write g as $g=g_1f_1+g_2f_2$ with g_1 and g_2 in D holomorphic and belonging to $\mathrm{BMO}(D)$ or $L^q(D)$, $q\in[1,+\infty)$.

In order to formulate our first result, we will need to compute the values of solutions of (1) and we will need to understand their interplay between different leafs of X_1 and X_2. This will be achieved using divided differences of $\frac{g}{f_1}$ on $X_2\setminus X_1$ and $\frac{g}{f_2}$ on $X_1\setminus X_2$, which we now define. For z a point in D and v a unit vector of $\mathbb{C}^n$, we set

$$\Lambda^{(1)}_{z,v}=\big\{\lambda\in\mathbb{C},|\lambda|<\tau\big(z,v,3\kappa\big|\rho(z)\big|\big)\text{ and }z+\lambda v\in X_2\setminus X_1\big\},$$

where $\tau(z,v,\varepsilon)$ is the maximal positive r such that the disc $\Delta_{z,v}(r)=\{z+\lambda v,|\lambda|<r\}$ is included in $D_{\rho(z)+\varepsilon}$. In particular, when v is the normal direction to $bD_{\rho(z)}$ at the point z, $\tau(z,v,\varepsilon)=\varepsilon$, and when v is tangent to $bD_{\rho(z)}$ at z, $\tau(z,v,\varepsilon)=\varepsilon^{\frac12}$. We notice that the points $z+\lambda v$, $\lambda\in\Lambda^{(1)}_{z,v}$, are the points of $X_2\setminus X_1$ which belong to the disc $\Delta_{z,v}(\tau(z,v,3\kappa|\rho(z)|))$, so they all belong to $D\cap(X_2\setminus X_1)$ provided that $\kappa<\frac13$.

For $z\in D\cap(X_2\setminus X_1)$, let us set $g^{(1)}(z)=\frac{g(z)}{f_1(z)}$ and for $z\in\mathbb{C}^n$, v a unit vector of $\mathbb{C}^n$ and $\lambda\in\mathbb{C}$ such that $z+\lambda v$ belongs to $X_2\setminus X_1$, let us put $g^{(1)}_{z,v}[\lambda]=g^{(1)}(z+\lambda v)$.

Assuming $g^{(1)}_{z,v}[\mu_1,\dots,\mu_k]$ to be well defined, we set for $\lambda_1,\dots,\lambda_{k+1}\in\mathbb{C}$ pairwise distinct in $\Lambda^{(1)}_{z,v}$:

$$g^{(1)}_{z,v}[\lambda_1,\dots,\lambda_{k+1}]:=\frac{g^{(1)}_{z,v}[\lambda_1,\dots,\lambda_k]-g^{(1)}_{z,v}[\lambda_2,\dots,\lambda_{k+1}]}{\lambda_1-\lambda_{k+1}}.$$

Lastly we define the following quantity:

$$c^{(1)}_\infty(g)=\sup\big(\big|g^{(1)}_{z,v}[\lambda_1,\dots,\lambda_k]\big|\tau\big(z,v,\big|\rho(z)\big|\big)^{k-1}\big),$$

where the supremum is taken over all $z\in D$, all $v\in\mathbb{C}^n$ with $|v|=1$, all $k\in\mathbb{N}^*$ and $\lambda_1,\dots,\lambda_k\in\Lambda^{(l)}_{z,v}$ pairwise distinct. We also define $\Lambda^{(2)}_{z,v}$, $g^{(2)}$, $g^{(2)}_{z,v}[\lambda_1,\dots,\lambda_k]$ and $c^{(2)}_\infty(g)$ analogously.

Our first main result gives necessary conditions in $\mathbb{C}^n$, $n \geq 2$, for the existence of g_1 and g_2 holomorphic and bounded such that $g = g_1 f_1 + g_2 f_2$ (see Theorem 6.4 for conditions with g_1 and g_2 in $L^q(D)$).

THEOREM 1.1. *Let D be a strictly convex domain of $\mathbb{C}^n$, $n \geq 2$, let f_1 and f_2 be two holomorphic functions defined on a neighbourhood of $\overline{D}$ and set $X_l = \{z, f_l(z) = 0\}$, $l = 1, 2$. Suppose that $X_l \cap bD$ is transverse for $l = 1$ and $l = 2$, and that $X_1 \cap X_2$ is a complete intersection. Let g_1, g_2 be two bounded holomorphic functions on D and set $g = g_1 f_1 + g_2 f_2$. Then*

(i) $c(g) = \sup_{z \in D} \frac{|g(z)|}{\max(|f_1(z)|, |f_2(z)|)}$ *is finite,*
(ii) $c_\infty^{(1)}(g)$ *and* $c_\infty^{(2)}(g)$ *are finite.*

The first necessary condition of Theorem 1.1 is obvious, because we trivially have that $c(g) \leq C \max(\|g_1\|_{L^\infty(D)}, \|g_2\|_{L^\infty(D)})$ for some universal positive constant C, and g_1 and g_2 are bounded.

The second condition may appear strange at first sight. Intuitively, it comes from the following fact. Assume that we can write g as $g = g_1 f_1 + g_2 f_2$ with g_1 and g_2 holomorphic and take z and $z + \lambda v$ on two distinct leaves of $X_2 \setminus X_1$. Now suppose that z gets close to a singularity of X_2 and to bD. Then, by transversality, λ will also get close to 0; the quantity $\frac{g_1(z+\lambda v) - g_1(z)}{\lambda}$ will thus be close to the derivative $\frac{\partial g_1}{\partial v}(z)$, which by Cauchy inequalities cannot grow faster than $\frac{\sup_D |g_1|}{\tau(z, v, |\rho(z)|)}$ if g_1 is bounded. It follows that the quantity $(\frac{g}{f_1}(z + \lambda v) - \frac{g}{f_1}(z))/\lambda \cdot \tau(z, v, |\rho(z)|)$ is bounded when g_1 is bounded. This can be generalised to higher orders of divided differences and this becomes the condition (ii) of Theorem 1.1.

With the additional hypothesis that $|f_1|^2 + |f_2|^2 \geq \varepsilon^2 > 0$, Condition (i) of Theorem 1.1 is shown to be sufficient or nearly sufficient in many of the known results like these of Carleson [10], Andersson and Carlsson [5], [6], [7], and Varopoulos [19]. In [10], working in $\mathbb{C}$ and assuming that g is bounded and that f_1 and f_2 are defined, bounded and holomorphic (only) on D and satisfy $|f_1|^2 + |f_2|^2 \geq \varepsilon^2 > 0$, Carleson proved that one can solve (1) with g_1 and g_2 bounded on D. In [5], [6], [7], [19], working in $\mathbb{C}^n$, $n \geq 2$, still assuming that g is bounded and that f_1 and f_2 are defined, bounded and holomorphic on D and satisfy $|f_1|^2 + |f_2|^2 \geq \varepsilon^2 > 0$, the authors proved that there exist g_1 and g_2 in the BMO space of bD which solve (1). However, when we do not make the assumption $|f_1|^2 + |f_2|^2 \geq \varepsilon^2 > 0$, this cannot be achieved if we only assume g to be bounded. For example, let us consider the ball $\mathbb{B}$ of radius 1 and centred at $(1, 0)$ in $\mathbb{C}^2$, $\rho(z) = |z_1|^2 + |z_2|^2 - 2 \operatorname{Re} z_1$, $f_1(z) = z_2^2$, $f_2(z) = z_2^2 - z_1^q$ and $g(z) = z_1^{\frac{q}{2}} z_2$ where $q \geq 3$ is an odd integer. Then $g(z) = z_2 z_1^{-\frac{q}{2}} f_1(z) - z_2 z_1^{-\frac{q}{2}} f_2(z)$, so g belongs to the ideal generated by f_1 and f_2, and $\frac{|g|}{|f_1| + |f_2|}$ is bounded on D by $\frac{3}{2}$, so $c(g)$ is finite. However, for small $\varepsilon > 0$, setting $z = (\varepsilon, 0)$, $v = (0,1)$, $\lambda_1 = \varepsilon^{\frac{q}{2}}$ and $\lambda_2 = -\varepsilon^{\frac{q}{2}}$, we have that

$$g^{(1)}_{z,v}[\lambda_1, \lambda_2] = \frac{\frac{g}{f_1}(z+\lambda_1 v) - \frac{g}{f_1}(z+\lambda_2 v)}{\lambda_1 - \lambda_2}\left|\rho(z)\right|^{\frac{1}{2}} = \varepsilon^{\frac{1-q}{2}}$$

which is unbounded when ε goes to zero. So $c^{(1)}_\infty(g)$ is not bounded and according to Theorem 1.1 we cannot write g as $g = g_1 f_1 + g_2 f_2$ with g_1 and g_2 bounded.

In our search for sufficient conditions on g to solve (1) with g_1 and g_2 holomorphic and belonging to the BMO space of D, one may consider the case of more regular holomorphic functions. For example, in [9], Bonneau, Cumenge and Zériahi consider the case of Lipschitz spaces. For f_1 and f_2 holomorphic in D, smooth in a neighbourhood of $\overline{D}$, maybe with common zeroes, and such that $\partial f_1 \wedge \partial f_2 \wedge \partial\rho$ does not vanish on $bD \cap X_1 \cap X_2$, they solve (1) with g_1 and g_2 in the BMO space of bD when g belongs to the Lipschitz space $C^{\frac{1}{2}}(\overline{D})$ and vanishes on $D \cap X_1 \cap X_2$. This result can be seen as a loss of regularity of $\frac{1}{2}$, which is optimal in their case. We could try to consider a more regular g and, perhaps at the cost of a huge loss of regularity, we could hope to get a BMO division. However, improving the regularity of g will not help in our case, as shown by the following example. We consider the functions $f_1(z) = z_1^3 - z_2^2$, $f_2(z) = z_2$ and $g(z) = z_1$ on $\mathbb{B} = \{z \in \mathbb{C}^2, \rho(z) = |z_1|^2 + |z_2|^2 - 2\operatorname{Re} z_1 < 0\}$. The function g belongs to the ideal of holomorphic functions on $\mathbb{B}$ generated by f_1 and f_2 because $g(z) = \frac{1}{z_1^2} f_1(z) + \frac{z_2}{z_1^2} f_2(z)$. However $\frac{g}{f_1}$ is trivially unbounded on $X_2 \cap \mathbb{B}$, so $c^{(1)}_\infty(g)$ is not bounded and Theorem 1.1 implies that (1) cannot be solved with g_1 and g_2 bounded on $\mathbb{B}$, although g is extremely regular. Moreover, since g belongs to any reasonable space, this example also shows that, without special assumptions on f_1 and f_2, it is hopeless to consider other spaces of functions like H^p or L^p or Besov spaces in order to get direct and nice generalisations of the theorems of Amar [2], Amar and Bruna [3], Amar and Menini [4], Andersson and Carlsson [5], [6], [7], Fàbrega and Ortega [11], Krantz and Li [12] or Skoda in [18].

Mixed conditions like $\frac{g}{f_1}$ and $\frac{g}{f_2}$ bounded on $D \cap X_2$ and $D \cap X_1$, respectively and g regular enough are not sufficient either. For example, the function $g(z) = z_1^2 z_2(z_2 z_1 - 1)$ is as regular as we may wish and g belongs to the ideal of holomorphic functions on $\mathbb{B}$ generated by $f_1(z) = z_1^5 - z_2^2$ and $f_2(z) = z_2^3 - z_1^4$ because $g(z) = (z_1^5 - z_2^2)\frac{z_2^2}{z_1^2} + (z_2^3 - z_1^4)\frac{z_2}{z_1^2}$. Moreover $\frac{g}{f_1}$ and $\frac{g}{f_2}$ are bounded on X_2 and X_1 respectively. However, for $z = (\varepsilon, 0)$, $v = (0,1)$, $\lambda_1 = \varepsilon^{\frac{5}{2}}$ and $\lambda_2 = -\varepsilon^{\frac{5}{2}}$, we have

$$g^{(2)}_{z,v}[\lambda_1, \lambda_2] = \frac{\frac{g}{f_2}(z+\lambda_1 v) - \frac{g}{f_2}(z+\lambda_2 v)}{\lambda_1 - \lambda_2}\left|\rho(z)\right|^{\frac{1}{2}} = \varepsilon^{-\frac{3}{2}}\sqrt{2-\varepsilon}.$$

So $c^{(2)}_\infty(g)$ is not bounded and again, Theorem 1.1 implies that (1) cannot be solved with g_1 and g_2 bounded.

According to these examples, it seems that divided differences are the key notion to obtain reasonable sufficient conditions for (1) to be solvable with g_1 and g_2 holomorphic and bounded. We will prove that they are indeed nearly sufficient in $\mathbb{C}^2$:

THEOREM 1.2. *Let D be a strictly convex domain of $\mathbb{C}^2$, let f_1 and f_2 be two holomorphic functions defined on a neighbourhood of $\overline{D}$ and set $X_l = \{z, f_l(z) = 0\}$, $l = 1, 2$. Suppose that $X_l \cap bD$ is transverse for $l = 1$ and $l = 2$, and that $X_1 \cap X_2$ is a complete intersection. Let g be a holomorphic function on D which belongs to the ideal of $\mathcal{O}(D)$ generated by f_1 and f_2 and such that*

(i) $c(g) = \sup_{z\in D} \frac{|g(z)|}{\max(|f_1(z)|,|f_2(z)|)}$ *is finite,*

(ii) $c_\infty^{(1)}(g)$ *and* $c_\infty^{(2)}(g)$ *are finite.*

Then there exist two holomorphic functions g_1 and g_2 which belong to the BMO *space of D and are such that $g_1 f_1 + g_2 f_2 = g$.*

We also have a similar result for $L^p(D)$-spaces, see Theorem 6.5.

In the previous papers dealing with corona type questions, there are two kinds of approaches. The first one is to find two smooth functions on D, $\tilde{g}_1$ and $\tilde{g}_2$, such that

$$\tilde{g}_1 f_1 + \tilde{g}_2 f_2 = g; \tag{2}$$

and to solve the equation

$$\overline{\partial}\varphi = \frac{\overline{f_1}\,\overline{\partial}\tilde{g}_2 - \overline{f_2}\,\overline{\partial}\tilde{g}_1}{|f_1|^2 + |f_2|^2}. \tag{3}$$

Then setting $g_1 = \tilde{g}_1 + \varphi f_2$ and $g_2 = \tilde{g}_2 - \varphi f_1$, g_1 and g_2 are holomorphic, we have $g = g_1 f_1 + g_2 f_2$ and, provided φ belongs to the appropriate space, g_1 and g_2 will belong to $\mathrm{BMO}(D)$, $H^p(D), \ldots$ So the problem is reduced to solving the Bezout equation (2) and then to solving the $\overline{\partial}$-equation (3) with an appropriate regularity. Let us mention that the usual choice for $\tilde{g}_i$ is simply

$$\tilde{g}_i = \frac{\overline{f_i} g}{|f_1|^2 + |f_2|^2}.$$

We point out that, even if it is not trivial a priori to check that $c_\infty^{(1)}(g)$ and $c_\infty^{(2)}(g)$ are finite with the only assumption that g is bounded on D, this classical choice of functions in Theorem 3.1 in Section 3, with the additional hypothesis that f_1 and f_2 are holomorphic in a neighbourhood of $\overline{D}$, allows us to retrieve a result of BMO type like those of Varopoulos in [19] and Andersson and Carlsson in [5], [6], [7].

In [6] and [9], the authors used an alternative technique. They constructed a division formula $g = f_1 T_1(g) + \cdots + f_k T_k(g)$ where for all i, T_i was a well chosen Berndtsson–Andersson integral operator, and under their respective assumptions, they proved that $T_i(g)$ belongs to the appropriate space.

In our case, Theorem 1.1 will be a corollary of the key result Theorem 3.1. In order to prove this theorem, we will first construct two currents T_1 and T_2 such that $f_1T_1 + f_2T_2 = 1$ on D and which have good properties (see Section 3). In Section 4, using these currents, we will construct two integral operators S_1 and S_2 such that if $\tilde{g}_1$ and $\tilde{g}_2$ are smooth functions with good growth conditions near bD which satisfy $\tilde{g}_1 f_1 + \tilde{g}_2 f_2 = g$, then $S_1(\tilde{g}_1, \tilde{g}_2)$ and $S_2(\tilde{g}_1, \tilde{g}_2)$ are holomorphic in D and satisfy $g = f_1 S_1(\tilde{g}_1, \tilde{g}_2) + f_2 S_2(\tilde{g}_1, \tilde{g}_2)$. It should be noticed that in our case, the integral operators depend on both $\tilde{g}_1$ and $\tilde{g}_2$ while in [6] and [9], the operators only depend on g. Moreover, contrary to what is done in [5], [7], [2], [4], [18], we do not solve a $\overline{\partial}$-equation in order to turn the smooth functions into holomorphic functions. In Section 5, we will finish the proof of Theorem 3.1 and prove that $S_j(\tilde{g}_1, \tilde{g}_2)$ belongs to $\mathrm{BMO}(D)$ or $L^q(D)$. Since in our case the usual choice $\tilde{g}_i = \frac{\overline{f_i} g}{|f_1|^2 + |f_2|^2}$ may not be a bounded function, we will have to construct new functions $\tilde{g}_1$ and $\tilde{g}_2$. This will be achieved thanks to the divided differences by a kind of interpolation method.

The paper is organised as follows. In Section 2, we recall some tools needed for the construction and the estimation of the division formula. Section 3 is devoted to the construction of the currents while Section 4 is devoted to the division formula itself. In Section 5, we prove that the currents lead to a division formula in $\mathrm{BMO}(D)$ or $L^q(D)$ spaces and finally in Section 6 we construct the smooth division formula using divided differences.

2. Notations and tools

2.1. Koranyi balls. The Koranyi balls centred at a point z in D have properties linked with distance from z to the boundary of D in a direction v. They were generalised in the case of convex domains of finite type by McNeal in [15] and [16]. A strictly convex domain being in particular a convex domain of type 2, we will adopt the formalism of convex domain of finite type.

The Koranyi balls in $\mathbb{C}^2$ are defined as follows. We call the coordinates system centred at ζ of basis η_ζ, v_ζ the Koranyi coordinates at ζ. We denote by (z_1^*, z_2^*) the coordinates of a point z in the Koranyi coordinates at ζ. The Koranyi ball centred in ζ of radius r is the set $\mathcal{P}_r(\zeta) := \{\zeta + \lambda\eta_\zeta + \mu v_\zeta, |\lambda| < r, |\mu| < r^{\frac{1}{2}}\}$.

Before we recall the properties of the Koranyi balls we will need, we adopt the following notation. We write $A \lesssim B$ if there exists some constant $c > 0$ such that $A \leq cB$. Each time we will mention on which parameters c depends. We will write $A \eqsim B$ if $A \lesssim B$ and $B \lesssim A$ both holds. The following propositions are part of well-known properties of Koranyi balls and McNeal polydiscs. The interested reader can find a proof of each statements in [15] in the case of convex domains of finite type, keeping in mind that a strictly convex domain is a convex domain of type 2.

PROPOSITION 2.1. *There exists a neighbourhood* $\mathcal{U}$ *of* bD *and positive real numbers* κ *and* c_1 *such that*

(i) *for all* $\zeta \in \mathcal{U} \cap D$, $\mathcal{P}_{4\kappa|\rho(\zeta)|}(\zeta)$ *is included in* D,
(ii) *for all* $\varepsilon > 0$, *all* $\zeta, z \in \mathcal{U}$, $\mathcal{P}_\varepsilon(\zeta) \cap \mathcal{P}_\varepsilon(z) \neq \emptyset$ *implies* $\mathcal{P}_\varepsilon(z) \subset \mathcal{P}_{c_1\varepsilon}(\zeta)$,
(iii) *for all* $\varepsilon > 0$ *sufficiently small, all* $z \in \mathcal{U}$, *all* $\zeta \in \mathcal{P}_\varepsilon(z)$ *we have* $|\rho(z) - \rho(\zeta)| \leq c_1\varepsilon$,
(iv) *for all* $\varepsilon > 0$, *all unit vectors* $v \in \mathbb{C}^n$, *all* $z \in \mathcal{U}$ *and all* $\zeta \in \mathcal{P}_\varepsilon(z)$, $\tau(z,v,\varepsilon) \eqsim \tau(\zeta,v,\varepsilon)$ *uniformly with respect to* ε, z *and* ζ.

For $\mathcal{U}$ given by Proposition 2.1 and z and ζ belonging to $\mathcal{U}$, we set $\delta(z,\zeta) = \inf\{\varepsilon > 0, \zeta \in \mathcal{P}_\varepsilon(z)\}$. Proposition 2.1 implies that δ is a pseudo-distance in the following sense.

PROPOSITION 2.2. *For* $\mathcal{U}$ *and* c_1 *given by Proposition* 2.1 *and for all* z, ζ *and* ξ *belonging to* $\mathcal{U}$ *we have*

$$\frac{1}{c_1}\delta(\zeta,z) \leq \delta(z,\zeta) \leq c_1\delta(\zeta,z)$$

and

$$\delta(z,\zeta) \leq c_1\big(\delta(z,\xi) + \delta(\xi,\zeta)\big).$$

2.2. Berndtsson–Andersson reproducing kernel in $\mathbb{C}^2$. Berndtsson–Andersson's kernel will be one of our most important ingredients in the construction of the functions g_1 and g_2 of Theorems 1.2 and 6.5. We now recall its definition for D a strictly convex domain of $\mathbb{C}^2$ of defining function ρ. We set $h_1(\zeta,z) = -\frac{1}{2}\frac{\partial\rho}{\partial\zeta_1}(\zeta)$, $h_2(\zeta,z) = -\frac{1}{2}\frac{\partial\rho}{\partial\zeta_2}(\zeta)$, $h = \sum_{i=1,2} h_i\, d\zeta_i$ and $\tilde{h} = \frac{1}{\rho}h$. For a $(1,0)$-form $\beta(\zeta,z) = \sum_{i=1,2}\beta_i(\zeta,z)\, d\zeta_i$ we set $\langle\beta(\zeta,z),\zeta - z\rangle = \sum_{i=1,2}\beta_i(\zeta,z)(\zeta_i - z_i)$. Then we define the Berndtsson–Andersson reproducing kernel by setting for an arbitrary positive integer N, $n = 1,2$ and all $\zeta, z \in D$:

$$P^{N,n}(\zeta,z) = C_{N,n}\left(\frac{1}{1 + \langle\tilde{h}(\zeta,z),\zeta - z\rangle}\right)^{N+n}(\overline{\partial}\tilde{h})^n,$$

where $C_{N,n} \in \mathbb{C}$ is a suitable constant. We also set $P^{N,n}(\zeta,z) = 0$ for all $z \in D$ and all $\zeta \notin D$. Then the following theorem holds true (see [8]).

THEOREM 2.3. *For all* $g \in \mathcal{O}(D) \cap C^\infty(\overline{D})$ *we have*

$$g(z) = \int_D g(\zeta)P^{N,2}(\zeta,z).$$

In order to find an upper bound for this kernel, we will need lower bound for $1 + \langle\tilde{h}(\zeta,z),\zeta - z\rangle$. This classical bound in the field is given by the following proposition. We include its proof for the reader convenience.

PROPOSITION 2.4. *The following inequality holds uniformly for all* ζ *and* z *in* D:

$$\big|\rho(\zeta) + \big\langle h(\zeta,z),\zeta - z\big\rangle\big| \gtrsim \delta(\zeta,z) + \big|\rho(\zeta)\big| + \big|\rho(z)\big|.$$

Proof. We write z as $z=\zeta+\lambda\eta_\zeta+\mu v_\zeta$ where η_ζ is the unit outer normal and where v_ζ belongs to $T^{\mathbb{C}}_\zeta bD_{\rho(\zeta)}$. With this notation, $\delta(\zeta,z)\eqsim|\lambda|+|\mu|^2$, $\operatorname{Re}\lambda\eqsim\operatorname{Re}\langle h(\zeta,z),\zeta-z\rangle$ and $\operatorname{Im}\lambda\eqsim\operatorname{Im}\langle h(\zeta,z),\zeta-z\rangle$.

Since ρ is convex, there exists c positive and small such that for all z and ζ in D

$$\begin{aligned}\rho(z)-\rho(\zeta)&\geq 2\operatorname{Re}\big(\partial\rho(\zeta)\cdot(z-\zeta)\big)+c|\zeta-z|^2\\&=4\operatorname{Re}\big\langle h(\zeta,z),\zeta-z\big\rangle+c|\zeta-z|^2.\end{aligned}\tag{4}$$

If $\operatorname{Re}\lambda<0$, we get from (4)

$$\begin{aligned}\big|\rho(\zeta)+\big\langle h(\zeta,z),\zeta-z\big\rangle\big|&\geq-\rho(\zeta)-\operatorname{Re}\big\langle h(\zeta,z),\zeta-z\big\rangle+\big|\operatorname{Im}\big\langle h(\zeta,z),\zeta-z\big\rangle\big|\\&\gtrsim-\rho(z)-\rho(\zeta)+c|\zeta-z|^2+|\lambda|\\&\gtrsim\delta(\zeta,z)+\big|\rho(\zeta)\big|+\big|\rho(z)\big|.\end{aligned}$$

If $\operatorname{Re}\lambda>0$, (4) now yields

$$\begin{aligned}&\big|\rho(\zeta)+\big\langle h(\zeta,z),\zeta-z\big\rangle\big|\\&\quad\gtrsim-\rho(\zeta)-2\operatorname{Re}\big\langle h(\zeta,z),\zeta-z\big\rangle+\operatorname{Re}\big\langle h(\zeta,z),\zeta-z\big\rangle+\big|\operatorname{Im}\big\langle h(\zeta,z),\zeta-z\big\rangle\big|\\&\quad\gtrsim-\rho(z)-\rho(\zeta)+c|\zeta-z|^2+|\lambda|\\&\quad\gtrsim\delta(\zeta,z)+\big|\rho(\zeta)\big|+\big|\rho(z)\big|.\end{aligned}$$

□

We will also need an upper bound for $\tilde{h}$ and thus for h. In order to get this bound, for a fixed $z\in D$, we write h in the Koranyi coordinates at z. We denote by (ζ_1^*,ζ_2^*) the Koranyi coordinates of ζ at z. We set $h_1^*=-\frac12\frac{\partial\rho}{\partial\zeta_1^*}(\zeta)$ and $h_2^*=-\frac12\frac{\partial\rho}{\partial\zeta_2^*}(\zeta)$ so that $h(\zeta,z)=\sum_{i=1,2}h_i^*(\zeta,z)\,d\zeta_i^*$. The following proposition is then a direct consequence of the smoothness of ρ.

PROPOSITION 2.5. *For all $\zeta\in\mathcal{P}_\varepsilon(z)$ we have uniformly with respect to z, ζ and ε*

(i) $|h_1^*(\zeta,z)|\lesssim1$, $|h_2^*(\zeta,z)|\lesssim\varepsilon^{\frac12}$,
(ii) $|\frac{\partial h_k^*}{\partial\bar\zeta_l^*}(\zeta,z)|$, $|\frac{\partial h_k^*}{\partial\zeta_l^*}(\zeta,z)|\lesssim1$ *for* $k,l\in\{1,2\}$.

3. A key result

In this section, we want to state the key result from which will follow the division theorems in the BMO and L^q spaces. Provided we have a "good" smooth division, this theorem will give the corresponding "good" holomorphic division.

THEOREM 3.1. *Let D be a strictly convex domain of $\mathbb{C}^2$, let f_1 and f_2 be two holomorphic functions defined on a neighbourhood of $\overline{D}$ and set $X_l=\{z,f_l(z)=0\}$, $l=1,2$. Suppose that $X_l\cap bD$ is transverse for $l=1$ and $l=2$, and that $X_1\cap X_2$ is a complete intersection.*

Then there exist two integers $k_1, k_2 \geq 1$ depending only on f_1 and f_2 such that if g is any holomorphic function on D which belongs to the ideal generated by f_1 and f_2 and for which there exist two C^∞ smooth functions $\tilde{g}_1$ and $\tilde{g}_2$ such that

(i) $g = \tilde{g}_1 f_1 + \tilde{g}_2 f_2$ *on* D,

(ii) *there exists* $N \in \mathbb{N}$ *such that* $|\rho|^N \tilde{g}_1$ *and* $|\rho|^N \tilde{g}_2$ *vanish to order* k_2 *on* bD,

(iii) *there exists* $q \in [1, +\infty]$ *such that for* $l = 1, 2$, $|\frac{\partial^{\alpha+\beta} \tilde{g}_l}{\partial \eta_\varsigma{}^\alpha \, \partial \overline{v_\varsigma}{}^\beta}||\rho|^{\alpha+\frac{\beta}{2}}$ *belongs to* $L^q(D)$ *for all nonnegative integers* α *and* β *with* $\alpha + \beta \leq k_1$,

then there exist two holomorphic functions g_1, g_2 *on* D *which belong to* $L^q(D)$ *if* $q < +\infty$ *and to* $\mathrm{BMO}(D)$ *if* $q = +\infty$, *such that* $g_1 f_1 + g_2 f_2 = g$ *on* D.

The number k_1 and k_2 are almost equal to the maximum of the multiplicities of the singularity of X_1 and X_2. The functions g_1 and g_2 will be obtained via integral operators acting on $\tilde{g}_1$ and $\tilde{g}_2$. These operators are a combination of a Berndtsson–Andersson kernel and of two $(2,2)$-currents T_1 and T_2 such that $f_1 T_1 + f_2 T_2 = 1$. As we will see in Section 4, a division formula can be constructed starting from any currents $\tilde{T}_1$ and $\tilde{T}_2$ such that $f_1 \tilde{T}_1 + f_2 \tilde{T}_2 = 1$. However, not all such currents will give operators such that g_1 and g_2 belongs to $L^q(D)$ or $\mathrm{BMO}(D)$; as we will see in this section, they have to be constructed taking into account the interplay between X_1 and X_2. We will also see that, if $\tilde{g}_1$ and $\tilde{g}_2$ are already holomorphic and satisfy the assumptions (i)–(iii) of Theorem 3.1, then $g_1 = \tilde{g}_1$ and $g_2 = \tilde{g}_2$.

Observe that in Theorem 3.1, we do not make any assumption on f_1 or f_2 except that the intersection $X_1 \cap bD$ and $X_2 \cap bD$ are transverse in the sense of tangent cones, and that $X_1 \cap X_2$ is a complete intersection. This later assumption can be removed provided we add a fourth assumption on $\tilde{g}_1$ and $\tilde{g}_2$. If we moreover assume that

(iv) $\frac{\partial^{\alpha+\beta} \tilde{g}_1}{\partial \eta_\varsigma{}^\alpha \, \partial \overline{v_\varsigma}{}^\beta} = 0$ on $X_2 \cap D$ and $\frac{\partial^{\alpha+\beta} \tilde{g}_2}{\partial \eta_\varsigma{}^\alpha \, \partial \overline{v_\varsigma}{}^\beta} = 0$ on $X_1 \cap D$ for all nonnegative integers α and β with $0 < \alpha + \beta \leq k_1$,

then Theorem 3.1 also holds whenever $X_1 \cap X_2$ is not complete. However, it then becomes very difficult to find $\tilde{g}_1$ and $\tilde{g}_2$ which satisfy this fourth assumption, except if $X_1 \cap X_2$ is actually complete. In Section 6, thanks to the assumptions on divided differences, we will construct the function $\tilde{g}_1$ and $\tilde{g}_2$ which satisfy the hypothesis of Theorem 3.1, but first, we construct the two currents T_1 and T_2.

If f_1 and f_2 are two holomorphic functions near the origin in $\mathbb{C}^n$, Mazzilli constructed in [14] two currents T and S such that $f_1 T = 1$, $f_2 S = \overline{\partial} T$ and $f_1 S = 0$ on a sufficiently small neighbourhood $\mathcal{U}$ of 0. He also proved that if T and S are any currents satisfying these three hypothesis, then any function g holomorphic on $\mathcal{U}$ can be written as $g = f_1 g_1 + f_2 g_2$ on $\mathcal{U}$ if and only if $g \overline{\partial} S = 0$. Moreover, g_1 and g_2 can be explicitly written down using T and S.

Here, when f_1 and f_2 are holomorphic on a domain D, we first want to obtain a decomposition $g=g_1f_1+g_2f_2$ on the whole domain D and then secondly we want to obtain growth estimates on g_1 and g_2. As a first approach, we could try to globalise the currents T and S of [14] in order to have a global decomposition. However, such an approach would fail to give the growth estimates we want.

In [14], f_1 plays a leading role and T is constructed independently of f_2, using only f_1. Then S is constructed using f_1 and f_2. If we assume for example that f_1 vanishes at a point ζ_0 near bD, because T is constructed independently of f_2, it seems difficult to prove that g_1 obtained using T is bounded except if we require that g vanishes at ζ_0 too; but considering $g=f_2$, we easily see that, in general, this condition is not necessary when one wants to write g as $g=g_1f_1+g_2f_2$ with g_1 and g_2 bounded for example. So the currents in [14] probably do not give a good decomposition.

Actually, it appears that the role of f_2 must be emphasised in the construction of the currents near a boundary point ζ_0 such that $f_1(\zeta_0)=0$ and $f_2(\zeta_0)\neq 0$, or more generally when f_2 is in some sense greater than f_1 and conversely. Following this idea, we construct two currents T_1 and T_2 such that $f_1T_1+f_2T_2=1$ on D. These currents are defined locally and using a suitable partition of unity we glue together the local currents and get a global current. We now define these local currents.

Let ε_0 be a small positive real number to be chosen later and let ζ_0 be a point in $\overline{D}$. We distinguish three cases.

First case: If ζ_0 belongs to $D_{-\varepsilon_0}$, that is, if ζ_0 is far from the boundary, we do not need to be careful. Using Weierstrass' preparation theorem when ζ_0 belongs to X_1, we write $f_1=u_{0,1}P_{0,1}$ where $u_{0,1}$ is a nonvanishing holomorphic function in a neighbourhood $\mathcal{U}_0\subset D_{-\frac{\varepsilon_0}{2}}$ of ζ_0 and $P_{0,1}(\zeta)=\zeta_2^{i_{0,1}}+\zeta_2^{i_{0,1}-1}a_{0,1}^{(1)}(\zeta_1)+\dots+a_{0,1}^{(i_{0,1})}(\zeta_1)$, $a_{0,1}^{(k)}$ holomorphic on $\mathcal{U}_0$ for all k. If ζ_0 does not belong to X_1, we set $P_{0,1}=1$, $i_{0,1}=0$, $u_{0,1}=f_1$ and we still have $f_1=u_{0,1}P_{0,1}$ with $u_{0,1}$ which does not vanish on some neighbourhood $\mathcal{U}_0$ of ζ_0.

For a smooth $(2,2)$-form φ compactly supported in $\mathcal{U}_0$ we set

$$\begin{aligned}\langle T_{0,1},\varphi\rangle &= \frac{1}{c_0}\int_{\mathcal{U}_0}\frac{\overline{P_{0,1}(\zeta)}}{f_1(\zeta)}\frac{\partial^{i_{0,1}}\varphi}{\partial\overline{\zeta}_2^{i_{0,1}}}(\zeta),\\ \langle T_{0,2},\varphi\rangle &= 0,\end{aligned}$$

where c_0 is a suitable constant. Integrating by parts, we get $f_1T_{0,1}+f_2T_{0,2}=1$ on $\mathcal{U}_0$ (see [14]).

Second case: If ζ_0 belongs to $bD\setminus(X_1\cap X_2)$, that is, if ζ_0 is "far" from $X_1\cap X_2$, without restriction we assume that $f_1(\zeta_0)\neq 0$. Let $\mathcal{U}_0$ be a neighbourhood of ζ_0 such that f_1 does not vanish in $\mathcal{U}_0$. As in the first case when $f_1(\zeta_0)\neq 0$, we set $P_{0,1}=1$, $i_{0,1}=0$, $u_{0,1}=f_1$ and for any smooth $(2,2)$-form φ compactly supported in $D\cap\mathcal{U}_0$ we put

$$\begin{aligned}\langle T_{0,1},\varphi\rangle &= \frac{1}{c_0}\int_{\mathcal{U}_0}\frac{\overline{P_{0,1}(\zeta)}}{f_1(\zeta)}\frac{\partial^{i_{0,1}}\varphi}{\partial\overline{\zeta}_2^{i_{0,1}}}(\zeta),\\ \langle T_{0,2},\varphi\rangle &= 0,\end{aligned}$$

where as previously c_0 is a suitable constant. Again, we have $f_1T_{0,1}+f_2T_{0,2}=1$ on $\mathcal{U}_0\cap D$.

Third case: If ζ_0 belongs to $X_1\cap X_2\cap bD$, the situation is more intricate. As in [1], for a small neighbourhood $\mathcal{U}_0$ of ζ_0, we cover $\mathcal{U}_0\cap D$ by a family of polydiscs $\mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$, $j\in\mathbb{N}$ and $k\in\{1,\dots,n_j\}$ such that:

(i) for all $j\in\mathbb{N}$, and all $k\in\{1,\dots,n_j\}$, $z_{j,k}$ belongs to $bD_{-(1-c\kappa)^j\varepsilon_0}$ where c is small positive real constant,
(ii) for all $j\in\mathbb{N}$, all $k,l\in\{1,\dots,n_j\}$, $k\neq l$, we have $\delta(z_{j,k},z_{j,l})\geq c\kappa(1-c\kappa)^j\varepsilon_0$,
(iii) for all $j\in\mathbb{N}$, all $z\in bD_{-(1-c\kappa)^j\varepsilon_0}$, there exists $k\in\{1,\dots,n_j\}$ such that $\delta(z,z_{j,k})<c\kappa(1-c\kappa)^j\varepsilon_0$,
(iv) $D\cap\mathcal{U}_0$ is included in $\bigcup_{j=0}^{+\infty}\bigcup_{k=1}^{n_j}\mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$,
(v) there exists $M\in\mathbb{N}$ such that for $z\in D\setminus D_{-\varepsilon_0}$, $\mathcal{P}_{4\kappa|\rho(z)|}(z)$ intersect at most M Koranyi balls $\mathcal{P}_{4\kappa|\rho(z_{j,k})|}(z_{j,k})$.

Such a family of polydiscs will be called a κ-covering.

We define on each polydisc $\mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$ two currents $T_{0,1}^{(j,k)}$ and $T_{0,2}^{(j,k)}$ such that $f_1T_{0,1}^{(j,k)}+f_2T_{0,2}^{(j,k)}=1$ as follows.

We denote by $\Delta_\xi(\varepsilon)$ the disc of radius ε centred at ξ and by $(\zeta_{0,1}^*,\zeta_{0,2}^*)$ the coordinates of ζ_0 in the Koranyi basis at $z_{j,k}$. In [1] were proved the next two propositions.

Proposition 3.2. *If $\kappa>0$ is small enough and if $\mathcal{P}_{4\kappa|\rho(z_{j,k})|}(z_{j,k})\cap X_l\neq\emptyset$, then $|\zeta_{0,1}^*|\geq 4\kappa|\rho(z_{j,k})|$.*

We assume κ so small that Proposition 3.2 holds for both X_1 and X_2 with the same κ. For $l=1$ or $l=2$, we denote by p_l the multiplicity of ζ_0 as a singularity of X_l. When $|\zeta_{0,1}^*|\geq 4\kappa|\rho(z_{j,k})|$ then X_l can be parametrised as follows (see [1]).

Proposition 3.3. *If $|\zeta_{0,1}^*|\geq 4\kappa|\rho(z_{j,k})|$, for $l=1$ and $l=2$, there exists p_l functions $\alpha_{l,1}^{(j,k)},\dots,\alpha_{l,p_l}^{(j,k)}$ holomorphic on $\Delta_0(4\kappa|\rho(z_{j,k})|)$, there exists $r>0$, depending neither on j nor on k, and there exists $u_l^{(j,k)}$ holomorphic on the ball of centre ζ_0 and radius r, bounded and bounded away from 0, such that:*

(i) *$\frac{\partial\alpha_{l,i}^{(j,k)}}{\partial\zeta_1^*}$ is bounded on $\Delta_0(4\kappa|\rho(z_{j,k})|)$ uniformly with respect to j and k,*
(ii) *for all $\zeta\in\mathcal{P}_{4\kappa|\rho(z_{j,k})|}(z_{j,k})$, $f_l(\zeta)=u_l^{(j,k)}(\zeta)\prod_{i=1}^{p_l}(\zeta_2^*-\alpha_{l,i}^{(j,k)}(\zeta_1^*))$.*

Now we define $T_{0,1}^{(j,k)}$ and $T_{0,2}^{(j,k)}$ with the following settings.

If $|\zeta_{0,1}^*| < 4\kappa|\rho(z_{j,k})|$, by Proposition 3.2, for $l=1$ or $l=2$, $\mathcal{P}_{4\kappa|\rho(z_{j,k}|}(z_{j,k}) \cap X_l = \emptyset$, which means that $z_{j,k}$ is "far" from X_1 and X_2. In this case, we set for $l=1$ and $l=2$:

$$\begin{aligned} I_l^{(j,k)} &:= \emptyset, \\ i_l^{(j,k)} &:= 0, \\ P_l^{(j,k)}(\zeta) &:= 1. \end{aligned}$$

If $|\zeta_{0,1}^*| \geq 4\kappa|\rho(z_{j,k})|$, then we may have $\mathcal{P}_{4\kappa|\rho(z_{j,k}|}(z_{j,k}) \cap X_l \neq \emptyset$ for $l=1$ or $l=2$. In that case we set for $l=1$ and $l=2$:

$$\begin{aligned} I_l^{(j,k)} &:= \Big\{ i, \exists z_1^* \in \mathbb{C}, \big|z_1^*\big| < 2\kappa\big|\rho(z_{j,k})\big| \text{ and} \\ &\qquad \big|\alpha_{l,i}^{(j,k)}\big(z_1^*\big)\big| < \left(\frac{5}{2}\kappa\big|\rho(z_{j,k})\big|\right)^{\frac{1}{2}} \Big\}, \\ i_l^{(j,k)} &:= \# I_l^{(j,k)}, \quad \text{the cardinal of } I_l^{(j,k)}, \\ P_l^{(j,k)}(\zeta) &:= \prod_{i \in I_l^{(j,k)}} \big(\zeta_2^* - \alpha_{i,l}^{(j,k)}\big(\zeta_1^*\big)\big). \end{aligned}$$

In both cases, we set

$$\mathcal{U}_1^{(j,k)} := \left\{ \zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k}), \left| \frac{f_1(\zeta)|\rho(z_{j,k})|^{\frac{i_1^{(j,k)}}{2}}}{P_1^{(j,k)}(\zeta)} \right| > \frac{1}{3} \left| \frac{f_2(\zeta)|\rho(z_{j,k})|^{\frac{i_2^{(j,k)}}{2}}}{P_2^{(j,k)}(\zeta)} \right| \right\},$$

$$\mathcal{U}_2^{(j,k)} := \left\{ \zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k}), \frac{2}{3} \left| \frac{f_2(\zeta)|\rho(z_{j,k})|^{\frac{i_2^{(j,k)}}{2}}}{P_2^{(j,k)}(\zeta)} \right| > \left| \frac{f_1(\zeta)|\rho(z_{j,k})|^{\frac{i_1^{(j,k)}}{2}}}{P_1^{(j,k)}(\zeta)} \right| \right\},$$

so that $\mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k}) = \mathcal{U}_1^{(j,k)} \cup \mathcal{U}_2^{(j,k)}$.

These open sets are designed in order to quantify where f_1 is "bigger" than f_2 and conversely. The idea is the following.

If i belongs to $I_l^{(j,k)}$ then $|\zeta_2^* - \alpha_{l,i}^{(j,k)}(\zeta_1^*)| \lesssim |\rho(z_{j,k})|^{\frac{1}{2}}$ for all $\zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$. Thus each zero of f_l in $\mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$ brings, in some sense, a factor $|\rho(z_{j,k})|^{\frac{1}{2}}$ in $f_l(\zeta)$. In the definition of $\mathcal{U}_l^{(j,k)}$, we take into account the zeros of f_1 and f_2 which are in the polydisc $\mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$ with the term $|\rho(z_{j,k})|^{\frac{i_1^{(j,k)}}{2}}$ and $|\rho(z_{j,k})|^{\frac{i_2^{(j,k)}}{2}}$. This means in particular that all the zeros in the polydisc are treated in the same way, we don't care if they are close from each others, from the boundary of the polydisc or not. The zeros which are outside the polydisc are taken into account by $\frac{f_l(\zeta)}{P_l^{(j,k)}(\zeta)}$, which will also measure how far they are from the polydisc.

Therefore, $\mathcal{U}_1^{(j,k)}$ is the open set where f_1 is bigger than f_2 for an order such that the zeros which are outside of the polydisc are taken into account with the term $\frac{f_l(\zeta)}{P_l^{(j,k)}(\zeta)}$ and the zeros which are inside with the term $|\rho(z_{j,k})|^{\frac{i_l^{(j,k)}}{2}}$, and conversely for $\mathcal{U}_2^{(j,k)}$.

For $l=1,2$ and for a smooth $(2,2)$-form φ compactly supported in $\mathcal{U}_l^{(j,k)}$ we set

$$\langle T_{0,l}^{(j,k)}, \varphi\rangle := \int_{\mathbb{C}^2} \frac{\overline{P_l^{(j,k)}(\zeta)}}{f_l(\zeta)} \frac{\partial^{i_l^{(j,k)}}\varphi}{\partial \overline{\zeta_2^*}^{i_l^{(j,k)}}}(\zeta).$$

Integrating $i_l^{(j,k)}$-times by parts, we get $f_l T_{0,l}^{(j,k)} = c_l^{(j,k)}$ on $\mathcal{U}_l^{(j,k)}$ where $c_l^{(j,k)}$ is an integer bounded by $i_l^{(j,k)}!$ (see [14]).

Now we glue together the currents $T_{0,l}^{(j,k)}$ in order to define the current $T_{0,l}$, $l=1,\,2$, such that $f_1T_{0,1}+f_2T_{0,2}=1$ on $D\cap\mathcal{U}_0$. Let $(\tilde\chi_{j,k})_{\substack{j\in\mathbb{N}\\ k\in\{1,\dots,n_j\}}}$ be a partition of unity subordinated to the covering $(\mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k}))_{\substack{j\in\mathbb{N}\\ k\in\{1,\dots,n_j\}}}$ of $\mathcal{U}_0\cap D$. Without restriction, we assume that $|\frac{\partial^{\alpha+\beta+\overline{\alpha}+\overline{\beta}}\tilde\chi_{j,k}}{\partial{\zeta_1^*}^{\alpha}\,\partial{\zeta_2^*}^{\beta}\,\partial\overline{\zeta_1^*}^{\overline{\alpha}}\,\partial\overline{\zeta_2^*}^{\overline{\beta}}}(\zeta)| \lesssim \frac{1}{|\rho(z_{j,k})|^{\alpha+\overline{\alpha}+\frac{\beta+\overline{\beta}}{2}}}$. Let also χ be a smooth function on $\mathbb{C}^2\setminus\{0\}$ such that $\chi(z_1,z_2)=1$ if $|z_1|>\frac{2}{3}|z_2|$ and $\chi(z_1,z_2)=0$ if $|z_1|<\frac{1}{3}|z_2|$ and let us define

$$\chi_1^{(j,k)}(\zeta) = \tilde\chi_{j,k}(\zeta)\cdot\chi\Bigg(\frac{f_1(\zeta)|\rho(z_{j,k})|^{\frac{i_1^{(j,k)}}{2}}}{P_1^{(j,k)}(\zeta)}, \frac{f_2(\zeta)|\rho(z_{j,k})|^{\frac{i_2^{(j,k)}}{2}}}{P_2^{(j,k)}(\zeta)}\Bigg),$$

$$\chi_2^{(j,k)}(\zeta) = \tilde\chi_{j,k}(\zeta)\cdot\Bigg(1-\chi\Bigg(\frac{f_1(\zeta)|\rho(z_{j,k})|^{\frac{i_1^{(j,k)}}{2}}}{P_1^{(j,k)}(\zeta)}, \frac{f_2(\zeta)|\rho(z_{j,k})|^{\frac{i_2^{(j,k)}}{2}}}{P_2^{(j,k)}(\zeta)}\Bigg)\Bigg).$$

For $l=1$ and $l=2$, the support of $\chi_l^{(j,k)}$ is included in $\mathcal{U}_l^{(j,k)}$ so we can put

$$T_{0,l} = \sum_{\substack{j\in\mathbb{N}\\ k\in\{1,\dots,n_j\}}} \frac{1}{c_l^{(j,k)}}\chi_l^{(j,k)}T_{0,l}^{(j,k)}$$

and we have $f_1T_{0,1}+f_2T_{0,2}=1$ on $\mathcal{U}_0\cap D$.

Now for all $\zeta_0\in bD\cup\overline{D_{-\varepsilon_0}}$ we have constructed a neighbourhood $\mathcal{U}_0$ of ζ_0 and two currents $T_{0,1}$ and $T_{0,2}$ such that $f_1T_{0,1}+f_2T_{0,2}=1$ on $\mathcal{U}_0\cap D$. If $\varepsilon_0>0$ is sufficiently small, we can cover $\overline{D}$ by finitely many open sets $\mathcal{U}_1,\dots,\mathcal{U}_n$. Let $\chi_1,\dots,\chi_n$ be a partition of unity subordinated to this family of open sets and $T_{1,1},\dots,T_{n,1}$ and $T_{1,2},\dots,T_{n,2}$ be the corresponding currents defined on $\mathcal{U}_1,\dots,\mathcal{U}_n$. We glue together this current and we set

$$T_1=\sum_{j=1}^n \chi_j T_{j,1} \quad \text{and} \quad T_2=\sum_{j=1}^n \chi_j T_{j,2},$$

so that $f_1T_1+f_2T_2=1$ on D. Moreover T_1 and T_2 are currents supported in $\overline{D}$, thus they are of finite order k_2 and we can apply T_1 and T_2 to functions of class C^{k_2} with support in $\overline{D}$. This gives k_2 of the Theorem 3.1.

4. The division formula

In this part, given any two currents T_1 and T_2 of order k_2 such that $f_1T_1+f_2T_2=1$, assuming that g is a holomorphic function on D which belongs to the ideal generated by f_1 and f_2, and which can be written as $g=\tilde{g}_1f_1+\tilde{g}_2f_2$, where $\tilde{g}_1$ and $\tilde{g}_2$ are two C^∞-smooth functions on D such that $|\rho|^N\tilde{g}_1$ and $|\rho|^N\tilde{g}_2$ vanish to order k_2 on bD for some $N\in\mathbb{N}$ sufficiently big, we write g as $g=g_1f_1+g_2f_2$ with g_1 and g_2 holomorphic on D. We point out that the formula we get is valid for any T_1 and T_2 of order k_2 such that $f_1T_1+f_2T_2=1$.

Under our assumptions, for $k=1$ and $k=2$ and all fixed $z\in D$, $\tilde{g}_1P^{N,k}(\cdot,z)$ and $\tilde{g}_2P^{N,k}(\cdot,z)$ can be extended by zero outside D and are of class C^{k_2} on $\mathbb{C}^2$. So we can apply T_1 and T_2 to $\tilde{g}_1P^{N,k}(\cdot,z)$ and $\tilde{g}_2P^{N,k}(\cdot,z)$.

For $l=1,2$, we denote by $b_l=b_{l,1}\,d\zeta_1+b_{l,2}\,d\zeta_2$ a $(1,0)$-form such that $f_l(z)-f_l(\zeta)=\sum_{i=1,2}b_{l,i}(\zeta,z)(z_i-\zeta_i)$. For the estimates, we will take $b_{l,i}(\zeta,z)=\int_0^1\frac{\partial f_l}{\partial\zeta_i}(\zeta+t(z-\zeta))\,dt$, but this is not necessary to get a division formula.

In order to construct the formula, we will need the following lemma which was proved in [13], Lemma 3.1.

LEMMA 4.1. *Let $Q=\sum_{i=1}^n Q_i\,d\zeta_i$ be a $(1,0)$ form of $\mathbb{C}^n$, let $H_1,\dots,H_p$ be p $(1,0)$-forms in $\mathbb{C}^n$ and let $W_1,\dots,W_{p-1}$ be $p-1$ $(0,1)$-forms in $\mathbb{C}^n$. Then the following equality holds*

$$\begin{aligned}
&\overline{\partial}\big(\langle Q,z-\zeta\rangle\big)(\overline{\partial}Q)^{n-p}\wedge H_p\wedge\bigwedge_{k=1}^{p-1}W_k\wedge H_k\\
&\quad=\frac{1}{n-p+1}\langle H_p,z-\zeta\rangle(\overline{\partial}Q)^{n-p+1}\wedge\bigwedge_{k=1}^{p-1}W_k\wedge H_k\\
&\qquad+\frac{1}{n-p+1}\sum_{l=1}^{p-1}\langle H_l,z-\zeta\rangle(\overline{\partial}Q)^{n-p+1}H_p\wedge W_l\wedge\bigwedge_{\substack{k=1\\k\neq l}}^{p-1}W_k\wedge H_k.
\end{aligned}$$

We now establish the division formula. From Theorem 2.3, we have for all $z \in D$:

$$g(z) = \int_D g(\zeta) P^{N,2}(\zeta, z)$$

and since $g = \tilde{g}_1 f_1 + \tilde{g}_2 f_2$:

$$\begin{aligned} g(z) = f_1(z) \int_D \tilde{g}_1(\zeta) P^{N,2}(\zeta, z) + f_2(z) \int_D \tilde{g}_2(\zeta) P^{N,2}(\zeta, z) \\ + \int_D \tilde{g}_1(\zeta) \big(f_1(\zeta) - f_1(z)\big) P^{N,2}(\zeta, z) \\ + \int_D \tilde{g}_2(\zeta) \big(f_2(\zeta) - f_2(z)\big) P^{N,2}(\zeta, z). \end{aligned} \tag{5}$$

Now from Lemma 4.1, there exists $\tilde{c}_{N,2}$ such that

$$\big(f_1(\zeta) - f_1(z)\big) P^{N,2}(\zeta, z) = \tilde{c}_{N,2} b_1(\zeta, z) \wedge \overline{\partial} P^{N,1}(\zeta, z)$$

and since by assumption $\tilde{g}_1 P^{N,1}$ vanishes on bD, Stokes' theorem yields

$$\begin{aligned} \int_D \tilde{g}_1(\zeta) \big(f_1(\zeta) - f_1(z)\big) P^{N,2}(\zeta, z) \\ = \tilde{c}_{N,2} \int_D \overline{\partial} \tilde{g}_1(\zeta) \wedge b_1(\zeta, z) \wedge P^{N,1}(\zeta, z). \end{aligned} \tag{6}$$

We now use the fact that $f_1 T_1 + f_2 T_2 = 1$ in order to rewrite this former integral:

$$\begin{aligned} \int_D \overline{\partial} \tilde{g}_1(\zeta) \wedge b_1(\zeta, z) \wedge P^{N,1}(\zeta, z) \\ = \big\langle f_1 T_1 + f_2 T_2, \overline{\partial} \tilde{g}_1 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \big\rangle \\ = \big\langle f_1 T_1, \overline{\partial} \tilde{g}_1 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \big\rangle \\ + f_2(z) \big\langle T_2, \overline{\partial} \tilde{g}_1 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \big\rangle \\ + \big\langle T_2, \big(f_2 - f_2(z)\big) \overline{\partial} \tilde{g}_1 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \big\rangle. \end{aligned} \tag{7}$$

Again from Lemma 4.1, there exists $\tilde{c}_{N,1}$ such that

$$\begin{aligned} \big(f_2(\zeta) - f_2(z)\big) b_1(\zeta, z) \wedge \overline{\partial} \tilde{g}_1 \wedge P^{N,1}(\zeta, z) - \big(f_1(\zeta) - f_1(z)\big) b_2(\zeta, z) \\ \wedge \overline{\partial} \tilde{g}_1 \wedge P^{N,1}(\zeta, z) = \tilde{c}_{N,1} b_1(\zeta, z) \wedge b_2(\zeta, z) \wedge \overline{\partial} \tilde{g}_1 \wedge \overline{\partial} P^{N,0}(\zeta, z). \end{aligned}$$

So

$$\begin{aligned} \big\langle T_2, \big(f_2 - f_2(z)\big) \overline{\partial} \tilde{g}_1 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \big\rangle \\ = -f_1(z) \big\langle T_2, \overline{\partial} \tilde{g}_1 \wedge b_2(\cdot, z) \wedge P^{N,1}(\cdot, z) \big\rangle \\ + \big\langle T_2, f_1 \overline{\partial} \tilde{g}_1 \wedge b_2(\cdot, z) \wedge P^{N,1}(\cdot, z) \big\rangle \\ + \tilde{c}_{N,1} \big\langle T_2, \overline{\partial} \tilde{g}_1 \wedge b_1(\cdot, z) \wedge b_2(\cdot, z) \wedge \overline{\partial} P^{N,0}(\cdot, z) \big\rangle. \end{aligned} \tag{8}$$

We plug together (6), (7) and (8) and their analogue for $\int_D (f_2(\zeta) - f_2(z)) g_2(\zeta) P^{N,2}(\zeta, z)$ in (5) and we get

$$
\begin{aligned}
g(z) = {} & f_1(z) \int_D \tilde{g}_1(\zeta) P^{N,2}(\zeta, z) - \tilde{c}_{N,2} f_1(z) \big\langle T_2, \overline{\partial} \tilde{g}_1 \wedge b_2(\cdot, z) \wedge P^{N,1}(\cdot, z) \big\rangle \\
& + \tilde{c}_{N,2} f_2(z) \big\langle T_2, \overline{\partial} \tilde{g}_1 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \big\rangle \\
& + f_2(z) \int_D \tilde{g}_2(\zeta) P^{N,2}(\zeta, z) - \tilde{c}_{N,2} f_2(z) \big\langle T_1, \overline{\partial} \tilde{g}_2 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \big\rangle \\
& + \tilde{c}_{N,2} f_1(z) \big\langle T_1, \overline{\partial} \tilde{g}_2 \wedge b_2(\cdot, z) \wedge P^{N,1}(\cdot, z) \big\rangle \\
& + \tilde{c}_{N,2} \big\langle T_1, (f_1 \overline{\partial} \tilde{g}_1 + f_2 \overline{\partial} \tilde{g}_2) \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \big\rangle \qquad (9) \\
& + \tilde{c}_{N,2} \big\langle T_2, (f_1 \overline{\partial} \tilde{g}_1 + f_2 \overline{\partial} \tilde{g}_2) \wedge b_2(\cdot, z) \wedge P^{N,1}(\cdot, z) \big\rangle \qquad (10) \\
& + \tilde{c}_{N,2} \tilde{c}_{N,1} \big\langle \overline{\partial} \tilde{g}_1 \wedge T_2 - \overline{\partial} \tilde{g}_2 \wedge T_1, b_1(\cdot, z) \wedge b_2(\cdot, z) \wedge \overline{\partial} P^{N,0}(\cdot, z) \big\rangle.
\end{aligned}
$$

Now since $\overline{\partial} g = f_1 \overline{\partial} \tilde{g}_1 + f_2 \overline{\partial} \tilde{g}_2 = 0$, the line (9) and (10) vanish. Therefore in order to get our division formula, it suffices to prove that $\overline{\partial}(\overline{\partial} \tilde{g}_1 \wedge T_2 - \overline{\partial} \tilde{g}_2 \wedge T_1) = 0$.

When $X_1 \cap X_2$ is not a complete intersection and when assumption (iv) in Section 3 is satisfied by $\tilde{g}_1$ and $\tilde{g}_2$, one can prove that $\overline{\partial} \tilde{g}_1 \wedge \overline{\partial} T_2 = 0$ and $\overline{\partial} \tilde{g}_2 \wedge \overline{\partial} T_1 = 0$.

When $X_1 \cap X_2$ is a complete intersection, we prove that for any $\zeta_0 \in D$ there exists a neighbourhood $\mathcal{U}_0$ of ζ_0 such that for all $(2,1)$-form φ, smooth and supported in $\mathcal{U}_0$, we have $\langle \overline{\partial} \tilde{g}_1 \wedge T_2 - \overline{\partial} \tilde{g}_2 \wedge T_1, \overline{\partial} \varphi \rangle = 0$.

Let ζ_0 be a point in D. By assumption on g, there exists a neighbourhood $\mathcal{U}_0$ of ζ_0 and two holomorphic functions γ_1 and γ_2 such that $g = \gamma_1 f_1 + \gamma_2 f_2$ on $\mathcal{U}_0$. We now use the following lemma whose proof is postponed to the end of this section.

LEMMA 4.2. *Let f_1 and f_2 be two holomorphic functions defined in a neighbourhood of 0 in $\mathbb{C}^2$, $X_1 = \{z, f_1(z) = 0\}$ and $X_2 = \{z, f_2(z) = 0\}$. We assume that $X_1 \cap X_2$ is a complete intersection and that 0 belongs to $X_1 \cap X_2$. Let φ_1 and φ_2 be two C^∞-smooth functions such that $f_1 \varphi_1 = f_2 \varphi_2$.*

Then, $\frac{\varphi_1}{f_2}$ and $\frac{\varphi_2}{f_1}$ are C^∞-smooth in a neighbourhood of 0.

Lemma 4.2 implies that the function $\psi = \frac{\tilde{g}_1 - \gamma_1}{f_2} = \frac{\gamma_2 - \tilde{g}_2}{f_1}$ is smooth on a perhaps smaller neighbourhood of ζ_0 still denoted by $\mathcal{U}_0$. Thus,

$$
\begin{aligned}
\langle \overline{\partial} \tilde{g}_1 \wedge T_2 - \overline{\partial} \tilde{g}_2 \wedge T_1, \overline{\partial} \varphi \rangle &= \big\langle \overline{\partial}(\tilde{g}_1 - \gamma_1) \wedge T_2 + \overline{\partial}(\gamma_2 - \tilde{g}_2) \wedge T_1, \overline{\partial} \varphi \big\rangle \\
&= \big\langle \overline{\partial}(f_2 \psi) \wedge T_2 + \overline{\partial}(f_1 \psi) \wedge T_1, \overline{\partial} \varphi \big\rangle \\
&= \langle f_2 T_2 + f_1 T_1, \overline{\partial} \psi \wedge \overline{\partial} \varphi \rangle \\
&= \int_{\mathcal{U}_0} \overline{\partial} \psi \wedge \overline{\partial} \varphi
\end{aligned}
$$

and since φ is supported in $\mathcal{U}_0$ we have $\int_{\mathcal{U}_0}\overline{\partial}\psi\wedge\overline{\partial}\varphi=-\int_{\mathcal{U}_0}d(\varphi\,\overline{\partial}\psi)=0$ and so

$$\langle\overline{\partial}\tilde{g}_1\wedge T_2-\overline{\partial}\tilde{g}_2\wedge T_1,\overline{\partial}\varphi\rangle=0.$$

Now we set

$$\begin{aligned}g_1(z)=&\int_D\tilde{g}_1(\zeta)P^{N,2}(\zeta,z)\\&+\tilde{c}_{N,2}\big(\big\langle T_1,\overline{\partial}\tilde{g}_2\wedge b_2(\cdot,z)\wedge P^{N,1}(\cdot,z)\big\rangle\\&-\big\langle T_2,\overline{\partial}\tilde{g}_1\wedge b_2(\cdot,z)\wedge P^{N,1}(\cdot,z)\big\rangle\big),\end{aligned}\tag{11}$$

$$\begin{aligned}g_2(z)=&\int_D\tilde{g}_2(\zeta)P^{N,2}(\zeta,z)\\&+\tilde{c}_{N,2}\big(\big\langle T_2,\overline{\partial}\tilde{g}_1\wedge b_1(\cdot,z)\wedge P^{N,1}(\cdot,z)\big\rangle\\&-\big\langle T_1,\overline{\partial}\tilde{g}_2\wedge b_1(\cdot,z)\wedge P^{N,1}(\cdot,z)\big\rangle\big)\end{aligned}\tag{12}$$

and we have

$$g=g_1f_1+g_2f_2$$

with g_1 and g_2 holomorphic on D. We notice that if $\tilde{g}_1$ and $\tilde{g}_2$ are already holomorphic functions then $g_1=\tilde{g}_1$ and $g_2=\tilde{g}_2$.

Proof of Lemma 4.2. Maybe after a unitary change of coordinates if needed, using Weierstrass' preparation theorem, we can assume that for $l=1,2$, the function f_l is given by $f_l(z,w)=z^{k_l}+a_1^{(l)}(w)z^{k_l-1}+\cdots+a_{k_l}^{(l)}(w)$ where $a_1^{(l)},\dots,a_{k_l}^{(l)}$ are holomorphic near 0 and vanish at 0. Moreover, since the intersection $X_1\cap X_2$ is transverse, f_1 and f_2 are relatively prime polynomials. Thus there exists two polynomials α_1 and α_2 with holomorphic coefficients in w and a function β of w not identically zero such that

$$\alpha_1(z,w)f_1(z,w)+\alpha_2(z,w)f_2(z,w)=\beta(w).$$

Multiplying this equality by φ_1 we get

$$f_2(\alpha_1\varphi_2+\alpha_2\varphi_1)=\beta\varphi_1.$$

We now prove that β divides the function $\psi:=\alpha_1\varphi_2+\alpha_2\varphi_1$.

If $\beta(0)\neq 0$, there is nothing to do. Otherwise, since β is not identically zero, there exists $k\in\mathbb{N}$ such that $\beta(w)=w^k\gamma(w)$ where $\gamma(0)\neq 0$.

For all $j\in\mathbb{N}$, we have

$$f_2(z,w)\frac{\partial^j\psi}{\partial\overline{w}^j}(z,w)=\beta(w)\frac{\partial\varphi_1}{\partial\overline{w}^j}(z,w)\tag{13}$$

and for $w=0$ and all z we thus get $\frac{\partial^j\psi}{\partial\overline{w}^j}(z,0)=0$.

By induction, we then deduce from (13) that $\frac{\partial^{i+j}\psi}{\partial w^i\,\partial\overline{w}^j}(z,0)=0$ for all $i\in\{0,\dots,k-1\}$ and all $j\in\mathbb{N}$. For any integer $n\geq k$, we therefore can write for all z and all w

$$\frac{\psi(z,w)}{w^k} = \sum_{\substack{k\le i+j\le n\\ i\ge k}} w^{i-k}\overline{w}^j \frac{\partial^{i+j}\psi}{\partial w^i\,\partial\overline{w}^j}(z,0) + \sum_{i+j=n+1} w^{i-k}\overline{w}^j \int_0^1 \frac{\partial^{n+1}\psi}{\partial w^i\,\partial\overline{w}^j}(z,tw)\,dt.$$

Now, it is easy to check by induction that the function $w\mapsto \frac{\overline{w}^{i+j}}{w^i}$ is of class C^{j-1} for all positive integer j and all nonnegative integer i. This implies that $\frac{\psi(z,w)}{w^k}$ is of class C^n for all positive integer n and therefore $\frac{\varphi_1}{f_2}=\frac{\psi}{\beta}$ is of class C^∞. $\square$

5. End of the proof of the key result

In this section, we will prove that the current T_1 and T_2 yield a good holomorphic division provided we have a good smooth division formula. According to the Definitions (11) and (12) of g_1 and g_2, in order to prove Theorem 3.1, for any k and l in $\{1,2\}$ and any $q\in[1,+\infty]$, we have to prove that if h is a smooth function such that, for all nonnegative integers α and β, $|\frac{\partial^{\alpha+\beta}h}{\partial\eta_\zeta{}^\alpha\,\partial v_\zeta{}^\beta}||\rho|^{\alpha+\frac{\beta}{2}}$ belongs to $L^q(D)$, then the function

$$z\mapsto \left\langle T_l, \overline{\partial}h\wedge b_k(\cdot,z)\wedge P^{N,1}(\cdot,z)\right\rangle$$

belongs to $L^q(D)$ if $q<\infty$ and to $\mathrm{BMO}(D)$ if $q=+\infty$.

As usually, since the modulus of the denominator in $P^{N,1}$ is greater than $|\rho(z)|+|\rho(\zeta)|+\delta(z,\zeta)$, the difficulties occurs when we integrate for ζ near z and when z is near bD. Moreover, by construction of T_1 and T_2, the main difficulty is when, in addition, z is near a point ζ_0 which belongs to $bD\cap X_1\cap X_2$ and we only consider that case.

So we assume that z belongs to the neighbourhood $\mathcal{U}_0$ of a point $\zeta_0\in bD\cap X_1\cap X_2$ and we use the same notations as in Section 3 for the construction of the currents. Moreover, without any restriction, we assume that the Koranyi basis at ζ_0 is the canonical basis of $\mathbb{C}^2$ and that ζ_0 is the origin of $\mathbb{C}^2$.

We will need an upper bound of $\frac{P_l^{(j,k)}}{f_l}\frac{\partial^{\alpha+\beta}f_l}{\partial\zeta_1^{*\alpha}\,\partial\zeta_2^{*\beta}}$ in order to estimate $\frac{P_l^{(j,k)}}{f_l}b_m$ and the derivatives of $\chi_l^{(j,k)}$. We set $Q_l^{(j,k)}=\frac{f_l}{P_l^{(j,k)}}$ and we begin with the following lemma.

LEMMA 5.1. *For all $j\in\mathbb{N}$, all $k\in\{1,\dots,n_j\}$, all α and β in $\mathbb{N}$, $l=1,2$, and all ζ in $\mathcal{P}_{2\kappa|\rho(z_{j,k})|}(z_{j,k})$, we have uniformly with respect to $j,k,l,$ and ζ*

$$\left|\frac{1}{Q_l^{(j,k)}(\zeta)}\frac{\partial^{\alpha+\beta}}{\partial\zeta_1^{*\alpha}\,\partial\zeta_2^{*\beta}}\left(Q_l^{(j,k)}(\zeta)\right)\right|\lesssim\left|\rho(z_{j,k})\right|^{-\alpha-\frac{\beta}{2}}.$$

Proof. We denote by $(\zeta^*_{0,1},\zeta^*_{0,2})$ the coordinates of ζ_0 in the Koranyi coordinates at $z_{j,k}$. The definition of $P_l^{(j,k)}$ forces us to distinguish three cases:

First case: If $|\zeta^*_{0,1}|>4\kappa|\rho(z_{j,k})|$, let $\alpha^{(j,k)}_{l,i}$, $i=1,\dots,p_l$, be the family of parametrisation given by Proposition 3.3. In this case, we actually seek an upper bound for

$$\frac{1}{\prod_{i\notin I_l^{(j,k)}}(\zeta_2^*-\alpha^{(j,k)}_{l,i}(\zeta_1^*))}\frac{\partial^{\alpha+\beta}}{\partial{\zeta_1^*}^\alpha\,\partial{\zeta_2^*}^\beta}\Bigg(\prod_{i\notin I_l^{(j,k)}}\left(\zeta_2^*-\alpha^{(j,k)}_{l,i}(\zeta_1^*)\right)\Bigg),$$

and it suffices to prove for all $i\notin I_l^{(j,k)}$ and all α and β that

$$\left|\frac{1}{\zeta_2^*-\alpha^{(j,k)}_{l,i}(\zeta_1^*)}\frac{\partial^{\alpha+\beta}}{\partial{\zeta_1^*}^\alpha\,\partial{\zeta_2^*}^\beta}\left(\zeta_2^*-\alpha^{(j,k)}_{l,i}(\zeta_1^*)\right)\right|\lesssim|\rho(z_{j,k})|^{-\alpha-\frac{\beta}{2}}. \tag{14}$$

By definition of $I_l^{(j,k)}$, we have $|\alpha^{(j,k)}_{l,i}(\zeta_1^*)|\geq(\frac{5}{2}\kappa|\rho(z_{j,k})|)^{\frac12}$ for all $\zeta_1^*\in\Delta_0(2\kappa|\rho(z_{j,k})|)$ so $|\zeta_2^*-\alpha^{(j,k)}_{l,i}(\zeta_1^*)|\gtrsim|\rho(z_{j,k})|^{\frac12}$ and (14) holds true for $\alpha=0$ and $\beta=1$.

According to Proposition 3.3, $\frac{\partial\alpha^{(j,k)}_{l,i}}{\partial\zeta^*}$ is uniformly bounded on $\Delta_0(4\kappa\times|\rho(z_{j,k})|)$. Cauchy's inequalities then yields $|\frac{\partial^\alpha\alpha^{(j,k)}_{l,i}}{\partial{\zeta_1^*}^\alpha}(\zeta_1^*)|\lesssim|\rho(z_{j,k})|^{1-\alpha}$. Since $|\zeta_2^*-\alpha^{(j,k)}_{l,i}(\zeta_1^*)|\gtrsim|\rho(z_{j,k})|^{\frac12}$, (14) holds true for $\alpha>0$ and $\beta=0$. Since the other cases are trivial, we are done in this case.

When $|\zeta^*_{0,1}|<4\kappa|\rho(z_{j,k})|$, we do not have the parametrisation of X_l given by Proposition 3.3 but according to Proposition 3.2, $\mathcal{P}_{4\kappa|\rho(z_{j,k})|}(z_{j,k})\cap X_l$ is empty, which means that any $\zeta\in\mathcal{P}_{2\kappa|\rho(z_{j,k})|}(z_{j,k})$ is far from X_l. We then have to distinguish two cases, depending on what "far" means. Before, we notice that, since $\mathcal{P}_{4\kappa|\rho(z_{j,k})|}(z_{j,k})\cap X_l=\emptyset$, $I_l^{(j,k)}$ is also empty and $P_l^{(j,k)}=1$.

Second case: If $|\zeta^*_{0,1}|<4\kappa|\rho(z_{j,k})|$ and $|\zeta^*_{0,2}|<(4\kappa|\rho(z_{j,k})|)^{\frac12}$, then we have $\delta(z_{j,k},\zeta_0)\lesssim|\rho(z_{j,k})|$ and thus for all $\zeta\in\mathcal{P}_{2\kappa|\rho(z_{j,k})|}(z_{j,k})$, $\delta(\zeta,\zeta_0)\lesssim|\rho(z_{j,k})|$. In particular, any ζ belonging to $\mathcal{P}_{2\kappa|\rho(z_{j,k})|}(z_{j,k})$ is almost at the same (pseudo-)distance from $z_{j,k}$ as from X_l.

For all $\varepsilon>0$ and all $\zeta\in\mathcal{P}_\varepsilon(\zeta_0)$, using Weierstrass Preparation theorem and a parametrisation of X_l, it is then easy to see that $|f_l(\zeta)|\lesssim\varepsilon^{\frac{p_l}{2}}$. Therefore, Cauchy's inequalities give

$$\left|\frac{\partial^{\alpha+\beta}f_l}{\partial{\zeta_1^*}^\alpha\,\partial{\zeta_2^*}^\beta}(\zeta)\right|\lesssim|\rho(z_{j,k})|^{\frac{p_l}{2}-\alpha-\frac{\beta}{2}}$$

for all $\zeta\in\mathcal{P}_{2\kappa|\rho(z_{j,k})|}(z_{j,k})$. Moreover, since $|\zeta^*_{0,1}|<4\kappa|\rho(z_{j,k})|$, on the one hand $f_l=Q_l^{(j,k)}$. On the other hand it follows from Proposition 3.2 that $\mathcal{P}_{4\kappa|\rho(z_{j,k})|}(z_{j,k})\cap X_l=\emptyset$. This yields $|f_l(\zeta)|\gtrsim|\rho(z_{j,k})|^{\frac{p_l}{2}}$ for all $\zeta\in\mathcal{P}_{2\kappa|\rho(z_{j,k})|}(z_{j,k})$, thus $|\frac{1}{Q_l^{(j,k)}(\zeta)}\frac{\partial^{\alpha+\beta}}{\partial{\zeta_1^*}^\alpha\,\partial{\zeta_2^*}^\beta}(Q_l^{(j,k)}(\zeta))|\lesssim|\rho(z_{j,k})|^{-\alpha-\frac{\beta}{2}}$.

Third case: If $|\zeta^*_{0,1}| < 4\kappa|\rho(z_{j,k})|$ and $|\zeta^*_{0,2}| \geq (4\kappa|\rho(z_{j,k})|)^{\frac{1}{2}}$, then all $\zeta \in \mathcal{P}_{3\kappa|\rho(z_{j,k})|}(z_{j,k})$ is far from ζ^*_0 and $Q_l^{(j,k)} = f_l$. We will see that $|f_l(\zeta)|$ is comparable to $|\zeta^*_{0,2}|^{p_l}$ for all $\zeta \in \mathcal{P}_{3\kappa|\rho(z_{j,k})|}(z_{j,k})$.

We set $a(z_{j,k}) = \frac{\partial\rho}{\partial\zeta_1}(z_{j,k})$, $b(z_{j,k}) = \frac{\partial\rho}{\partial\zeta_2}(z_{j,k})$ and

$$P(z_{j,k}) = \frac{1}{\sqrt{|a(z_{j,k})|^2 + |b(z_{j,k})|^2}} \begin{pmatrix} \overline{a(z_{j,k})} & \overline{b(z_{j,k})} \\ -b(z_{j,k}) & a(z_{j,k}) \end{pmatrix}.$$

Then we have $\zeta^* = P(z_{j,k})(\zeta - z_{j,k})$ and moreover $|a(z_{j,k})| \eqsim 1$ and $b(z_{j,k})$ tends to 0 when $z_{j,k}$ goes to ζ_0, hence, $b(z_{j,k})$ is arbitrary small provided $\mathcal{U}_0$ is sufficiently small.

Therefore, if $\mathcal{U}_0$ is sufficiently small, for all $\zeta \in \mathcal{P}_{3\kappa|\rho(z_{j,k})|}(z_{j,k})$,

$$\begin{aligned}|\zeta_2| &\geq \frac{|a(z_{j,k})||\zeta^*_{0,2}| - |b(z_{j,k})||\zeta^*_{0,1}| - |b(z_{j,k})||\zeta^*_1| - |a(z_{j,k})||\zeta^*_2|}{\sqrt{|a(z_{j,k})|^2 + |b(z_{j,k})|^2}} \\ &\gtrsim \left|\zeta^*_{0,2}\right|.\end{aligned}$$

We also trivially have $|\zeta_2| \lesssim |\zeta^*_{0,2}|$ and so $|\zeta_2| \eqsim |\zeta^*_{0,2}|$. On the other hand

$$\begin{aligned}|\zeta_1| &\leq \frac{1}{\sqrt{|a(z_{j,k})|^2 + |b(z_{j,k})|^2}} \left(\left|a(z_{j,k})\right|\left(\left|\zeta^*_{0,1}\right| + \left|\zeta^*_1\right|\right) + \left|b(z_{j,k})\right|\left(\left|\zeta^*_{0,2}\right| + \left|\zeta^*_2\right|\right)\right) \\ &\leq 6\kappa\left|\rho(z_{j,k})\right| + \left|b(z_{j,k})\right|\left(\left|\zeta^*_{0,2}\right| + \left(2\kappa\left|\rho(z_{j,k})\right|\right)^{\frac{1}{2}}\right) \\ &\leq c\left|\zeta^*_{0,2}\right|,\end{aligned}$$

where c depends neither on $z_{j,k}$ nor on ζ and is arbitrarily small provided $\mathcal{U}_0$ is small enough.

Now let $\alpha \in \mathbb{C}$ be such that $f_l(\zeta_1, \alpha) = 0$. Since the intersection $X_l \cap bD$ is transverse, there exists a positive constant C depending neither on ζ, nor on α, nor on j and nor on k such that $|\alpha| \leq C|\zeta_1|$.

Therefore if $\mathcal{U}_0$ is small enough, $|\alpha| \leq \frac{1}{2}|\zeta_2|$. For all $\zeta \in \mathcal{P}_{3\kappa|\rho(z_{j,k})|}(z_{j,k})$, this yields

$$\begin{aligned}\left|f_l(\zeta)\right| &\eqsim \prod_{\alpha/f_l(\zeta_1,\alpha)=0} |\zeta_2 - \alpha| \\ &\eqsim \left|\zeta^*_{0,2}\right|^{p_l}.\end{aligned}$$

Cauchy's inequalities then give for all $\zeta \in \mathcal{P}_{2\kappa|\rho(z_{j,k})|}(z_{j,k})$

$$\left|\frac{\partial^{\alpha+\beta} f_l}{\partial{\zeta^*_1}^\alpha\, \partial{\zeta^*_2}^\beta}(\zeta)\right| \lesssim \left|\zeta^*_{0,2}\right|^{p_l} \left|\rho(z_{j,k})\right|^{-\alpha-\frac{\beta}{2}},$$

and since $Q_l^{(j,k)} = f_l$, we are done in this case and the lemma is shown. □

Lemma 5.1 yields an upper bound for the derivatives of $\chi_l^{(j,k)}$.

COROLLARY 5.2. *For all $j\in\mathbb{N}$, all $k\in\{1,\dots,n_j\}$, all α and β in $\mathbb{N}$, $l=1,2$ and all $\zeta\in\mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$, we have uniformly with respect to j,k,l and ζ*

$$\left|\frac{\partial^{\alpha+\beta}\chi_l^{(j,k)}}{\partial\overline{\zeta}_1^{*\alpha}\,\partial\overline{\zeta}_2^{*\beta}}(\zeta)\right|\lesssim\left|\rho(z_{j,k})\right|^{-\alpha-\frac{\beta}{2}}.$$

Proof. Since by construction $|\frac{\partial^{\alpha+\beta}\tilde{\chi}_{j,k}}{\partial\overline{\zeta}_1^{*\alpha}\,\partial\overline{\zeta}_2^{*\beta}}(\zeta)|\lesssim|\rho(z_{j,k})|^{-\alpha-\frac{\beta}{2}}$, we only have to consider $\frac{\partial^{\alpha+\beta}}{\partial\zeta_1^{*\alpha}\,\partial\zeta_2^{*\beta}}\chi(\frac{f_1(\zeta)}{P_1^{(j,k)}(\zeta)}|\rho(z_{j,k})|^{i_1^{(j,k)}},\frac{f_2(\zeta)}{P_2^{(j,k)}(\zeta)}|\rho(z_{j,k})|^{i_2^{(j,k)}})$.

The derivative $\frac{\partial^{\gamma+\delta}\chi}{\partial z_1^\gamma\,\partial z_2^\delta}(z_1,z_2)$ is bounded up to a uniform multiplicative constant by $\frac{1}{|z_1|^\gamma|z_2|^\delta}$ when $\frac{1}{3}|z_2|<|z_1|<\frac{2}{3}|z_2|$ and is zero otherwise.

So we can estimate $|\frac{\partial^{\alpha+\beta}\chi_l^{(j,k)}}{\partial\overline{\zeta_1^*}^\alpha\,\partial\overline{\zeta_2^*}^\beta}|$ by a sum of products of $|\frac{1}{Q_l^{(j,k)}}\frac{\partial^{\tilde{\gamma}+\tilde{\delta}}Q_l^{(j,k)}}{\partial\overline{\zeta}_1^{*\tilde{\gamma}}\,\partial\overline{\zeta}_2^{*\tilde{\delta}}}|$ where the sum of the $\tilde{\gamma}$'s equals α and the sum of the $\tilde{\delta}$'s equals β. Lemma 5.1 then gives the wanted estimates. $\square$

COROLLARY 5.3. *For any smooth function h, we can write*

$$\frac{\partial^{i_l^{(j,k)}}}{\partial\overline{\zeta_2^*}^{i_l^{(j,k)}}}\left(\chi_l^{(j,k)}(\zeta)\,\overline{\partial}h(\zeta)\wedge P^{N,1}(\zeta,z)\right)=\psi_1^{(j,k,l)}(\zeta,z)\,d\zeta_1^*+\psi_2^{(j,k,l)}(\zeta,z)\,d\zeta_2^*$$

with $\psi_1^{(j,k,l)}$ and $\psi_2^{(j,k,l)}$ two $(0,2)$-forms supported in $\mathcal{U}_l^{(j,k)}$ satisfying uniformly with respect to j,k,z and $\zeta\in\mathcal{U}_l^{(j,k)}$:

$$\left|\psi_1^{(j,k,l)}(\zeta,z)\right|\lesssim\left|\rho(z_{j,k})\right|^{-\frac{i_l^{(j,k)}}{2}-\frac{5}{2}}\left(\frac{|\rho(z_{j,k})|}{|\rho(z_{j,k})|+|\rho(z)|+\delta(z_{j,k},z)}\right)^N\tilde{h}(\zeta),$$

$$\left|\psi_2^{(j,k,l)}(\zeta,z)\right|\lesssim\left|\rho(z_{j,k})\right|^{-\frac{i_l^{(j,k)}}{2}-2}\left(\frac{|\rho(z_{j,k})|}{|\rho(z_{j,k})|+|\rho(z)|+\delta(z_{j,k},z)}\right)^N\tilde{h}(\zeta),$$

and, for ∇_z a differential operators of order 1 acting on z,

$$\left|\nabla_z\psi_1^{(j,k,l)}(\zeta,z)\right|\lesssim\left|\rho(z_{j,k})\right|^{-\frac{i_l^{(j,k)}}{2}-\frac{7}{2}}\left(\frac{|\rho(z_{j,k})|}{|\rho(z_{j,k})|+|\rho(z)|+\delta(z_{j,k},z)}\right)^N\tilde{h}(\zeta),$$

$$\left|\nabla_z\psi_2^{(j,k,l)}(\zeta,z)\right|\lesssim\left|\rho(z_{j,k})\right|^{-\frac{i_l^{(j,k)}}{2}-3}\left(\frac{|\rho(z_{j,k})|}{|\rho(z_{j,k})|+|\rho(z)|+\delta(z_{j,k},z)}\right)^N\tilde{h}(\zeta),$$

where

$$\tilde{h}(\zeta)=\max_{n\in\{0,\dots,i_l^{(j,k)}\}}\left(\left|\frac{\partial^{n+1}h}{\partial\overline{\zeta_2^*}^{n+1}}(\zeta)\left|\rho(\zeta)\right|^{\frac{n+1}{2}}\right|,\left|\frac{\partial^{n+1}h}{\partial\overline{\zeta_1^*}\,\partial\overline{\zeta_2^*}^n}(\zeta)\left|\rho(\zeta)\right|^{\frac{n}{2}+1}\right|\right).$$

Proof. Propositions 2.4 and 2.5 imply that

$$\frac{\partial^n}{\partial \overline{\zeta_2^*}^n} P^{N,1}(\zeta,z) = \sum_{p,q=1,2} \tilde{\psi}_{p,q}^{(n,N)}(\zeta,z)\, d\zeta_p^* \wedge d\overline{\zeta_q^*},$$

where

$$\left|\tilde{\psi}_{p,q}^{n,N}(\zeta,z)\right| \lesssim \left(\frac{|\rho(\zeta)|}{|\rho(\zeta)|+|\rho(z)|+\delta(\zeta,z)}\right)^N \left|\rho(\zeta)\right|^{-\frac{1}{p}-\frac{1}{q}-\frac{n}{2}}.$$

From Proposition 2.1, if κ is small enough, we have for all $\zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$, $\frac{1}{2}|\rho(z_{j,k})| \leq |\rho(\zeta)|$ and thus, provided κ is small enough:

$$\begin{aligned}\left|\rho(\zeta)\right| + \delta(\zeta,z) &\geq \frac{1}{2}\left|\rho(z_{j,k})\right| + \frac{1}{c_1}\delta(z,z_{j,k}) - \delta(z_{j,k},\zeta)\\ &\gtrsim \left|\rho(z_{j,k})\right| + \delta(z,z_{j,k})\end{aligned}$$

and so $|\tilde{\psi}_{p,q}^{n,N}(\zeta,z)| \lesssim (\frac{|\rho(z_{j,k})|}{|\rho(z_{j,k})|+|\rho(z)|+\delta(z_{j,k},z)})^N |\rho(z_{j,k})|^{-\frac{1}{p}-\frac{1}{q}-\frac{n}{2}}$. This inequality and Corollary 5.2 now yield the two first estimates. The two others can be shown in the same way. $\square$

In order to estimate $\frac{\overline{P_l^{(j,k)}}}{f_l} b_m$, we need the following lemma.

LEMMA 5.4. *For all $j \in \mathbb{N}$, all $k \in \{1,\dots,n_j\}$, all α and β in $\mathbb{N}$, $l = 1,2$ and all $\zeta \in \mathcal{P}_{2\kappa|\rho(z_{j,k})|}(z_{j,k})$ we have uniformly with respect to j,k,l and ζ*

$$\left|\frac{\partial^{\alpha+\beta}}{\partial {\zeta_1^*}^\alpha\, \partial {\zeta_2^*}^\beta}\left(\prod_{i \in I_l^{(j,k)}} \left(\zeta_2^* - \alpha_{l,i}^{(j,k)}(\zeta_1^*)\right)\right)\right| \lesssim \left|\rho(z_{j,k})\right|^{\frac{i_l^{(j,k)}}{2}-\alpha-\frac{\beta}{2}}.$$

Proof. For every $i \in I_l^{(j,k)}$, there exists a complex number $z_1^* \in \Delta_0(2\kappa \times |\rho(z_{j,k})|)$ such that $|\alpha_{l,i}^{(j,k)}(z_1^*)| < \frac{5}{2}\kappa|\rho(z_{j,k})|^{\frac{1}{2}}$. Since $|\frac{\partial \alpha_{l,i}^{(j,k)}}{\partial \zeta_1^*}|$ is uniformly bounded on $\Delta_0(4\kappa|\rho(z_{j,k})|)$, for all $\zeta \in \mathcal{P}_{4\kappa|\rho(z_{j,k})|}(z_{j,k})$, we have $\prod_{i \in I_l^{(j,k)}} |\zeta_2^* - \alpha_{l,i}^{(j,k)}(\zeta_1^*)| \lesssim |\rho(z_{j,k})|^{\frac{i_l^{(j,k)}}{2}}$. Cauchy's inequalities then give the results. $\square$

As a direct corollary of Lemmas 5.1 and 5.4 we get the following corollary.

COROLLARY 5.5. *For all $j \in \mathbb{N}$, all $k \in \{1,\dots,n_j\}$, all α and β in $\mathbb{N}$, $l = 1,2$ and all $\zeta \in \mathcal{P}_{2\kappa|\rho(z_{j,k})|}(z_{j,k})$ we have uniformly with respect to j,k,l and ζ*

$$\left|\frac{P_l^{(j,k)}(\zeta)}{f_l(\zeta)} \frac{\partial^{\alpha+\beta} f_l}{\partial {\zeta_1^*}^\alpha\, \partial {\zeta_2^*}^\beta}(\zeta)\right| \lesssim \left|\rho(z_{j,k})\right|^{\frac{i_l^{(j,k)}}{2}-\alpha-\frac{\beta}{2}}.$$

In the following corollary, we give estimates for $l,m\in\{1,2\}$ of $\frac{P_l^{(j,k)}}{f_l}b_m$, which do not depend on m thanks to the covering $\mathcal{U}_1^{(j,k)}$, $\mathcal{U}_2^{(j,k)}$ of the polydisc $\mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$.

Corollary 5.6. *For $l,m\in\{1,2\}$, we can write $\frac{P_l^{(j,k)}}{f_l}b_m=\varphi_1^{(j,k,l,m)}\,d\zeta_1^*+\varphi_2^{(j,k,l,m)}\,d\zeta_2^*$ with $\varphi_1^{(j,k,l,m)}$ and $\varphi_2^{(j,k,l,m)}$ satisfying for all $\zeta\in\mathcal{U}_l^{(j,k)}$*

$$\left|\varphi_1^{(j,k,l,m)}(\zeta,z)\right|\lesssim\sum_{0\leq\alpha+\beta\leq\max(p_1,p_2)}\left|\rho(z_{j,k})\right|^{\frac{i_l^{(j,k)}}{2}-1}\left|\frac{\delta(\zeta,z)}{\rho(z_{j,k})}\right|^{\alpha+\frac{\beta}{2}},$$

$$\left|\varphi_2^{(j,k,l,m)}(\zeta,z)\right|\lesssim\sum_{0\leq\alpha+\beta\leq\max(p_1,p_2)}\left|\rho(z_{j,k})\right|^{\frac{i_l^{(j,k)}}{2}-\frac{1}{2}}\left|\frac{\delta(\zeta,z)}{\rho(z_{j,k})}\right|^{\alpha+\frac{\beta}{2}},$$

and for all differential operators ∇_z of order 1 acting on z,

$$\left|\nabla_z\varphi_1^{(j,k,l,m)}(\zeta,z)\right|\lesssim\sum_{0\leq\alpha+\beta\leq\max(p_1,p_2)}\left|\rho(z_{j,k})\right|^{\frac{i_l^{(j,k)}}{2}-2}\left|\frac{\delta(\zeta,z)}{\rho(z_{j,k})}\right|^{\alpha+\frac{\beta}{2}},$$

$$\left|\nabla_z\varphi_2^{(j,k,l,m)}(\zeta,z)\right|\lesssim\sum_{0\leq\alpha+\beta\leq\max(p_1,p_2)}\left|\rho(z_{j,k})\right|^{\frac{i_l^{(j,k)}}{2}-\frac{3}{2}}\left|\frac{\delta(\zeta,z)}{\rho(z_{j,k})}\right|^{\alpha+\frac{\beta}{2}},$$

uniformly with respect to ζ,z,j and k.

Proof. Without restriction, we assume $l=1$ and for $m=1,2$, we write $b_m(\zeta,z)=b_{m,1}^*(\zeta,z)\,d\zeta_1^*+b_{m,2}^*(\zeta,z)\,d\zeta_2^*$ where $b_{m,n}^*=\int_0^1\frac{\partial f_m}{\partial\zeta_n^*}(\zeta+t(z-\zeta))\,dt$. So

$$\begin{aligned}&b_{m,n}^*(\zeta,z)\\&\quad=\sum_{0\leq\alpha+\beta\leq\max(p_1,p_2)}\frac{1}{\alpha+\beta+1}\frac{\partial^{\alpha+\beta+1}f_m}{\partial\zeta_n^*\,\partial\zeta_1^{*\alpha}\,\partial\zeta_2^{*\beta}}(\zeta)\left(z_1^*-\zeta_1^*\right)^{\alpha}\left(z_2^*-\zeta_2^*\right)^{\beta}\\&\qquad+o\left(|z-\zeta|^{\max(p_1,p_2)}\right)\end{aligned}$$

and Corollary 5.5 yields for all $\zeta\in\mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$:

$$\left|\overline{\frac{P_1^{(j,k)}(\zeta)}{f_1(\zeta)}}b_{1,1}(\zeta,z)\right|\lesssim\sum_{0\leq\alpha+\beta\leq\max(p_1,p_2)}\left|\rho(z_{j,k})\right|^{\frac{i_1^{(j,k)}}{2}-1}\left|\frac{\delta(\zeta,z)}{\rho(z_{j,k})}\right|^{\alpha+\frac{\beta}{2}}$$

uniformly with respect to z,ζ,j and k. The proof of the inequality for $|\overline{\frac{P_1^{(j,k)}(\zeta)}{f_1(\zeta)}}b_{1,2}(\zeta,z)|$ is exactly the same. The one for $|\overline{\frac{P_1^{(j,k)}(\zeta)}{f_1(\zeta)}}b_{2,1}(\zeta,z)|$ uses the definition of $\mathcal{U}_1^{(j,k)}$.

On $\mathcal{U}_1^{(j,k)}$, we have $|\frac{P_1^{(j,k)}}{f_1}| \lesssim |\frac{P_2^{(j,k)}}{f_2}||\rho(z_{j,k})|^{\frac{i_1^{(j,k)}-i_2^{(j,k)}}{2}}$ and again Corollary 5.5 yields uniformly with respect to z, ζ, j and k

$$\left|\frac{\overline{P_1^{(j,k)}(\zeta)}}{f_1(\zeta)} b_{2,1}(\zeta,z)\right| \lesssim \left|\frac{P_2^{(j,k)}(\zeta)}{f_2(\zeta)} b_{2,1}(\zeta,z)\right| \left|\rho(z_{j,k})\right|^{\frac{i_1^{(j,k)}-i_2^{(j,k)}}{2}}$$

$$\lesssim \sum_{0\le\alpha+\beta\le\max(p_1,p_2)} \left|\rho(z_{j,k})\right|^{\frac{i_1^{(j,k)}}{2}-1} \left|\frac{\delta(\zeta,z)}{\rho(z_{j,k})}\right|^{\alpha+\frac{\beta}{2}}.$$

Again, the inequality for $|\frac{\overline{P_1^{(j,k)}(\zeta)}}{f_1(\zeta)} b_{2,2}(\zeta,z)|$ can be obtained in the same way. □

Corollaries 5.3 and 5.6 imply for some N' arbitrarily large, provided N is large enough, and for all $\zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$ that

$$\left|\frac{\overline{P_l^{(j,k)}}(\zeta)}{f_l(\zeta)} b_m(\zeta,z) \wedge \frac{\partial^{i_l^{(j,k)}}}{\partial\overline{\zeta_2^*}^{i_l^{(j,k)}}}\left(\chi_l^{(j,k)}(\zeta)\overline{\partial}h(\zeta) \wedge P^{N,1}(\zeta,z)\right)\right|$$

$$\le \left|\rho(z_{j,k})\right|^{-3}\left(\frac{|\rho(z_{j,k})|}{|\rho(z_{j,k})| + |\rho(z)| + \delta(z_{j,k},z)}\right)^{N'} \tilde{h}(\zeta)$$

and for ∇_z a differential of order 1

$$\left|\nabla_z\left(\frac{\overline{P_l^{(j,k)}}(\zeta)}{f_l(\zeta)} b_m(\zeta,z) \wedge \frac{\partial^{i_l^{(j,k)}}}{\partial\overline{\zeta_2^*}^{i_l^{(j,k)}}}\left(\chi_l^{(j,k)}(\zeta)\,\overline{\partial}h(\zeta) \wedge P^{N,1}(\zeta,z)\right)\right)\right|$$

$$\le \left|\rho(z)\right|^{-1}\left|\rho(z_{j,k})\right|^{-3}\left(\frac{|\rho(z_{j,k})|}{|\rho(z_{j,k})| + |\rho(z)| + \delta(z_{j,k},z)}\right)^{N'} \tilde{h}(\zeta),$$

where $\tilde{h}(\zeta) = \max_{n\in\{0,\dots,i_l^{(j,k)}\}}(|\frac{\partial^{n+1}h}{\partial\overline{\zeta_2^*}^{n+1}}(\zeta)|\rho(\zeta)|^{\frac{n+1}{2}}|, |\frac{\partial^{n+1}h}{\partial\overline{\zeta_1^*}\,\partial\overline{\zeta_2^*}^{n}}(\zeta)|\rho(\zeta)|^{\frac{n}{2}+1}|)$, which gives k_1 of Theorem 3.1. Now we conclude as in the proof of Theorem 1.1 of [1] that Theorem 3.1 holds true.

6. Local division

6.1. Local holomorphic division. In this subsection, we will prove two theorems which enables us to go from local smooth division to global smooth division.

THEOREM 6.1. *When $n=2$, let g be a holomorphic function defined on D. Assume that $X_1 \cap X_2$ is a complete intersection and that there exist $\kappa > 0$, a real number $q \ge 1$ and a locally finite covering $(\mathcal{P}_{\kappa|\rho(\zeta_j)|}(\zeta_j))_{j\in I}$ of D such that for all $j \in I$, there exist two function $\hat{g}_1^{(j)}$ and $\hat{g}_2^{(j)}$, C^∞-smooth on $\mathcal{P}_{\kappa|\rho(\zeta_j)|}(\zeta_j)$, which satisfy*

(a) $g = \hat{g}_1^{(j)} f_1 + \hat{g}_2^{(j)} f_2$ *on* $\mathcal{P}_{\kappa|\rho(\zeta_j)|}(\zeta_j)$;

(b) $\sum_{j\in I}\int_{\mathcal{P}_{\kappa|\rho(\zeta_j)|}(\zeta_j)}|\frac{\partial^{\alpha+\beta}\hat{g}_l^{(j)}}{\partial\overline{\zeta_1^*}^{\alpha}\,\partial\overline{\zeta_2^*}^{\beta}}(z)|\rho(\zeta_j)|^{\alpha+\frac{\beta}{2}}|^q\,dV(z)<\infty$ *for* $l=1$ *and* $l=2$ *and all integers* α *and* β;

(c) *for* $l=1$ *and* $l=2$, *for all nonnegatives integers* $\alpha,\overline{\alpha},\beta$ *and* $\overline{\beta}$, *there exist* $N\in\mathbb{N}$ *and* $c>0$ *such that* $|\rho(\zeta_j)|^N\sup_{\mathcal{P}_{\kappa|\rho(\zeta_j)|}(\zeta_j)}|\frac{\partial^{\alpha+\overline{\alpha}+\beta+\overline{\beta}}\hat{g}_l^{(j)}}{\partial\zeta_1^{*\alpha}\,\partial\zeta_2^{*\beta}\,\partial\overline{\zeta_1^*}^{\overline{\alpha}}\,\partial\overline{\zeta_2^*}^{\overline{\beta}}}|\leq c$, *for all* j.

Then there exist two smooth functions $\tilde{g}_1$ *and* $\tilde{g}_2$ *which satisfy* (i)–(iii) *of Theorem* 3.1 *with* q.

Proof. It suffices to glue together all the $\hat{g}_1^{(j)}$ and $\hat{g}_2^{(j)}$ using a suitable partition of unity. Let $(\chi_j)_{j\in\mathbb{N}}$ be a partition of unity subordinated to $(\mathcal{P}_{\kappa|\rho(\zeta_j)|}(\zeta_j))_{j\in\mathbb{N}}$ such that for all j and all $\zeta\in\mathcal{P}_{\kappa|\rho(\zeta_j)|}(\zeta_j)$, we have $|\frac{\partial^{\alpha+\overline{\alpha}+\beta+\overline{\beta}}\chi_j}{\partial z_1^{*\alpha}\,\partial z_2^{*\beta}\,\partial\overline{z_1^*}^{\overline{\alpha}}\,\partial\overline{z_2^*}^{\overline{\beta}}}(\zeta)|\lesssim\frac{1}{|\rho(\zeta_j)|^{\alpha+\overline{\alpha}+\frac{\beta+\overline{\beta}}{2}}}$, uniformly with respect to ζ_j and ζ. We set $\tilde{g}_1=\sum_j\chi_j\hat{g}_1^{(j)}$ and $\tilde{g}_2=\sum_j\chi_j\hat{g}_2^{(j)}$ and thus we get the two functions defined on D which satisfy (i), (ii) and (iii) by construction. □

We have for $q=+\infty$ the following result.

THEOREM 6.2. *Let* D *be a strictly convex domain of* $\mathbb{C}^2$, f_1 *and* f_2 *be two holomorphic functions defined on a neighbourhood of* $\overline{D}$ *and set* $X_l=\{z,f_l(z)=0\}$, $l=1,2$. *Suppose that* $X_1\cap bD$ *and* $X_2\cap bD$ *are transverse, and that* $X_1\cap X_2$ *is a complete intersection.*

Let g *be a function holomorphic on* D *and assume that there exists* $\kappa>0$ *such that for all* $z\in D$, *there exist two functions* $\hat{g}_1$ *and* $\hat{g}_2$, *depending on* z, C^∞*-smooth on* $\mathcal{P}_{\kappa|\rho(z)|}(z)$, *such that*

(a) $g=\hat{g}_1f_1+\hat{g}_2f_2$ *on* $\mathcal{P}_{\kappa|\rho(z)|}(z)$;

(b) *for all nonnegative integers* α, β, $\overline{\alpha}$ *and* $\overline{\beta}$, *there exist* $c>0$, *not depending on* z, *such that* $\sup_{\mathcal{P}_{\kappa|\rho(z)|}(z)}|\frac{\partial^{\alpha+\overline{\alpha}+\beta+\overline{\beta}}\hat{g}_l}{\partial z_1^{*\alpha}\,\partial z_2^{*\beta}\,\partial\overline{z_1^*}^{\overline{\alpha}}\,\partial\overline{\zeta_2^*}^{\overline{\beta}}}|\leq c|\rho(z)|^{-\alpha-\frac{\beta}{2}}$ *for* $l=1$ *and* $l=2$.

Then there exist two smooth functions $\tilde{g}_1$ *and* $\tilde{g}_2$ *which satisfy the assumptions* (i)–(iii) *of Theorem* 3.1 *for* $q=+\infty$.

The proof of Theorem 6.2 is exactly the same than the proof of Theorem 6.1 so we omit it.

6.2. Divided differences and division. We first prove a lemma we will need in this section.

LEMMA 6.3. *Let* α *and* β *be two functions defined on a subset* $\mathcal{U}$ *of* $\mathbb{C}$. *Then, for all* $z_1,\dots,z_n$ *pairwise distinct points of* $\mathcal{U}$ *we have*

$$(\alpha\cdot\beta)[z_1,\dots,z_n]=\sum_{k=1}^{n}\alpha[z_1,\dots,z_k]\cdot\beta[z_k,\dots,z_n].$$

Proof. We prove the lemma by induction on n, the case $n=1$ being trivial. We assume the lemma proved for n points, $n \geq 1$. Let $z_1, \dots, z_{n+1}$ be $n+1$ points of $\mathcal{U}$. Then

$$\begin{aligned}
&(\alpha \cdot \beta)[z_1, \dots, z_{n+1}] \\
&\quad = \frac{(\alpha \cdot \beta)[z_1, z_3, \dots, z_{n+1}] - (\alpha \cdot \beta)[z_2, \dots, z_{n+1}]}{z_1 - z_2} \\
&\quad = \frac{1}{z_1 - z_2} \left(\sum_{k=3}^{n+1} \alpha[z_1, z_3, \dots, z_k] \beta[z_k, \dots, z_{n+1}] + \alpha[z_1] \beta[z_3, \dots, z_{n+1}] \right) \\
&\qquad - \frac{1}{z_1 - z_2} \sum_{k=2}^{n+1} \alpha[z_2, \dots, z_k] \beta[z_k, \dots, z_{n+1}] \\
&\quad = \sum_{k=3}^{n+1} \frac{\alpha[z_1, z_3, \dots, z_k] - \alpha[z_2, \dots, z_k]}{z_1 - z_2} \beta[z_k, \dots, z_{n+1}] \\
&\qquad + \frac{\alpha[z_1] - \alpha[z_2]}{z_1 - z_2} \beta[z_2, \dots, z_{n+1}] \\
&\qquad + \alpha[z_1] \frac{\beta[z_1, z_3, \dots, z_{n+1}] - \beta[z_2, \dots, z_{n+1}]}{z_1 - z_2}.
\end{aligned}$$

□

6.2.1. *The* L^∞–BMO-*case.* In this subsection, we prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. The first point is trivial and we only prove the second one for $l=1$. Let $\lambda_1, \dots, \lambda_k$ be k pairwise distinct elements of $\Lambda^{(1)}_{z,v}$. For all i we have $g^{(1)}_{z,v}[\lambda_i] = g_1(z + \lambda_i v)$ because $f_2(z + \lambda_i v) = 0$. Therefore, $g^{(1)}_{z,v}[\lambda_1, \dots, \lambda_k] = (g_1)_{z,v}[\lambda_1, \dots, \lambda_k]$. By [17]

$$g^{(1)}_{z,v}[\lambda_1, \dots, \lambda_k] = \frac{1}{2i\pi} \int_{|\lambda| = \tau(z,v,4\kappa|\rho(z)|)} \frac{g_1(z + \lambda v)}{\prod_{i=1}^k (\lambda - \lambda_i)} \, d\lambda,$$

it follows that

$$\left| g^{(1)}_{z,v}[\lambda_1, \dots, \lambda_k] \right| \lesssim \tau\big(z, v, |\rho(z)|\big)^{-k+1} \sup_{b\Delta_{z,v}(\tau(z,v,4\kappa|\rho(z)|))} |g_1|.$$

Therefore $c^{(1)}_\infty(g) \lesssim \sup_{b\Delta_{z,v}(\tau(z,v,4\kappa|\rho(z)|))} |g_1|$, and since g_1 is bounded, $c^{(1)}_\infty(g)$ is finite. □

Now we prove Theorem 1.2, that is that these conditions are sufficient in $\mathbb{C}^2$ in order to get a BMO division.

Proof of Theorem 1.2. It suffices to construct, for all z near bD, two smooth functions $\hat{g}_1$ and $\hat{g}_2$ on $\mathcal{P}_{\kappa|\rho(z)|}(z)$ which satisfy (a) and (b) of Theorem 6.2 and then to apply Theorem 3.1 with the function $\tilde{g}_1$ and $\tilde{g}_2$ given by Theorem 6.2.

Let ζ_0 be a point in bD. If $f_1(\zeta_0)\neq 0$, then f_1 does not vanish on a neighbourhood $\mathcal{U}_0$ of ζ_0. Then we can define $\hat{g}_1=\frac{g}{f_1}$, $\hat{g}_2=0$ which obviously satisfy (a) and (b) for all $z\in D$ close to ζ_0. We proceed analogously if $f_2(\zeta_0)\neq 0$.

If ζ_0 belongs to $X_1\cap X_2\cap bD$, since the intersection $X_1\cap X_2$ is complete, without restriction we can choose a neighbourhood $\mathcal{U}_0$ of ζ_0 such that $X_1\cap X_2\cap\mathcal{U}_0=\{\zeta_0\}$. Then we fix some point z in $\mathcal{U}_0$ and we construct $\hat{g}_1$ and $\hat{g}_2$ on $\mathcal{P}_{\kappa|\rho(z)|}(z)$ which satisfy (a) and (b) of Theorem 6.2. We denote by p_1 and p_2 the multiplicity of ζ_0 as singularity of f_1 and f_2 respectively. We also denote by $(\zeta^*_{0,1},\zeta^*_{0,2})$ the coordinates of ζ_0 in the Koranyi coordinates at z.

If $|\zeta^*_{0,1}|<4\kappa|\rho(z)|$, then for $l=1$ and $l=2$ we set $I_l=\emptyset$, $i_l=0$, $P_l(\zeta)=1$ and $Q_l(\zeta)=f_l(\zeta)$.

Otherwise, we use the parametrisation $\alpha_{1,i}$, $i\in\{1,\dots,p_1\}$, of X_1 and $\alpha_{2,i}$, $i\in\{1,\dots,p_2\}$, of X_2 given by Proposition 3.3. We denote by I_l the set

$$I_l=\left\{i,\exists z_1^*\in\Delta_0\big(2\kappa|\rho(z)|\big)\text{ such that }\big|\alpha_{l,i}(z_1^*)\big|\le\left(\frac{5}{2}\kappa\big|\rho(z)\big|\right)^{\frac{1}{2}}\right\},$$

$i_l=\#I_l$, $P_l(\zeta)=\prod_{i\in I_l}(\zeta_2^*-\alpha_{l,i}(\zeta_1^*))$ and $Q_l(\zeta)=\frac{f_l}{P_l}$.

Our first goal is to find $\tilde{h}_1$ and $\tilde{h}_2$ in $C^\infty(\mathcal{P}_{\kappa|\rho(z)|}(z))$ such that $g=\tilde{h}_1P_1+\tilde{h}_2P_2$ on $\mathcal{P}_{\kappa|\rho(z)|}(z)$ and which moreover satisfy good estimates. The function g belong to the ideal of $\mathcal{O}(\mathcal{P}_{4\kappa|\rho(z)|}(z))$ generated by f_1 and f_2 and so there exist h_1 and h_2 holomorphic in $\mathcal{P}_{4\kappa|\rho(z)|}(z)$ such that $g=P_1h_1+P_2h_2$. Moreover, we observe that necessarily $\tilde{h}_2(\zeta)=h_2(\zeta)=\frac{g(\zeta)}{P_2(\zeta)}$ for all ζ such that $P_1(\zeta)=0$ and $P_2(\zeta)\neq 0$, but we also notice that h_2 may not satisfy good estimates like uniform boundedness for example. Thus, we already know $\tilde{h}_2(\zeta)$ for all ζ such that $P_1(\zeta)=0$ and $P_2(\zeta)\neq 0$ and by interpolation, we will reconstruct a "good" $\tilde{h}_2$ in the whole polydisc $\mathcal{P}_{\kappa|\rho(z)|}(z)$. We point out that we do not directly divide by f_1 and f_2 because if we do so, we are not able to handle the error term we get during the interpolation procedure.

If $i_1=0$, we set $\hat{h}_2=0$. Otherwise, without restriction we assume that $I_1=\{1,\dots,i_1\}$ and for $k\le i_1$ and ζ_1^* such that $P_2(z+\zeta_1^*\eta_z+\alpha_{1,i}(\zeta_1^*)v_z)\neq 0$, we introduce

$$h^{(2)}_{1,\dots,k}(\zeta_1^*):=\left(\frac{g}{P_2}\right)_{z+\zeta_1^*\eta_z,v_z}\big[\alpha_{1,1}(\zeta_1^*),\dots,\alpha_{1,k}(\zeta_1^*)\big]\tag{15}$$

and

$$\hat{h}_2(\zeta)=\sum_{k=1}^{i_2}h^{(2)}_{1,\dots,k}(\zeta_1^*)\prod_{i=1}^{k-1}\big(\zeta_2^*-\alpha_{1,i}(\zeta_1^*)\big).$$

We define $\hat{h}_1$ analogously. Since $X_1\cap X_2\cap\mathcal{U}_0=\{\zeta_0\}$, $\hat{h}_1$ and $\hat{h}_2$ are defined on $\mathcal{P}_{4\kappa|\rho(z)|}(z)$. Moreover, $\hat{h}_2(\zeta_1^*,\cdot)$ is the polynomial which interpolates $h_2(\zeta_1^*,\cdot)$

at the points $\alpha_{1,1}(\zeta_1^*),\dots,\alpha_{1,i_1}(\zeta_1^*)$. Therefore, we get from [17]

$$h_2(\zeta)=\hat{h}_2(\zeta)+P_1(\zeta)e_1(\zeta) \tag{16}$$

with

$$e_1(\zeta)=\frac{1}{2i\pi}\int_{|\xi|=(4\kappa|\rho(z)|)^{\frac12}}\frac{h_2(\zeta_1^*,\xi)}{P_1(\zeta_1^*,\xi)\cdot(\xi-\zeta_2^*)}\,d\xi. \tag{17}$$

We have an analogous expression for h_1 and we point out that (16), (17) and theirs analogue for g_1 also holds if $i_1=0$ or $i_2=0$.

This yields

$$g(\zeta)=P_1(\zeta)\hat{h}_1(\zeta)+P_2(\zeta)\hat{h}_2(\zeta)+P_1(\zeta)P_2(\zeta)e(\zeta), \tag{18}$$

where

$$\begin{aligned}e(\zeta)&=e_1(\zeta)+e_2(\zeta)\\&=\frac{1}{2i\pi}\int_{|\xi|=(4\kappa|\rho(z)|)^{\frac12}}\frac{g(\zeta_1^*,\xi)}{P_1(\zeta_1^*,\xi)\cdot P_2(\zeta_1^*,\xi)\cdot(\xi-\zeta_2^*)}\,d\xi.\end{aligned}$$

If we were trying to divide by f_1 and f_2 directly instead of dividing by P_1 and P_2, in the error term above, we wouldn't get g but $h_1P_1+h_2P_2$ that we cannot handle.

Of course, $\hat{h}_2$ will be a part of the function $\tilde{h}_2$ we are looking for and so we first look for an upper bound for $\hat{h}_2$ using our assumption on the divided differences of $g^{(2)}=\frac{g}{f_2}$.

Fact 1: $\hat{h}_2$ satisfies for all $\zeta\in\mathcal{P}_{2\kappa|\rho(z)|}(z)$, uniformly with respect to z and ζ

$$\big|\hat{h}_2(\zeta)\big|\lesssim c_\infty^{(2)}(g)\sup_{|\xi|=(4\kappa|\rho(z)|)^{\frac12}}\big|Q_2\big(z+\zeta_1^*\eta_z+\xi v_z\big)\big|. \tag{19}$$

Indeed: We have by Lemma 6.3

$$\begin{aligned}h^{(2)}_{1,\dots,k}\big(\zeta_1^*\big)&=\left(\frac{g}{P_2}\right)_{z+\zeta_1^*\eta_z,v_z}\big[\alpha_{1,1}\big(\zeta_1^*\big),\dots,\alpha_{1,k}\big(\zeta_1^*\big)\big]\\&=\big(g^{(2)}Q_2\big)_{z+\zeta_1^*\eta_z,v_z}\big[\alpha_{1,1}\big(\zeta_1^*\big),\dots,\alpha_{1,k}\big(\zeta_1^*\big)\big]\\&=\sum_{j=1}^{k}g^{(2)}_{z+\zeta_1^*\eta_z,v_z}\big[\alpha_{1,1}\big(\zeta_1^*\big),\dots,\alpha_{1,j}\big(\zeta_1^*\big)\big]\\&\qquad\times(Q_2)_{z+\zeta_1^*\eta_z,v_z}\big[\alpha_{1,j}\big(\zeta_1^*\big),\dots,\alpha_{1,k}\big(\zeta_1^*\big)\big].\end{aligned}$$

From Montel's theorem [17] on divided differences in $\mathbb{C}$ and from Cauchy's inequalities, since $\tau(z,v_z,4\kappa|\rho(z)|)\eqsim(4\kappa|\rho(z)|)^{\frac12}$, it follows that

$$\begin{aligned}&\big|(Q_2)_{z+\zeta_1^*\eta_z,v_z}\big[\alpha_{1,j}\big(\zeta_1^*\big),\dots,\alpha_{1,k}\big(\zeta_1^*\big)\big]\big|\\&\quad\lesssim\big|\rho(z)\big|^{\frac{j-k}{2}}\sup_{|\xi|=(4\kappa|\rho(z)|)^{\frac12}}\big|Q_2\big(z+\zeta_1^*\eta_z+\xi v_z\big)\big|.\end{aligned}$$

With the assumption $c_\infty^{(2)}(g)<\infty$, this gives for all $\zeta_1^*\in\Delta_0(2\kappa|\rho(z)|)$:

$$\left|h^{(2)}_{1,\dots,k}\big(\zeta_1^*\big)\right|\lesssim c_\infty^{(2)}(g)\big|\rho(z)\big|^{\frac{1-k}{2}}\sup_{|\xi|=(4\kappa|\rho(z)|)^{\frac12}}\left|Q_2\big(z+\zeta_1^*\eta_z+\xi v_z\big)\right| \tag{20}$$

and so (19) holds true.

Of course we have the analogous estimate for $\hat h_1$. Now we have to handle the error term in (18). Since there is a factor P_1P_2 in front of e in (18), we can put P_2e either with $\hat h_1$ in $\tilde h_1$ or we can put P_1e with $\hat h_2$ in $\tilde h_2$. But in order to have a good upper bound for $\tilde h_1$ and $\tilde h_2$, we have to cut it in two pieces in a suitable way. This will be done analogously to the construction of the currents. Let

$$\mathcal U_1:=\left\{\zeta\in\mathcal P_{\kappa|\rho(z)|}(z),\left|\frac{f_1(\zeta)|\rho(z)|^{\frac{i_1}{2}}}{P_1(\zeta)}\right|>\frac13\left|\frac{f_2(\zeta)|\rho(z)|^{\frac{i_2}{2}}}{P_2(\zeta)}\right|\right\},$$
$$\mathcal U_2:=\left\{\zeta\in\mathcal P_{\kappa|\rho(z)|}(z),\frac23\left|\frac{f_2(\zeta)|\rho(z)|^{\frac{i_2}{2}}}{P_2(\zeta)}\right|>\left|\frac{f_1(\zeta)|\rho(z)|^{\frac{i_1}{2}}}{P_1(\zeta)}\right|\right\}.$$

Let also χ be a smooth function on $\mathbb C^2\setminus\{0\}$ such that $\chi(z_1,z_2)=1$ if $|z_1|>\frac23|z_2|$ and $\chi(z_1,z_2)=0$ if $|z_1|<\frac13|z_2|$.

We set $\chi_1(\zeta)=\chi\big(\frac{f_1(\zeta)|\rho(z)|^{\frac{i_1}{2}}}{P_1(\zeta)},\frac{f_2(\zeta)|\rho(z)|^{\frac{i_2}{2}}}{P_2(\zeta)}\big)$, $\chi_2(\zeta)=1-\chi_1(\zeta)$ and lastly we define

$$\tilde h_1(\zeta)=\hat h_1(\zeta)+\chi_1(\zeta)P_2(\zeta)e(\zeta),$$
$$\tilde h_2(\zeta)=\hat h_2(\zeta)+\chi_2(\zeta)P_1(\zeta)e(\zeta).$$

And now we look for an upper bound for $P_1(\zeta)e(\zeta)$ on $\mathcal U_1$.

Fact 2: For all ζ belonging to $\mathcal P_{4\kappa|\rho(z)|}(z)$, we have uniformly with respect to ζ and z

$$\big|P_1(\zeta)e(\zeta)\big|\lesssim c(g)\Big(\big|\rho(z)\big|^{\frac{i_1-i_2}{2}}\sup_{\mathcal P_{4\kappa|\rho(z)|}(z)}|Q_1|+\sup_{\mathcal P_{4\kappa|\rho(z)|}(z)}|Q_2|\Big). \tag{21}$$

Proof: For $l=1$ and $l=2$, for all $i\in I_l$ and for all $\zeta_1^*\in\Delta_0(4\kappa|\rho(z)|)$ we have, from Proposition 3.3, $|\alpha_{l,i}(\zeta_1^*)|\le(3\kappa|\rho(z)|)^{\frac12}$ provided κ is small enough. Hence $|P_l(\zeta)|\lesssim|\rho(z)|^{\frac{i_l}{2}}$ for all $\zeta\in\mathcal P_{4\kappa|\rho(z)|}(z)$, and with assumption (i), we get for all $\zeta\in\mathcal P_{4\kappa|\rho(z)|}(z)$

$$\begin{aligned}\big|g(\zeta)\big|&\le c(g)\big(\big|f_1(\zeta)\big|+\big|f_2(\zeta)\big|\big)\\&\lesssim c(g)\big(\big|\rho(z)\big|^{\frac{i_1}{2}}\big|Q_1(\zeta)\big|+\big|\rho(z)\big|^{\frac{i_2}{2}}\big|Q_2(\zeta)\big|\big).\end{aligned}$$

This yields for all $\zeta\in\mathcal P_{\kappa|\rho(z)|}(z)$

$$\big|e(\zeta)\big|\lesssim c(g)\Big(\big|\rho(z)\big|^{-\frac{i_2}{2}}\sup_{\mathcal P_{4\kappa|\rho(z)|}(z)}|Q_1|+\big|\rho(z)\big|^{-\frac{i_1}{2}}\sup_{\mathcal P_{4\kappa|\rho(z)|}(z)}|Q_2|\Big)$$

from which (21) follows.

Therefore we have the identity $g=P_1\tilde h_1+P_2\tilde h_2$ and upper bounds for $\tilde h_2$ using (19) and (21), the corresponding one for $\tilde h_1$ being also true of course. But our final goal is to write g as $g=\hat g_1 f_1+\hat g_2 f_2$. So we put $\hat g_1=\frac{\tilde h_1}{Q_1}$ and $\hat g_2=\frac{\tilde h_2}{Q_2}$ so that $g=\hat g_1 f_1+\hat g_2 f_2$. Moreover, from (19) and (21), and since χ_2 has support in $\mathcal{U}_2$, it follows for $\zeta\in\mathcal{P}_{\kappa|\rho(z)|}(z)$

$$\begin{aligned}\left|\hat g_2(\zeta)\right| \leq \left(c_\infty^{(2)}(g)+c(g)\right)\frac{1}{Q_2(\zeta)}\sup_{\mathcal{P}_{4\kappa|\rho(z)|}(z)}|Q_2| \\ +c(g)\frac{1}{Q_1(\zeta)}\sup_{\mathcal{P}_{4\kappa|\rho(z)|}(z)}|Q_1|.\end{aligned}\tag{22}$$

Therefore, in order to prove that $\tilde g_2$ is bounded, we will have to prove that $\frac{Q_l(\xi)}{Q_l(\zeta)}$ is bounded for $\zeta\in\mathcal{P}_{\kappa|\rho(z)|}(z)$ and $\xi\in\mathcal{P}_{4\kappa|\rho(z)|}(z)$. This is the aim of the following Fact 3.

Fact 3: For $l=1$ and $l=2$, $\zeta\in\mathcal{P}_{2\kappa|\rho(z)|}(z)$ and $\xi\in\mathcal{P}_{4\kappa|\rho(z)|}(z)$, we have uniformly with respect to z, ζ and ξ:

$$\left|\frac{Q_l(\xi)}{Q_l(\zeta)}\right|\lesssim 1.\tag{23}$$

The proof of Fact 3 is analogous to the proof of Lemma 5.1. Without any restriction, we assume $l=2$.

First case: If $|\zeta_{0,1}^*|>4\kappa|\rho(z)|$, then we have the parametrisation of X_2 and it suffices to prove for $i\notin I_2$ that $|\frac{\xi_2^*-\alpha_{2,i}^*(\xi_1^*)}{\zeta_2^*-\alpha_{2,i}^*(\zeta_1^*)}|\lesssim 1$.

If $|\alpha_{2,i}(\xi_1^*)|\geq|\rho(z)|^{\frac12}$, since from Proposition 3.3 $\frac{\partial\alpha_{2,i}}{\partial\zeta_1^*}$ is bounded, $|\alpha_{2,i}(\zeta_1^*)|\geq\frac12|\rho(z)|^{\frac12}$ and $|\alpha_{2,i}(\zeta_1^*)|\geq\frac12|\alpha_{2,i}(\xi_1^*)|$, so $|\frac{\xi_2^*-\alpha_{2,i}^*(\xi_1^*)}{\zeta_2^*-\alpha_{2,i}^*(\zeta_1^*)}|\lesssim 1$ is satisfied.

If $|\alpha_{2,i}(\xi_1^*)|\leq|\rho(z)|^{\frac12}$, then $|\xi_2^*-\alpha_{2,i}^*(\xi_1^*)|\lesssim|\rho(z)|^{\frac12}$ and since by definition of I_2, $|\alpha_{2,i}(\zeta_1^*)|\geq\frac52\kappa|\rho(z)|^{\frac12}$ for all $\zeta_1^*\in\Delta_0(2\kappa|\rho(z)|)$, we have $|\zeta_2^*-\alpha_{2,i}(\zeta_1^*)|\gtrsim\kappa|\rho(z)|^{\frac12}$ for all $\zeta\in\mathcal{P}_{2\kappa|\rho(z)|}(z)$ and so the inequality $|\frac{\xi_2^*-\alpha_{2,i}^*(\xi_1^*)}{\zeta_2^*-\alpha_{2,i}^*(\zeta_1^*)}|\lesssim 1$ holds true.

Second case: If $|\zeta_{0,1}^*|<4\kappa|\rho(z)|$ and $|\zeta_{0,2}^*|<(4\kappa|\rho(z)|)^{\frac12}$, then $\delta(\xi,\zeta_0)\lesssim\delta(\xi,z)+\delta(z,\zeta_0)\lesssim|\rho(z)|$ and as in the proof of Lemma 5.1, $|Q_2(\xi)|=|f_2(\xi)|\lesssim|\rho(z)|^{\frac{p_2}{2}}$. From Proposition 3.2, $\mathcal{P}_{4\kappa|\rho(z)|}(z)\cap X_2=\emptyset$ so $|f_2(\zeta)|\gtrsim|\rho(\zeta)|^{\frac{p_2}{2}}$ and again we are done in this case.

Third case: If $|\zeta_{0,1}^*|<4\kappa|\rho(z)|$ and $|\zeta_{0,2}^*|\geq(4\kappa|\rho(z)|)^{\frac12}$, then as in the third case of the proof of Lemma 5.1, $f_2(\xi)$ and $f_2(\zeta)$ are comparable to $|\zeta_{0,2}^*|^{p_2}$. Again we are done in this case and Fact 3 is proved.

From (22) and (23), we get that $\hat g_2$ is uniformly bounded. However, assumption (b) of Theorem 6.2 is a little stronger and we need that the derivatives

$\frac{\partial^{\alpha+\beta+\overline{\alpha}+\overline{\beta}}\hat{g}_2}{\partial\zeta_1^{*\alpha}\,\partial\zeta_2^{*\beta}\,\partial\overline{\zeta_1^*}^{\overline{\alpha}}\,\partial\overline{\zeta_2^*}^{\overline{\beta}}}$ of $\hat{g}_2$ do not explode faster than $|\rho(z)|^{\alpha+\frac{\beta}{2}}$ is $\mathcal{P}_{\kappa|\rho(z)|}(z)$ for all $\alpha,\beta,\overline{\alpha}$ and $\overline{\beta}$.

Actually, inequality (19) and Cauchy's inequalities imply that, for all $\zeta\in\mathcal{P}_{\kappa|\rho(z)|}(z)$, $|\frac{\partial^{\alpha+\beta}\hat{h}_2}{\partial\zeta_1^{*\alpha}\,\partial\zeta_2^{*\beta}}(\zeta)|\lesssim|\rho(z)|^{-\alpha-\frac{\beta}{2}}c_\infty^{(2)}(g)\sup_{|\xi|=(4\kappa|\rho(z)|)^{\frac12}}|Q_2(z+\zeta_1^*\eta_z+\xi v_z)|$. With Lemma 5.1 and (23), we get $|\frac{\partial^{\alpha+\beta}}{\partial\zeta_1^{*\alpha}\,\partial\zeta_2^{*\beta}}(\frac{\hat{h}_2}{Q_2})|\lesssim|\rho(z)|^{-\alpha-\frac{\beta}{2}}c_\infty^{(2)}(g)$.

Applying the same process with (21) to eP_1, we get

$$\begin{aligned}&\left|\frac{\partial^{\alpha+\beta}eP_1}{\partial\zeta_1^{*\alpha}\,\partial\zeta_2^{*\beta}}(\zeta)\right|\\&\quad\lesssim\big|\rho(z)\big|^{-\alpha-\frac{\beta}{2}}c(g)\Big(\big|\rho(z)\big|^{\frac{i_1-i_2}{2}}\sup_{\mathcal{P}_{4\kappa|\rho(z)|}(z)}|Q_1|+\sup_{\mathcal{P}_{4\kappa|\rho(z)|}(z)}|Q_2|\Big).\end{aligned}$$

Again Lemma 5.1 and (23) yield

$$\left|\frac{\partial^{\alpha+\beta+\overline{\alpha}+\overline{\beta}}}{\partial\zeta_1^{*\alpha}\,\partial\zeta_2^{*\beta}\,\partial\overline{\zeta}^{\overline{\alpha}}\,\partial\overline{\zeta}^{\overline{\beta}}}(\zeta)\left(\chi_2\frac{eP_1}{Q_2}\right)\right|\lesssim\big|\rho(z)\big|^{-\alpha-\overline{\zeta}-\frac{\beta+\overline{\beta}}{2}}c(g).$$

Therefore, $\hat{g}_2$ satisfies (b) of Theorem 6.2 and of course, $\hat{g}_1$ also does. □

6.3. The L^q-case. The assumption, under which a function g holomorphic on D can be written as $g=g_1f_1+g_2f_2$ with g_1 and g_2 being holomorphic on D and belonging to $L^q(D)$, uses a κ-covering $(\mathcal{P}_{\kappa|\rho(z_j)|}(z_j))_{j\in\mathbb{N}}$ in addition to the divided differences.

By transversality of X_1 and bD, and of X_2 and bD, for all j there exists w_j in the complex tangent plane to $bD_{\rho(z_j)}$ such that π_j, the orthogonal projection on the hyperplane orthogonal to w_j passing through z_j, is a covering of X_1 and X_2. We denote by $w_1^*,\dots,w_n^*$ an orthonormal basis of $\mathbb{C}^n$ such that $w_1^*=\eta_{z_j}$ and $w_n^*=w_j$ and we set $\mathcal{P}'_\varepsilon(z_j)=\{z'=z_j+z_1^*w_1^*+\dots+z_{n-1}^*w_{n-1}^*, |z_1^*|<\varepsilon$ and $|z_k^*|<\varepsilon^{\frac12}, k=2,\dots,n-1\}$. We put

$$\begin{aligned}&c^{(l)}_{q,\kappa,(z_j)_{j\in\mathbb{N}}}(g)\\&\quad=\sum_{j=0}^{\infty}\int_{z'\in\mathcal{P}'_{2\kappa|\rho(z_j)|}(z_j)}\sum_{\substack{\lambda_1,\dots,\lambda_k\in\Lambda_{z',w_n^*}\\ \lambda_i\neq\lambda_l\text{ for }i\neq l}}\big|\rho(z_j)\big|^{q\frac{k-1}{2}+1}\\&\qquad\times\big|g^{(l)}_{z',w_n^*}[\lambda_1,\dots,\lambda_k]\big|^q\,dV_{n-1}(z'),\end{aligned}$$

where dV_{n-1} is the Lebesgue measure in $\mathbb{C}^{n-1}$ and $g^{(l)}=\frac{g}{f_l}$, $l=1$ or $l=2$.

Now we prove the following necessary conditions:

Theorem 6.4. *Let g_1 and g_2 belonging to $L^q(D)$, $q\in[1,+\infty[$, be two holomorphic functions on D and set $g=g_1f_1+g_2f_2$. Then*

(i) $\frac{g}{\max(|f_1|,|f_2|)}$ *belongs to* $L^q(D)$ *and*

$$\left\|\frac{g}{\max(|f_1|,|f_2|)}\right\|_{L^q(D)} \lesssim \max\big(\|g_1\|_{L^q(D)}, \|g_2\|_{L^q(D)}\big).$$

(ii) *For* $l=1$ *or* $l=2$ *and any* κ*-covering* $(\mathcal{P}_{\kappa|\rho(z_j)|}(z_j))_j$*, we have*

$$c^{(l)}_{q,\kappa,(z_j)_j}(g) \lesssim \|g_l\|^q_{L^q(D)}.$$

Proof. The point (i) is trivial and we only prove (ii). As in the proof of Theorem 1.1, for all $j\in\mathbb{N}$, all $z'\in\mathcal{P}'_{\kappa|\rho(z_j)|}(z_j)$ and all $r\in[\frac{7}{2}\kappa|\rho(z_j)|^{\frac12}, 4\kappa|\rho(z_j)|^{\frac12}]$, we have

$$g^{(l)}_{z',w_n^*}[\lambda_1,\dots,\lambda_k] = \frac{1}{2i\pi}\int_{|\lambda|=r}\frac{g_l(z'+\lambda w_n^*)}{\prod_{i=1}^k(\lambda-\lambda_i)}\,d\lambda.$$

After integration for $r\in[(7/2\kappa|\rho(z_j)|)^{\frac12},(4\kappa|\rho(z_j)|)^{\frac12}]$, Jensen's inequality yields

$$\big|g^{(l)}_{z',w_n^*}[\lambda_1,\dots,\lambda_k]\big|^q \lesssim \big|\rho(z_j)\big|^{\frac{1-k}{2}q-1}\int_{|\lambda|\le(4\kappa|\rho(z_j)|)^{\frac12}}\big|g_l\big(z'+\lambda w_n^*\big)\big|^q\,dV_1(\lambda).$$

Now we integrate the former inequality for $z'\in\mathcal{P}'_{\kappa|\rho(z_j)|}(z_j)$ and get

$$\begin{aligned}&\int_{z'\in\mathcal{P}'_{\kappa|\rho(z_j)|}(z_j)}\big|g^{(l)}_{z',w_n^*}[\lambda_1,\dots,\lambda_k]\big|^q\big|\rho(z_j)\big|^{\frac{k-1}{2}q+1}\,dV_{n-1}(z)\\ &\quad\lesssim\int_{z\in\mathcal{P}_{4\kappa|\rho(z_j)|}(z_j)}\big|g_l(z)\big|^q\,dV_n(z).\end{aligned}$$

Since $(\mathcal{P}_{\kappa|\rho(z_j)|}(z_j))_{j\in\mathbb{N}}$ is a κ-covering, we deduce from this inequality that $c^{(l)}_{q,\kappa,(z_j)_{j\in\mathbb{N}}}(g)\lesssim\|g_l\|^q_{L^q(D)}$. $\square$

THEOREM 6.5. *Let* g *be a holomorphic function on* D *belonging to the ideal generated by* f_1 *and* f_2*, such that* $c^{(l)}_{q,\kappa,(z_j)_j}(g)$ *is finite and such that* $\frac{g}{\max(|f_1|,|f_2|)}$ *belongs to* $L^q(D)$*.*

Then there exist two holomorphic functions g_1 *and* g_2 *which belong to* $L^q(D)$ *and such that* $g=g_1f_1+g_2f_2$*.*

Proof. We aim to apply Theorem 6.1. For all j in $\mathbb{N}$, in order to construct on $\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$ two functions $\tilde g_1^{(j)}$ and $\tilde g_2^{(j)}$ which satisfy the assumption of Theorem 6.1, we proceed as in the proof of Theorem 1.2. The main difficulty occurs, as in the proof of Theorem 1.2, when we are near a point ζ_0 which belongs to $X_1\cap X_2\cap bD$. We denote by $(\zeta^*_{0,1},\zeta^*_{0,2})$ the coordinates of ζ_0 in the Koranyi coordinates at z_j. If $|\zeta^*_{0,1}|<4\kappa|\rho(z_{j_0})|$, we set $i_{1,j}=i_{2,j}=0$, $I_{1,j}=I_{2,j}=\emptyset$, $P_{1,j}=P_{2,j}=1$, $Q_{1,j}=f_1$ and $Q_{2,j}=f_2$. Otherwise, we use the parametrisation $\alpha^{(j)}_{1,i}$, $i\in\{1,\dots,p_1^{(j)}\}$ of X_1 and $\alpha^{(j)}_{2,i}$, $i\in\{1,\dots,p_2^{(j)}\}$ of X_2 given by Proposition 2.2 and for $l=1$ and $l=2$, we still denote by $I_{l,j}$ the set

$I_{l,j}=\{i, \exists z_1^*\in\Delta_0(2\kappa|\rho(z_j)|)$ such that $|\alpha_{l,i}^{(j)}(z_1^*)|\leq(\frac{5}{2}\kappa|\rho(z_j)|)^{\frac12}\}$, $i_{l,j}=\#I_{l,j}$, $P_{l,j}(\zeta)=\prod_{i\in I_{l,j}}(\zeta_2^*-\alpha_{l,i}^{(j)}(\zeta_1^*))$ and $Q_{l,j}=\frac{f_l}{P_{l,j}}$. We define $\hat h_1^{(j)}$ and $\hat h_2^{(j)}$ as $\hat h_1$ and $\hat h_2$ in the proof of Theorem 1.2. Instead of defining $e_1^{(j)}$ and $e_2^{(j)}$ by integrals over the set $\{|\xi|=(4\kappa|\rho(z_j)|)^{\frac12}\}$ as we defined e_1 and e_2 in the proof of Theorem 1.2, here we integrate over $\{(\frac72\kappa|\rho(z_j)|)^{\frac12}\leq|\xi|\leq(4\kappa|\rho(z_j)|)^{\frac12}\}$ and set

$$e^{(j)}(z)=\frac{1}{2\pi(2-\sqrt{\frac72})\sqrt{\kappa|\rho(z_j)|}}\cdot\int_{\{(\frac72\kappa|\rho(z_j)|)^{\frac12}\leq|\xi|\leq(4\kappa|\rho(z_j)|)^{\frac12}\}}\frac{g(z_1^*,\xi)}{P_{1,j}(z_1^*,\xi)P_{2,j}(z_1^*,\xi)(z_2^*-\xi)}\,dV(\xi).$$

We therefore have for all j and all $z\in\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$:

$$g(z)=\tilde h_1^{(j)}(z)P_{1,j}(z)+\tilde h_2^{(j)}(z)P_{2,j}(z)+P_{1,j}(z)P_{2,j}(z)e^{(j)}(z).$$

We split $\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$ in two parts as in Theorem 1.2 and set

$$\mathcal{U}_1^{(j)}:=\left\{\zeta\in\mathcal{P}_{\kappa|\rho(z_j)|}(z_j),\left|\frac{f_1(\zeta)|\rho(z_j)|^{\frac{i_{1,j}}{2}}}{P_{1,j}(\zeta)}\right|>\frac13\left|\frac{f_2(\zeta)|\rho(z_j)|^{\frac{i_{2,j}}{2}}}{P_2(\zeta)}\right|\right\},$$
$$\mathcal{U}_2^{(j)}:=\left\{\zeta\in\mathcal{P}_{\kappa|\rho(z_j)|}(z_j),\frac23\left|\frac{f_2(\zeta)|\rho(z_j)|^{\frac{i_{2,j}}{2}}}{P_{2,j}(\zeta)}\right|>\left|\frac{f_1(\zeta)|\rho(z_j)|^{\frac{i_{1,j}}{2}}}{P_{1,j}(\zeta)}\right|\right\}.$$

We still denote by χ a smooth function on $\mathbb{C}^2\setminus\{0\}$ such that $\chi(z_1,z_2)=1$ if $|z_1|>\frac23|z_2|$ and $\chi(z_1,z_2)=0$ if $|z_1|<\frac13|z_2|$; and we set $\chi_1^{(j)}(\zeta)=\chi(\frac{f_1(\zeta)|\rho(z_j)|^{\frac{i_{1,j}}{2}}}{P_1^{(j)}(\zeta)},\frac{f_2(\zeta)|\rho(z_j)|^{\frac{i_{2,j}}{2}}}{P_2^{(j)}(\zeta)})$, $\chi_2^{(j)}(\zeta)=1-\chi_1^{(j)}(\zeta)$ and

$$\tilde g_1^{(j)}(z)=\frac{1}{Q_1^{(j)}(z)}\big(\hat h_1^{(j)}(z)+\chi_1^{(j)}(z)P_{2,j}(z)e^{(j)}(z)\big),$$
$$\tilde g_2^{(j)}(z)=\frac{1}{Q_2^{(j)}(z)}\big(\hat h_2^{(j)}(z)+\chi_2^{(j)}(z)P_{1,j}(z)e^{(j)}(z)\big).$$

Therefore $g=\tilde g_1^{(j)}f_1+\tilde g_2^{(j)}f_2$ on $\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$ and in order to apply Theorem 6.1, the assumptions (b) and (c) are left to be shown.

As in the proof of Fact 1, it follows from Lemma 6.3 and (23) that

$$\left|\frac{1}{Q_{2,j}(z)}\hat h_2^{(j)}(z)\right|\lesssim\sum_{k=1}^{i_{2,j}}|\rho(z_j)|^{\frac{k-1}{2}}\left|g^{(2)}_{z_j+z_1^*\eta_{z_j},v_{z_j}}\left[\alpha_{1,1}(z_1^*),\ldots,\alpha_{1,k}(z_1^*)\right]\right|$$

uniformly with respect to $z\in\mathcal{P}_{2\kappa|\rho(z_j)|}(z_j)$ and $j\in\mathbb{N}$ and therefore

$$\sum_{j\in\mathbb{N}}\int_{\mathcal{P}_{2\kappa|\rho(z_j)|}(z_j)}\left|\frac{1}{Q_{2,j}(z)}\hat h_2^{(j)}(z)\right|^q dV(z)\lesssim c^{(l)}_{q,\kappa,(z_j)}(g).\tag{24}$$

In particular $\frac{\hat{h}_2^{(j)}}{Q_{2,j}}$ is an holomorphic function with L^q-norm on $\mathcal{P}_{2\kappa|\rho(z_j)|}(z_j)$ lower than $(c^{(2)}_{q,\kappa,(z_j)}(g))^{\frac{1}{q}}$. Thus Cauchy's inequalities imply, for all $\alpha,\beta\in\mathbb{N}$ and all $z\in\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$, that

$$\left|\frac{\partial^{\alpha+\beta}}{\partial z_1^{*\alpha}\,\partial z_2^{*\beta}}\left(\frac{1}{Q_{2,j}}\hat{h}_2^{(j)}(z)\right)\right|\lesssim\left(c^{(l)}_{q,\kappa,(z_j)}(g)\right)^{\frac{1}{q}}\left|\rho(z_j)\right|^{-\frac{3}{q}-\alpha-\frac{\beta}{2}}. \tag{25}$$

Since $\frac{g}{\max(|f_1|,|f_2|)}$ belongs to $L^q(D)$, g itself belongs to $L^q(D)$ and so

$$\int_{\mathcal{P}_{2\kappa|\rho(z_j)|}(z_j)}\left|e^{(j)}(z)\right|^q dV(z)\lesssim\left|\rho(z_j)\right|^{-q\frac{i_{1,j}+i_{2,j}}{2}}\int_{\mathcal{P}_{4\kappa|\rho(z_j)|}(z_j)}\left|g(z)\right|^q dV(z).$$

In particular, for all α and β and all $z\in\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$, we have

$$\left|\frac{\partial^{\alpha+\beta}e^{(j)}}{\partial z_1^{*\alpha}\,\partial z_2^{*\beta}}(z)\right|\lesssim\left|\rho(z_j)\right|^{-\frac{3}{q}-\frac{i_{1,j}+i_{2,j}}{2}-\alpha-\frac{\beta}{2}}. \tag{26}$$

The inequalities (25) and (26) imply that the hypothesis (c) of Theorem 6.1 is satisfied by $\tilde{g}_2^{(j)}$ for some large N, the same is also true for $\tilde{g}_1^{(j)}$.

Now, for z belonging to $\mathcal{U}_2^{(j)}$, we get from (23):

$$\begin{aligned}&\left|\frac{P_1^{(j)}(z)e^{(j)}(z)}{Q_2^{(j)}(z)}\right|\\ &\quad\lesssim\frac{1}{|\rho(z_j)|}\int_{(\frac{7}{2}\kappa|\rho(z_j)|)^{1/2}\leq|\xi|\leq(4\kappa|\rho(z_j)|)^{\frac{1}{2}}}\frac{|g(\zeta_1^*,\xi)|}{\max(|f_1(\zeta_1^*,\xi)|,|f_2(\zeta_1^*,\xi)|)}\,dV(\xi)\end{aligned}$$

and so

$$\begin{aligned}&\int_{\mathcal{U}_2\cap\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)}\left|\frac{P_1^{(j)}(z)e^{(j)}(z)}{Q_2^{(j)}(z)}\right|^q dV(z)\\ &\quad\lesssim\int_{\mathcal{P}_{4\kappa|\rho(z_j)|}(z_j)}\left(\frac{|g(\zeta_1^*,\xi)|}{\max(|f_1(\zeta_1^*,\xi)|,|f_2(\zeta_1^*,\xi)|)}\right)^q dV(\xi).\end{aligned}$$

Since $(\mathcal{P}_{\kappa|\rho(z_j)|}(z_j))_{j\in\mathbb{N}}$ is a κ-covering, this yields:

$$\sum_{j\in\mathbb{N}}\int_{\mathcal{U}_2\cap\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)}\left|\frac{P_1^{(j)}(z)e^{(j)}(z)}{Q_2^{(j)}(z)}\right|^q dV(z)\lesssim\left\|\frac{g}{\max(|f_1|,|f_2|)}\right\|^q_{L^q(D)}. \tag{27}$$

Moreover, for all $\alpha,\beta\in\mathbb{N}$, $|\frac{\partial^{\alpha+\beta}\chi_2^{(j)}}{\partial\overline{\zeta_1^*}^{\alpha}\,\partial\overline{\zeta_2^*}^{\beta}}(z)|\lesssim|\rho(z_j)|^{-\alpha-\frac{\beta}{2}}$, (24) and (27) imply that $(\tilde{g}_2^{(j)})_{j\in\mathbb{N}}$ satisfy the assumption (b) of Theorem 6.1 that we can therefore apply. □

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