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EICHLER INTEGRALS FOR MAASS CUSP FORMS OF HALF-INTEGRAL WEIGHT

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ABSTRACT. In this paper, we define and discuss Eichler integrals for Maass cusp forms of half-integral weight on the full modular group. We discuss nearly periodic functions associated to the Eichler integrals, introduce period functions for such Maass cusp forms, and show that the nearly periodic functions and the period functions are closely related. Those functions are extensions of the periodic functions and period functions for Maass cusp forms of weight 0 on the full modular group introduced by Lewis and Zagier.

1. Introduction

Recall that modular cusp forms of weight $k \in 2\mathbb{N}$ (for the group $\mathrm{SL}(2,\mathbb{Z})$) are holomorphic functions u_{h} from the upper half-plane $\mathbb{H} = \{z = x + iy; x, y \in \mathbb{R}, y > 0\}$ to $\mathbb{C}$, satisfying the $u_{\mathrm{h}}(z+1) = u_{\mathrm{h}}(z)$ and $u_{\mathrm{h}}(-1/z) = z^k u_{\mathrm{h}}(z)$, and vanish as $y \to \infty$. More details can be found in, for example, [10] and [15].

In the context of the Eichler–Shimura theorem, we attach to each modular cusp form a polynomial p of degree $\leq k-2$. One way to define it is by the following integral transformation:

$$p(\zeta) := \int_0^{i\infty} (\zeta - z)^{k-2} u_{\mathrm{h}}(z)\,\mathrm{d}z \quad (\zeta \in \mathbb{C}). \tag{1.1}$$

This integral transformation goes back to Eichler in [6]. One important property is that each period polynomial satisfies the identities

$$\begin{aligned} p(\zeta) + \zeta^{k-2} p\left(\frac{-1}{\zeta}\right) &= 0 \quad \text{and} \\ p(\zeta) + (\zeta+1)^{k-2} p\left(\frac{-1}{\zeta+1}\right) + \zeta^{k-2} p\left(\frac{-\zeta-1}{\zeta}\right) &= 0 \end{aligned} \tag{1.2}$$

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for each $\zeta \in \mathbb{C}$. Some more details on the Eichler–Shimura theorem can be found in for example, [10, Chapters V and VI], [9, Section 1.1] and [14].

There exists another way to define the period polynomial, involving a variant of the above integral. Consider the integral transformation

$$f_{\mathrm{h}}(\zeta) := \int_{\zeta}^{i\infty} (\zeta - z)^{k-2} u_{\mathrm{h}}(z)\,\mathrm{d}z \quad (\zeta \in \mathbb{H}), \tag{1.3}$$

defined only on the upper half-plane $\mathbb{H}$. f_{h} is obviously a holomorphic function; and it easily seen that f_{h} is periodic. The period polynomial p now appears in the calculation

$$\begin{aligned} f_{\mathrm{h}}(\zeta) - \zeta^{k-2} f_{\mathrm{h}}\left(\frac{-1}{\zeta}\right) &= \int_{\zeta}^{i\infty} (\zeta - z)^{k-2} u_{\mathrm{h}}(z)\,\mathrm{d}z - \int_{\zeta}^{0} (\zeta - z)^{k-2} u_{\mathrm{h}}(z)\,\mathrm{d}z \\ &= \int_{0}^{i\infty} (\zeta - z)^{k-2} u_{\mathrm{h}}(z)\,\mathrm{d}z = p(\zeta) \quad (\zeta \in \mathbb{H}). \end{aligned}$$

One important extension of the Eichler–Shimura isomorphism was done by Lewis and Zagier in [11]. They found a one-to-one correspondence between Maass cusp forms of weight 0 (for $\mathrm{SL}(2,\mathbb{Z})$) and functions called period functions.

Part of their main result is the following theorem.

THEOREM ([11]). *Let s be a complex number with $\mathrm{Re}(s) = \frac{1}{2}$. There is an isomorphism between the following three function spaces*:

(1) *The space of Maass cusp forms of weight* 0 *with eigenvalue $s(1-s)$ for* $\mathrm{SL}(2,\mathbb{Z})$: Maass cusp forms *of* weight 0 *for* $\mathrm{SL}(2,\mathbb{Z})$ *are real-analytic functions $u : \mathbb{H} \to \mathbb{C}$, satisfying the transformation properties $u(z+1) = u(z)$ and $u(-1/z) = u(z)$, are eigenfunctions of the hyperbolic Laplacian $\Delta_0 = -y^2(\partial_x^2 + \partial_y^2)$ with eigenvalue $s(1-s)$, and vanish as $y \to \infty$.*
(2) *The space of holomorphic functions f on $\mathbb{C} \setminus \mathbb{R}$, satisfying $f(z+1) = f(z)$ and bounded by $|\mathrm{Im}(z)|^{-A}$ for some $A > 0$, such that the function $f(z) - z^{-2s} f(-1/z)$ extends holomorphically across the positive real axis and is bounded by a multiple of* $\min\{1, |z|^{-1}\}$ *in the right half-plane.*
(3) *The space of holomorphic solutions $\psi : \mathbb{C}' \to \mathbb{C}$ of the three-term functional equation*

$$\psi(\zeta) = \psi(\zeta+1) + (\zeta+1)^{2s} \psi\left(\frac{\zeta}{\zeta+1}\right)$$

on $\mathbb{C}' = \mathbb{C} \setminus \mathbb{R}_{\le 0}$ which satisfy the growth condition

$$\psi(\zeta) = \begin{cases} \mathcal{O}(\frac{1}{|\zeta|}) & \text{as } \zeta \to \infty,\ \mathrm{Re}(\zeta > 0) \text{ and} \\ \mathcal{O}(1) & \text{as } \zeta \to 0,\ \mathrm{Re}(\zeta > 0). \end{cases}$$

In analogy to the case of modular cusp forms, Lewis and Zagier also introduce two integral transformations in [11, Chapter II] from Maass cusp forms to period functions (which are the functions ψ above) and the periodic holomorphic functions f on $\mathbb{C} \setminus \mathbb{R}$.

In this paper, we extend the integral transformations of [11] to the context of Maass forms of half-integral weights with a multiplier system. Our main result is the following theorem.

THEOREM 1.1. *Let ν be a purely imaginary complex number, i.e. $\nu \in i\mathbb{R}$, $k \in \frac{1}{2}\mathbb{Z}$ a weight and v a compatible multiplier as defined in Section 2.2.*

For each Maass cusp form u of weight k, eigenvalue $\frac{1}{4} - \nu^2$ and multiplier v for $\mathrm{SL}(2, \mathbb{Z})$, see Definition 3.1 for details, there exist the following functions:

(1) *A holomorphic function $f : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$, which is nearly periodic, that is, $f(z+1) = af(z)$ for some $a \in \mathbb{C}$ with $|a| = 1$, such that $f(z) - v\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right) z^{2\nu-1} f(-1/z)$ extends holomorphically across the positive real axis and is bounded by a multiple of $\min\{1, |z|^{-1}\}$ in the right half-plane.*

(2) *A holomorphic solution $P : \mathbb{C}' \to \mathbb{C}$ of the three-term functional equation*

$$P(\zeta) = v\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)^{-1} P(\zeta+1) + v\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right)^{-1} (\zeta+1)^{2\nu-1} P\left(\frac{\zeta}{\zeta+1}\right)$$

on $\mathbb{C}'$ which satisfies the growth condition

$$P(\zeta) = \begin{cases} \mathcal{O}\left(\frac{1}{|\zeta|}\right) & \textit{as } \zeta \to \infty,\ \mathrm{Re}(\zeta > 0) \textit{ and} \\ \mathcal{O}(1) & \textit{as } \zeta \to 0,\ \mathrm{Re}(\zeta > 0). \end{cases}$$

We call such a function P a period function.

The proof of this theorem follows ideas presented above for modular cusp forms for positive even weight: We use a Maass–Selberg differential form to define the kernel of integral transformations similarly as $(\zeta - z)^{k-2} u_{\mathrm{h}}(z)\, \mathrm{d}z$ is used above. We then describe properties of the determined nearly periodic functions and period functions, that is, the images of our integral transformations. Summarizing, we introduce the arrows of the following diagram and show that the diagram commutes:

$$\begin{tikzcd}
\text{Maass cusp forms of half-integer weight} \arrow[d] \arrow[dr] & \\
\text{Nearly periodic functions} \arrow[r, leftrightarrow] & \text{Period functions}
\end{tikzcd}$$

It is our hope that the results of this paper form a first step towards a working Eichler–Shimura theory for Maass cusp forms of half-integral weight (for $\mathrm{SL}(2, \mathbb{Z})$) since we establish one direction of a possible bijection between Maass cusp forms and period functions. We will discuss this and some other related questions briefly in Section 9.

The paper is organized as follows: Section 2 contains the preliminaries, like defining properly the group $\mathrm{SL}(2,\mathbb{Z})$ and its linear fractional transformations, the multiplier systems and the slash and double-slash notations. In Section 3, we define the Maass cusp forms for half-integral weight. Section 4 introduces the R-function and the Maass–Selberg form. Sections 5 and 6, contain the definitions of the integral transformations from Maass cusp forms to nearly periodic functions on one hand and to period functions on the other hand. These sections contain also our main result in a more detailed version. In Section 7, we collect those results and prove Theorem 1.1. We use Section 8, to compare and relate our integral transforms to the one appearing in the setting of the classical modular cusp forms. The remaining Section 9 contains a short discussion and outlook.

2. Preliminaries

2.1. The matrix group $\mathrm{SL}(2,\mathbb{Z})$ and its linear fractional transformations. Let $\mathrm{SL}(2,\mathbb{R})$ denote the group of 2×2 matrices with real entries and determinant 1. The subgroup $\mathrm{SL}(2,\mathbb{Z})\subset \mathrm{SL}(2,\mathbb{R})$ denotes the *full modular group*, that is the subgroup of matrices with integer entries. It is generated by

$$S=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \quad\text{and}\quad T=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}. \tag{2.1}$$

These satisfy

$$S^2=(ST)^3=(-\mathbf{1}), \tag{2.2}$$

where $(-\mathbf{1})\in \mathrm{SL}(2,\mathbb{Z})$ is the matrix with -1 on the diagonal and 0 on the off-diagonal entries. $\mathbf{1}$ denotes the identity matrix. We denote

$$T':=TST=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}. \tag{2.3}$$

Note that

$$ST=\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix} \quad\text{and}\quad STST=T^{-1}S=\begin{pmatrix}-1 & -1\\ 1 & 0\end{pmatrix}. \tag{2.4}$$

The group $\mathrm{SL}(2,\mathbb{R})$ acts on the upper half-plane $\mathbb{H}=\{z\in\mathbb{C};\mathrm{Im}(z)>0\}$ and its boundary $\mathbb{P}_{\mathbb{R}}=\mathbb{R}\cup\{\infty\}$ and the lower half-plane $\mathbb{H}^-=\{z\in\mathbb{C};\mathrm{Im}(z)<0\}$ by *fractional linear transformations*

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} z:=\begin{cases}\frac{a}{c} & \text{if } z=\infty,\\ \infty & \text{if } z=-\frac{d}{c} \text{ with } c\neq 0, \text{ and}\\ \frac{az+b}{cz+d} & \text{otherwise.}\end{cases} \tag{2.5}$$

We also need the following μ-function:

$$\mu:\ \mathrm{SL}(2,\mathbb{R})\times\mathbb{C}\to\mathbb{C};\quad \mu\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix},z\right):=cz+d. \tag{2.6}$$

Obviously μ satisfies the cocycle-relation

$$\mu(\gamma\delta, z) = \mu(\gamma, \delta z)\mu(\delta, z)$$

for every $\gamma, \delta \in \mathrm{SL}(2,\mathbb{R})$.

Moreover, we have

$$\begin{aligned} &\mathrm{Im}(\gamma z) = \frac{\mathrm{Im}(z)}{|\mu(\gamma,z)|^2}, \qquad \frac{\mathrm{d}}{\mathrm{d}z}\gamma z = \frac{1}{(\mu(\gamma,z))^2}, \\ &\frac{\mathrm{d}}{\mathrm{d}\bar{z}}\gamma\bar{z} = \frac{1}{(\mu(\gamma,\bar{z}))^2} \quad \text{and} \quad \gamma\zeta - \gamma z = \frac{\zeta - z}{\mu(\gamma,\zeta)\mu(\gamma,z)} \end{aligned} \tag{2.7}$$

for every $\gamma \in \mathrm{SL}(2,\mathbb{R})$.

2.2. Multiplier systems. We call a function $v :\ \mathrm{SL}(2,\mathbb{Z}) \to \mathbb{C}_{\neq 0}$ *multiplier* or *multiplier system* compatible with the half-integral *weight* k if v satisfies

$$v(\gamma\delta)e^{ik\arg(\mu(\gamma\delta,z))} = v(\gamma)v(\delta)e^{ik\arg(\mu(\gamma,\delta z))}e^{ik\arg(\mu(\delta,z))} \tag{2.8}$$

for every $\gamma, \delta \in \mathrm{SL}(2,\mathbb{Z})$ and $z \in \mathbb{H}$.

REMARK 2.1. (1) The range of $\arg(\cdot)$ is $-\pi < \arg(z) \leq \pi$ for all $z \in \mathbb{C}_{\neq 0}$.
(2) Condition (2.8) implies that the system of equations

$$f(\gamma z) = v(\gamma)e^{ik\arg(\mu(\gamma,z))}f(z) \quad \big(z \in \mathbb{H}, \gamma \in \mathrm{SL}(2,\mathbb{Z})\big), \tag{2.9}$$

allows non-zero solutions $f : \mathbb{H} \to \mathbb{C}$.
(3) We have in particular

$$v\big((-\mathbf{1})\big) = e^{-ik\pi}, \tag{2.10}$$

since (2.9) with $\gamma = (-\mathbf{1})$ implies $v((-\mathbf{1}))e^{ik\arg(-1)} = 1$, if f does not vanish everywhere.

2.3. The slash and double-slash notations. We define arbitrary powers z^s with (possibly complex) s by using the standard branch of the logarithm: $z^s = |z|^s e^{is\arg(z)}$ with $\arg(z) \in (-\pi, \pi]$ for every $z \in \mathbb{C}_{\neq 0}$.

We introduce the *slash* and the *double-slash* notations. Let $k \in \frac{1}{2}\mathbb{Z}$, $\nu \in \mathbb{C}$ and v be a multiplier. For $f : \mathbb{H} \to \mathbb{C}$ and $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, we define

$$\begin{aligned} &\big(f|_k^v\gamma\big)(z) := e^{-ik\arg(\mu(\gamma,z))}v(\gamma)^{-1}f(\gamma z) \quad \text{and} \\ &\big(f\|_\nu^v\gamma\big)(z) := v(\gamma)^{-1}\big(\mu(\gamma,z)\big)^{2\nu-1}f(\gamma z) \end{aligned} \tag{2.11}$$

for every $z \in \mathbb{H}$. For example (2.9), reads as $f|_k^v\gamma = f$.

We define the slash and double-slash notations also for functions $f : \mathbb{H}^- \to \mathbb{C}$ on the lower half-plane, since $\gamma z \in \mathbb{H}^-$ in (2.5) for every $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ and $z \in \mathbb{H}^-$.

As slight abuse of notation, we also use the slash notation

$$\big(f|_k^1\gamma\big)(z) = e^{-ik\arg(\mu(\gamma,z))}f(\gamma z) \tag{2.12}$$

for matrices $\gamma \in \mathrm{SL}(2,\mathbb{R})$.

Consider the subset $\mathrm{SL}(2,\mathbb{Z})^+ \subset \mathrm{SL}(2,\mathbb{Z})$, containing all matrices $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ with only non-negative entries, that is, all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ satisfying $a,b,c,d \geq 0$. These matrices have the property that they map the cut-plane $\mathbb{C}' = \mathbb{C} \setminus (-\infty, 0]$ into itself: for every $z \in \mathbb{C}'$ and $\gamma \in \mathrm{SL}(2,\mathbb{Z})^+$ we have $\gamma z \in \mathbb{C}'$. The slash and double-slash notations in (2.11) are also well defined for functions $f : \mathbb{C}' \to \mathbb{H}$ and all $\gamma \in \mathrm{SL}(2,\mathbb{Z})^+$. For given real z, we may even extend the slash and double-slash notations to certain matrices $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ which satisfy $\mu(\gamma, z) > 0$ on occasion.

Remark 2.2. The slash notation in (2.11) can be viewed as a group operation of $\mathrm{SL}(2,\mathbb{Z})$ on the space of functions on the upper half-plane. Indeed, relation (2.8) implies that $f|_k^v(\gamma\delta) = (f|_k^v\gamma)|_k^v\delta$ holds.

The double-slash notation in (2.11) is, as the name indicates, an abbreviation for the given expression. It is not a group operation in general. The same is true for the slash-notation except in the case mentioned above.

3. Maass cusp forms of half-integral weight and Maass operators

In this section, we define Maass cusp form and useful Maass operators that will be used later. As usual, we write $z = x + iy$ for complex z with real part x and imaginary part y.

Definition 3.1. Let k be a half-integral weight and $v :\ \mathrm{SL}(2,\mathbb{Z}) \to \mathbb{C}_{\neq 0}$ a compatible multiplier system. A *Maass cusp form* of weight k and multiplier v for $\mathrm{SL}(2,\mathbb{Z})$ is a real-analytic function $u : \mathbb{H} \to \mathbb{C}$ satisfying

(1) $u|_k^v\gamma = u$ for every $\gamma \in \mathrm{SL}(2,\mathbb{Z})$,

(2) u is an eigenfunction of the Laplace operator Δ_k with eigenvalue λ, that is, $\Delta_k u = \lambda u$ where

$$\Delta_k = -y^2\left(\partial_x^2 + \partial_y^2\right) + iky\partial_x \tag{3.1}$$

with $z = x + iy \in \mathbb{H}$,

(3) u satisfies the growth condition $u(z) = \mathcal{O}(y^c)$ as $y \to \infty$ for every $c \in \mathbb{R}$.

It is known, see, for example [2], that the eigenvalue λ is real. It is convenient to write $\lambda = \frac{1}{4} - \nu^2$ with suitable *spectral parameter* $\nu \in \mathbb{R} \cup i\mathbb{R}$.

In the following lemma, we extend the definition of the Maass cusp form to the lower half-plane by considering the conjugate of the form defined on the upper half-plane.

Lemma 3.2. *Let u be a Maass cusp form of weight k, multiplier system v and eigenvalue λ. Defining $\tilde{u} : \mathbb{H}^- \to \mathbb{C};\ z \mapsto \tilde{u}(z) := u(\bar{z})$ for a Maass cusp form u defines a real-analytic function on the lower half-plane which satisfies*

(1) *$\tilde{u}|_{-k}^v\gamma = \tilde{u}$ for every $\gamma \in \mathrm{SL}(2,\mathbb{Z})$,*

(2) *$\tilde{u}$ is an eigenfunction of the Laplace operator Δ_{-k} with eigenvalue λ, and*

(3) *$\tilde{u}$ satisfies the growth condition $\tilde{u}(z) = \mathcal{O}(|y|^c)$ as $y \to -\infty$ for every $c \in \mathbb{R}$.*

Proof. Using the identity $\arg(\bar\zeta) = -\arg(\zeta)$, $\zeta \in \mathbb{C} \setminus (-\infty, 0]$, the transformation property follows immediately:

$$\tilde u(\gamma z) = u(\gamma \bar z) = e^{ik \arg(\mu(\gamma,\bar z))} v(\gamma) u(\bar z) = e^{i(-k)\arg(\mu(\gamma,z))} v(\gamma)\tilde u(z)$$

for every $z \in \mathbb{H}^-$ and $\gamma \in \mathrm{SL}(2,\mathbb{Z})$. The substitution $y \mapsto -y$ (i.e. $z \mapsto \bar z$) gives

$$\begin{aligned}
&\Delta_{-k}\tilde u(z)\\
&\quad = \left[-y^2\left(\partial_x^2 + \partial_y^2\right) + i(-k)y\partial_x\right] u(\bar z)\\
&\quad = \left[-(-y)^2\left(\partial_x^2 + (-1)^2\partial_y^2\right) + i(-k)(-y)\partial_x\right] u(z) \quad \text{(using } y \mapsto -y\text{)}\\
&\quad = \Delta_k u(z) = \lambda u(z) = \lambda u(\bar z) \quad \text{(using } y \mapsto -y\text{)}\\
&\quad = \lambda \tilde u(z)
\end{aligned}$$

for $z \in \mathbb{H}^-$. This shows the second property. The growth condition follows directly from the definition $\tilde u(z) = u(\bar z)$. □

The raising and lowering *Maass operators* acting on the space of cusp forms of given weight k, multiplier v, and eigenvalue λ are given by

$$\mathrm{E}_k^{\pm} = \pm 2iy\partial_x + 2y\partial_y \pm k. \tag{3.2}$$

Equivalently, it is sometimes convenient to write

$$\mathrm{E}_k^{+} = 4iy\partial_z + k \quad \text{and} \quad \mathrm{E}_k^{-} = -4iy\partial_{\bar z} - k. \tag{3.3}$$

They satisfy the identity

$$\mathrm{E}_{k\mp 2}^{\pm}\,\mathrm{E}_k^{\mp} = -4\Delta_k - k(k\mp 2). \tag{3.4}$$

Thus if $u : \mathbb{H} \to \mathbb{C}$ is an eigenfunction of Δ_k with spectral value ν, i.e. $\Delta_k u = (\frac{1}{4} - \nu^2)u$, then u satisfies

$$\mathrm{E}_{k\mp 2}^{\pm}\,\mathrm{E}_k^{\mp} u = (1 + 2\nu \mp k)(-1 + 2\nu \pm k)u. \tag{3.5}$$

The slash notation defined in (2.11) commutes with the Laplace operator,

$$\Delta_k\left(f|_k^v\gamma\right) = (\Delta_k f)|_k^v\gamma, \tag{3.6}$$

and interacts as follows with the Maass-operators

$$\mathrm{E}_k^{\pm}\left(f|_k^v\gamma\right) = \left(\mathrm{E}_k^{\pm} f\right)|_{k\pm 2}^v\gamma \quad \left(\gamma \in \mathrm{SL}(2,\mathbb{Z})\right) \tag{3.7}$$

for every $k \in \mathbb{R}$ and u real-analytic.

4. The Maass–Selberg differential form and the R-function

We need to define the Maass–Selberg differential form that will be used later to define the kernel of the associated integrals of the Maass cusp forms. First, we define what is known as the R-function. It will play an important role in the construction of the kernel.

4.1. The R-function. We define $h(z) := \operatorname{Im}(z)$ for $z \in \mathbb{H}$. For $k \in \frac{1}{2}\mathbb{Z}$ and $\nu \in \mathbb{C}$, it is easy to see that $h(z)$ is real-analytic and positive for $z \in \mathbb{H}$, and that h satisfies the differential equations

$$\Delta_k h^{\frac{1}{2}-\nu} = \left(\frac{1}{4} - \nu^2\right) h^{\frac{1}{2}-\nu} \quad \text{and} \quad \mathrm{E}_k^{\pm}\, h^{\frac{1}{2}-\nu} = (1 - 2\nu \pm k) h^{\frac{1}{2}-\nu}. \tag{4.1}$$

Define

$$R_{k,\nu}(z,\zeta) := \left(\frac{\sqrt{\zeta - z}}{\sqrt{\zeta - \overline{z}}}\right)^{-k} \left(\frac{|\operatorname{Im}(z)|}{(\zeta - z)(\zeta - \overline{z})}\right)^{\frac{1}{2}-\nu} \tag{4.2}$$

for $\zeta, z \in \mathbb{C}$ such that

$$\zeta - z, \zeta - \overline{z} \notin \mathbb{R}_{\leq 0} \tag{4.3}$$

holds.

Remark 4.1. The square roots $\sqrt{\zeta - z}$ and $\sqrt{\zeta - \overline{z}}$ on the right-hand side of (4.2) are well defined since we require that $\zeta - z$ and $\zeta - \overline{z}$ are in $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. The square roots in this situation, interpreted as principal square roots, are holomorphic.

The R-function has the following properties:

Proposition 4.2. (1) *The function*

$$z \mapsto R_{k,\nu}(z,\zeta)$$

is smooth in the real and imaginary part of z if (4.3) *holds.*

(2) *The map*

$$\zeta \mapsto R_{k,\nu}(z,\zeta)$$

is holomorphic on $\mathbb{C} \setminus \{z - r, \overline{z} - r; r \geq 0\}$.

(3) *$R_{k,\nu}$ has the form*

$$\begin{aligned} R_{k,\nu}(z,\zeta) &= e^{-ik \arg(\zeta - z)} \left(\frac{\operatorname{Im}(z)}{(\zeta - z)(\zeta - \overline{z})}\right)^{\frac{1}{2}-\nu} \\ &= \left(h^{\frac{1}{2}-\nu}|_k^1 \begin{pmatrix} 0 & 1 \\ -1 & \zeta \end{pmatrix}\right)(z) \end{aligned} \tag{4.4}$$

for real ζ and $z \in \mathbb{H}$.

(4) *Assume the usual restriction* (4.3) *for z and ζ. The function*

$$\mathbb{H} \to \mathbb{C}; \quad z \mapsto R_{k,\nu}(z,\zeta)$$

satisfies the differential equations

$$\begin{aligned} \Delta_k R_{k,\nu}(\cdot,\zeta) &= \left(\frac{1}{4} - \nu^2\right) R_{k,\nu}(\cdot,\zeta) \quad \textit{and} \\ \mathrm{E}_k^{\pm}\, R_{k,\nu}(\cdot,\zeta) &= (1 - 2\nu \pm k) R_{k\pm 2,\nu}(\cdot,\zeta). \end{aligned} \tag{4.5}$$

The function

$$\mathbb{H}^- \to \mathbb{C}; \quad z \mapsto R_{k,\nu}(z,\zeta)$$

satisfies

$$\Delta_{-k} R_{k,\nu}(\cdot,\zeta) = \left(\frac{1}{4} - \nu^2\right) R_{k,\nu}(\cdot,\zeta) \quad \textit{and}$$

$$\mathrm{E}^{\pm}_{-k} R_{k,\nu}(\cdot,\zeta) = (1-2\nu \pm k) R_{k\pm 2,\nu}(\cdot,\zeta).$$

Proof. (1) For fixed $\zeta \in \mathbb{C}$, assume that $z \in \mathbb{C}$ satisfies (4.3). It is then obvious from (4.2) that the function $z \mapsto R_{k,\nu}(z,\zeta)$ is smooth in the real and the imaginary parts of z. (Observe that the values under the square-roots are never negative by condition (4.3).)

(2) Fix $z \in \mathbb{C}$ this time. Again, it is obvious from (4.2) that the function $\zeta \mapsto R_{k,\nu}(z,\zeta)$ is holomorphic for all $\zeta \in \mathbb{C}$ satisfying condition (4.3). Noticing that ζ satisfies (4.3) is equivalent to the fact that ζ is an element of the "two-cut plane" $\mathbb{C} \setminus (z + \mathbb{R}_{\leq 0} \cup \overline{z} + \mathbb{R}_{\leq 0})$. This shows the second part of the proposition.

(3) The first equality follows by rewriting the right-hand side of (4.2) using the identity

$$\frac{\sqrt{\zeta - z}}{\sqrt{\zeta - \overline{z}}} = \frac{\zeta - z}{\sqrt{\zeta - z}\sqrt{\zeta - \overline{z}}} = e^{i \arg(\zeta - z)}, \tag{4.6}$$

which is correct under the given assumptions $\zeta \in \mathbb{R}$ and $z \in \mathbb{H}$. The second equality follows by the slash notation in (2.12).

(4) We assume real ζ and $z \in \mathbb{H}$ for the moment. Combining the last expression of $R_{k,\nu}(z,\zeta)$ in (4.4) with (3.6) and then with (4.1) gives

$$\begin{aligned}
\Delta_k R_{k,\nu}(\cdot,\zeta) &= \Delta_k \left(h^{\frac{1}{2}-\nu} \big|^1_k \begin{pmatrix} 0 & 1 \\ -1 & \zeta \end{pmatrix} \right) \\
&= \left(\Delta_k h^{\frac{1}{2}-\nu}\right) \big|^1_k \begin{pmatrix} 0 & 1 \\ -1 & \zeta \end{pmatrix} \\
&= \left(\frac{1}{4} - \nu^2\right) h^{\frac{1}{2}-\nu} \big|^1_k \begin{pmatrix} 0 & 1 \\ -1 & \zeta \end{pmatrix} \\
&= \left(\frac{1}{4} - \nu^2\right) R_{k,\nu}(\cdot,\zeta).
\end{aligned}$$

Analogously, just using (3.7) instead of (3.6) shows that

$$\mathrm{E}^{\pm}_{k} R_{k,\nu}(\cdot,\zeta) = (1-2\nu \pm k) R_{k\pm 2,\nu}(\cdot,\zeta).$$

Next, using the substitution arguments in the proof of Lemma 3.2 shows that $R_{k,\nu}(z,\zeta)$ satisfies the stated properties for $z \in \mathbb{H}^-$.

The last step is to extend ζ from real values to complex values. This can be done since ζ is just a constant for the differential operators. This shows that the stated differential equations hold also for complex ζ as long as z and ζ satisfy the condition (4.3). $\square$

REMARK 4.3. The R-function appeared first in [11], where Lewis and Zagier introduced

$$R_{0,-\frac{1}{2}}(z,\zeta) = \frac{y}{(x-\zeta)^2+y^2} = \frac{i}{2}\left(\frac{1}{z-\zeta} - \frac{1}{\overline{z}-\zeta}\right)$$

in [11, p. 211, above (2.6)]. Their notation for the above expression was $R_\zeta(z)$.

Before we describe the transformation law of $R_{k,\nu}(z,\zeta)$, we need one trivial auxiliary lemma which will allow us to perform a certain factorization in the proof of the forthcoming Lemma 4.6.

LEMMA 4.4. *Let $z, w \in \mathbb{C}'$ and $\alpha \in \mathbb{C}$ be complex numbers satisfying either*

(1) $z \in \mathbb{R}_{>0}$ *or*
(2) *the product* $zw \in \mathbb{R}_{>0}$.

Then the identity $(zw)^\alpha = z^\alpha w^\alpha$ *holds.*

Proof. (1) Assume that z is real and positive and $w \in \mathbb{C}'$. This ensures $\arg(w) = \arg(zw) \in (-\pi,\pi)$. Writing w in its polar coordinates gives

$$(zw)^\alpha = \left(\left(z|w|\right)e^{i\arg(w)}\right)^\alpha = z^\alpha\left(|w|e^{i\arg(w)}\right)^\alpha = z^\alpha w^\alpha.$$

(2) Assume that zw is real and positive and both $z, w \in \mathbb{C}'$. This ensures $\arg(z) = -\arg(w) \in (-\pi,\pi)$. Writing z and w in its polar coordinates gives

$$\begin{aligned}(zw)^\alpha z^{-\alpha} &= \left(|zw|\right)^\alpha z^{-\alpha} \\ &= \left(|z|e^{i\arg(z)}|w|e^{i\arg(w)}\right)^\alpha \left(|z|^{-1}e^{-i\arg(z)}\right)^\alpha \\ &= \left(|z|e^{i\arg(z)}|w|e^{i\arg(w)}\right)^\alpha \left(|z|^{-1}e^{-i\arg(z)}\right)^\alpha \quad \text{(using case (1))} \\ &= \left(|z|e^{i\arg(z)}|w|e^{i\arg(w)}|z|^{-1}e^{-i\arg(z)}\right)^\alpha \\ &= \left(|w|e^{i\arg(w)}\right)^\alpha = w^\alpha,\end{aligned}$$

where we used that $zw \in \mathbb{R}_{>0}$, $z^{-1} \in \mathbb{C}'$ implies

$$\left(zwz^{-1}\right)^\alpha = (zw)^\alpha\left(z^{-1}\right)^\alpha$$

as shown above.

The identity $(zw)^\alpha = z^\alpha w^\alpha$ holds in both situations. $\square$

REMARK 4.5. It is important that $z, w \notin \mathbb{R}_{<0}$ in the second case of the above auxiliary lemma. If z, w are both real and negative, then zw itself is positive. Due to the choice involved in the argument function, see Remark 2.1, the arguments $\arg(z) = \pi$ and $\arg(w) = \pi$ are not anymore of opposite sign: $\arg(z) \neq -\arg(w)$. Hence, the factorization in the proof of the auxiliary lemma does not work anymore.

In what follows, we show the transformation law of $R_{k,\nu}(\zeta,z)$.

LEMMA 4.6. *Let* $\gamma\in\mathrm{SL}(2,\mathbb{Z})$, $\zeta,z\in\mathbb{C}$ *satisfying* (4.3) *and* $\mu(\gamma,\zeta),\mu(\gamma,z)\in\mathbb{C}'$ *with* $\mathrm{Re}(\mu(\gamma,\zeta))>0$. *Moreover, assume that* ζ *and* z *satisfy one of the following three conditions*:

(1) $\mu(\gamma,\zeta)\in\mathbb{R}_{>0}$,
(2) $\zeta\in\mathbb{H}$ *and* $\gamma z\in\gamma\zeta+i\mathbb{R}_{>0}$ *or*
(3) $\zeta\in\mathbb{H}^-$ *and* $\gamma\bar z\in\gamma\bar\zeta+i\mathbb{R}_{>0}$.

Then, the function $(\zeta,z)\mapsto R_{k,\nu}(z,\zeta)$ *satisfies the transformation formula*

$$R_{k,\nu}(\gamma z,\gamma\zeta)=e^{ik\arg(\mu(\gamma,z))}\big(\mu(\gamma,\zeta)\big)^{1-2\nu}R_{k,\nu}(z,\zeta). \tag{4.7}$$

Proof. Take $\gamma\in\mathrm{SL}(2,\mathbb{Z})$ and $\zeta,z\in\mathbb{C}$ satisfying (4.3) and $\mu(\gamma,z)\in\mathbb{C}'$.

One key observation is the fact that the factorization

$$\left(\frac{|\mathrm{Im}(\gamma z)|}{(\gamma\zeta-\gamma z)(\gamma\zeta-\gamma\bar z)}\right)^{\frac12-\nu}=\big(\mu(\gamma,\zeta)\big)^{1-2\nu}\left(\frac{|\mathrm{Im}(z)|}{(\zeta-z)(\zeta-\bar z)}\right)^{\frac12-\nu} \tag{4.8}$$

holds if ζ and z satisfy one of the additional assumptions.

First, we assume $\mu(\gamma,\zeta)\in\mathbb{R}_{>0}$. Using identities in (2.7) and Lemma 4.4 gives

$$\begin{aligned}\left(\frac{|\mathrm{Im}(\gamma z)|}{(\gamma\zeta-\gamma z)(\gamma\zeta-\gamma\bar z)}\right)^{\frac12-\nu}&=\left(\frac{\frac{|\mathrm{Im}(z)|}{\mu(\gamma,z)\mu(\gamma,\bar z)}}{\frac{\zeta-z}{\mu(\gamma,\zeta)\mu(\gamma,z)}\,\frac{\zeta-\bar z}{\mu(\gamma,\zeta)\mu(\gamma,\bar z)}}\right)^{\frac12-\nu}\\&=\left(\frac{|\mathrm{Im}(z)|}{\frac{\zeta-z}{\mu(\gamma,\zeta)}\,\frac{\zeta-\bar z}{\mu(\gamma,\zeta)}}\right)^{\frac12-\nu}\\&=\big(\mu(\gamma,\zeta)\big)^{1-2\nu}\left(\frac{|\mathrm{Im}(z)|}{(\zeta-z)(\zeta-\bar z)}\right)^{\frac12-\nu}.\end{aligned}$$

Next, we assume the second case $\zeta\in\mathbb{H}$ and $\gamma z\in\gamma\zeta+i\mathbb{R}_{>0}$. In particular, we have that $\gamma\zeta-\gamma z$ and also $\gamma\zeta-\gamma\bar z$ are non-vanishing purely imaginary:

$$\gamma\zeta-\gamma z=-it\quad\text{and}\quad\gamma\zeta-\gamma z=it'$$

for some $t,t'\in\mathbb{R}_{\geq0}$. We have in fact $t'=t+2\,\mathrm{Re}(\gamma\zeta)$. Hence, the expression

$$\frac{|\mathrm{Im}(\gamma z)|}{(\gamma\zeta-\gamma z)(\gamma\zeta-\gamma\bar z)}=\frac{|\mathrm{Im}(\gamma z)|}{(-it)it'}=\frac{|\mathrm{Im}(\gamma z)|}{tt'}$$

is positive. On the other hand, we have

$$\frac{|\mathrm{Im}(\gamma z)|}{(\gamma\zeta-\gamma z)(\gamma\zeta-\gamma\bar z)}=\frac{\frac{|\mathrm{Im}(z)|}{\mu(\gamma,z)\mu(\gamma,\bar z)}}{\frac{\zeta-z}{\mu(\gamma,\zeta)\mu(\gamma,z)}\,\frac{\zeta-\bar z}{\mu(\gamma,\zeta)\mu(\gamma,\bar z)}}=\frac{|\mathrm{Im}(z)|}{(\zeta-z)(\zeta-\bar z)}\big(\mu(\gamma,\zeta)\big)^2.$$

We may apply Lemma 4.4 to $\mu(\gamma,\zeta)^2$ and $\frac{|\mathrm{Im}(z)|}{(\zeta-z)(\zeta-\bar{z})}$ since the assumption $\mathrm{Re}(\mu(\gamma,\zeta))>0$ implies $|\arg(\mu(\gamma,\zeta))|\le\frac{\pi}{2}$ and hence $|\arg((\mu(\gamma,\zeta))^2)|<\pi$:

$$\begin{aligned}\left(\frac{|\mathrm{Im}(\gamma z)|}{(\gamma\zeta-\gamma z)(\gamma\zeta-\gamma\bar{z})}\right)^{\frac{1}{2}-\nu} &= \left(\frac{|\mathrm{Im}(z)|}{\frac{\zeta-z}{\mu(\gamma,\zeta)}\frac{\zeta-\bar{z}}{\mu(\gamma,\zeta)}}\right)^{\frac{1}{2}-\nu}\\ &= \big(\mu(\gamma,\zeta)\big)^{1-2\nu}\left(\frac{|\mathrm{Im}(z)|}{(\zeta-z)(\zeta-\bar{z})}\right)^{\frac{1}{2}-\nu}.\end{aligned}$$

Finally, we assume the third case $\zeta\in\mathbb{H}^-$ and $\gamma z\in\gamma\zeta+i\mathbb{R}_{>0}$. We show that (4.8) holds by interchanging the role of z and $\bar{z}$ in the calculation above.

This proves the identity (4.8) for all three cases.

We need also the identity

$$\left(\frac{\sqrt{\gamma\zeta-\gamma z}}{\sqrt{\gamma\zeta-\gamma\bar{z}}}\right)^{-k} = e^{ik\arg(\mu(\gamma,z))}\left(\frac{\sqrt{\zeta-z}}{\sqrt{\zeta-\bar{z}}}\right)^{-k}. \tag{4.9}$$

Indeed, it follows also by applying (2.7) and the assumption $\mu(\gamma,z)\in\mathbb{C}'$ as the following calculation shows:

$$\begin{aligned}\left(\frac{\sqrt{\gamma\zeta-\gamma z}}{\sqrt{\gamma\zeta-\gamma\bar{z}}}\right)^{-k} &= \left(\frac{\sqrt{\frac{\zeta-z}{\mu(\gamma,\zeta)\mu(\gamma,z)}}}{\sqrt{\frac{\zeta-\bar{z}}{\mu(\gamma,\zeta)\mu(\gamma,\bar{z})}}}\right)^{-k}\\ &= \left(\frac{\sqrt{\mu(\gamma,\bar{z})}}{\sqrt{\mu(\gamma,z)}}\right)^{-k}\left(\frac{\sqrt{\zeta-z}}{\sqrt{\zeta-\bar{z}}}\right)^{-k}\\ &= e^{ik\arg(\mu(\gamma,z))}\left(\frac{\sqrt{\zeta-z}}{\sqrt{\zeta-\bar{z}}}\right)^{-k}.\end{aligned}$$

We also used (4.6) and that $k\in\frac{1}{2}\mathbb{Z}$ is real.

To finally prove the lemma, we combine the identities (4.8) and (4.9). We have

$$\begin{aligned}&R_{k,\nu}(\gamma z,\gamma\zeta)\\ &\quad= \left(\frac{\sqrt{\gamma\zeta-\gamma z}}{\sqrt{\gamma\zeta-\gamma\bar{z}}}\right)^{-k}\left(\frac{|\mathrm{Im}(\gamma z)|}{(\gamma\zeta-\gamma z)(\gamma\zeta-\gamma\bar{z})}\right)^{\frac{1}{2}-\nu}\\ &\quad= e^{ik\arg(\mu(\gamma,z))}\big(\mu(\gamma,\zeta)\big)^{1-2\nu}\left(\frac{\sqrt{\zeta-z}}{\sqrt{\zeta-\bar{z}}}\right)^{-k}\left(\frac{|\mathrm{Im}(z)|}{(\zeta-z)(\zeta-\bar{z})}\right)^{\frac{1}{2}-\nu}\\ &\quad= e^{ik\arg(\mu(\gamma,z))}\big(\mu(\gamma,\zeta)\big)^{1-2\nu}R_{k,\nu}(z,\zeta).\end{aligned}$$ □

REMARK 4.7. The second assumption on ζ and z in Lemma 4.6 is: $\zeta\in\mathbb{H}$ and $\gamma z\in\gamma\zeta+i\mathbb{R}_{>0}$. This is equivalent to saying that z lies in the open geodesic ray connecting $\zeta\in\mathbb{H}$ to the cusp $\gamma^{-1}(i\infty)$. The third assumption can be rephrased analogously: The third condition is equivalent with saying that $\bar{z}$ lies in the geodesic ray connecting $\bar{\zeta}\in\mathbb{H}$ to the cusp $\gamma^{-1}(i\infty)$.

4.2. The Maass–Selberg differential form. We recall differential forms presented in [12] and observe the action of Maass raising and lowering operators applied to those differential forms.

Let f,g be real-analytic functions and write again $z=x+iy$. We define

$$\{f,g\}^+(z)=f(z)g(z)\frac{\mathrm{d}z}{y}\quad\text{and}\quad \{f,g\}^-(z)=f(z)g(z)\frac{\mathrm{d}\bar{z}}{y}. \tag{4.10}$$

We extend the slash-notation to linear combinations of the differential forms $\{f,g\}^\pm$: We define

$$\begin{aligned}\{f,g\}^+|_k^v\gamma(z)&=e^{-ik\arg(\mu(\gamma,z))}v(\gamma)^{-1}f(\gamma z)g(\gamma z)\frac{\mathrm{d}(\gamma z)}{\mathrm{Im}(\gamma z)},\\ \{f,g\}^-|_k^v\gamma(z)&=e^{-ik\arg(\mu(\gamma,z))}v(\gamma)^{-1}f(\gamma z)g(\gamma z)\frac{\mathrm{d}\overline{(\gamma z)}}{\mathrm{Im}(\gamma z)},\end{aligned} \tag{4.11}$$

and extend it in the obvious way to linear combinations.

Lemma 4.8. *Let v be a multiplier and $k,q\in\frac{1}{2}\mathbb{Z}$.*

(1) *We have for any matrix $\gamma\in\mathrm{SL}(2,\mathbb{Z})$ that*

$$\{f,g\}^\pm|_{k+q\mp2}^v\gamma=\begin{cases}\{f|_k^v\gamma,g|_q^1\gamma\}^\pm & \textit{and}\\ \{f|_k^1\gamma,g|_q^v\gamma\}^\pm.\end{cases} \tag{4.12}$$

(2) $\{f,g\}^\pm=\{g,f\}^\pm$.

Proof. The last property follows directly from the definition in (4.10).

We prove (4.12) by direct calculation, using the identities in (2.7). For example, we have

$$\begin{aligned}\{f,g\}^+|_{k+q-2}^v(z)&=v(\gamma)^{-1}e^{(-k-q+2)i\arg(\mu(\gamma,z))}f(\gamma z)g(\gamma z)\frac{\mathrm{d}(\gamma z)}{\mathrm{Im}(\gamma z)}\\ &=v(\gamma)^{-1}e^{(-k-q)i\arg(\mu(\gamma,z))}f(\gamma z)g(\gamma z)\frac{\mathrm{d}z}{\mathrm{Im}(z)}\\ &=\left\{f|_k^v\gamma,g|_q^1\gamma\right\}^+(z)\end{aligned}$$

for all $z\in\mathbb{H}$. The other identities follow analogously. □

Combining the property (4.12) of the 1-forms in (4.10) with Maass operators, we see that

$$\left\{\mathrm{E}_k^\pm\left(f|_k^v\gamma\right),g|_q^1\gamma\right\}^\pm=\left\{\mathrm{E}_k^\pm\left(f|_k^1\gamma\right),g|_q^v\gamma\right\}^\pm=\left\{\mathrm{E}_k^\pm f,g\right\}^\pm|_{k+q}^v\gamma \tag{4.13}$$

for every $\gamma\in\mathrm{SL}(2,\mathbb{R})$ and $k,q\in\frac{1}{2}\mathbb{Z}$. We also have the relations

$$\begin{aligned}\left\{\mathrm{E}_k^+ f,g\right\}^+&=-\left\{f,\mathrm{E}_{-k}^+ g\right\}^++4i\partial_z(fg)\,\mathrm{d}z\quad\text{and}\\ \left\{\mathrm{E}_k^- f,g\right\}^-&=-\left\{f,\mathrm{E}_{-k}^- g\right\}^--4i\partial_{\bar{z}}(fg)\,\mathrm{d}\bar{z}\end{aligned} \tag{4.14}$$

and

$$\left\{\mathrm{E}_k^- f, g\right\}^+ = -\left\{f, \mathrm{E}_{-k}^- g\right\}^+ - 4i\partial_{\bar z}(fg)\,\mathrm{d}z \quad \text{and}$$
$$\left\{\mathrm{E}_k^+ f, g\right\}^- = -\left\{f, \mathrm{E}_{-k}^+ g\right\}^- + 4i\partial_z(fg)\,\mathrm{d}\bar z.$$

We are now able to define the Maass–Selberg form.

Definition 4.9. Let f, g be real-analytic and $k \in \frac{1}{2}\mathbb{Z}$. We define the *Maass–Selberg form* η_k by

$$\eta_k(f,g) = \left\{\mathrm{E}_k^+ f, g\right\}^+ - \left\{f, \mathrm{E}_{-k}^- g\right\}^-. \tag{4.15}$$

Lemma 4.10. *Let f, g be real-analytic and $k \in \frac{1}{2}\mathbb{Z}$. The Maass–Selberg form has the following properties*:

(1) *We have the equations*

$$\eta_k(f,g) + \eta_{-k}(g,f) = 4i \cdot \mathrm{d}(fg) \tag{4.16}$$

and

$$\eta_k(f,g) - \eta_{-k}(g,f) = 4\left([g f_y - f g_y]\,\mathrm{d}x + \left[f g_x - g f_x + \frac{ik}{y} fg \right] \mathrm{d}y \right),$$

where f_x denotes $\partial_x f$, $f_y = \partial_y f$, $g_x = \partial_x g$ and $g_y = \partial_y g$, respectively.

(2) *If there exists a $\lambda \in \mathbb{R}$ such that f and g satisfy $\Delta_k f = \lambda f$ and $\Delta_{-k} g = \lambda g$, then the Maass–Selberg form is closed.*

(3) *We have that*

$$\eta_k(f,g)|_0^v \gamma = \eta_k\left(f|_k^v \gamma, g|_{-k}^1 \gamma\right) = \eta_k\left(f|_k^1 \gamma, g|_{-k}^v \gamma\right) \tag{4.17}$$

for any multiplier system v.

(4) *Let $\nu \in \mathbb{C}$ and assume that f and g are eigenfunctions of the operators Δ_k and Δ_{-k}, respectively, both with eigenvalue $\frac{1}{4} - \nu^2$. Then, we have that*

$$\begin{aligned} \eta_{k+2}\left(\mathrm{E}_k^+ f, \mathrm{E}_{-k}^- g\right) = {} & (1+2\nu+k)(1-2\nu+k)\eta_k(f,g) \\ & + 4i\,\mathrm{d}\left(\left(\mathrm{E}_k^+ f\right)\left(\mathrm{E}_{-k}^- g\right)\right). \end{aligned} \tag{4.18}$$

Proof. Recall that $\partial_z = \frac{1}{2}\partial_x - \frac{i}{2}\partial_y$, $\partial_{\bar z} = \frac{1}{2}\partial_x + \frac{i}{2}\partial_y$, $\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y$, $\mathrm{d}\bar z = \mathrm{d}x - i\,\mathrm{d}y$ and $\mathrm{d}f = \partial_z f\,\mathrm{d}z + \partial_{\bar z} f\,\mathrm{d}\bar z$ for any function f smooth in x and y.

(1) We prove the first item via direct computation:

$$\begin{aligned} \eta_k(f,g) + \eta_{-k}(g,f) &= -4i\left[(f g_z + g f_z)\,\mathrm{d}z + (f g_{\bar z} + g f_{\bar z})\,\mathrm{d}\bar z\right] \\ &= 4i\,\mathrm{d}(fg) \end{aligned}$$

and

$$\begin{aligned} \eta_k(f,g) - \eta_{-k}(g,f) &= 4i\left[\left(g f_z - f g_z - \frac{ik}{2y} fg \right) \mathrm{d}z - \left(g f_{\bar z} - f g_{\bar z} - \frac{ik}{2y} fg \right) \mathrm{d}\bar z \right] \\ &= 4\left[(g f_y - f g_y)\,\mathrm{d}x + \left(f g_x - g f_x + i\frac{k}{y} fg \right) \mathrm{d}y \right], \end{aligned}$$

where we used (3.3) for the Maass operators.

(2) We want to show that $\mathrm{d}\eta_k(f,g)=0$ under the given conditions. Since

$$2\eta_k(f,g)=\big(\eta_k(f,g)+\eta_{-k}(g,f)\big)+\big(\eta_k(f,g)-\eta_{-k}(g,f)\big),$$

and since we already proved the first part of the lemma, it is enough to show that $\eta_k(f,g)-\eta_{-k}(g,f)$ is closed. We find, after some computation, that

$$\mathrm{d}\big(\eta_k(f,g)-\eta_{-k}(g,f)\big)=[f\Delta_{-k}g-g\Delta_k f]\frac{\mathrm{d}x\wedge\mathrm{d}y}{y^2}.$$

(3) This follows directly from (4.13).

(4) It follows directly from the equations in (4.14) that

$$\begin{aligned}\eta_{k+2}\big(\mathrm{E}_k^+ f,\mathrm{E}_{-k}^- g\big)&=4i\,\mathrm{d}\big[\big(\mathrm{E}_k^+ f\big)\big(\mathrm{E}_{-k}^- g\big)\big]\\&\quad-\big\{\mathrm{E}_k^+ f,\mathrm{E}_{-k-2}^+\mathrm{E}_{-k}^- g\big\}^+ +\big\{\mathrm{E}_{k+2}^-\mathrm{E}_k^+ f,\mathrm{E}_{-k}^- g\big\}^+.\end{aligned}$$

We may apply (3.5), since we assume that f and g are eigenfunctions of the Laplace operators Δ_k and Δ_{-k}, respectively, with the same eigenvalue. The statement in the lemma follows. □

REMARK 4.11. The first three items of Lemma 4.10 are generalizations of the lemma given in [11, Chapter II, Section 2]. We have that

$$\eta_0(f,g)=[f,g]$$

where $[\cdot,\cdot]$ is defined in [11, Chapter II, Section 2]. Our form $\{\cdot,\cdot\}^\pm$ in (4.10) differs from the form $\{\cdot,\cdot\}$ in [11, Chapter II, Section 2, (2.5)], contrary to the notational resemblance.

LEMMA 4.12. *Let f and g be smooth functions (in x and y) on $\mathbb{H}\cup\mathbb{H}^-$ satisfying $f(z)=f(\bar z)$ and $g(z)=g(\bar z)$.*

(1) *We can extend the Maass–Selberg form to smooth (in x and y) functions f,g defined on the lower half-plane.*

(2) *The Maass–Selberg form satisfies*

$$\eta_k(f,g)(z)=\eta_k(g,f)(\bar z). \tag{4.19}$$

Proof. (1) All differentials and other components used in the definition of the Maass–Selberg-form are well defined for smooth (in x and y) functions on the lower half-plane $\mathbb{H}^-$. Hence, the extension makes sense.

(2) This follows by direct calculations. First, use the relations (4.15), (4.10) and (3.2) followed by (3.3) to rewrite everything depending on the pair $(z,\bar z)$ respectively, (x,y). Then, use the substitution $(z,\bar z)\mapsto(\bar z,z)$ respectively, $(x,y)\mapsto(x,-y)$. As final step, use the above relations in reverse order. □

4.3. Everything combined. We will now insert the function $R_{k,\nu}$ into the Maass–Selberg form and use the form to define nearly periodic functions.

LEMMA 4.13. *Let v be a multiplier which is compatible with the half-integral weight k, $\nu \in \mathbb{C}$, u a Maass cusp form with weight k multiplier v and eigenvalue $\frac{1}{4} - \nu^2$. Moreover, let $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ and $\zeta, z \in \mathbb{C}$ satisfying the assumptions of Lemma 4.6.*

It follows that

$$\begin{aligned} \eta_k\big(u, R_{-k,\nu}(\cdot, \gamma\zeta)\big)|_0^v \gamma(z) &= \big(\mu(\gamma,\zeta)\big)^{1-2\nu} \eta_k\big(u, R_{-k,\nu}(\cdot,\zeta)\big)(z) \quad \textit{and} \\ \eta_{-k}\big(R_{-k,\nu}(\cdot,\gamma\zeta), u\big)|_0^v \gamma(z) &= \big(\mu(\gamma,\zeta)\big)^{1-2\nu} \eta_{-k}\big(R_{-k,\nu}(\cdot,\zeta), u\big)(z). \end{aligned} \tag{4.20}$$

Proof. We show the second identity:

$$\begin{aligned} &\eta_{-k}\big(R_{-k,\nu}(\cdot,\gamma\zeta), u\big)|_0^v \gamma(z) \\ &\quad = \eta_{-k}\big(R_{-k,\nu}(\cdot,\gamma\zeta)|_{-k}^1 \gamma, u|_k^v \gamma\big)(z) \quad \text{using (4.17)} \\ &\quad = \eta_{-k}\big(R_{-k,\nu}(\cdot,\gamma\zeta)|_{-k}^1 \gamma, u\big)(z) \\ &\quad = \big(\mu(\gamma,\zeta)\big)^{1-2\nu} \eta_k\big(R_{-k,\nu}(\cdot,\zeta), u\big)(z) \quad \text{using (4.7)}. \end{aligned}$$

The use of the transformation formula (4.7) in the calculation above is allowed since z and ζ satisfy the assumptions of Lemma 4.6 and since $\mathrm{E}_{-k}^{+} R_{-k,\nu}(z,\zeta)$ appearing in the construction of $\eta_{\pm k}(R_{-k,\nu}(\cdot,\zeta), u)(z)$ satisfies (4.5).

The first identity follows by the same arguments. □

5. Nearly periodic functions

Let u be a Maass cusp form of weight k, multiplier v, and spectral value ν as defined in and below Definition 3.1.

We define on $\mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$;

$$\zeta \mapsto f(\zeta) := \begin{cases} \int_\zeta^{i\infty} \eta_{-k}(R_{-k,\nu}(\cdot,\zeta), u)(z) & \text{if } \mathrm{Im}(\zeta) > 0 \text{ and} \\ -\int_\zeta^{-i\infty} \eta_k(R_{-k,\nu}(\cdot,\zeta), \tilde{u})(z) & \text{if } \mathrm{Im}(\zeta) < 0, \end{cases} \tag{5.1}$$

where $\tilde{u}(z) = u(\bar{z})$ as defined in Lemma 3.2. The path of integration is the geodesic ray connecting ζ and the cusp $i\infty$ respectively, $-i\infty$ in the upper respectively, lower half-plane.

REMARK 5.1. (1) We have made a choice in Definition 5.1 between integrating over the forms

$$\text{(1a)} \quad -\eta_k\big(u, R_{-k,\nu}(\cdot,\zeta)\big) \quad \text{or} \quad \text{(1b)} \quad \eta_{-k}\big(R_{-k,\nu}(\cdot,\zeta), u\big)$$

on $\mathbb{H}$ and

$$\text{(2a)} \quad -\eta_k\big(R_{-k,\nu}(\cdot,\zeta), \tilde{u}\big) \quad \text{or} \quad \text{(2b)} \quad \eta_{-k}\big(\tilde{u}, R_{-k,\nu}(\cdot,\zeta)\big)$$

on $\mathbb{H}^-$.

We will see later in Remark 6.2 that each choice leads to the same period function on $\mathbb{H}$ prospectively $\mathbb{H}^-$. Remark 5.4 compares our choice to the situation discussed in [11].

(2) The reason, why we extended Maass cusp forms and the R-function to the lower half-plane $\mathbb{H}^-$ and also extended the Maass–Selberg-form η_k to functions on the lower half-plane, see, for example, Lemma 3.2, Proposition 4.2 and Lemma 4.12 respectively, is the second case of (5.1). We want to be able to integrate along the geodesic ray $\zeta - i\mathbb{R}_{>0}$ connecting $\zeta \in \mathbb{H}^-$ and $-i\infty$ in the lower half-plane. In our opinion, this representation illustrates better how the function f is defined on $\mathbb{H}^-$ compared to the (on $\mathbb{H}^-$ equivalent) integral representation in Lemma 5.3.

LEMMA 5.2. *For $|\mathrm{Re}(\nu)| < \frac{1}{2}$, the integration in (5.1) is well-defined along the geodesic paths connecting ζ to $i\infty$ in $\mathbb{H}$ respectively ζ to $-i\infty$ in $\mathbb{H}^-$.*

Proof. The singularity of $R_{k,\nu}(z,\zeta)$ for $z \to \zeta \in \mathbb{H}$ respectively $z \to \bar{\zeta} \in \mathbb{H}$ is of the form $(\zeta - z)^{\nu-\frac{1}{2}}$ respectively, $(\zeta - \bar{z})^{\nu-\frac{1}{2}}$. The whole integrand has at most the same singularity since $R_{k,\nu}$ is an eigenfunction of the Maass operators, see (4.5). The weight argument $\arg(\zeta - z)$ has also a well defined limit. Hence, the integration is well defined for ν values satisfying $|\mathrm{Re}(\nu)| < \frac{1}{2}$. □

LEMMA 5.3. *For $\zeta \in \mathbb{H}^-$ we have*

$$f(\zeta) = -\int_{\bar{\zeta}}^{i\infty} \eta_k\big(u, R_{-k,\nu}(\cdot,\zeta)\big)(z). \tag{5.2}$$

Proof. For $\zeta \in \mathbb{H}^-$ we have

$$\begin{aligned} f(\zeta) &= -\int_{\zeta}^{-i\infty} \eta_k\big(R_{-k,\nu}(\cdot,\zeta), \tilde{u}\big)(z) \\ &= -\int_{\zeta}^{-i\infty} \eta_k\big(u, R_{-k,\nu}(\cdot,\zeta)\big)(\bar{z}) \quad \text{using (4.19)} \\ &= -\int_{\bar{\zeta}}^{i\infty} \eta_k\big(u, R_{-k,\nu}(\cdot,\zeta)\big)(z), \end{aligned}$$

where we used $u(z) = \tilde{u}(\bar{z})$ for $z \in \mathbb{H}$ in Lemma 3.2. □

REMARK 5.4. For $k = 0$, we can compare the definition of f in (5.1) to the one in [11, p. 212] since we have $\eta_0(f,g) = [f,g]$, see Remark 4.11. We find that [11] uses exactly the opposite choice: They use $\int_{\zeta}^{i\infty} \eta_0(u, R_{-0,\nu}(\cdot,\zeta))$ for $\zeta \in \mathbb{H}$ (compared to $\int_{\zeta}^{i\infty} \eta_{-0}(R_{-0,\nu}(\cdot,\zeta), u)$ in (5.1)). On the lower half plane they use $-\int_{\bar{\zeta}}^{i\infty} \eta_{-0}(R_{-0,\nu}(\cdot,\zeta), u)$ for $\zeta \in \mathbb{H}^-$ (compared to our integral $-\int_{\bar{\zeta}}^{i\infty} \eta_0(u, R_{-0,\nu}(\cdot,\zeta))$ in (5.2)).

In the following lemma, we describe the transformation property of the function $f(\zeta)$.

LEMMA 5.5. *Let $|\mathrm{Re}(\nu)| < \frac{1}{2}$. The function f defined in (5.1) satisfies*

$$f\|_\nu^v \gamma(z) = v(\gamma)^{-1}\big(\mu(\gamma,\zeta)\big)^{2\nu-1} f(\gamma\zeta) \tag{5.3}$$
$$= \begin{cases} \int_\zeta^{\gamma^{-1}i\infty} \eta_{-k}(R_{-k,\nu}(\cdot,\zeta),u)(z) & \text{if } \zeta\in\mathbb{H} \text{ and} \\ -\int_{\bar\zeta}^{\gamma^{-1}i\infty} \eta_k(u, R_{-k,\nu}(\cdot,\zeta))(z) & \text{if } \zeta\in\mathbb{H}^- \end{cases}$$

for every $\zeta\in\mathbb{H}\cup\mathbb{H}^-$ and $\gamma\in\mathrm{SL}(2,\mathbb{Z})$ satisfying $\mathrm{Re}(\mu(\gamma,\zeta))>0$. The path of integration on the right-hand side is the geodesic ray connecting ζ respectively, $\bar\zeta$ and $\gamma^{-1}(i\infty)$.

Proof. Let $\zeta\in\mathbb{H}$ and $\gamma\in\mathrm{SL}(2,\mathbb{Z})$ such that $\mathrm{Re}(\mu(\gamma,\zeta))>0$ holds. We get

$$f(\gamma\zeta) = \int_{\gamma\zeta}^{i\infty} \eta_{-k}\big(R_{-k,\nu}(\cdot,\gamma\zeta),u\big)(z) = \int_\zeta^{\gamma^{-1}i\infty} \eta_{-k}\big(R_{-k,\nu}(\cdot,\gamma\zeta),u\big)(\gamma z)$$
$$= v(\gamma)\int_\zeta^{\gamma^{-1}i\infty} \eta_{-k}\big(R_{-k,\nu}(\cdot,\gamma\zeta),u\big)|_0^v\gamma(z).$$

Now, we would like to apply Lemma 4.13. Therefore, we must check, if all z of the integration path satisfy the conditions on ζ and z as given in Lemma 4.6. Remark 4.7 implies that we have to verify if the integration path is the geodesic ray connecting ζ and $\gamma^{-1}(i\infty)$. This is indeed the case. Hence, the second condition of Lemma 4.6 is satisfied since we assume $\mathrm{Re}(\mu(\gamma,\zeta))>0$. Using the transformation formula (4.20) in Lemma 4.13 gives

$$f(\gamma\zeta) = v(\gamma)\int_\zeta^{\gamma^{-1}i\infty} \eta_{-k}\big(R_{-k,\nu}(\cdot,\gamma\zeta),u\big)|_0^v\gamma(z)$$
$$= v(\gamma)\big(\mu(\gamma,\zeta)\big)^{1-2\nu}\int_\zeta^{\gamma^{-1}i\infty} \eta_{-k}\big(R_{-k,\nu}(\cdot,\zeta),u\big)(z).$$

The same calculation for $\zeta\in\mathbb{H}^-$, using the integral for f in (5.2), gives

$$v(\gamma)^{-1}\big(\mu(\gamma,\zeta)\big)^{2\nu-1} f(\gamma\zeta) = -\int_{\bar\zeta}^{\gamma^{-1}i\infty} \eta_k\big(u, R_{-k,\nu}(\cdot,\zeta)\big)(z). \qquad \square$$

DEFINITION 5.6. We call a function g *nearly periodic* if there exists an $a\in\mathbb{C}$ with $|a|=1$ such that $g(z+1)=ag(z)$ holds for all z.

We check that $f(\zeta)$ is nearly periodic as application of Lemma 5.5: For $\zeta\in\mathbb{H}$, we find

$$v(T)^{-1}f(T\zeta) = \int_\zeta^{T^{-1}i\infty} \eta_{-k}\big(R_{-k,\nu}(\cdot,\zeta),u\big)(z)$$
$$= \int_\zeta^{i\infty} \eta_{-k}\big(R_{-k,\nu}(\cdot,\zeta),u\big)(z)$$
$$= f(\zeta),$$

where we use the invariance of the cusp $i\infty$ under translation and the trivial fact that $\mathrm{Re}(\mu(T,\zeta))=1$. The same arguments hold for $\zeta\in\mathbb{H}^-$. Hence, we just proved the following lemma.

LEMMA 5.7. *The function f in defined in (5.1) satisfies*

$$v(T)^{-1}f(\zeta+1)=f(\zeta)\quad \text{for every } \zeta\in\mathbb{C}\setminus\mathbb{R}. \tag{5.4}$$

Written in the double-slash notation (2.11), we have

$$f\|_\nu^v T=f\quad \text{on } \mathbb{C}\setminus\mathbb{R}. \tag{5.5}$$

Similar to [11, Proposition 2], we continue to prove an algebraic correspondence between f and a solution of a suitable three-term equation on $\mathbb{C}\setminus\mathbb{R}$.

LEMMA 5.8. *Assume that k and ν satisfy $e^{\mp\pi i(2\nu-1)}\neq e^{\pi ik}$. Put*

$$c_\pm^\star=1-e^{\pi ik}e^{\pm\pi i(2\nu-1)}. \tag{5.6}$$

Then, there exists a bijection between nearly periodic functions f satisfying (5.4) and solutions P of the three-term equation

$$\begin{aligned}P(\zeta)&=v(T)^{-1}P(\zeta+1)+v\big(T'\big)^{-1}(\zeta+1)^{2\nu-1}P\left(\frac{\zeta}{\zeta+1}\right)\\ &\textit{i.e. } P\|_\nu^v\big(\mathbf{1}-T-T'\big)(\zeta)=0\quad \text{for } \zeta\in\mathbb{C}\setminus\mathbb{R}.\end{aligned} \tag{5.7}$$

The bijection is given by the formulas:

$$\begin{aligned}c_\pm^\star f(\zeta)&=P(\zeta)+v(S)^{-1}\zeta^{2\nu-1}P(S\zeta)\quad \big(\mathrm{Im}(\zeta)\gtrless 0\big)\\ &=P\|_\nu^v(\mathbf{1}+S)(\zeta)\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}P(\zeta)&=f(\zeta)-v(S)^{-1}\zeta^{2\nu-1}f(S\zeta)\quad (\zeta\in\mathbb{C}\setminus\mathbb{R})\\ &=f\|_\nu^v(\mathbf{1}-S)(\zeta).\end{aligned} \tag{5.9}$$

REMARK 5.9. Observe at a formal level that P, as defined in (5.9), satisfies the three-term functional equation (5.7):

$$\begin{aligned}&P\|_\nu^v\big(\mathbf{1}-T-T'\big)\\ &\quad=f\|_\nu^v(\mathbf{1}-S)\|_\nu^v(\mathbf{1}-T-TST)\quad \big(TST=T' \text{ by } (2.3)\big)\\ &\quad=f\|_\nu^v(\mathbf{1}-S-T+ST-TST+STST)\\ &\quad=f\|_\nu^v\big(\mathbf{1}-S-T+ST-TST+T^{-1}S\big)\quad \big(STST=T^{-1}S \text{ by } (2.4)\big)\\ &\quad=f\|_\nu^v\big((\mathbf{1}-T)+\big(T^{-1}S-S\big)+(ST-TST)\big)\\ &\quad=f\|_\nu^v(\mathbf{1}-T)\|_\nu^v\big(\mathbf{1}+T^{-1}S+ST\big)\\ &\quad=0.\end{aligned}$$

However, the calculation is only formal, since the double-slash notation just hides the weight factors and the multipliers. In general, we do not know whether they match since the double-slash notation is not a group action. We have to check them on each occasion.

Proof of Lemma 5.8. Let $z \in \mathbb{C} \setminus \mathbb{R}$. First, we compute

$$v(S)^{-1} v(S)^{-1} \zeta^{2\nu-1} \left(\frac{-1}{\zeta} \right)^{2\nu-1}.$$

We have

$$\zeta^{2\nu-1} \left(\frac{-1}{\zeta} \right)^{2\nu-1} = e^{(2\nu-1)i(\arg(\zeta)+\arg(\frac{-1}{\zeta}))} = e^{\pm \pi i(2\nu-1)} \quad \big(\operatorname{Im}(\zeta) \gtrless 0\big)$$

since $\arg(\zeta)+\arg(-\frac{1}{\zeta}) = \pm\pi$ for $\operatorname{Im}(\zeta) \gtrless 0$. The choices $+$ and $>$, respectively $-$ and $<$ correspond. The consistency relation (2.8) for multipliers implies

$$v(S)v(S) = e^{-ik\pi}.$$

Hence

$$v(S)^{-1} v(S)^{-1} \zeta^{2\nu-1} \left(\frac{-1}{\zeta} \right)^{2\nu-1} = e^{\pi i k} e^{\pm \pi i(2\nu-1)} \tag{5.10}$$

holds.

Next, we show that (5.8) and (5.9) are inverses of each other. On one hand, we have

$$\begin{aligned}
c_\pm^\star f(\zeta) &= P(\zeta) + v(S)^{-1} \zeta^{2\nu-1} P\left(\frac{-1}{\zeta} \right) \\
&= f(\zeta) - v(S)^{-1} \zeta^{2\nu-1} f\left(\frac{-1}{\zeta} \right) \\
&\qquad + v(S)^{-1} \zeta^{2\nu-1} \left[f\left(\frac{-1}{\zeta} \right) - v(S)^{-1} \left(\frac{-1}{\zeta} \right)^{2\nu-1} f(\zeta) \right] \\
&= f(\zeta) \left[1 - v(S)^{-1} v(S)^{-1} \zeta^{2\nu-1} \left(\frac{-1}{\zeta} \right)^{2\nu-1} \right] \\
&= f(\zeta) \left[1 - e^{\pi i k} e^{\pm \pi i(2\nu-1)} \right] \quad \big(\text{for } \operatorname{Im}(\zeta) \gtrless 0\big).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
c_\pm^\star P(\zeta) &= c_\pm^\star \left[f(\zeta) - v(S)^{-1} \zeta^{2\nu-1} f\left(\frac{-1}{\zeta} \right) \right] \\
&= P(\zeta) + v(S)^{-1} \zeta^{2\nu-1} P\left(\frac{-1}{\zeta} \right) \\
&\qquad - v(S)^{-1} \zeta^{2\nu-1} \left[P\left(\frac{-1}{\zeta} \right) + v(S)^{-1} \left(\frac{-1}{\zeta} \right)^{2\nu-1} P(\zeta) \right] \\
&= P(\zeta) \left[1 - e^{\pi i k} e^{\pm \pi i(2\nu-1)} \right] \quad \big(\text{for } \operatorname{Im}(\zeta) \gtrless 0\big),
\end{aligned}$$

using the same argument calculations as above.

We now show, that f being nearly periodic corresponds to P satisfying the three-term equation.

Let f be a nearly periodic function satisfying (5.4) and P function given by (5.9). Then, we find (for $\zeta \in \mathbb{H} \cup \mathbb{H}^-$)

$$\begin{aligned}
&P\|_\nu^v \big(\mathbf{1} - T - T'\big)(\zeta) \\
&\quad = P(\zeta) - v(T)^{-1} P(T\zeta) - v\big(T'\big)^{-1} (\zeta+1)^{2\nu-1} P\big(T'\zeta\big) \\
&\quad = \big(f(\zeta) - v(S)^{-1} \zeta^{2\nu-1} f(S\zeta)\big) \\
&\qquad - v(T)^{-1} \big(f(T\zeta) - v(S)^{-1} (T\zeta)^{2\nu-1} f(ST\zeta)\big) \\
&\qquad - v\big(T'\big)^{-1} (\zeta+1)^{2\nu-1} \\
&\qquad \cdot \big(f\big(T'\zeta\big) - v(S)^{-1} \big(T'\zeta\big)^{2\nu-1} f\big(ST'\zeta\big)\big) \\
&\quad = \big(f(\zeta) - v(T)^{-1} f(T\zeta)\big) \\
&\qquad + \big(v\big(T'\big)^{-1} v(S)^{-1} (\zeta+1)^{2\nu-1} \big(T'\zeta\big)^{2\nu-1} f(STST\zeta) \\
&\qquad - v(S)^{-1} \zeta^{2\nu-1} f(S\zeta)\big) \\
&\qquad + \big(v(T)^{-1} v(S)^{-1} (T\zeta)^{2\nu-1} f(ST\zeta) - v\big(T'\big)^{-1} (\zeta+1)^{2\nu-1} f(TST\zeta)\big)
\end{aligned}$$

(f is nearly periodic and $STST = T^{-1}S$)

$$\begin{aligned}
&\quad = 0 + v(S)^{-1} \zeta^{2\nu-1} \\
&\qquad \cdot \left(v\big(T'\big)^{-1} \frac{1}{\zeta^{2\nu-1}} (\zeta+1)^{2\nu-1} \big(T'\zeta\big)^{2\nu-1} f\big(T^{-1}S\zeta\big) - f(S\zeta) \right) \\
&\qquad + v(T) v\big(T'\big)^{-1} (\zeta+1)^{2\nu-1} \\
&\qquad \cdot \big(v\big(T'\big) v(T)^{-2} v(S)^{-1} f(ST\zeta) - v(T)^{-1} f(TST\zeta)\big) \\
&\quad = v(S)^{-1} \zeta^{2\nu-1} \big(v(T) f\big(T^{-1}S\zeta\big) - f(S\zeta)\big) \\
&\qquad + v(T) v\big(T'\big)^{-1} (\zeta+1)^{2\nu-1} \big(f(ST\zeta) - v(T)^{-1} f(TST\zeta)\big) \\
&\quad = 0.
\end{aligned}$$

We used several times multiplier identities based on the consistency relation (2.8). Hence, P satisfies the three-term equation (5.7).

Conversely, let us assume that the function P satisfies the three-term equation (5.7) on $\mathbb{C} \setminus \mathbb{R}$. We have to show that f attached by (5.8) is indeed nearly periodic. Applying the three-term equation to P in ζ and $ST\zeta = \frac{-1}{\zeta+1}$ we obtain:

$$\begin{aligned}
0 &= \big(P\|_\nu^v \big[-\mathbf{1} + T + T'\big]\big)\|_\nu^v [\mathbf{1} - ST](\zeta) \\
&= \big[-P(\zeta) + P\|_\nu^v T(\zeta) + P\|_\nu^v T'(\zeta)\big] - v(ST)^{-1} (\zeta+1)^{2\nu-1}
\end{aligned}$$

$$
\begin{aligned}
&\cdot\left[-P\left(\frac{-1}{\zeta+1}\right)+P\|_\nu^v T\left(\frac{-1}{\zeta+1}\right)+P\|_\nu^v T'\left(\frac{-1}{\zeta+1}\right)\right]\\
={}&\left[-P(z)+v(T)^{-1}P(\zeta+1)+v\big(T'\big)^{-1}(\zeta+1)^{2\nu-1}P\left(\frac{\zeta}{\zeta+1}\right)\right]\\
&-v(ST)^{-1}(\zeta+1)^{2\nu-1}\left[-P\left(\frac{-1}{\zeta+1}\right)+v(T)^{-1}P\left(\frac{\zeta}{\zeta+1}\right)\right.\\
&\left.+v\big(T'\big)^{-1}\left(\frac{-1}{\zeta+1}+1\right)^{2\nu-1}P\left(\frac{\frac{-1}{\zeta+1}}{\frac{-1}{\zeta+1}+1}\right)\right]\\
={}&-\left[P(\zeta)+v(ST)^{-1}v\big(T'\big)^{-1}(\zeta+1)^{2\nu-1}\left(\frac{\zeta}{\zeta+1}\right)^{2\nu-1}P\left(\frac{-1}{\zeta}\right)\right]\\
&+\left[v(T)^{-1}P(\zeta+1)+v(ST)^{-1}(\zeta+1)^{2\nu-1}P\left(\frac{-1}{\zeta+1}\right)\right]\\
&+\left[v\big(T'\big)^{-1}(\zeta+1)^{2\nu-1}P\big(T'\zeta\big)-v(ST)^{-1}v(T)^{-1}(\zeta+1)^{2\nu-1}P\left(\frac{\zeta}{\zeta+1}\right)\right]\\
={}&-\left[P(\zeta)+v(S)^{-1}z^{2\nu-1}P\left(\frac{-1}{\zeta}\right)\right]\\
&+v(T)^{-1}\left[P(T\zeta)+v(S)^{-1}(T\zeta)^{2\nu-1}P\left(\frac{-1}{T\zeta}\right)\right]+0\\
={}&-c_\pm^\star f(\zeta)+v(T)^{-1}c_\pm^\star f(\zeta+1)\quad\big(\text{for }\mathrm{Im}(\zeta)\gtrless 0\big),
\end{aligned}
$$

using again multiplier identities derived from the consistency relation (2.8). This shows that if P satisfies the three-term equation then f is nearly periodic. $\square$

6. Period functions

6.1. Period functions by integral transforms. We follow [12, Section 2.3], which is an extension of [11, Chapter II, Section 2] to real weights, and define the following integral transformation of a Maass cusp form.

Definition 6.1. Let $\zeta\in(0,\infty)$ and $\nu\in\mathbb{C}$, a compatible multiplier v and a weight $k\in\frac{1}{2}\mathbb{Z}$. Let u be a Maass cusp form of weight k, multiplier v and eigenvalue $\frac{1}{4}-\nu^2$.

We associate a function $P_{k,\nu}:(0,\infty)\to\mathbb{C};\ \zeta\mapsto P_{k,\nu}(\zeta)$ to the cusp form u by the integral transform

$$P_{k,\nu}(\zeta)=\int_0^{i\infty}\eta_{-k}\big(R_{-k,\nu}(\cdot,\zeta),u\big)(z), \tag{6.1}$$

where the path of integration is the upper imaginary axis, i.e., the geodesic connecting 0 and $i\infty$.

The integral transform above is well defined, as the following arguments show. Let $\zeta \in (0,\infty)$ and consider the function $R_{-k,\nu}(z,\zeta)$. The construction of $R_{-k}(\cdot,\zeta)$ implies polynomial growth for $\mathrm{Im}(z)\to\infty$ and $\mathrm{Im}(z)\downarrow 0$. The Maass cusp form u decays quicker than any polynomial at cusps, see Definition 3.1. Hence, the integral $\int_0^{i\infty}\eta_{-k}(R_{-k,\nu}(\cdot,\zeta),u)(z)$ is well defined.

REMARK 6.2. The definition of $P_{k,\nu}$ in [12, Definition 41] is

$$P_{k,\nu}(\zeta)=\int_0^{i\infty}\eta_k\big(u,R_{-k,\nu}(\cdot,\zeta)\big)(z)$$

which seems to differ from the one we use in (6.1). However, u and $R_{-k,\nu}$ are eigenfunctions of Δ_k and Δ_{-k}, respectively. This implies that the Maass–Selberg form is closed, see Lemma 4.10, and we have

$$\begin{aligned}&\int_0^{i\infty}\eta_{-k}\big(R_{-k,\nu}(\cdot,\zeta),u\big)(z)\\&\quad=\int_0^{i\infty}\eta_k\big(u,R_{-k,\nu}(\cdot,\zeta)\big)(z)+\int_0^{i\infty}\mathrm{d}\big(u(\cdot)R_{-k,\nu}(\cdot,\zeta)\big).\end{aligned}$$

Due to u being cuspidal, and hence vanishing in 0 and $i\infty$, we have

$$\int_0^{i\infty}\mathrm{d}\big(u(\cdot)R_{-k,\nu}(\cdot,\zeta)\big)=0.$$

Hence, the definitions of $P_{k,\nu}$ in (6.1) and in [12, Definition 41] agree. This also shows that the choice mentioned in Remark 5.1 does not matter for the period functions.

LEMMA 6.3. *Let k,v,ν and u be as in Definition 6.1, let $\zeta\in(0,\infty)$ and $\gamma\in\mathrm{SL}(2,\mathbb{Z})$ such that $\mu(\gamma,\zeta)>0$ and $\gamma(0,\infty)\subset(0,\infty)$. The function $P_{k,\nu}$ defined in (6.1) satisfies*

$$\big(P_{k,\nu}\|_\nu^v\gamma\big)(\zeta)=\int_{\gamma^{-1}0}^{\gamma^{-1}\infty}\eta_{-k}\big(R_{-k,\nu}(\cdot,\zeta),u\big)(z), \tag{6.2}$$

where the path of integration is the geodesic connecting $\gamma^{-1}0$ and $\gamma^{-1}\infty$.

Proof. We have

$$\begin{aligned}\big(P_{k,\nu}\|_\nu^v\gamma\big)(\zeta)&=v(\gamma)^{-1}\big(\mu(\gamma\zeta)\big)^{2\nu-1}\int_0^{\infty}\eta_{-k}\big(R_{-k,\nu}(\cdot,\gamma\zeta),u\big)(z)\\&=\int_0^{\infty}\eta_{-k}\big(R_{-k,\nu}(\cdot,\zeta),u\big)\big(\gamma^{-1}z\big)\quad\text{using Lemma 4.13}\\&=\int_{\gamma^{-1}0}^{\gamma^{-1}\infty}\eta_{-k}\big(R_{-k,\nu}(\cdot,\zeta),u\big)(z).\end{aligned}$$

The use of Lemma 4.13 is valid since ζ and $\mu(\gamma,\zeta)$ are both positive reals. The path of integration of the last integral is the geodesic connecting $\gamma^{-1}0$

and $\gamma^{-1}\infty$ and lies in the upper left quadrant $\{z \in \mathbb{C}; \mathrm{Re}(z) \le 0, \mathrm{Im}(z) \ge 0\}$ of $\mathbb{C}$. $\square$

We show next that $P_{k,\nu}$ satisfies the *three-term equation* on $\mathbb{R}_+$.

LEMMA 6.4. *Let ν, k and v as in Definition 6.1 and u a Maass cusp form with weight k compatible multiplier v and eigenvalue $\frac{1}{4}-\nu^2$. The function $P_{k,\nu}$ satisfies the three-term equation*

$$0 = P_{k,\nu}\|_\nu^v \big(\mathbf{1} - T - T'\big) \quad \textit{on } (0,\infty). \tag{6.3}$$

Proof. Let $\zeta > 0$. Lemma 6.3 allows us to write

$$\begin{aligned} 0 &= \left(\int_0^\infty - \int_{-1}^\infty - \int_0^{-1}\right) \eta_{-k}\big(R_{-k,\nu}(\cdot,\zeta), u\big)(z) \\ &= P_{k,\nu}(\zeta) - v(T)^{-1} P_{k,\nu}(T\zeta) - v\big(T'\big)^{-1} (\zeta+1)^{2\nu-1} P_{k,\nu}\big(T'\zeta\big) \\ &= P_{k,\nu}\|_\nu^v \big(\mathbf{1} - T - T'\big)(\zeta). \end{aligned}$$

$\square$

The next step is to extend $P_{k,\nu}(\zeta)$ to the right half plane $\{\zeta \in \mathbb{C}; \mathrm{Re}(\zeta) > 0\}$. Let ζ be in the right half-plane and recall that $R_{-k,\nu}(z,\zeta)$ is holomorphic in ζ if $\mathrm{Re}(z) \le 0$. Hence, the function $P_{k,\nu}(\zeta)$, given by the integral transform (6.1) extends holomorphically to $\{\zeta \in \mathbb{C}; \mathrm{Re}(\zeta) > 0\}$. It is easily checked that $P_{k,\nu}(\zeta+1)$ and $P_{k,\nu}(\frac{\zeta}{\zeta+1})$ have also holomorphic extensions to this right half-plane.

The last step is to extend $P_{k,\nu}$ to the cut plane $\mathbb{C}' = \mathbb{C} \setminus (-\infty, 0]$. Assume $\mathrm{Re}(\zeta) > 0$ for the moment. Since the differential form $\eta_k(u, R_{-k,\nu}(\cdot,\zeta))$ is closed, see Lemma 4.10, we replace vertical path of integration in (6.1) by a path which connects 0 and $i\infty$ in the upper left quadrant and which passes to the left of either ζ or $\bar{\zeta}$. We then may move ζ to any point for which either ζ or $\bar{\zeta}$ is still right of the new integration path. This procedure extends $P_{k,\nu}$ to a holomorphic function on $\mathbb{C}'$.

Summarizing we have the following theorem.

THEOREM 6.5. *Under the assumptions of Definition 6.1, the function $P_{k,\nu}$ associated to u by (6.1) extends to a holomorphic function on the cut plane $\mathbb{C}'$ which satisfies the three-term equation (6.3) on $\mathbb{R}_{>0}$. It also satisfies the growth conditions*

$$P_{k,\nu}(\zeta) = \begin{cases} \mathcal{O}\big(z^{\max\{0, 2\,\mathrm{Re}(\nu)-1}\big) & \textit{as } \mathrm{Im}(z) = 0, \zeta \downarrow 0 \textit{ and} \\ \mathcal{O}\big(z^{\min\{0, 2\,\mathrm{Re}(\nu)-1}\big) & \textit{as } \mathrm{Im}(z) = 0, \zeta \to \infty. \end{cases} \tag{6.4}$$

Proof. The first part of the proposition follows from the discussion above.

The cusp form u is bounded on $\mathbb{H}$ since a cusp form vanishes at all cusps $\mathbb{Q} \cup i\infty$ and u is real-analytic on $\mathbb{H}$. Also, $\mathrm{E}_k^+ u$ is bounded since the Maass operator maps cusp forms of weight k to cusp forms of weight $k+2$. Applying successively (6.1), (4.15), (4.10), (4.5) and (4.4) we find

$$\begin{aligned}
&P_{k,\nu}(\zeta)\\
&= \int_0^{i\infty} \left[\big(\mathrm{E}^+_{-k} R_{-k,\nu}(\cdot,\zeta)\big)(z) u(z) \frac{\mathrm{d}z}{y} - R_{-k,\nu}(z,\zeta) \big(\mathrm{E}^-_k u\big)(z) \frac{\mathrm{d}\bar{z}}{y} \right]\\
&= \int_0^{i\infty} \left[(1-2\nu-k) R_{2-k,\nu}(z,\zeta) u(z) \frac{\mathrm{d}z}{y} - R_{-k,\nu}(z,\zeta) \big(\mathrm{E}^-_k u\big)(z) \frac{\mathrm{d}\bar{z}}{y} \right]\\
&= i \int_0^{\infty} e^{ik\arg(\zeta-iy)} \left(\frac{y}{(\zeta-iy)(\zeta+iy)} \right)^{\frac{1}{2}-\nu}\\
&\quad \times \big[(1-2\nu-k) e^{2i\arg(\zeta-iy)} u(iy) - \big(\mathrm{E}^-_k u\big)(iy)\big] \frac{\mathrm{d}y}{y}
\end{aligned}$$

for $\zeta > 0$. Using the notation $f(z) \ll g(z)$ for $f(z) = \mathcal{O}(g(z))$, we find the estimate

$$\begin{aligned}
\big|P_{k,\nu}(\zeta)\big| \ll \int_0^{\infty} &\left| \frac{y}{\zeta^2+y^2} \right|^{\frac{1}{2}-\mathrm{Re}(\nu)}\\
&\cdot \max\Big\{ \big|\big(\mathrm{E}^-_k u\big)(iy)\big|, \big|(1-2\nu-k) u(iy)\big| \Big\} \frac{\mathrm{d}y}{y}
\end{aligned} \tag{6.5}$$

for $\zeta > 0$. The integral converges since u and hence $u(iy)$ and $(\mathrm{E}^-_k u)(iy)$ decay quickly as $y \to \infty$ and as $y \downarrow 0$.

Using the estimate

$$\frac{y}{\zeta^2+y^2} \le \zeta^{-2} y$$

in (6.5) gives

$$P_{k,\nu}(\zeta) = \mathcal{O}\big(\zeta^{2\,\mathrm{Re}(\nu)-1}\big) \quad \text{for every } \zeta > 0.$$

We have

$$P_{k,\nu}(\zeta) = \mathcal{O}(1) \quad \text{for every } \zeta > 0$$

if we use

$$\frac{y}{\zeta^2+y^2} \le y^{-1}$$

in (6.2). This proves the stated growth condition. □

6.2. Period functions and nearly periodic functions. Let us start with a Maass cusp form u of weight k, multiplier v and spectral value ν as in Definition 3.1. We associated in Section 5 a nearly periodic function f by the integral transform (5.1):

$$\begin{aligned}
&\mathbb{C} \setminus \mathbb{R} \to \mathbb{C};\\
&\zeta \mapsto f(\zeta) := \begin{cases} \int_\zeta^{i\infty} \eta_{-k}(R_{-k,\nu}(\cdot,\zeta), u)(z) & \text{if } \zeta \in \mathbb{H} \text{ and}\\ -\int_\zeta^{-i\infty} \eta_k(R_{-k,\nu}(\cdot,\zeta), \tilde{u})(z) & \text{if } \zeta \in \mathbb{H}^-. \end{cases}
\end{aligned} \tag{5.1}$$

Then, we attached a period-function P by (5.9):

$$P = f\|_\nu^v(\mathbf{1} - S) \quad \big(\text{on } \mathbb{H} \cup \mathbb{H}^-\big) \tag{5.9}$$

which satisfies the three-term equation

$$0 = P\|_\nu^v\big(\mathbf{1} - T - T'\big) \quad \big(\text{on } \mathbb{H} \cup \mathbb{H}^-\big). \tag{5.7}$$

On the other hand, we have the integral transformation (6.1) from the Maass cusp form u to the period function $P_{k,\nu}$:

$$P_{k,\nu}(\zeta) = \int_0^{i\infty} \eta_{-k}\big(R_{-k,\nu}(\cdot,\zeta), u\big)(z) \quad (\text{on } \mathbb{R}_{>0}) \tag{6.1}$$

which satisfies the three-term equation

$$0 = P\|_\nu^v\big(\mathbf{1} - T - T'\big) \quad (\text{on } \mathbb{R}_{>0}) \tag{6.3}$$

and extends to $\mathbb{C}'$ (Theorem 6.5).

Are both directions compatible? In other words, do we get the same function P on $\mathbb{H} \cup \mathbb{H}^-$, regardless of using the intermediate periodic function via (5.1) and (5.9) of taking the formula (6.1)?

Lemma 6.6. *Let k, v, ν and u be as in Definition 6.1 with $|\mathrm{Re}(\nu)| < \frac{1}{2}$. The maps*

$$u \overset{(5.1)}{\longmapsto} f \overset{(5.9)}{\longmapsto} P \quad \textit{and} \quad u \overset{(6.1)}{\longmapsto} P_{k,\nu}$$

give rise to the same function $P = P_{k,\nu}$ on $\{\zeta \in \mathbb{C}; \mathrm{Re}(\zeta) > 0, \mathrm{Im}(\zeta) \neq 0\}$.

Proof. For $\zeta \in \mathbb{H}$ with $\mathrm{Re}(\zeta) > 0$ in the upper half-plane, we find

$$\begin{aligned} P_{k,\nu}(\zeta) &= \int_0^{i\infty} \eta_{-k}\big(R_{-k,\nu}(\cdot,\zeta), u\big)(z) \\ &= \int_\zeta^{i\infty} \eta_{-k}\big(R_{-k,\nu}(\cdot,\zeta), u\big)(z) + \int_0^{\zeta} \eta_{-k}\big(R_{-k,\nu}(\cdot,\zeta), u\big)(z) \\ &= \int_\zeta^{i\infty} \eta_{-k}\big(R_{-k,\nu}(\cdot,\zeta), u\big)(z) - \int_\zeta^{S^{-1}i\infty} \eta_{-k}\big(R_{-k,\nu}(\cdot,\zeta), u\big)(z) \\ &= f(\zeta) - v(S)^{-1}\zeta^{2\nu-1} f(S\zeta) \quad \text{using Lemma 5.5} \\ &= P(\zeta). \end{aligned}$$

A similar calculation holds for $\zeta \in \mathbb{H}^-$ with $\mathrm{Re}(\zeta) > 0$:

$$\begin{aligned} P_{k,\nu}(\zeta) &= -\int_0^{i\infty} \eta_k\big(u, R_{-k,\nu}(\cdot,\zeta)\big)(z) \quad \text{using Lemma 4.16} \\ &= -\int_{\overline{\zeta}}^{i\infty} \eta_k\big(u, R_{-k,\nu}(\cdot,\zeta)\big)(z) - \int_0^{\overline{\zeta}} \eta_k\big(u, R_{-k,\nu}(\cdot,\zeta)\big)(z) \\ &= f(\zeta) - v(S)^{-1}\zeta^{2\nu-1} f(S\zeta) \quad \text{using Lemma 5.5} \\ &= P(\zeta). \end{aligned}$$

□

THEOREM 6.7. *The function P given by (5.9) on $\mathbb{H}\cup\mathbb{H}^-$ is holomorphic, extends holomorphically to the cut-plane $\mathbb{C}'=\mathbb{C}\setminus(-\infty,0]$, satisfies the three-term-equation $0=P\|_\nu^v(\mathbf{1}-T-T')$ on $\mathbb{C}'$, and satisfies the growth condition* (6.4).

Proof. Lemma 6.6 shows that P agrees on $\{\zeta\in\mathbb{C};\mathrm{Re}(\zeta)>0,\mathrm{Im}(\zeta)\neq 0\}$ with $P_{k,\nu}$ given by (6.1). The latter extends holomorphically to $\mathbb{C}'$ and satisfies the growth condition (6.4) by Proposition 6.5. ☐

7. Proof of Theorem 1.1

Our main theorem is basically proven in Section 5 and Section 6. We just have to collect all parts.

The map $u\mapsto f$ from Maass cusp forms to nearly periodic functions is defined in (5.1). That f is nearly periodic is shown in Lemma 5.7 and Theorem 6.7 shows the remaining part.

The bijection $f\leftrightarrow P$ is due to Lemma 5.8.

The map $u\mapsto P_{k,\nu}$ from Maass cusp forms to period functions is given in (6.1). The properties of $P_{k,\nu}$ are described in Theorem 6.5.

Lemma 6.6, cumulating in Theorem 6.7, shows that the period function P obtained via $u\overset{(5.1)}{\mapsto}f\overset{(5.8)}{\mapsto}P$ and via $u\overset{(6.1)}{\mapsto}P_{k,\nu}$ are the same.

This concludes the proof of Theorem 1.1.

8. Period functions and period polynomials

In the following section, we compare the integral transformation (6.1) and the classical Eichler integral in (1.1) for holomorphic cusp forms.

Let u_{h} be a modular cusp form of weight $k\in 2\mathbb{N}$ as defined in the introduction. We attach a Maass cusp form $u:\mathbb{H}\to\mathbb{C}$ to u_{h} by

$$u(z):=\mathrm{Im}(z)^{\frac{k}{2}}u_{\mathrm{h}}(z). \tag{8.1}$$

As shown in [13, Section 3.2], u is indeed a Maass cusp form of weight k, trivial multiplier $v\equiv 1$ and eigenvalue $\frac{k}{2}(1-\frac{k}{2})$. Hence, u has spectral values $\nu\in\{\frac{k-1}{2},\frac{1-k}{2}\}$.

The following proposition compares the period functions attached to u and the period polynomial attached to u_h. It is based on [12, Proposition 49].

PROPOSITION 8.1. *Let u be the Maass cusp form in* (8.1)*, which is derived from a modular cusp form u_{h} of weight $k\in 2\mathbb{N}$.*

(1) *The function $P_{k,\frac{1-k}{2}}$ associated to u by* (6.1) *vanishes everywhere.*
(2) *The function $P_{k,\frac{k-1}{2}}$ associated to u by* (6.1) *restricted to the right half-plane $\{\zeta\in\mathbb{C};\mathrm{Re}(\zeta)>0\}$ is a multiple of the period polynomial p associated to u_{h} by* (1.1)*: $P_{k,\frac{k-1}{2}}=(2-2k)p$.*

Proof. Since
$$\left(\frac{k-1}{2}\right)\left(\frac{1-k}{2}\right)=\frac{k}{2}\left(1-\frac{k}{2}\right),$$
we see that $\frac{k-1}{2}$ and $\frac{1-k}{2}$ are spectral values of u. Moreover, $\mathrm{E}_k^-\,u=0$ as shown in [13, Section 3.2].

Let $P_{k,\nu}$ be the period function associated to u via (6.1):
$$P_{k,\nu}(\zeta)=\int_0^{i\infty}\eta_{-k}\big(R_{-k,\nu}(\cdot,\zeta),u\big)(z).$$
Using (4.15) and (4.10), we then get
$$P_{k,\nu}(\zeta)=\int_0^{i\infty}\left[\big(\mathrm{E}_{-k}^+R_{-k,\nu}(\cdot,\zeta)\big)(z)u(z)\frac{\mathrm{d}z}{y}-R_{-k,\nu}(z,\zeta)\big(\mathrm{E}_k^-u\big)(z)\frac{\mathrm{d}\bar z}{y}\right].$$
Recalling that $\mathrm{E}_{-k}^+\,R_{-k,\nu}=(1-2\nu-k)R_{2-k,\nu}$ in (4.5) and $\mathrm{E}_k^-\,u=0$ above, we find
$$P_{k,\nu}(\zeta)=(1-2\nu-k)\int_0^{i\infty}R_{2-k,\nu}(z,\zeta)u(z)\frac{\mathrm{d}z}{y}\tag{8.2}$$
for $\mathrm{Re}(\zeta)>0$.

To prove the first part of the proposition, we assume $\nu=\frac{1-k}{2}$. Then, the factor $1-2\nu-k$ in (8.2) vanishes, implying $P_{k,\frac{1-k}{2}}=0$.

To prove the second part of the proposition, we assume $\nu=\frac{k-1}{2}$. By (4.2), we have
$$\begin{aligned}R_{2-k,\frac{k-1}{2}}(z,\zeta)&=\left(\frac{\sqrt{\zeta-z}}{\sqrt{\zeta-\bar z}}\right)^{k-2}\left(\frac{|\mathrm{Im}(z)|}{(\zeta-z)(\zeta-\bar z)}\right)^{\frac{2-k}{2}}\\&=(\zeta-z)^{k-2}\big|\mathrm{Im}(z)\big|^{\frac{2-k}{2}}\end{aligned}\tag{8.3}$$
for every ζ and z with $\zeta-z,\zeta-\bar z\neq\mathbb{R}_{\le 0}$. Combining this with (8.1) in (8.2) gives
$$P_{k,\frac{k-1}{2}}(\zeta)=(2-2k)\int_0^{i\infty}(\zeta-z)^{k-2}u_{\mathrm{h}}(z)\,\mathrm{d}z=(2-2k)p(\zeta)$$
for at least all ζ in the right half-plane. □

Can we also recover the periodic function f_h? The answer is given in the following proposition.

Proposition 8.2. *Let u be the Maass cusp form in* (8.1), *which is derived from a modular cusp form u_{h} of weight $k\in 2\mathbb{N}$.*

(1) *The integral transformation* (5.1) *defining $f(\zeta)$ for $\zeta\in\mathbb{H}$ is well-defined for both spectral values $\nu\in\{\frac{1-k}{2},\frac{k-1}{2}\}$.*
(2) *The function f associated to u by* (5.1) *with weight k and spectral value $\nu=\frac{1-k}{2}$ vanishes everywhere.*

(3) *The function f associated to u by* (5.1) *with weight k and spectral value $\nu = \frac{k-1}{2}$ and restricted to the upper half-plane $\mathbb{H}$ is a multiple of f_{h} associated to u_{h} by* (1.3): $f = (2-2k) f_{\mathrm{h}}$.

Proof. We follow the arguments of the proof of Proposition 8.1. For $\zeta \in \mathbb{H}$ is the nearly periodic function f associated to u given by (5.1). Using (4.15) and (4.10), we then get

$$f(\zeta) = \int_{\zeta}^{i\infty} \left[\left(\mathrm{E}_{-k}^{+} R_{-k,\nu}(\cdot,\zeta)\right)(z) u(z) \frac{\mathrm{d}z}{y} - R_{-k,\nu}(z,\zeta) \left(\mathrm{E}_{k}^{-} u\right)(z) \frac{\mathrm{d}\bar{z}}{y} \right].$$

Recalling $\mathrm{E}_{-k}^{+} R_{-k,\nu} = (1-2\nu-k) R_{2-k,\nu}$ in (4.5) and $\mathrm{E}_{k}^{-} u = 0$, we find

$$f(\zeta) = (1-2\nu-k) \int_{\zeta}^{i\infty} R_{2-k,\nu}(z,\zeta) u(z) \frac{\mathrm{d}z}{y} \tag{8.4}$$

for $\zeta \in \mathbb{H}$.

To prove the second part of the proposition, we assume $\nu = \frac{1-k}{2}$. Then, the factor $1-2\nu-k$ in (8.4) vanishes, implying $P_{k,\frac{1-k}{2}} = 0$.

To prove the third part of the proposition, we assume $\nu = \frac{k-1}{2}$. Using (8.3) and (8.1) in (8.4) gives

$$f(\zeta) = (2-2k) \int_{\zeta}^{i\infty} (\zeta - z)^{k-2} u_{\mathrm{h}}(z) \, \mathrm{d}z = (2-2k) f_{\mathrm{h}}(\zeta)$$

for every $\zeta \in \mathbb{H}$.

The first part follows also from the above calculations. Even if the calculations above are a priori formal, the well-definiteness of the results show that the original integral transforms are also well defined. We are just adding cleverly zeros. □

9. Discussion and outlook

In this paper, we introduced and discussed Eichler integrals attached to Maass cusp forms of half-integral weight. We also introduced the corresponding period functions. This generalizes on one hand the classical case of period polynomials and periodic functions associated to holomorphic modular cusp forms, as shown in Section 8. On the other hand, our results fit neatly with the also known case of Maass cusp forms of weight 0 and associated periodic and period functions, discussed in [11].

Obvious remaining questions are:

(1) For half-integral weight, do the period functions (i.e., the space of holomorphic solutions of the three-term equation (5.7) which satisfy the growth condition (6.4)) bijectively correspond to Maass cusp forms? We only show one direction.
(2) Can we use the introduced period functions to describe a "Eichler–Shimura-cohomology" for the half-integral or real weight case?

(3) Does everything also hold for real or complex weights and/or non-cuspidal forms? For example, can we extend the results to the general Maass wave forms introduced in [13]?
(4) What can we say about Eichler–Shimura theory of harmonic Maass wave forms?
(5) How are period functions and L-series related.

The first question is positively answered for Maass cusp forms of weight 0 in [11] and for real weights in [12]. Also, Bruggeman, Lewis and Zagier discuss recently the case of Maass forms (of weight 0 and of polynomial growth in the cusps) and is associated group cohomology in [3]. Deitmar and Hilgert discuss the situation for subgroups of finite index and weight 0 in $\mathrm{SL}(2,\mathbb{Z})$ in [5]. A recent result by Deitmar discusses the situation for Maass wave forms of higher order in [4].

To our knowledge, the second and third question are still open for general Maass wave forms with complex weight. Our results extend trivially to the case of Maass cusp forms with real weight (by just replacing half-integer with real everywhere). The third question is also positively answered in [8] for generalized modular forms (introduced in [7]).

The fourth question is answered in [1]. They show an Eichler–Shimura-type result for harmonic Maass wave forms, see e.g. [1, Theorem 1.2].

The last question is also discussed in [12], generalizing the first part of [11].

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