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ON ABEL SUMMABILITY OF JACOBI POLYNOMIALS SERIES, THE WATSON KERNEL AND APPLICATIONS

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Abstract. In this paper, we return to the study of the Watson kernel for the Abel summability of Jacobi polynomial series. These estimates have been studied for over more than 40 years. The main innovations are in the techniques used to get the estimates that allow us to handle the cases $0 < \alpha$ as well as $-1 < \alpha < 0$, with essentially the same methods. To that effect, we use an integral superposition of Natanson kernels, and the A. P. Calderón-Kurtz, B. Muckenhoupt A_p-weight theory. We consider also a generalization of a theorem due to Zygmund in the context of Borel measures. The proofs are different from the ones given in (Sobre la conjugación y sumabilidad de series de Jacobi (1971) Universidad de Buenos Aires, *Studia Math.* **49** (1974) 217–224, *Colloq. Math.* **30** (1974) 277–288 and *Illinois J. Math.* **41** (1997) 237–265). We will discuss in detail the Calderón–Zygmund decomposition for nonatomic Borel measures in $\mathbb{R}$. We prove that the Jacobi measure is doubling and following (*Studia Math.* **57** (1976) 297–306), we study the A_p weight theory in the context of Abel summability of Jacobi expansions. We consider power weights of the form $(1-x)^{\overline{\alpha}}, (1+x)^{\overline{\beta}},\ -1 < \overline{\alpha} < 0, -1 < \overline{\beta} < 0$. Finally, as an application of the weight theory we obtain L^p estimates for the maximal operator of Abel summability of Jacobi function expansions for suitable values of p.

1. Introduction

Given $\alpha, \beta > -1$, consider the Jacobi measure $\mathcal{J}^{\alpha,\beta}$ on $[-1,1]$, defined as

$$\mathcal{J}^{\alpha,\beta}(dx) = \omega_{\alpha,\beta}(x)\,dx = (1-x)^{\alpha}(1+x)^{\beta}\,dx. \tag{1.1}$$

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The Jacobi polynomials of parameters α, β, $\{P_n^{\alpha,\beta}\}_{n\geq 0}$ are orthogonal polynomials with respect to the measure $\mathcal{J}^{\alpha,\beta}$,

$$\int_{-1}^{1} P_n^{\alpha,\beta}(x) P_m^{\alpha,\beta}(x) \mathcal{J}^{\alpha,\beta}(dx) = 0, \quad \text{if } n \neq m, \tag{1.2}$$

with

$$\begin{aligned}\int_{-1}^{1} \left[P_n^{\alpha,\beta}(x)\right]^2 \mathcal{J}^{\alpha,\beta}(dx) &= \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha\beta+1)} \\ &= h_n^{(\alpha,\beta)}.\end{aligned} \tag{1.3}$$

The normalization is given by

$$P_n^{\alpha,\beta}(1) = \binom{n+\alpha}{n}. \tag{1.4}$$

The main reference for Jacobi polynomials is [19], see also [1], [2], [13] and [14].

The Jacobi functions are defined, for each n, as

$$F_n^{(\alpha,\beta)}(x) = P_n^{\alpha,\beta}(x)(1-x)^{\alpha/2}(1+x)^{\beta/2}. \tag{1.5}$$

Therefore, from (1.2) one gets that the Jacobi functions $\{F_n^{(\alpha,\beta)}\}$ are orthogonal on $[-1,1]$ with respect to the Lebesgue measure,

$$\int_{-1}^{1} F_n^{\alpha,\beta}(x) F_m^{\alpha,\beta}(x)\, dx = 0, \quad \text{if } n \neq m.$$

For any $f \in L^2([-1,1], \mathcal{J}^{\alpha,\beta})$, we consider its Fourier–Jacobi polynomial expansion

$$\sum_{n=0}^{\infty} \hat{f}^{(\alpha,\beta)}(n) P_n^{\alpha,\beta}(x), \tag{1.6}$$

where

$$\hat{f}^{(\alpha,\beta)}(n) = \frac{1}{h_n^{(\alpha,\beta)}} \int_{-1}^{1} f(y) P_n^{\alpha,\beta}(y) \mathcal{J}^{\alpha,\beta}(dy),$$

is the nth Fourier–Jacobi polynomial coefficient.[1] Then its partial sum of f, $s_m^{\alpha,\beta}(f,x)$, can be written as

$$s_m^{\alpha,\beta}(f,x) = \int_{-1}^{1} \mathcal{K}_m^{\alpha,\beta}(x,y) f(y) \mathcal{J}^{\alpha,\beta}(dy), \tag{1.7}$$

where

$$\mathcal{K}_m^{\alpha,\beta}(x,y) = \sum_{n=0}^{m} \frac{P_n^{\alpha,\beta}(x) P_n^{\alpha,\beta}(y)}{h_n^{(\alpha,\beta)}}.$$

[1] In [8], the expansions are considered with respect to the orthonormal family.

The kernel $\mathcal{K}_m^{\alpha,\beta}$ is called the *Dirichlet–Jacobi kernel.* Moreover, by orthogonality, we get

$$\int_{-1}^{1} \mathcal{K}_m^{\alpha,\beta}(x,y)\mathcal{J}^{\alpha,\beta}(dy) = 1.$$

Given the Jacobi polynomial series expansion of f, (1.6), let us consider its Abel summability

$$f^{\alpha,\beta}(r,x) = \sum_{n=0}^{\infty} r^n \hat{f}^{(\alpha,\beta)}(n) P_n^{\alpha,\beta}(x), \quad 0 < r < 1. \tag{1.8}$$

Using a classical argument and the estimate (see [19], (7.32.1))

$$\left|P_n^{\alpha,\beta}(x)\right| \le C n^{q+1/2}, \tag{1.9}$$

where $q = \max(\alpha,\beta) \ge -1/2$, it is easy to see that the series (1.8) converges uniformly and absolutely on $[-1,1]$. Therefore $f^{\alpha,\beta}(r,x)$ has the integral representation,

$$f^{\alpha,\beta}(r,x) = \int_{-1}^{1} K^{\alpha,\beta}(r,x,y) f(y) \mathcal{J}^{\alpha,\beta}(dy), \tag{1.10}$$

for $f \in L^1([-1,1], J_{\alpha,\beta})$. Here

$$K^{\alpha,\beta}(r,x,y) = \sum_{n=0}^{\infty} r^n \frac{P_n^{\alpha,\beta}(x) P_n^{\alpha,\beta}(y)}{h_n^{(\alpha,\beta)}}. \tag{1.11}$$

$K^{\alpha,\beta}$ is called the *Watson kernel.* Observe that the kernel is symmetric in x and y, i.e. $K^{(\alpha,\beta)}(r,x,y) = K^{(\alpha,\beta)}(r,y,x)$. The positivity of this kernel was initially proved by G. Gasper, see [11], [12]. The case of Abel summability for Gegenbauer expansions was considered by B. Muckenhoupt and E. Stein in their landmark paper in 1965 [16].

Analogously, for any $f \in L^2([-1,1])$ we consider its Fourier–Jacobi function expansion

$$\sum_{n=0}^{\infty} \tilde{f}^{(\alpha,\beta)}(n) F_n^{\alpha,\beta}(x), \tag{1.12}$$

where

$$\tilde{f}^{(\alpha,\beta)}(n) = \frac{1}{h_n^{(\alpha,\beta)}} \int_{-1}^{1} f(y) F_n^{\alpha,\beta}(y)\, dy,$$

is the nth Fourier–Jacobi function coefficient. Then its partial sum $\tilde{s}_m^{\alpha,\beta}(f,x)$, can be written as

$$\tilde{s}_m^{\alpha,\beta}(f,x) = \int_{-1}^{1} \tilde{\mathcal{K}}_m^{\alpha,\beta}(x,y) f(y)\, dy, \tag{1.13}$$

where

$$\begin{aligned}\tilde{\mathcal{K}}_m^{\alpha,\beta}&(x,y)\\ &=\sum_{n=0}^{m}\frac{F_n^{\alpha,\beta}(x)F_n^{\alpha,\beta}(y)}{h_n^{(\alpha,\beta)}}\\ &=\sum_{n=0}^{m}\frac{P_n^{\alpha,\beta}(x)P_n^{\alpha,\beta}(y)}{h_n^{(\alpha,\beta)}}\\ &\quad\times(1-x)^{\alpha/2}(1-y)^{\alpha/2}(1+x)^{\beta/2}(1+y)^{\beta/2}.\end{aligned}$$

Now, given the Jacobi function series expansion of f (1.12), consider its Abel summability

$$\tilde{f}^{\alpha,\beta}(r,x)=\sum_{n=0}^{\infty}r^n\tilde{f}^{(\alpha,\beta)}(n)F_n^{\alpha,\beta}(x),\quad 0<r<1. \tag{1.14}$$

Therefore, we also get the integral representation,

$$\tilde{f}^{\alpha,\beta}(r,x)=\int_{-1}^{1}\tilde{K}^{\alpha,\beta}(r,x,y)f(y)\,dy, \tag{1.15}$$

for $f\in L^1([-1,1])$, here

$$\begin{aligned}\tilde{K}^{\alpha,\beta}&(r,x,y)\\ &=\sum_{n=0}^{\infty}r^n\frac{F_n^{\alpha,\beta}(x)F_n^{\alpha,\beta}(y)}{h_n^{(\alpha,\beta)}}\\ &=\sum_{n=0}^{\infty}r^n\frac{P_n^{\alpha,\beta}(x)P_n^{\alpha,\beta}(y)}{h_n^{(\alpha,\beta)}}(1-x)^{\alpha/2}(1-y)^{\alpha/2}(1+x)^{\beta/2}(1+y)^{\beta/2}\\ &=K^{\alpha,\beta}(r,x,y)(1-x)^{\alpha/2}(1-y)^{\alpha/2}(1+x)^{\beta/2}(1+y)^{\beta/2}.\end{aligned}$$

$\tilde{K}^{\alpha,\beta}$ is called the *modified Watson kernel* for Jacobi functions.

From the previous representation and (1.15) we get,

$$\begin{aligned}\tilde{f}^{\alpha,\beta}(r,x)&=(1-x)^{\alpha/2}(1+x)^{\beta/2}\\ &\quad\times\int_{-1}^{1}K^{\alpha,\beta}(r,x,y)(1-y)^{\alpha/2}(1+y)^{\beta/2}f(y)\,dy.\end{aligned} \tag{1.16}$$

In 1936 Watson obtained the following representation for $K^{\alpha,\beta}(r,x,y)$, see [10], page 272, and also [21],

$$\begin{aligned}K^{\alpha,\beta}(r,x,y)&=r^{(1-\alpha-\beta)/2}\\ &\quad\times\frac{d}{dr}\left(k^{1+\alpha+\beta}\int_0^{\pi/2}\frac{\sec^{2+\alpha+\beta}\omega\cos(\alpha-\beta)\omega}{Z_1^{\alpha}Z_2^{\beta}Y}\,d\omega\right),\end{aligned} \tag{1.17}$$

where $k=\frac{1}{2}(r^{1/2}+r^{-1/2}), s=k\sec\omega$,

$$
\begin{aligned}
Y &= \left(\left(\frac{x-y}{2}\right)^2+\left(s^2-1\right)\left(s^2-xy\right)\right)^{1/2},\\
Z_1 &= s^2-\frac{1}{2}(x+y)+Y, \quad \text{and}\\
Z_2 &= s^2+\frac{1}{2}(x+y)+Y.
\end{aligned}
$$

The integral in (1.17) can be proved that is convergent only if $\alpha+\beta>-1$; since $s\geq 2$, $Y^2\sim s^4, Z_1\sim s^2, Z_2\sim s^2$, then taking the change of variable $s=k\sec\omega$,

$$\int_0^{\pi/2}\frac{\sec^{2+\alpha+\beta}\omega\cos(\alpha-\beta)\omega}{Z_1^\alpha Z_2^\beta Y}\,d\omega\leq k^{-(2+\alpha+\beta)}\int_k^\infty\frac{s^{\alpha+\beta+1}}{Z_1^\alpha Z_2^\beta Y}\frac{k\,ds}{s\sqrt{s^2-k^2}}.$$

Assuming that $1/2<r<1$, and then $1<k<3/2<2$, for $2<s<\infty$,

$$
\begin{aligned}
\int_2^\infty\frac{s^{\alpha+\beta+1}}{Z_1^\alpha Z_2^\beta Y}\frac{k\,ds}{s\sqrt{s^2-k^2}} &\sim C\int_k^\infty\frac{s^{\alpha+\beta+1}}{s^{2\alpha}s^{2\beta}s^2}\frac{ds}{\sqrt{s^2-k^2}}\\
&= C\int_k^\infty\frac{1}{s^{\alpha+\beta+2}}\,ds=C(\alpha,\beta)<\infty,
\end{aligned}
$$

therefore

$$
\begin{aligned}
\int_0^{\pi/2}\frac{\sec^{2+\alpha+\beta}\omega\cos(\alpha-\beta)\omega}{Z_1^\alpha Z_2^\beta Y}\,d\omega &\leq Ck^{-(1+\alpha+\beta)}\int_k^2\frac{s^{\alpha+\beta+1}}{s^{2\alpha}s^{2\beta}s^2}\frac{ds}{\sqrt{s^2-k^2}}\\
&= C(\alpha,\beta).
\end{aligned}
$$

The Watson kernel is good for localization. The deficits of this representation are:

- First, the integral is only convergent for $\alpha+\beta>-1$,
- It is not clear from the representation that the kernel is positive.

There is another representation of the Watson kernel obtained by W. N. Bailey in 1939 ([4], page 102, see also [3], page 11),

$$
\begin{aligned}
K^{(\alpha,\beta)}(r,x,y) &= \frac{\Gamma(\alpha+\beta+2)(1-r)}{2^{\alpha+\beta+2}\Gamma(\alpha+1)\Gamma(\beta+1)(1+r)^{\alpha+\beta+2}}\\
&\quad\times\sum_n\sum_m\frac{(\frac{(\alpha+\beta+2)}{2})_{m+n}(\frac{(\alpha+\beta+3)}{2})_{m+n}}{m!n!(\alpha+1)_m(\beta+1)_n}\left(\frac{a^2}{k^2}\right)^m\left(\frac{b^2}{k^2}\right)^n\\
&= \frac{\Gamma(\alpha+\beta+2)(1-r)}{2^{\alpha+\beta+2}\Gamma(\alpha+1)\Gamma(\beta+1)(1+r)^{\alpha+\beta+2}}\\
&\quad\times F_4\left(\frac{(\alpha+\beta+2)}{2},\frac{(\alpha+\beta+3)}{2};\alpha+1,\beta+1;\frac{a^2}{k^2},\frac{b^2}{k^2}\right),
\end{aligned}
$$

with $a = \frac{\sqrt{(1-x)(1-y)}}{2}, b = \frac{\sqrt{(1+x)(1+y)}}{2}$, and as before $k = \frac{1}{2}(r^{-1/2} + r^{1/2})$. F_4 is the Appell hypergeometric function in two variables,

$$F_4(\alpha, \beta; \gamma, \gamma'; x, y) = \sum_n \sum_m \frac{(\alpha)_{m+n}(\beta)_{m+n}}{m!n!(\gamma)_m(\gamma')_n} x^m y^n.$$

Let us observe that the condition for absolute convergence of the F_4 function is $|x|^{1/2} + |y|^{1/2} < 1$, see [20], and therefore the expression above for $K^{(\alpha,\beta)}(r,x,y)$ converges absolutely if $\frac{a}{k} + \frac{b}{k} < 1$ and that there is not restriction on α, β, i.e., it is valid for any $\alpha > -1, \beta > -1$.

Observe that by direct inspection of Bailey's representation it is clear that

$$K^{(\alpha,\beta)}(r,x,y) \geq 0.$$

From the uniform convergence of the Jacobi polynomials series and the fact that the system is complete, it can be proved, using the orthogonality, that

$$\int_{-1}^{1} K^{(\alpha,\beta)}(r,x,y) \mathcal{J}^{\alpha,\beta}(dy) = 1.$$

On the other hand, by Hölder's inequality, it is easy to see that for $1 \leq p \leq \infty$,

$$\left\| f^{\alpha,\beta}(r,\cdot) \right\|_{p,\alpha,\beta} \leq \|f\|_{p,\alpha,\beta}, \tag{1.18}$$

where

$$\|f\|_{p,\alpha,\beta} = \left(\int_{-1}^{1} \left| f(x) \right|^p \mathcal{J}^{\alpha,\beta}(dy) \right)^{1/p},$$

is the L^p norm with respect to the Jacobi measure $\mathcal{J}^{\alpha,\beta}(dy)$.

Moreover, we have the strong L^p-convergence of the Abel sum. We will present an elementary and direct proof of this result, without further restriction that $\alpha > -1, \beta > -1$.

LEMMA 1.1. *For $\alpha > -1, \beta > -1$*

$$\left\| f^{\alpha,\beta}(r,\cdot) - f \right\|_{p,\alpha,\beta} \to 0, \quad \textit{as } r \to 1, \tag{1.19}$$

for $1 < p \leq \infty$.

Proof. The proof will be done in cases, for different values of p.

(i) For the case $p = 2$, using Parserval's identity, the positivity of the kernel $K^{(\alpha,\beta)}(r,x,y)$ and the completeness of $\{P_n^{\alpha,\beta}\}$, we have immediately that for $f \in L^2(\mathcal{J}^{\alpha,\beta})$,

$$\left\| f^{\alpha,\beta}(r,\cdot) - f \right\|_{2,\alpha,\beta} = \sum_{n=0}^{\infty} \left(r^{2n} - 1 \right) \left| \hat{f}^{(\alpha,\beta)}(n) \right|^2 \to 0,$$

as $r \to 1$.

(ii) For $p \neq 2$, fix $\lambda > 0$, and let $f \in L^p(\mathcal{J}^{\alpha,\beta})$. Without any loss of generality we may assume $f \geq 0$. Write f as $f = f_1 + f_2$ with $|f_1| \leq \lambda$, $f_1 \in L^2(\mathcal{J}^{\alpha,\beta})$ and let us take λ big enough that $\|f_2\|_p < \varepsilon$.

- Now if $2 < p \leq \infty$, then $|\frac{f_1}{\lambda}| \leq 1$ implies $|\frac{f_1}{\lambda}|^p \leq |\frac{f_1}{\lambda}|^2$,

$$\begin{aligned}\left\|f_1^{\alpha,\beta}(r,\cdot) - f_1\right\|_{p,\alpha,\beta}^p &= 2^p\lambda^p\left\|\frac{1}{2}\left(\frac{f_1}{\lambda}\right)^{\alpha,\beta}(r,\cdot) - \frac{1}{2}\left(\frac{f_1}{\lambda}\right)\right\|_{p,\alpha,\beta}^p \\ &\leq 2^p\lambda^p\left\|\frac{1}{2}\left(\frac{f_1}{\lambda}\right)^{\alpha,\beta}(r,\cdot) - \frac{1}{2}\left(\frac{f_1}{\lambda}\right)\right\|_{2,\alpha,\beta}^2 \\ &= 2^{p-2}\lambda^{p-2}\left\|f_1^{\alpha,\beta}(r,\cdot) - f_1\right\|_{2,\alpha,\beta}^2 \to 0\end{aligned}$$

as $r \to 1$, from the previous case. Now from (1.18)

$$\begin{aligned}\left\|f_2^{\alpha,\beta}(r,\cdot) - f_2\right\|_{p,\alpha,\beta}^p &\leq 2^p\left(\left\|f_2^{\alpha,\beta}(r,\cdot)\right\|_{p,\alpha,\beta}^p + \|f_2\|_{p,\alpha,\beta}^p\right) \\ &\leq 2^{p+1}\|f_2\|_{p,\alpha,\beta}^p < 2^{p+1}\varepsilon^p.\end{aligned}$$

- Finally, for $1 \leq p < 2$, since $|f_1| \leq \lambda$, using Hölder's inequality, we have

$$\left\|f_1^{\alpha,\beta}(r,\cdot) - f_1\right\|_{p,\alpha,\beta}^p \leq C\left\|f_1^{\alpha,\beta}(r,\cdot) - f_1\right\|_{2,\alpha,\beta}^2.$$

The inequality for f_2 is obtained similarly as in the previous case. □

The maximal function $f^*_{\alpha,\beta}$ for the Abel summability of the Jacobi polynomial expansions is defined as

$$f^*_{\alpha,\beta}(x) = \sup_{0<r<1}\left|f^{\alpha,\beta}(r,x)\right| = \sup_{0<r<1}\left|\int_{-1}^1 K^{\alpha,\beta}(r,x,y)f(y)\mathcal{J}^{\alpha,\beta}(dy)\right|. \tag{1.20}$$

We will give an alternative proof, as a consequence of the main result of this paper, that, for $\alpha+\beta > -1$, $f^*_{\alpha,\beta}$ is weak-$(1,1)$ continuous with respect to $\mathcal{J}^{\alpha,\beta}$, that is,

$$\mathcal{J}^{\alpha,\beta}\left\{f^*_{\alpha,\beta} > \lambda\right\} \leq \frac{C_{\alpha,\beta}}{\lambda}\|f\|_{1,\alpha,\beta}. \tag{1.21}$$

On the other hand, using Bailey's representation it is almost trivial to get, for $\alpha > -1, \beta > -1$

$$\left\|f^{\alpha,\beta}(r,\cdot)\right\|_\infty \leq C\|f\|_\infty. \tag{1.22}$$

It follows then

$$\left\|f^*_{\alpha,\beta}\right\|_\infty \leq C\|f\|_\infty. \tag{1.23}$$

Therefore, by interpolation we get, for $1 < p < \infty$,

$$\left\|f^*_{\alpha,\beta}\right\|_{p,\alpha,\beta} \leq C\|f\|_{p,\alpha,\beta}. \tag{1.24}$$

For more details on the Jacobi maximal function can be found in [5], [6] and [8].

After this paper was completed, we learned that there is some overlapping with results obtained by A. Nowak, P. Sjögren and collaborators, see [17]. It is important to note that th Calderón–Zygmund decomposition for Jacobi measures was introduced in 1971 by Luis Cafarelli, as an auxiliary result, in his doctoral dissertation, see [5].

2. Estimates of the Watson kernel

By the product rule in the Watson representation (1.17),

$$K^{\alpha,\beta}(r,x,y) = r^{(1-\alpha-\beta)/2}\frac{d}{dr}\left(k^{1+\alpha+\beta}\int_0^{\pi/2}\frac{\sec^{2+\alpha+\beta}\omega\cos(\alpha-\beta)\omega}{Z_1^\alpha Z_2^\beta Y}\,d\omega\right)$$

we get four kernels A, B, C, D defined in the following way,

$$A = r^{(1-\alpha-\beta)/2}\frac{d}{dr}\big(k^{1+\alpha+\beta}\big)\int_0^{\pi/2}\frac{\sec^{2+\alpha+\beta}\omega\cos(\alpha-\beta)\omega}{Z_1^\alpha Z_2^\beta Y}\,d\omega,$$

$$B = r^{(1-\alpha-\beta)/2}k^{1+\alpha+\beta}\int_0^{\pi/2}\frac{d}{dr}\big(Y^{-1}\big)\frac{\sec^{2+\alpha+\beta}\omega\cos(\alpha-\beta)\omega}{Z_1^\alpha Z_2^\beta}\,d\omega,$$

$$C = r^{(1-\alpha-\beta)/2}k^{1+\alpha+\beta}\int_0^{\pi/2}\frac{d}{dr}\big(Z_1^{-\alpha}\big)\frac{\sec^{2+\alpha+\beta}\omega\cos(\alpha-\beta)\omega}{Z_2^\beta Y}\,d\omega,$$

$$D = r^{(1-\alpha-\beta)/2}k^{1+\alpha+\beta}\int_0^{\pi/2}\frac{d}{dr}\big(Z_2^{-\beta}\big)\frac{\sec^{2+\alpha+\beta}\omega\cos(\alpha-\beta)\omega}{Z_1^\alpha Y}\,d\omega.$$

Then we have, see [6], pages 282–283 or [8], Lemma 4.1, pages 245–249,

LEMMA 2.1. *We have the following estimate for the Watson kernel,*

$$K^{\alpha,\beta}(r,x,y) \le C(\alpha,\beta)\big(1+L(r,x,y)\big), \tag{2.1}$$

where $C(\alpha,\beta)$ is a positive constant, depending on α,β only, $L(r,x,y)$ is the integral

$$L(r,x,y) = (1-r)\int_k^2 \frac{(s-\min(x,y))^{1-\alpha}}{((x-y)^2+(s-1)(s-\min(x,y)))^{3/2}}\frac{ds}{(s-k)^{1/2}},$$

where $k = \frac{1}{2}(r^{1/2}+r^{-1/2}), 0 \le x \le 1$.

For the proof of this lemma, the following estimates will be needed, for detail see Appendix in [8]. Let $1 \le s \le 2, 0 \le x \le 1, |y| \le 1$. Then:

(i) $s^2 - \min(x,y) \le 4(s-\min(x,y))$;
(ii) $s-\min(x,y) \le 2(s-xy) \le 4(s-\min(x,y))$;
(iii) $C_1((x-y)^2+(s-1)(s-\min(x,y))) \le Y^2$ and $Y^2 \le C_2((x-y)^2+(s-1)(s-\min(x,y)))$;
(iv) $s^2-\min(x,y) \le Z_1 \le C(s^2-\min(x,y))$;
(v) $1 \le s^2+\max(x,y) \le Z_2 \le C$;
(vi) If $\varphi(x,r) = (k-1)^{1/2}(k-x)^{1/2}$, then $k-1 \le \phi(x,r) \le k-x$, for $k>1$;
(vii) $C_1(1-r)^2 \le k-1 \le C_2(1-r^2)$, if $0<r_0<r<1$.

Here C, C_1, C_2 denote positive constants. From these estimates, observe that:

- By (iii), $Y^2 \sim ((x-y)^2+(s-1)(s-\min(x,y)))$.

- By (iv), $Z_1 \sim (s^2 - \min(x, y))$.
- By (v), Z_2 is essentially a constant.

Observe that if $-1 < x < 0$ similar estimates hold, just changing the role of α and β. For details of the proof of Lemma 1 see [8], Lemma 4.1.

Finally in [6], pages 284–286 and [8], Lemma 4.1, page 254, the following estimate for L was obtained.

LEMMA 2.2.

$$L(r, x, y) \leq C_{\alpha,\beta} \sum_{n=0}^{\infty} \frac{1}{2^{n/2}} \frac{1}{\mathcal{J}^{\alpha,\beta}(I_n(x, r))} \chi_{I_n(x,r)}, \tag{2.2}$$

where $I_n(x, t) = [x - 2^n \varphi(x, r), x + 2^n \varphi(x, r)] \cap [-1, 1]$, $\chi_{I_n(x,r)}$ *is its characteristic function and* $\varphi(x, r) = (k - 1)^{1/2}(k - x)^{1/2}$.

Our aim in this paper to get another estimate for $L(r, x, y)$ using superposition of Natanson's kernels. The following technical result, see (5.1) and (5.2) of [8], is needed, for completeness the proof will be given.

LEMMA 2.3. *For* k *chosen as above, there exist constants* C_1 *and* C_2 *independent of* r *such that,*

$$(1 - r) \int_k^2 \frac{1}{(s - k)^{1/2}(s - 1)}\, ds < C_1 \tag{2.3}$$

and

$$(1 - r) \int_k^2 \frac{1}{(s - k)^{1/2}(s - 1)^{1/2}(s - x)^{1/2}}\, ds < C_2. \tag{2.4}$$

Proof. Let us prove first (2.3). Observe that, by the estimate (vii) we have $(k - 1) \sim (1 - r)^2$ i.e. $(k - 1)^{1/2} \sim (1 - r)$. Then, integrating by parts,

$$\begin{aligned}(k - 1)^{1/2} &\int_k^2 \frac{1}{(s - k)^{1/2}(s - 1)}\, ds \\ &= (k - 1)^{1/2} \left[2(2 - k)^{1/2} + \int_k^2 \frac{(s - k)^{1/2}}{(s - 1)^2}\, ds \right],\end{aligned}$$

and

$$\begin{aligned}(k - 1)^{1/2} \int_k^2 \frac{(s - k)^{1/2}}{(s - 1)^2}\, ds &\leq (k - 1)^{1/2} \int_k^2 \frac{1}{(s - 1)^{3/2}}\, ds \\ &= (k - 1)^{1/2} \int_k^2 \frac{1}{(s - k + k - 1)^{3/2}}\, ds \\ &\leq \int_k^2 \frac{1}{(k - 1)} \frac{1}{(|\frac{s-k}{k-1}| + 1)^{3/2}}\, ds \\ &= \frac{1}{\lambda} \int_k^2 k_1\left(\frac{s - k}{\lambda}\right) ds < C,\end{aligned}$$

where $\lambda = (k-1)$ and the Poisson type kernel $k_1(x) = \frac{1}{(|x|+1)^{3/2}}$. Observe that $\int_{-\infty}^{\infty} k_1(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{(|x|+1)^{3/2}}\,dx = 4$.

The second estimate (2.4) follows immediately from (2.3). $\square$

The following technical result is also needed for the proof of the coming Theorem 2.6.

LEMMA 2.4. *For any* $\eta > 1$

$$\sup_{0<|a|<1} \frac{1}{[(z+a)^2+1]^\eta} \leq \frac{C}{[z^2+1]^\eta}.$$

Proof. Let us consider two cases:

- If $|z| > 3$ that is, $\frac{|z|}{3} > 1$, then for $0 < |a| < 1$

$$|z+a| \geq |z| - |a| > \frac{2|z|}{3} + \left(\frac{|z|}{3} - 1\right) \geq \frac{2|z|}{3}, \quad \text{so}$$

$$|z+a|^2 \geq \frac{4|z|^2}{9}.$$

Thus

$$\frac{1}{[(z+a)^2+1]^\eta} \leq \frac{1}{[\frac{4|z|^2}{9}+1]^\eta}$$
$$\leq \frac{1}{[\frac{4|z|^2}{9}+\frac{4}{9}]^\eta} = \frac{(\frac{9}{4})^\eta}{[|z|^2+1]^\eta} = \frac{C}{[|z|^2+1]^\eta}.$$

- If $|z| < 3$, then

$$\frac{1}{[(z+a)^2+1]^\eta} \leq 1, \quad \text{and} \quad \frac{1}{10^\eta} \leq \frac{1}{[z^2+1]^\eta} \leq 1,$$

thus

$$\frac{1}{[(z+a)^2+1]^\eta} \leq 1 \leq \frac{10^\eta}{[z^2+1]^\eta} = \frac{C}{[z^2+1]^\eta}.$$

$\square$

DEFINITION 2.5. Given $-\infty \leq a < b \leq \infty$, a Borel measure μ with support in (a,b), a nonnegative kernel $K(r,x,y)$ depending on a parameter r, that satisfies

(i) K it is monotone increasing in y, for $a < y < x$, monotone decreasing in y, for $b > y > x$

(ii)

$$\int_a^b K(r,x,y)\mu(dy) \leq M,$$

where M is independent of x and r,

is called a *Natanson's kernel* with respect to μ.

THEOREM 2.6. *For $\alpha > -1$, the expression*

$$L := (1-r)\int_k^2 \frac{(s-\min(x,y))^{1-\alpha}}{((x-y)^2+(s-1)(s-\min(x,y)))^{3/2}}\frac{ds}{(s-k)^{1/2}} \tag{2.5}$$

is bounded by a superposition of a family of Natanson's kernels integrated with respect to the parameter s.[2] *Calling $K^*(x,y,r,\alpha)$ that bounding kernel, we have that $\int_0^1 K^*(x,y,r,\alpha)(1-y)^\alpha\,dy$ is bounded from above independent from x,r.*

Proof. We will consider two cases,

(i) Case $\alpha \geq 0$.

(i-1) If $x \leq y < 1$: Calling (L,I) the corresponding part of L in this range we have

$$\begin{aligned}(L,I) &= (1-r)\int_k^2 \frac{(s-x)^{1-\alpha}}{((x-y)^2+(s-1)(s-x))^{3/2}}\frac{ds}{(s-k)^{1/2}}\\ &\leq (1-r)\int_k^2 \frac{1}{(s-k)^{1/2}}\frac{(s-x)^{-\alpha}}{(s-1)}\\ &\qquad\times\frac{1}{[(s-1)(s-x)]^{1/2}}\frac{1}{((\frac{x-y}{[(s-1)(s-x)]^{1/2}})^2+1)^{3/2}}\,ds.\end{aligned}$$

Considering the Poisson type kernel

$$k_2(x) = \frac{1}{(x^2+1)^{3/2}},$$

we get, for $\lambda = [(s-1)(s-x)]^{1/2}$,

$$\begin{aligned}(L,I) &\leq (1-r)\int_k^2 \frac{(s-x)^{-\alpha}}{(s-k)^{1/2}(s-1)}\frac{1}{\lambda}k_2\left(\frac{x-y}{\lambda}\right)ds\\ &\leq (1-r)\int_k^2 \frac{(s-x)^{-\alpha}}{(s-k)^{1/2}(s-1)}K_\lambda(x-y)\,ds\end{aligned}$$

with

$$\begin{aligned}K_\lambda(x-y) &= \frac{1}{[(s-1)(s-x)]^{1/2}}\frac{1}{((\frac{x-y}{[(s-1)(s-x)]^{1/2}})^2+1)^{3/2}}\\ &= \frac{1}{\lambda}k_2\left(\frac{x-y}{\lambda}\right).\end{aligned}$$

For fixed x, $K_\lambda(x-y)$ is a Natanson's kernel and moreover,

$$\frac{1}{(s-x)^\alpha}K_\lambda(x-y)$$

[2] We are following similar notation as in [8].

is a Natanson's kernel (since $\frac{1}{(s-x)^\alpha}$ does not depend on y). Hence, this integral with respect to s is also a Natanson kernel.

Now if we integrate (L, I) with respect to the measure $\mu_\alpha(dy) = (1-y)^\alpha\, dy$, noticing that on this range, $(s-x)^{-\alpha} \le (1-y)^{-\alpha}$,

$$\int_{-\infty}^{\infty} k_2(x)\, dx = \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^{3/2}}\, dx = 2,$$

using also Lemma 2.3, we get

$$\begin{aligned}
&\int_x^1 (L, I)(1-y)^\alpha\, dy \\
&\quad\le \int_0^1 (1-r) \int_k^2 \frac{(s-x)^{-\alpha}}{(s-k)^{1/2}(s-1)} K_\lambda(x-y)\, ds (1-y)^\alpha\, dy \\
&\quad\le \int_0^1 (1-r) \int_k^2 \frac{1}{(s-k)^{1/2}(s-1)}\, ds K_\lambda(x-y)\, dy \le C.
\end{aligned}$$

(i-2) If $0 < y < x$: Calling (L, II) the corresponding part of L in this range, using the same notation as in (i-1), we have

$$(L, II) \le (1-r) \int_k^2 \frac{(s-y)^{1-\alpha}}{(s-k)^{1/2}(s-1)(s-x)} K_\lambda(x-y)\, ds.$$

Now, writing

$$\begin{aligned}
(s-y)^{1-\alpha} &= (s-y)(s-y)^{-\alpha} \\
&= \big[(s-x)+(x-y)\big](s-y)^{-\alpha},
\end{aligned}$$

we get two terms,

$$\begin{aligned}
(L, II) \le{}& (1-r) \int_k^2 \frac{(s-y)^{-\alpha}}{(s-k)^{1/2}(s-1)} K_\lambda(x-y)\, ds \\
&+ (1-r) \int_k^2 \frac{(s-y)^{-\alpha}}{(s-k)^{1/2}(s-1)(s-x)} (x-y) K_\lambda(x-y)\, ds \\
={}& (L, II1) + (L, II2).
\end{aligned}$$

The first term is analogous to case (i-1). But in this range

$$(s-y)^{-\alpha} \le (1-y)^{-\alpha},$$

and

$$\begin{aligned}
&(x-y) K_\lambda(x-y) \\
&\quad\le \frac{|x-y|}{[(s-1)(s-x)]^{1/2}} \frac{1}{\big((\frac{x-y}{[(s-1)(s-x)]^{1/2}})^2+1\big)^{3/2}}
\end{aligned}$$

$$\leq C\frac{[[\frac{|x-y|}{[(s-1)(s-x)]^{1/2}}]^2+1]^{1/2}}{((\frac{x-y}{[(s-1)(s-x)]^{1/2}})^2+1)^{3/2}}$$
$$= C\big[(s-1)(s-x)\big]^{1/2}\frac{1}{[(s-1)(s-x)]^{1/2}}\frac{1}{((\frac{x-y}{[(s-1)(s-x)]^{1/2}})^2+1)}.$$

Hence, by considering the Poisson type kernel $k_3(x)=\frac{1}{(x^2+1)}$, we get, for $\lambda=[(s-1)(s-x)]^{1/2}$,

$$\begin{aligned}(x-y)K_\lambda(x-y) &\leq C\big[(s-1)(s-x)\big]^{1/2}\frac{1}{\lambda}k_3\left(\frac{x-y}{\lambda}\right)\\ &= C\big[(s-1)(s-x)\big]^{1/2}K^*_\lambda(x-y).\end{aligned}$$

Finally, for fixed x, $(1-y)^{-\alpha}K_\lambda(x-y)$ is increasing for $y<x$, so is the case for $(1-y)^{-\alpha}K^*_\lambda(x-y)$. Thus, they are (one-sided) Natanson's kernels.

Therefore, (L,II) is then dominated by the sum of the kernels, for $y<x$

$$(1-r)\int_k^2\frac{1}{(s-k)^{1/2}(s-1)(1-y)^\alpha}K_\lambda(x-y)\,ds,$$

and

$$(1-r)\int_k^2\frac{[(s-1)(s-x)]^{1/2}}{(s-k)^{1/2}(s-1)(s-x)(1-y)^\alpha}K^*_\lambda(x-y)\,ds.$$

Then, integrating (L,II) respect to the measure $\mu_\alpha(dy)=(1-y)^\alpha\,dy$, on this range, using (2.3), we get the uniform boundedness,

$$\begin{aligned}&\int_0^x(L,II)(1-y)^\alpha\,dy\\ &\quad\leq\int_0^1(1-r)\int_k^2\frac{(1-y)^{-\alpha}}{(s-k)^{1/2}(s-1)}K_\lambda(x-y)\,ds(1-y)^\alpha\,dy\\ &\qquad+\int_0^1(1-r)\int_k^2\frac{[(s-1)(s-x)]^{1/2}(1-y)^{-\alpha}}{(s-k)^{1/2}(s-1)(s-x)}K^*_\lambda(x-y)\,ds(1-y)^\alpha\,dy\\ &\quad\leq\int_0^1(1-r)\int_k^2\frac{1}{(s-k)^{1/2}(s-1)}K_\lambda(x-y)\,ds\,dy\\ &\qquad+\int_0^1(1-r)\int_k^2\frac{[(s-1)(s-x)]^{1/2}}{(s-k)^{1/2}(s-1)(s-x)}K^*_\lambda(x-y)\,ds\,dy\\ &\quad< C.\end{aligned}$$

(ii) Case $-1<\alpha<0$.

(ii-1) If $x \leq y < 1$: Let (L, III), be the corresponding part of L in this range, as

$$(L, III) \leq (1-r) \int_k^2 \frac{(s-x)^{1-\alpha}}{(s-k)^{1/2}(s-1)(s-x)} \times \frac{1}{[(s-1)(s-x)]^{1/2}} \frac{1}{((\frac{x-y}{[(s-1)(s-x)]^{1/2}})^2+1)^{3/2}}\, ds$$
$$= (1-r) \int_k^2 \frac{(s-x)^{-\alpha}}{(s-k)^{1/2}(s-1)} K_\lambda(x-y)\, ds.$$

This kernel is decreasing for $y > x$. We need a bound for the integral with respect to the measure $\mu_\alpha(dy) = (1-y)^\alpha\, dy$. Now, let us write

$$x - y = \big[x + (1-s) - y\big] + (s-1),$$

and obtain

$$\frac{x-y}{[(s-1)(s-x)]^{1/2}} = \frac{x+(1-s)-y}{[(s-1)(s-x)]^{1/2}} + \frac{s-1}{[(s-1)(s-x)]^{1/2}}.$$

If $a = \frac{s-1}{[(s-1)(s-x)]^{1/2}}$, then we dominate K_λ by

$$K_\lambda(x-y) \leq \sup_{a, |a| \leq 1} \frac{1}{[(s-1)(s-x)]^{1/2}} \frac{1}{[(\frac{(x+1-s)-y}{[(s-1)(s-x)]^{1/2}} + a)^2 + 1]^{3/2}}.$$

Using Lemma 2.4, with $\eta = 3/2$, we get

$$\sup_{a, |a|<1} \frac{1}{[(z+a)^2+1]^{3/2}} \leq \frac{C}{[z^2+1]^{3/2}}.$$

Since $\alpha < 0$, $\psi(y) = (1-y)^\alpha$ is an A_1-Muckenhoupt weight with respect to the Lebesgue measure, see [9], and therefore,

$$\int_x^1 \frac{(1-y)^\alpha}{[(s-1)(s-x)]^{1/2}} \frac{dy}{[(\frac{x+(1-s)-y}{[(s-1)(s-x)]^{1/2}})^2+1]^{3/2}} \leq C M\psi\big(x+(1-s)\big),$$

where $M\psi$ is the Hardy–Littlewood maximal function of ψ is a A_1-weight, we get

$$M\psi\big(x+(1-s)\big) \leq C\psi\big(x+(1-s)\big) = C\big[1 - \big(x+(1-s)\big)\big]^\alpha$$
$$= C(s-x)^\alpha.$$

Thus, by estimate (2.3)

$$\int_x^1 (L, III)(1-y)^\alpha\, dy \leq C(1-r) \int_k^2 \frac{(s-x)^{-\alpha}}{(s-k)^{1/2}(s-1)} \times \int_x^1 K_\lambda(x-y)(1-y)^\alpha\, dy\, ds$$
$$\leq C.$$

(ii-2) Finally, if $0<y<x$: Let (L,IV) be the corresponding part of L in this range,

$$\begin{aligned}(L,IV) &\leq (1-r)\int_k^2 \frac{1}{(s-k)^{1/2}(s-1)(s-x)} \\ &\qquad \times \frac{(s-y)^{1-\alpha}}{[(s-1)(s-x)]^{1/2}((\frac{x-y}{[(s-1)(s-x)]^{1/2}})^2+1)^{3/2}}\,ds \\ &= (1-r)\int_k^2 \frac{1}{(s-k)^{1/2}(s-1)(s-x)} K_\lambda(x-y)\,ds.\end{aligned}$$

Now, since $\alpha<0$

$$(s-y)^{1-\alpha} \leq C_\alpha\big[(s-x)^{1-\alpha}+(x-y)^{1-\alpha}\big],$$

we split the integral into two terms. The first term

$$(L,IV,1) = C_\alpha(1-r)\int_k^2 \frac{(s-x)^{-\alpha}}{(s-k)^{1/2}(s-1)} K_\lambda(x-y)\,ds.$$

The first term can be handled in a similar way as in the case (ii-1), taking

$$x-y = \big[(x+1-s)-y\big]+(s-1),$$

and using again Lemma 2.4, with $\eta=3/2$. We get as before,

$$\begin{aligned}\int_x^1 (L,IV,1)(1-y)^\alpha\,dy &\leq C(1-r)\int_k^2 \frac{(s-x)^{-\alpha}}{(s-k)^{1/2}(s-1)} \\ &\qquad\times \int_x^1 K_\lambda(x-y)(1-y)^\alpha\,dy\,ds \\ &\leq C.\end{aligned}$$

For the second term,

$$(L,IV,2) = C_\alpha(1-r)\int_k^2 \frac{(x-y)^{-\alpha}}{(s-k)^{1/2}(s-1)(s-x)} K_\lambda(x-y)\,ds,$$

the numerator can be rewritten as

$$\begin{aligned}&(x-y)^{1-\alpha} \\ &\quad= \big[(s-1)(s-x)\big]^{(1-\alpha)/2}\left(\frac{x-y}{[(s-1)(s-x)]^{1/2}}\right)^{1-\alpha} \\ &\quad\leq \big[(s-1)(s-x)\big]^{(1-\alpha)/2}\left[\left(\frac{x-y}{[(s-1)(s-x)]^{1/2}}\right)^2+1\right]^{(1-\alpha)/2}.\end{aligned}$$

Then

$$\begin{aligned}&(x-y)^{-\alpha}K_\lambda(x-y) \\ &\quad\leq \frac{[(s-1)(s-x)]^{(1-\alpha)/2}}{[(s-1)(s-x)]^{1/2}}\,\frac{[(\frac{x-y}{[(s-1)(s-x)]^{1/2}})^2+1]^{(1-\alpha)/2}}{[(\frac{x-y}{[(s-1)(s-x)]^{1/2}})^2+1]^{3/2}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{[(s-1)(s-x)]^{(1-\alpha)/2}}{[(s-1)(s-x)]^{1/2}} \frac{1}{[(\frac{x-y}{[(s-1)(s-x)]^{1/2}})^2+1]^{3/2-(1-\alpha)/2}} \\
&= \big[(s-1)(s-x)\big]^{(1-\alpha)/2} K_\lambda^{**}(x-y),
\end{aligned}
$$

where

$$
\begin{aligned}
K_\lambda^{**}(x-y) &= \frac{1}{[(s-1)(s-x)]^{1/2}} \frac{1}{[(\frac{x-y}{[(s-1)(s-x)]^{1/2}})^2+1]^{3/2-(1-\alpha)/2}} \\
&= \frac{1}{\lambda} k_4\left(\frac{x-y}{\lambda}\right)
\end{aligned}
$$

with $\lambda = [(s-1)(s-x)]^{1/2}$, and

$$
k_4(x) = \frac{1}{(x^2+1)^{3/2-(1-\alpha)/2}}.
$$

Since $\frac{3}{2} - \frac{1-\alpha}{2} = 1+\alpha/2 > 1/2$ k_4 is a Poisson type kernel.

Finally, as

$$
\begin{aligned}
&(s-1)^{1/2}(s-x)^{1/2}(s-1)^{-\alpha/2}(s-x)^{-\alpha/2} \\
&\quad \le (s-1)^{1/2}(s-x)^{1/2}(s-x)^{-\alpha}
\end{aligned}
$$

then $(L, IV, 2)$ is dominated by,

$$
(1-r)\int_k^2 \frac{(s-x)^{-\alpha}}{(s-k)^{1/2}(s-1)^{1/2}(s-x)^{1/2}} K_\lambda^{**}(x-y)\, ds.
$$

Then, this is analogous to the case (ii-1), but with the kernel k_4 and therefore, using estimate (2.3), and weight theory we get that $(L, IV, 2)$ is bounded. □

3. Applications

We are going to obtain several consequences from Theorem 2.6.

First, we consider a result due to A. Zygmund (see [22], Vol. I, Lemma 7.1, pages 154–155).

Theorem 3.1 (Zygmund). *Given $-\infty \le a < b \le \infty$, a Borel measure μ with support in (a,b) and a kernel $K(r,x,\cdot)$ depending on a parameter r, satisfying the following conditions*

$$
\int_a^b \big|K(r,x,y)\big|\mu(dy) \le M_1 \tag{3.1}
$$

and

$$
\int_x^b \mu(x,y) V_2\big(K(r,x,dy)\big) \le M_2, \qquad \int_a^x \mu(y,x) V_2\big(K(r,x,dy)\big) \le M_2. \tag{3.2}
$$

Here, M_1, M_2 *are constants independent of* x *and* r, $\mu(x,y) = \int_{x\wedge y}^{x\vee y} \mu(du)$ *and* $V_2(K(r,x,\cdot))$ *is the (first) variation of the kernel* $K(r,x,y)$ *in the variable* y*, namely,*

$$V_2\big(K(r,x,\cdot)\big) = \sup \sum_i \big|K(r,x,y_i) - K(r,x,y_{i-1})\big|,$$

where the supremum is taken over all partitions of $[a,b]$ *and the integrals are considered in the Lebesgue–Stieltjes sense.*

Then, for $f \in L^1(\mu)$*, we have*

$$\left|\int_a^b K(r,x,y) f(y) \mu(dy)\right| \le M f_\mu^*(x), \tag{3.3}$$

where M *depends only on* M_1, M_2 *and*

$$f_\mu^*(x) = \sup_{x\in I} \frac{1}{\mu(I)} \int_I f(y)\mu(dy),$$

is the noncentered Hardy–Littlewood maximal function for f *with respect to the measure* μ.

A kernel $K(r,x,y)$ satisfying properties (3.1) and (3.2) will be called *Zygmund's kernels.*

Proof of Theorem 3.1. Using the integration by parts formula for Stieltjes integrals, we have

$$\begin{aligned}\int_x^b K(r,x,y)\mu(dy) &= \left(\int_x^b \mu(du)\right) K(r,x,b) - \int_x^b \left(\int_x^y \mu(du)\right) K(r,x,dy)\\ &= \mu(x,b)K(r,x,b) - \int_x^b \mu(x,y) K(r,x,dy).\end{aligned}$$

Therefore, by hypothesis

$$\begin{aligned}\big|\mu(x,b)K(r,x,b)\big| &\le \int_x^b \big|K(r,x,y)\big|\mu(dy) + \int_x^b \mu(x,y)K(r,x,dy)\\ &\le \int_x^b \big|K(r,x,y)\big|\mu(dy) + \int_x^b \mu(x,y) V_2\big(K(r,x,dy)\big)\\ &\le M_1 + M_2.\end{aligned}$$

Now, for $f \in L^1(\mu)$ and using again the integration by parts formula,

$$\begin{aligned}&\int_x^b f(y)K(r,x,y)\mu(dy)\\ &\quad= \left(\int_x^b f(y)\mu(dy)\right) K(r,x,b) - \int_x^b \left(\int_x^y f(y)\mu(dy)\right) K(r,x,dy)\\ &\quad= \left(\int_x^b f(y)\mu(dy)\right) K(r,x,b) - \int_x^b \left(\int_x^b f(y)\mu(dy)\right) K(r,x,dy)\end{aligned}$$

$$= \left(\frac{1}{\mu(x,b)}\int_x^b f(y)\mu(dy)\right)\mu(x,b)K(r,x,b) - \int_x^b \left(\frac{1}{\mu(x,y)}\int_x^b f(y)\mu(dy)\right)\mu(x,y)K(r,x,dy).$$

Thus,

$$\begin{aligned}
&\left|\int_x^b f(y)K(r,x,y)\mu(dy)\right| \\
&\quad\le f_\mu^*(x)\big|\mu(x,b)K(r,x,b)\big| + f_\mu^*(x)\int_x^b \mu(x,y)V_2\big(K(r,x,dy)\big) \\
&\quad\le (M_1+M_2)f_\mu^*(x) + M_2 f_\mu^*(x) = (M_1+2M_2)f_\mu^*(x). \qquad \square
\end{aligned}$$

In particular, Zygmund's result implies Natanson's lemma (see [15], Theorem 1),

COROLLARY 3.2 (Natanson). *Given* $-\infty \le a < b \le \infty$, *a Borel measure* μ *with support in* (a,b) *and* $K(r,x,y)$ *be a Natanson kernel. Then, for* $f \in L^1(\mu)$, *we have*

$$\left|\int_a^b K(r,x,y)f(y)\mu(dy)\right| \le Mf_\mu^*(x). \tag{3.4}$$

Proof. We need to check that K satisfies the conditions of Zygmund's result. Condition (3.1) is trivially satisfied and (3.2) are easily obtained from the monotonicity conditions. $\square$

In particular, Poisson type kernels are Natanson's kernels and therefore they satisfy the conditions of Zygmund's lemma.

Now, as a consequence of Theorem 2.6 and using Zygmund's lemma we have the following theorem.

THEOREM 3.3. *Let* $\alpha+\beta > -1$ *and* $f \in L^1(\mathcal{J}^{\alpha,\beta})$, *define the operator*

$$\begin{aligned}
J_\alpha f(x) = \int_0^1 (1-r)\int_k^2 &\frac{(s-\min(x,y))^{1-\alpha}}{((x-y)^2+(s-1)(s-\min(x,y)))^{3/2}} \\
&\times \frac{ds}{(s-k)^{1/2}}(1-y)^\alpha f(y)\,dy.
\end{aligned} \tag{3.5}$$

Then,

$$J_\alpha f(x) \le C f^*_{\mathcal{J}^{\alpha,\beta}}(x), \tag{3.6}$$

where $f^*_{\mathcal{J}^{\alpha,\beta}}$ *is the (noncentered) Hardy–Littlewood maximal function with respect to the Jacobi measure* $\mathcal{J}^{\alpha,\beta}$.

Proof. The idea of the proof is as follows. By using Theorem 2.6 (for both cases $-1<\alpha<0$ and $0\le\alpha$) and applying Zygmund's result to the bounding kernel (which is a combination of Natanson's kernels) for the measure $\mu_\alpha(dy)=(1-y)^\alpha$, we get the desire estimate for $J_\alpha f$.

We need to analyze two cases:

(i) Case $\alpha\ge 0$. In this case we have, from the estimates in (i-1) and (i-2) in the proof of Theorem 2.6, with the same notation used there,

$$
\begin{aligned}
J_\alpha f(x) &\le (1-r)\int_k^2 \frac{(s-x)^{-\alpha}}{(s-k)^{1/2}(s-1)}\int_x^1 K_\lambda(x-y)f(y)(1-y)^\alpha\,dy\,ds\\
&\quad + (1-r)\int_k^2 \frac{(s-y)^{-\alpha}}{(s-k)^{1/2}(s-1)}\int_0^x K_\lambda(x-y)f(y)(1-y)^\alpha\,dy\,ds\\
&\quad + (1-r)\int_k^2 \frac{(s-y)^{-\alpha}}{(s-k)^{1/2}(s-1)(s-x)}\\
&\quad \times\int_0^x (x-y)K_\lambda(x-y)f(y)(1-y)^\alpha\,dy\,ds\\
&\le (1-r)\int_k^2 \frac{(s-x)^{-\alpha}}{(s-k)^{1/2}(s-1)}\int_x^1 K_\lambda(x-y)f(y)(1-y)^\alpha\,dy\,ds\\
&\quad + (1-r)\int_k^2 \frac{(1-y)^{-\alpha}}{(s-k)^{1/2}(s-1)}\int_0^x K_\lambda(x-y)f(y)(1-y)^\alpha\,dy\,ds\\
&\quad + (1-r)\int_k^2 \frac{(1-y)^{-\alpha}}{(s-k)^{1/2}(s-1)(s-x)}\\
&\quad \times\int_0^x K^*_\lambda(x-y)f(y)(1-y)^\alpha\,dy\,ds.
\end{aligned}
$$

Since $K_\lambda(x-\cdot)$ and $K^*_\lambda(x-\cdot)$ are Natanson's kernels, applying Zygmund's result with respect to the measure $\mu(dy)=(1-y)^\alpha\,dy$ and Lemma 2.3, we get

$$J_\alpha f(x)\le C f^*_{\mathcal{J}^{\alpha,\beta}}(x).$$

(ii) Case $-1<\alpha<0$. In this case we have, from the estimates in (ii-1) and (ii-2) in the proof of Theorem 2.6

$$
\begin{aligned}
J_\alpha f(x) &\le C(1-r)\int_k^2 \frac{(s-x)^{-\alpha}}{(s-k)^{1/2}(s-1)}\\
&\qquad \times\int_x^1 K_\lambda(x-y)f(y)(1-y)^\alpha\,dy\,ds\\
&\qquad + C_\alpha(1-r)\int_k^2 \frac{(s-x)^{-\alpha}}{(s-k)^{1/2}(s-1)}\\
&\qquad \times\int_0^x K_\lambda(x-y)f(y)(1-y)^\alpha\,dy\,ds
\end{aligned}
$$

$$+ (1-r)\int_k^2 \frac{(s-x)^{-\alpha}}{(s-k)^{1/2}(s-1)^{1/2}(sx)^{1/2}} \times \int_0^x K_\lambda^{**}(x-y)f(y)(1-y)^\alpha\,dy\,ds.$$

Since $K_\lambda(x-\cdot)$ and $K_\lambda^{**}(x-\cdot)$ are Natanson's kernels, applying Zygmund's result with respect to the measure $\mu(dy) = (1-y)^\alpha\,dy$ and Lemma 2.3, we get

$$J_\alpha f(x) \le C f^*_{\mathcal{J}^{\alpha,\beta}}(x). \qquad \square$$

Observation. Notice that there is another operator

$$J_{\alpha,\beta}f(x) = \int_{-1}^0 (1-r)\int_k^2 \frac{(s-\min(x,y))^{1-\alpha}}{((x-y)^2+(s-1)(s-\min(x,y)))^{3/2}} \times \frac{ds}{(s-k)^{1/2}}(1+y)^\beta f(y)\,dy. \tag{3.7}$$

With analogous arguments as in the previous result, for $\alpha+\beta > -1$, we have immediately

$$J_{\alpha,\beta}f(x) \le C f^*_{\mathcal{J}^{\alpha,\beta}}(x). \tag{3.8}$$

Therefore, by the continuity properties of $f^*_{\mathcal{J}^{\alpha,\beta}}$, we have the following corollary.

Corollary 3.4. *For $\alpha, \beta > -1$, the operators J_α and $J_{\alpha,\beta}$ are weak-$(1,1)$ continuous with respect to the Jacobi measure $\mathcal{J}^{\alpha,\beta}$.*

Then, using the inequality (2.1) and the two previous results, we get that Jacobi maximal function $f^*_{\alpha,\beta}$ (see (3.13)) is weak $(1,1)$ with respect to the Jacobi measure.

Now, let us consider a Calderón–Zygmund's decomposition for a nonatomic Borel measure μ on $\mathbb{R}$, implicit in Cafarelli's doctoral dissertation [5] (compare with the classical case, see [18]).

Theorem 3.5 (Calderón–Zygmund). *Given $-\infty \le a < b \le \infty$, a nonatomic Borel measure μ with support on (a,b), $\lambda > 0$ and $f \in L^1(\mu)$, $f \ge 0$, then there exists a family of nonoverlapping intervals $\{I_k\}$*

(i) $\lambda < \frac{1}{\mu(I_k)}\int_{I_k} f(y)\mu(dy) \le 2\lambda$,
(ii) $|f(x)| \le \lambda$, *a.e.* μ, *for* $x \notin \bigcup_k I_k$.

Proof.

- If $\frac{1}{\mu(a,b)}\int_a^b f(y)\mu(dy) > \lambda$ then

$$\mu(a,b) < \frac{1}{\lambda}\int_a^b f(y)\mu(dy) = \frac{1}{\lambda}\|f\|_1,$$

and then there is nothing to prove.

- If $\frac{1}{\mu(a,b)}\int_a^b f(y)\mu(dy) \le \lambda$, then consider two intervals, $I_{0,1}, I_{0,2}$ with disjoint interiors such that $(a,b) = I_{0,1} \cup I_{0,2}$ and $\mu(I_{0,1}) = \mu(I_{0,2}) = \frac{1}{2}\mu(a,b)$. Let us observe that we can not have that the inequality
$$\frac{1}{\mu(I_{0,i})}\int_{I_{0,i}} f(y)\mu(dy) > \lambda,$$
hold for both $i=1$ and $i=2$ since otherwise,
$$\begin{aligned}&\frac{1}{\mu(a,b)}\int_{(a,b)} f(y)\mu(dy)\\ &\quad = \frac{2}{\mu(I_{0,1})}\int_{I_{0,1}} f(y)\mu(dy) + \frac{2}{\mu(I_{0,2})}\int_{I_{0,2}} f(y)\mu(dy) > 4\lambda,\end{aligned}$$
which is a contradiction, then we have that at least one of then (or even both) satisfy
$$\frac{1}{\mu(I_{0,i})}\int_{I_{0,i}} f(y)\mu(dy) \le \lambda.$$
In that case consider again two intervals, $I_{i,1}, I_{i,2}$ with disjoint interiors such that $I_{0,i} = I_{i,1} \cup I_{i,2}$ and $\mu(I_{i,1}) = \mu(I_{i,2}) = \frac{1}{2}\mu(I_{0,i}) = \frac{1}{4}\mu(a,b)$ and iterate the previous argument. If we have
$$\frac{1}{\mu(I_{0,i})}\int_{I_{0,i}} f(y)\mu(dy) > \lambda,$$
then
$$\begin{aligned}\frac{1}{\mu(I_{0,i})}\int_{I_{0,i}} f(y)\mu(dy) &\le \frac{1}{\mu(I_{0,i})}\int_{(a,b)} f(y)\mu(dy)\\ &= \frac{2}{\mu(a,b)}\int_{(a,b)} f(y)\mu(dy) \le 2\lambda.\end{aligned}$$
Set $I_{0,i}$ aside, it will be one of our chosen interval I_k.

This infinite recursion will give us a family $\{I_k\}$ such that,
$$\lambda < \frac{1}{\mu(I_k)}\int_{I_k} f(y)\mu(dy) \le 2\lambda.$$

Set $G_\lambda = \bigcup_{k=1}^{\infty} I_k$, then
$$\begin{aligned}\mu(G_\lambda) = \sum_{k=1}^{\infty}\mu(I_k) &< \frac{1}{\lambda}\sum_{k=1}^{\infty}\int_{I_k} f(y)\mu(dy) \le \frac{1}{\lambda}\int_{G_\lambda} f(y)\mu(dy)\\ &\le \frac{1}{\lambda}\int_{\mathbb{R}} f(y)\mu(dy) = \frac{1}{\lambda}\|f\|_{1,\mu}.\end{aligned}$$
Let us observe that if $x \notin \bigcup_k I_k$ then there is an infinite family of intervals I containing x such that
$$\frac{1}{\mu(I)}\int_I f(y)\mu(dy) \le \lambda,$$

then by Lebesgue differentiation theorem, see Lemma 7 of [7], we get $|f(x)| \leq \lambda$ a.e. μ, $x \notin \bigcup_k I_k$.

Now, set $\mu_k = \frac{1}{\mu(I_k)} \int_{I_k} f(y)\mu(dy)$ we can write $f = g + b$ where,

$$g(x) = f\chi_{\mathbb{R}-G_\lambda}(x) + \sum_k \mu_k \chi_{I_k}(x)$$

and

$$b(x) = f(x) - g(x) = \sum_k \big(f(x) - \mu_k\big)\chi_{I_k}(x).$$

g, b are called that good and bad part of f, respectively. Observe that $g \leq 2\lambda$ in G_λ, the bad part is only nonzero in G_λ and $\int_{I_k} b(y)\mu(dy) = 0$.

If $G^*_\lambda = \bigcup_{k=1}^\infty I^*_k$ where $I^*_k = 3I_k$ meaning that I^*_k is the union of I_k with two other intervals (one to the right and one to the left of it) with the same μ measure, that is, $I^*_k = I'_k \cup I_k \cup I''_k$, with $\mu(I'_k) = \mu(I_k) = \mu(I''_k)$, then

$$\mu\big(G^*_\lambda\big) = \sum_{k=1}^\infty \mu\big(I^*_k\big) = 3\sum_{k=1}^\infty \mu(I_k) \leq \frac{3}{\lambda}\|f\|_{1,\mu}. \qquad \square$$

We can use Calderón–Zygmund decomposition for a kernel $K(r, x, y)$ that satisfies the conditions of Zygmund's theorem.

Proposition 3.6. *Given a nonatomic Borel measure μ, with support in (a, b), and $K(r, x, y)$ be a Zygmund's kernel. Then for $f \in L^1(\mu)$ and $x \notin G^*_\lambda$,*

$$\sup_r \left| \int_a^b K(r, x, y) f(y)\mu(dy) \right| \leq C\lambda. \tag{3.9}$$

Proof. We know by Zygmund's lemma that

$$\left| \int_a^b K(r, x, y) f(y)\mu(dy) \right| \leq M f^*_\mu(x).$$

Now, using Calderón–Zygmund decomposition for $f = g + b$, we get

$$\int_a^b K(r, x, y) f(y)\mu(dy) = \int_a^b K(r, x, y) g(y)\mu(dy) + \int_a^b K(r, x, y) b(y)\mu(dy)$$

and as $|g| < 2\lambda$, a.e. μ, by (3.1),

$$\left| \int_a^b K(r, x, y) g(y)\mu(dy) \right| < 2M_1\lambda.$$

If $x \notin G^*_\lambda$, using integration by parts, where $I_k = (a_k, b_k)$

$$\begin{aligned} &\left| \int_a^b K(r, x, y) b(y)\mu(dy) \right| \\ &\quad = \left| \sum_k \int_{I_k} \big(f(y) - \mu_k\big) K(r, x, y)\mu(dy) \right| \end{aligned}$$

$$= \left| \sum_k \int_{a_k}^{b_k} \bigl(f(y) - \mu_k\bigr) K(r,x,y)\mu(dy) \right|$$
$$= \left| \sum_k \int_{a_k}^{b_k} \left(\int_{a_k}^{y} \bigl(f(u) - \mu_k\bigr)\mu(du) \right) K(r,x,dy) \right|$$

as $\int_{a_k}^{b_k} (f(u) - \mu_k)\mu(du) = 0$, using that $x \notin G_\lambda^*$, and (3.2)

$$\begin{aligned} \left| \int_a^b K(r,x,y)b(y)\mu(dy) \right| &\leq C\lambda \sum_k \int_{I_k} \mu(I_k) V_2\bigl(K(r,x,dy)\bigr) \\ &\leq C\lambda \sum_k \int_{I_k} \mu(x,y) V_2\bigl(K(r,x,dy)\bigr) \\ &\leq C\lambda \int_{G_\lambda} \mu(x,y) V_2\bigl(K(r,x,dy)\bigr) \\ &\leq C\lambda \int_x^b \mu(x,y) V_2\bigl(K(r,x,dy)\bigr) \leq C\lambda M_2. \end{aligned}$$

Thus, for $x \notin G_\lambda^*$

$$\sup_r \left| \int_a^b K(r,x,y)f(y)\mu(dy) \right| \leq C\lambda. \qquad \square$$

This result could be extended to the case of measures that do have atoms.

The following result was proved implicitly by L. Cafarelli in [5],

THEOREM 3.7. *For $\alpha, \beta > -1$, the Jacobi measure $\mathcal{J}^{\alpha,\beta}$ is a doubling measure.*

Proof. Let us consider first the measure $\mu(dy) = y^a$, in $[0,1], a > -1$. Then we will see that μ is a doubling measure on $[0,1]$.

Let $k \geq 2$ and $I_{k,j} = [k2^{-j}, (k+1)2^{-j}]$ a dyadic interval. Observe that

$$\mu(I_{k,j}) = \int_{k2^{-j}}^{(k+1)2^{-j}} y^a\, dy = \frac{2^{-j(a+1)}}{a+1} \bigl[(k+1)^{a+1} - k^{a+1}\bigr].$$

Now let us consider $3I_{k,j}$ the interval with the same center $(k+1/2)2^{-j}$ and 3 times the length of $I_{k,j}$ that is, $3I_{k,j} = [(k-1)2^{-j}, (k+2)2^{-j}]$, then

$$\mu(3I_{k,j}) = \int_{(k-1)2^{-j}}^{(k+2)2^{-j}} y^a\, dy = \frac{2^{-j(a+1)}}{a+1} \bigl[(k+2)^{a+1} - (k-1)^{a+1}\bigr].$$

Thus,

$$\begin{aligned} \frac{\mu(3I_{k,j})}{\mu(I_{k,j})} &= \frac{(k+2)^{a+1} - (k-1)^{a+1}}{(k+1)^{a+1} - k^{a+1}} \\ &= \frac{(1+\frac{2}{k})^{a+1} - (1-\frac{1}{k})^{a+1}}{(1+\frac{1}{k})^{a+1} - 1}. \end{aligned}$$

It can be proved that the quotient $\frac{\mu(3I_{k,j})}{\mu(I_{k,j})}$ is increasing in k for $a\in(0,1)$ and decreasing for $a\in(-1,0)\cup(1,\infty)$. By L'Hopital rule,

$$\lim_{k\to\infty}\frac{\mu(3I_{k,j})}{\mu(I_{k,j})}=\lim_{k\to\infty}\frac{2(1+\frac{2}{k})^a+(1-\frac{1}{k})^a}{(1+\frac{1}{k})^a}=3.$$

Therefore if $a\in(0,1)$,

$$C_a=\frac{3^{a+1}}{2^{a+1}-1}\leq\frac{\mu(3I_{k,j})}{\mu(I_{k,j})}\leq 3$$

and elsewhere

$$3\leq\frac{\mu(3I_{k,j})}{\mu(I_{k,j})}\leq\frac{3^{a+1}}{2^{a+1}-1}=C_a.$$

Similarly, using the same arguments, we can prove that μ is also a doubling measure on $[-1,0]$.

Now observe that, by a change of variable, on $[0,1]$ the measure $y^a\,dy$ is equivalent to $(1-y)^a\,dy$, in the following sense

$$\int_0^1 f(y)(1-y)^\alpha\,dy=\int_0^1 f(1-u)u^\alpha\,du=\int_0^1\overline{f(u)}u^\alpha\,du,$$

and clearly there is a one-to-one correspondence between f and $\overline{f}$. Similarly, on $[-1,0]$ the measure $y^a\,dy$ is equivalent to $(1+y)^a\,dy$.

Finally, as a consequence of the previous results we have that the Jacobi measure $\mathcal{J}^{\alpha,\beta}(dy)=(1-y)^\alpha(1+y)^\beta\,dy$ in $(0,1)$ is equivalent to $y^\alpha\,dy$ and is equivalent to $y^\beta\,dy$ in $(-1,0)$. Therefore, $\mathcal{J}^{\alpha,\beta}$ is then a doubling measure on $[-1,1]$. $\square$

Now that we know that the Jacobi measure $\mathcal{J}^{\alpha,\beta}$ is a doubling measure we can use the result of A. P. Calderón [7], in order to get the A_p weight theory for $\mathcal{J}^{\alpha,\beta}$. Remember a function $\omega>0$, is an A_p weight with respect to $\mathcal{J}^{\alpha,\beta}$, $\omega\in A_p$, if

$$\left[\frac{1}{\mathcal{J}^{\alpha,\beta}(B)}\int_B\omega(y)\mathcal{J}^{\alpha,\beta}(dy)\right]\times\left[\frac{1}{\mathcal{J}^{\alpha,\beta}(B)}\int_B\omega(y)^{-1/(p-1)}\mathcal{J}^{\alpha,\beta}(dy)\right]^{p-1}\leq C_p, \tag{3.10}$$

for $1<p<\infty$ and

$$M_{\mathcal{J}^{\alpha,\beta}}\omega(x)\leq C_1\omega(x), \tag{3.11}$$

for $p=1$. For a complete exposition of the A_p weight theory see, for instance, the book of J. Duoandikoetxea [9].

In what follows, we will use the following notation for a measure $\mu(dx)=g(x)\,dx$,

$$\int_a^b\mu(dy)=\int_a^b g(y)\,dy=G(b)-G(a).$$

We want to consider some interesting A_1 weights for the Jacobi measure. Observe that by the factorization result (see Duoandikoetxea [9], Proposition 7.2, page 136) they are like building blocks for A_p weights for $p>1$. First of all, we need the following technical result.

LEMMA 3.8. *Let μ be a nonnegative Borel measure on $[0,1)$ and absolutely continuous that is, $\mu(dx)=g(x)\,dx$ where g is nonnegative and continuous. Then if f is a nonincreasing nonnegative function, then*

$$\frac{1}{G(x)-G(a)}\int_x^a f(y)g(y)\,dy$$

is also nonincreasing function. The same result is true for a nonnegative Borel measure μ on $(-1,0]$.

Proof. Since

$$\begin{aligned}\frac{d}{dx}&\left(\frac{1}{G(a)-G(x)}\int_x^a f(y)g(y)\,dy\right)\\ &=\frac{-f(x)g(x)(G(a)-G(x))+(\int_x^a f(y)g(y)\,dy)g(x)}{(G(a)-G(x))^2}\\ &=\frac{g(x)(-f(x)(G(a)-G(x))+\int_x^a f(y)g(y)\,dy)}{(G(a)-G(x))^2}\leq 0,\end{aligned}$$

as $g\geq 0$ and $f(x)\int_x^a g(y)\,dy\geq\int_x^a f(y)g(y)\,dy$. Therefore, the quotient is nonincreasing as claimed. □

We will use the previous result to consider lateral maximal functions. If we consider the left lateral maximal function of nonincreasing nonnegative function f,

$$f_-^*(a)=\sup_{0\leq x\leq a}\frac{1}{G(a)-G(x)}\int_x^a f(y)g(y)\,dy,$$

we have, by Lemma 3.8,

$$f_-^*(a)=\frac{1}{G(a)-G(0)}\int_0^a f(y)g(y)\,dy=\frac{1}{G(a)}\int_0^a f(y)g(y)\,dy,$$

as $G(0)=0$.

By analogous argument, we have that for a nonincreasing nonnegative function f, its the right lateral maximal function equals,

$$f_+^*(a)=\sup_{0\leq a\leq x}\frac{1}{G(x)-G(a)}\int_a^x f(y)g(y)\,dy=f(a_+).$$

The case of a general nonnegative Borel measure μ can be obtained using Helly's selection principle.

Now, let us consider the A_1 weights for the Jacobi measure.

LEMMA 3.9. (i) *For* $1 < \alpha < \infty$, *let us consider the power measure* $\mu_\alpha(dx) = x^\alpha\,dx$ *on* $[0,1)$, *then the function* $\omega_{\overline{\alpha}}(x) = x^{\overline{\alpha}}, -1 < \overline{\alpha} < 0, \alpha + \overline{\alpha} > -1$ *is a* A_1 *weight with respect to* μ_α.

(ii) *Similarly, considering the power measure* $\mu_\beta(dx) = x^\beta\,dx$ *on* $[-1,0)$, *then the function* $\omega_{\overline{\beta}}(x) = x^{\overline{\beta}}, -1 < \overline{\beta} < 0, \beta + \overline{\beta} > -1$ *is an* A_1 *weight with respect to* μ_β.

Proof. By previous considerations, the left maximal function with respect to μ_α is equal to,

$$\begin{aligned}\frac{C}{x^{\alpha+1}}\int_0^x t^\alpha t^{\overline{\alpha}}\,dt &= \frac{C}{x^{\alpha+1}}\int_0^x t^{\alpha+\overline{\alpha}}\,dt\\ &= \frac{C}{x^{\alpha+1}}x^{\alpha+\overline{\alpha}+1} = Cx^{\overline{\alpha}},\end{aligned}$$

and the right maximal functions simply $x^{\overline{\alpha}}$, thus the function $\omega_{\overline{\alpha}}(x) = x^{\overline{\alpha}}, -1 < \overline{\alpha} < 0, \alpha + \overline{\alpha} > -1$ is an A_1 weight with respect to the measure μ_α.

Similarly, on $[-1,0)$ $\omega_{\overline{\beta}}(x) = x^{\overline{\beta}}, -1 < \overline{\beta} < 0, \beta + \overline{\beta} > -1$ is an A_1 weight with respect to the measure $\mu_\beta(dx) = x^\beta\,dx, 1 < \beta < \infty$. □

Now we have the following result for the Jacobi measure. This result extends the set of weights that were considered in [8], where only positive power were considered.

THEOREM 3.10. *Given the Jacobi measure* $\mathcal{J}^{\alpha,\beta}(dx) = (1-x)^\alpha(1+x)^\beta\,dx$ *on* $[-1,1]$, *the functions*

$$\omega_{\overline{\alpha},\overline{\beta}}(x) = (1-x)^{\overline{\alpha}}(1+x)^{\overline{\beta}}, \tag{3.12}$$

are A_1 *weights with respect to* $\mathcal{J}^{\alpha,\beta}$ *for* $\alpha + \overline{\alpha} > -1, \beta + \overline{\beta} > -1$.

Proof. By Lemma 3.9 and similar arguments as above, the function $\omega_{\overline{\alpha}}(x) = (1-x)^{\overline{\alpha}}, -1 < \overline{\alpha} < 0, \alpha + \overline{\alpha} > -1$ is an A_1 weight with respect to the measure $\nu_\alpha(dx) = (1-x)^\alpha\,dx$ and similarly, the function $\omega_{\overline{\beta}}(dx) = (1+x)^{\overline{\beta}}\,dx, -1 < \overline{\beta} < 0, \beta + \overline{\beta} > -1 > -1$ is an A_1 weight with respect to the measure $\nu_\beta(dx) = (1+x)^\beta\,dx$ on $[-1,0)$ and from there we get our result immediately. □

Finally, as a consequence of Theorem 3.10, we have the following result for Abel summability of Jacobi function expansions. The maximal function $\tilde{f}^*_{\alpha,\beta}$ for Jacobi function expansions is defined as

$$\tilde{f}^*_{\alpha,\beta}(x) = \sup_{0<r<1}\left|\tilde{f}^{\alpha,\beta}(r,x)\right| = \sup_{0<r<1}\left|\int_{-1}^1 \tilde{K}^{\alpha,\beta}(r,x,y)f(y)\,dy\right|. \tag{3.13}$$

Then,

THEOREM 3.11. *For $\alpha<0,\beta<0$ such that $\alpha+\beta>-1$, the Abel summability of Jacobi function expansions we have for* $\max[\frac{2}{2-|\alpha|},\frac{2}{2-|\beta|}]<p<\min[\frac{2}{|\alpha|},\frac{2}{|\beta|}]$,

$$\big\|\tilde{f}_*^{\alpha,\beta}\big\|_p\le C\|f\|_p.$$

Proof. Let us consider only the case of the interval $[0,1]$ with $\alpha<0$, the case $[-1,0]$ is totally analogous. From (1.16) we have, by the maximal inequality of the Hardy–Littlewood function M_{ν_α} with respect to the measure $\nu_\alpha(dx)=(1-x)^\alpha\,dx$,

$$\begin{aligned}
&\int_0^1\tilde{K}^{\alpha,\beta}(r,x,y)f(y)\,dy\\
&\quad\le C_\beta(1-x)^{\alpha/2}\int_0^1 K^{\alpha,\beta}(r,x,y)(1-y)^{\alpha/2}f(y)\,dy\\
&\quad= C_\beta(1-x)^{\alpha/2}\int_0^1 K^{\alpha,\beta}(r,x,y)\big[(1-y)^{-\alpha/2}f(y)\big](1-y)^{\alpha}\,dy\\
&\quad\le C_\beta M_{\nu_\alpha}\big((1-\cdot)^{-\alpha/2}f\big)(x)(1-x)^{\alpha/2}.
\end{aligned}$$

Since this bound is independent of r, we get

$$\sup_{0<r<1}\int_0^1\tilde{K}^{\alpha,\beta}(r,x,y)f(y)\,dy\le C_\beta M_{\nu_\alpha}\big((1-\cdot)^{-\alpha/2}f\big)(x)(1-x)^{\alpha/2}.$$

Therefore, by the L^2 continuity of M_{ν_α} with respect to the measure ν_α,

$$\begin{aligned}
\big\|\tilde{f}_*^{\alpha,\beta}\big\|_2&\le\int_0^1\bigg[\sup_{0<r<1}\int_0^1\tilde{K}^{\alpha,\beta}(r,x,y)f(y)\,dy\bigg]^2dx\\
&\le C_\beta\int_0^1\big[M_{\nu_\alpha}\big((1-\cdot)^{-\alpha/2}f\big)\big]^2(x)(1-x)^\alpha\,dx\\
&\le C_\beta\int_0^1\big[(1-x)^{-\alpha/2}f(x)\big]^2(1-x)^\alpha\,dx\\
&\le C_\beta\int_0^1\big[f(y)\big]^2\,dy=C\|f\|_2^2.
\end{aligned}$$

Thus,

$$\big\|\tilde{f}_*^{\alpha,\beta}\big\|_2\le C\|f\|_2.$$

Analogously, for the L^p inequality. If $p>2$,

$$\begin{aligned}
\big\|\tilde{f}_*^{\alpha,\beta}\big\|_p&\le\int_0^1\bigg[\sup_{0<r<1}\int_0^1\tilde{K}^{\alpha,\beta}(r,x,y)f(y)\,dy\bigg]^p dx\\
&\le C_\beta\int_0^1\big[M_{\nu_\alpha}\big((1-\cdot)^{-\alpha/2}f\big)\big]^p(x)(1-x)^{p\alpha/2}\,dx
\end{aligned}$$

and observe that

$$(1-x)^{p\alpha/2} = (1-x)^{p\alpha/2-\alpha+\alpha} = (1-x)^{\delta+\alpha}, \quad \delta = p\alpha/2-\alpha = \alpha(p-2)/2;$$

$(1-x)^{\delta+\alpha}$ is a $A_p(\nu_\alpha)$-weight if $\delta+\alpha = \alpha p/2 > -1$, i.e. $p < 2/|\alpha|$, and therefore, by the L^p continuity of M_{ν_α} with respect to the measure ν_α,

$$\begin{aligned}\int_0^1 \Big[\int_0^1 \tilde{K}^{\alpha,\beta}(r,x,y)f(y)\,dy\Big]^p dx &\leq C_\beta \int_0^1 (1-x)^{-\alpha p/2}\big[f(x)\big]^p (1-x)^{\alpha p/2}\,dx \\ &= C_\beta \int_0^1 \big[f(x)\big]^p\,dx = C_\beta \|f\|_p^p.\end{aligned}$$

If $1<p<2$, $(1-x)^{p\alpha/2-\alpha}$ is a $A_p(\nu_\alpha)$-weight if and only if

$$\begin{aligned}(1-x)^{(p\alpha/2-\alpha)(-1/(p-1))} &= (1-x)^{(-p\alpha/2(p-1)+\alpha/(p-1))} \\ &= (1-x)^{(-q\alpha/2+q\alpha/p)},\end{aligned}$$

is a $A_q(\nu_\alpha)$-weight, $\frac{1}{p}+\frac{1}{q}=1$, see [9]. But

$$-q\alpha/2+q\alpha/p = -q\alpha/2+q\alpha/p = q\alpha(1/p-1/2) = \gamma,$$

and therefore $(1-x)^{(p\alpha/2-\alpha)(-1/(p-1))} = (1-x)^{\gamma}$ is a $A_q(\nu_\alpha\,dy)$-weight for $q>2$. Then

$$\begin{aligned}\int_0^1 \Big[\int_0^1 \tilde{K}^{\alpha,\beta}(r,x,y)f(y)\,dy\Big]^q dx &\leq C_\beta \int_0^1 (1-x)^{-\alpha q/2}\big[f(x)\big]^q (1-x)^{\alpha q/2}\,dx \\ &= C_\beta \int_0^1 \big[f(x)\big]^q\,dx = C_\beta \|f\|_q^q.\end{aligned}$$

From the previous case, the condition $p<2/|\alpha|$ holds if and only if $q > \frac{2}{2-|\alpha|}$, so the general condition for p is

$$\frac{2}{2-|\alpha|} < p < \frac{2}{|\alpha|}.$$

Finally, the bilateral condition in $[-1,1]$ is then

$$\max\left[\frac{2}{2-|\alpha|}, \frac{2}{2-|\beta|}\right] < p < \min\left[\frac{2}{|\alpha|}, \frac{2}{|\beta|}\right]. \qquad \square$$

Observation. For the case of α,β positive, the previous result was obtained in [8].

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, 9th printing, Dover, New York, 1966. MR 0208797

[2] G. E. Andrews, R. Askey and R. Roy, *Special functions*, Encyclopedia of Mathematics and Its Applications, Cambridge University Press, Cambridge, 1999. MR 1688958

[3] R. Askey, *Orthogonal polynomials and special functions*, Regional Conf. in Applied Math., vol. 21, SIAM, Philadelphia, 1975. MR 0481145

[4] W. N. Bailey, *Generalized hypergeometric series*, Cambridge Tracts in Mathematics and Mathematical Physics, vol. 32, Stechert-Hafner, New York, 1964. MR 0185155

[5] L. Cafarelli, *Sobre la conjugación y sumabilidad de series de Jacobi*, Ph.D. thesis, Facultad de Ciencias Exactas, Universidad de Buenos Aires, Argentina, 1971.

[6] L. Cafarelli and C. P. Calderón, *On Abel summability of multiple Jacobi series*, Colloq. Math. **30** (1974), 277–288. MR 0372525

[7] A. P. Calderón, *Inequalities for a maximal function relative to a metric*, Studia Math. **57** (1976), 297–306. MR 0442579

[8] C. P. Calderón and V. Vera de Serio, *On Abel summability of Jacobi type series*, Illinois J. Math. **41** (1997), no. 2, 237–265. MR 1441676

[9] J. Duoandikoetxea, *Fourier analysis*, Graduate Studies in Mathematics, vol. 29, Amer. Math. Soc., Providence, RI, 2001. MR 1800316

[10] A. Edéryi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher trascendental functions III*, Based on notes by H. Bateman, McGraw-Hill, New York, 1955. MR 0698781

[11] G. Gasper, *Positivity and convolution structure for Jacobi series*, Ann. of Math. (2) **93** (1971), 112–118. MR 0284628

[12] G. Gasper, *Banach algebras for Jacobi series and positivity of a kernel*, Ann. of Math. (2) **95** (1972), 261–280. MR 0310536

[13] D. Jackson, *Fourier series and orthogonal polynomials*, Dover, New York, 2004. Reprint from the 1941 original. MR 2098657

[14] N. N. Lebedev, *Special functions and their applications*, Dover, New York, 1972. MR 0350075

[15] B. Muckenhoupt, *Poisson integrals for Hermite and Laguerre expansion*, Trans. Amer. Math. Soc. **139** (1969), 231–242. MR 0249917

[16] B. Muckenhoupt and E. M. Stein, *Classical expansions and their relations to conjugate harmonic functions*, Trans. Amer. Math. Soc. **118** (1965), 17–92. MR 0199636

[17] A. Nowak and P. Sjögren, *Calderón–Zygmund operators related to Jacobi expansions*, J. Fourier Anal. Appl. **18** (2012), 717–749. MR 2984366

[18] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, 1970. MR 0290095

[19] G. Szegö, *Orthogonal polynomials*, Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, 1959. MR 0106295

[20] Z. X. Wang and D. R. Guo, *Special functions*, World Scientific, Singapore, 1989. MR 1034956

[21] G. N. Watson, *A treatise on the theory of Bessel functions*, 2nd ed., Cambridge Univ. Press, Cambridge, 1980. MR 1349110

[22] A. Zygmund, *Trigonometric series*, vols. I, II, 2nd ed., Cambridge Univ. Press, Cambridge, 1959. MR 0107776

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