

A PSEUDO-DAUGAVET PROPERTY FOR NARROW PROJECTIONS IN LORENTZ SPACES

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ABSTRACT. Let X be a rearrangement-invariant space. An operator $T : X \rightarrow X$ is called narrow if for each measurable set A and each $\varepsilon > 0$ there exists $x \in X$ with $x^2 = \chi_A$, $\int x d\mu = 0$ and $\|Tx\| < \varepsilon$. In particular, all compact operators are narrow. We prove that if X is a Lorentz function space $L_{w,p}$ on $[0,1]$ with $p > 2$, then there exists a constant $k_X > 1$ such that for every narrow projection P on $L_{w,p}$ $\|\text{Id} - P\| \geq k_X$. This generalizes earlier results on L_p and partially answers a question of E. M. Semenov. Moreover, we prove that every rearrangement-invariant function space X with an absolutely continuous norm contains a complemented subspace isomorphic to X which is the range of a narrow projection and a non-narrow projection. This gives a negative answer to a question of A. Plichko and M. Popov.

1. Introduction

We study a question of E. M. Semenov, which asks whether for every separable rearrangement-invariant function space X ($X \neq L_2$) on $[0, 1]$ there exists a constant $k_X > 1$ such that for every rich subspace $Y \subsetneq X$ and every projection $P : X \xrightarrow{\text{onto}} Y$ we have

$$(1.1) \quad \|P\| \geq k_X.$$

A subspace $Y \subset X$ is said to be rich if the quotient map $Q : X \rightarrow X/Y$ is narrow, where narrow operators are a generalization of compact operators on rearrangement-invariant function spaces (see Definitions 2.1 and 2.3). Narrow operators were formally introduced by Plichko and Popov [22], but they have been used earlier in the theory of Banach spaces; see, e.g., [10, Section 9]. They were systematically studied, in particular, in [22], [11], and [13].

Natural examples of rich subspaces include subspaces of finite co-dimension. See [22, §10] for a discussion of properties of rich subspaces. Here we just

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mention that if $Y \subset X$ is rich and an operator T on X is compact then the subspace $(I+T)(Y) \subset X$ is rich, where I denotes the identity operator on X ; see [22, Proposition 10.11].

Results related to the above question of E. M. Semenov go back to Lozanovskii [20] and Benyamini and Lin [5] who proved that for every p , $1 \leq p < \infty$, $p \neq 2$, there exists a function $\varphi_p : (0, \infty) \rightarrow (0, \infty)$ such that for every nonzero compact operator T on $L_p[0, 1]$ we have

$$(1.2) \quad \|I - T\| \geq 1 + \varphi_p(\|T\|).$$

When $p = 1$ or $p = \infty$, we can take $\varphi_p(t) = t$ and the above statement is called the *Daugavet equation*; cf., e.g., [12]. If $p = 2$, we may have $\|I - T\| = 1$ for many compact operators T , e.g., for all orthogonal projections of finite rank. If $1 < p < \infty$, $p \neq 2$, then $\varphi_p(t) < t$ for all $t \in (0, \infty)$. Following [21] we say that a Banach space X satisfies a *pseudo-Daugavet property* if there exists a function $\varphi_X : (0, \infty) \rightarrow (0, \infty)$ such that (1.2) is satisfied for every nonzero compact operator T on X . Both the Daugavet property and the pseudo-Daugavet property have important geometric implications. For example, spaces with the Daugavet property cannot be reflexive and the pseudo-Daugavet property is related to the problem of best compact approximation in X , i.e., to the question whether every bounded linear operator on X has an element of best approximation in the class of compact operators on X ; see [2] for a thorough introduction to this subject and for references (cf. also [5] and [3]).

Popov [23] proved that (1.1) is valid in L_p , $1 \leq p < \infty$, $p \neq 2$, i.e., that for each p , $1 \leq p < \infty$, $p \neq 2$, there exists a constant $k_p > 1$ such that for every rich subspace $Y \subsetneq L_p$ and every projection $P : L_p \xrightarrow{\text{onto}} Y$ we have

$$(1.3) \quad \|P\| \geq k_p.$$

This result has very important applications to the study of the geometric structure of spaces L_p ; in particular, it follows from (1.3) that every “well” complemented (i.e., with constant of complementation smaller than $\max\{k_p, k_{p'}\}$, where $1/p + 1/p' = 1$) subspace of L_p is isomorphic to L_p [22].

Franchetti [7] found the exact value of the constant k_p . He proved that $k_p = \|I - A\|_p$, where A is the rank one projection defined by

$$(1.4) \quad Ax \stackrel{\text{def}}{=} \left(\int_{\Omega} x(s) d\mu(s) \right) \cdot \mathbf{1}.$$

(Note here that the projection A is well defined in any r.i. space with finite measure.) The norm of $\|I - A\|_p$ for $p \in (1, \infty)$ has been evaluated by Franchetti [6] and, independently, by Oskolkov (unpublished).

Plichko and Popov [22, Theorem 9.7] generalized (1.2) to all narrow operators on L_p , $1 \leq p < \infty$, $p \neq 2$. Later Oikhberg [21] proved (1.2) for compact operators on non-commutative spaces L_p , $1 < p < \infty$, $p \neq 2$. The first results

related to (1.1) and (1.2) (but not the Daugavet equation) for spaces other than L_p seem to be the following:

THEOREM 1.1 ([14, Theorem 4.3]). *Suppose that X is a separable real order-continuous Köthe function space on (Ω, μ) , where μ is nonatomic and finite. Let $Y \subset X$ be a subspace of codimension one and P be any projection from X onto Y . Then*

$$\|P\| > 1,$$

unless X contains a band isometric to L_2 , i.e., unless there exists a set $B \subset \Omega$ and a nonnegative measurable function w with $\text{supp } w = B$ such that for any $x \in X$ with $\text{supp } x \subset B$

$$\|x\|_X = \left(\int |x|^2 w d\mu \right)^{1/2}.$$

THEOREM 1.2 ([8, Theorem 1]). *Let X be a real r.i. space on (Ω, Σ, μ) , where μ is nonatomic and $\mu(\Omega) = 1$. If X is not isometrically isomorphic to L_2 , then*

$$\|I - A\|_X > 1,$$

where A is the rank one projection defined in (1.4).

THEOREM 1.3 ([24, Theorem 4]). *Let X be a separable real nonatomic r.i. space on $[0, 1]$ which is not isometric to L_2 . Let $Y \subsetneq X$ be any subspace of finite codimension and P be any projection from X onto Y . Then*

$$\|P\| > 1.$$

E. M. Semenov also communicated to us that he has proved (1.1) for every r.i. space X on $[0, 1]$ ($X \neq L_2$) and every projection P from X onto a rich subspace $Y \subsetneq X$ with the additional condition that $P(\chi_{[0,1]}) = 0$ (unpublished).

In the present paper we prove that (1.1) is valid if X is a Lorentz space $L_{p,w}[0, 1]$ with $p > 2$ (and no restrictions on w); see Theorem 3.1.

Our result is valid in both the complex and the real case. We have not attempted to find the exact value of k_X . E. M. Semenov suggested that the result of Franchetti mentioned above (that $k_p = \|I - A\|_p$) should generalize to all r.i. spaces X .

In the final section we study duals of narrow operators. In [22] it was proved that in general the conjugate operator T^* to a narrow operator $T : E \rightarrow E$ need not be narrow (for any r.i. space E with E^* having an absolutely continuous norm to consider the notion of narrow operators). This naturally leads us to study properties of operators which are conjugate to narrow operators. We call such operators **-narrow* (see Definition 5.3). We prove that **-narrowness* of an operator on a reflexive r.i. space is a property of the image under it of

the unit ball (Proposition 5.5). On the other hand, we also show that the notion of narrow operators cannot be formulated in terms of the image. Namely we prove that if E is an r.i. space with an absolutely continuous norm, then there exists a complemented subspace E_0 of E that is isomorphic to E and for which there are two projections onto E_0 , one of which is narrow and the second is not narrow (Theorem 5.8). This answers (negatively) a question posed by A. Plichko and M. Popov [22, Question 2, p. 71].

2. Preliminaries

Let us suppose that Ω is a Polish space and that μ is a σ -finite Borel measure on Ω . We use the term Köthe space in the sense of [17, p. 28]. Thus a *Köthe function space* X on (Ω, μ) is a Banach space of (equivalence classes of) locally integrable Borel functions f on Ω such that:

- (1) If $|f| \leq |g|$ a.e. and $g \in X$ then $f \in X$ with $\|f\|_X \leq \|g\|_X$.
- (2) If A is a Borel set of finite measure then $\chi_A \in X$.

We say that X is *order-continuous* if whenever $f_n \in X$ with $f_n \downarrow 0$ a.e. then $\|f_n\|_X \downarrow 0$. X has the *Fatou property* if whenever $0 \leq f_n \in X$ with $\sup \|f_n\|_X < \infty$ and $f_n \uparrow f$ a.e. then $f \in X$ with $\|f\|_X = \sup \|f_n\|_X$.

A *rearrangement-invariant function space (r.i. space)* is a Köthe function space on $([0, 1], \mu)$, where μ is the Lebesgue measure, which satisfies the following conditions:

- (1) Either X is order-continuous or X has the Fatou property.
- (2) If $\tau : [0, 1] \rightarrow [0, 1]$ is any measure-preserving invertible Borel automorphism then $f \in X$ if and only if $f \circ \tau \in X$ and $\|f\|_X = \|f \circ \tau\|_X$.
- (3) $\|\chi_{[0,1]}\|_X = 1$.

Next we recall the definition of the Lorentz spaces. These were introduced by Lorentz [18], [19] in connection with some problems of harmonic analysis and interpolation theory and have since been extensively studied by many authors.

If f is a measurable function, we define the *non-increasing rearrangement* of f to be

$$f^*(t) = \inf\{s : \mu(|f| > s) \leq t\}.$$

Notice that when f is a simple function, $f = \sum_{k=1}^m a_k \chi_{A_k}$, then f^* is also a simple function and the range of f^* equals $\{|a_k| : k = 1, \dots, m\}$.

If $1 \leq p < \infty$, and if $w : (0, 1) \rightarrow (0, \infty)$ is a non-increasing function, we define the *Lorentz norm* of a measurable function f to be

$$\|f\|_{w,p} = \left(\int_{[0,1]} w(t) f^*(t)^p dt \right)^{1/p}.$$

We define the *Lorentz space* $L_{w,p}([0, 1], \mu)$ to be the space of those measurable functions f for which $\|f\|_{w,p}$ is finite. These spaces are generalizations of the

L_p spaces: if $w(x) = 1$ for all $0 \leq x < 1$, then $L_{w,p} = L_p$ with equality of norms. Lorentz spaces are one of the most important examples of r.i. spaces.

Narrow operators generalize the notion of compact operators on r.i. spaces.

DEFINITION 2.1. Let X be an r.i. space and Y be any Banach space. We say that an operator $T : X \rightarrow Y$ is *narrow* if for each measurable set A and each $\varepsilon > 0$ there exists $x \in X$ with $x^2 = \chi_A$, $\int x d\mu = 0$ and $\|Tx\| < \varepsilon$.

In fact, in the above definition the condition that $\int x d\mu = 0$ can be omitted [22, Proposition 8.1]. Every compact operator is narrow [22, Proposition 8.2].

Plichko and Popov proved the following lemma, which will be useful in our proof.

LEMMA 2.2. [22, Lemma 8.1] *Let $T : X \rightarrow Y$ be narrow. Then for every $\varepsilon > 0$, every measurable set A and every integer $n \geq 1$ there exists a partition $A = A' \cup A''$ into measurable subsets with $\mu(A') = 2^{-n}\mu(A)$ and $\mu(A'') = (1 - 2^{-n})\mu(A)$ such that $\|Th\| < \varepsilon$, where $h = (2^n - 1)\chi_{A'} - \chi_{A''}$.*

DEFINITION 2.3. Let X be an r.i. space. A subspace $Y \subset X$ is called *rich* if the quotient map $Q : X \rightarrow X/Y$ is narrow.

In other words, $Y \subset X$ is rich if for every measurable set A and every $\varepsilon > 0$ there exist $y \in Y$ and $x \in X$ such that $x^2 = \chi_A$, $\int x d\mu = 0$ and $\|x - y\| < \varepsilon$.

In particular, every subspace of finite codimension is rich.

3. Main result

THEOREM 3.1. *Suppose $L_{p,w}$ is a Lorentz space on $[0, 1]$ with $p > 2$. Then there exists $\varrho_p > 1$ such that for every nontrivial projection P from $L_{p,w}$ onto a rich subspace*

$$\|P\| \geq \varrho_p.$$

In the proof of Theorem 3.1 we will use the following two propositions:

PROPOSITION 3.2. *Suppose $L_{p,w}$ is a Lorentz space on $[0, 1]$ with $p > 2$. Then there exist $\delta_p \in (0, 1/8)$, $\lambda_p = \lambda_p(\delta_p, p) \in (\delta_p/(\delta_p - 4), 0)$ and $\gamma_p = \gamma_p(\lambda_p, \delta_p, p) \in (0, 1)$ such that*

$$\gamma_p + \frac{1}{2}|\lambda_p|\delta_p < 1$$

which satisfy the following property:

For every simple function $x = \sum_{k=1}^m a_k \chi_{A_k}$ such that $1 \leq \|x\|_{p,w} \leq 1 + \frac{3}{2}\delta_p$ and

$$\frac{|a_i|}{|a_j|} \notin (3 - \delta_p, 3)$$

for all $i, j = 1, \dots, m$, and for every partition $A_k = B_k \sqcup C_k$ with $\mu(B_k) = (1/4)\mu(A_k)$ we have

$$\left\| \lambda_p x + \sum_{k=1}^m a_k (3\chi_{B_k} - \chi_{C_k}) \right\|_{p,w} \leq \gamma_p \left\| \sum_{k=1}^m a_k (3\chi_{B_k} - \chi_{C_k}) \right\|_{p,w} .$$

PROPOSITION 3.3. *Let X be an r.i. space. Given a simple function $x = \sum_{k=1}^m a_k \chi_{A_k} \in X$ and $\delta \in (0, 1/8)$ there exists a simple function $x^\# = x^\#(\delta) = \sum_{k=1}^m a_k^\# \chi_{A_k}$ such that for all $i, j = 1, \dots, m$,*

$$\frac{|a_i^\#|}{|a_j^\#|} \notin (3 - \delta, 3)$$

and $\|x - x^\#\| < (3/2)\delta$, $\|x\| \leq \|x^\#\| < (1 + (3/2)\delta)\|x\|$.

Let us first show that Theorem 3.1 is indeed a consequence of Propositions 3.2 and 3.3.

Proof of Theorem 3.1. Fix $\varepsilon > 0$. Since P is a non-trivial projection, there exists a simple function $x = \sum_{k=1}^m a_k \chi_{A_k}$, $a_k \neq 0$, with $\|x\| = 1$ and $\|Px\| < \varepsilon$. (Note that since we will always work in $L_{p,w}$ we will drop the subscript and simply use $\|\cdot\|$ to mean $\|\cdot\|_{p,w}$ throughout this proof.)

Let δ_p be as defined in the statement of Proposition 3.2. Since $\delta_p \in (0, 1/8)$, by Proposition 3.3, there exists a simple function $x^\# = \sum_{k=1}^m a_k^\# \chi_{A_k}$ with $1 \leq \|x^\#\| < 1 + (3/2)\delta_p$, $\|x - x^\#\| < (3/2)\delta_p$, and such that

$$(3.1) \quad \frac{|a_i^\#|}{|a_j^\#|} \notin (3 - \delta_p, \delta_p),$$

for all $i, j = 1, \dots, m$.

Since $I - P$ is narrow, by Lemma 2.2, for each k , $1 \leq k \leq m$, there exists a partition $A_k = B_k \sqcup C_k$ such that $\mu(B_k) = (1/4)\mu(A_k)$ and

$$\|(I - P)(3\chi_{B_k} - \chi_{C_k})\| < \frac{\varepsilon}{|a_k^\#|m}.$$

Then for $z = \sum_{k=1}^m a_k^\# (3\chi_{B_k} - \chi_{C_k})$ we obtain

$$\|(I - P)z\| < \varepsilon,$$

and

$$\|z\| \leq 3\|x^\#\| < 3 + \frac{9}{2}\delta_p.$$

Moreover, by (3.1) and Proposition 3.2 we conclude that

$$\|\lambda_p x^\# + z\| \leq \gamma_p \|z\|,$$

where λ_p and γ_p are constants defined in Proposition 3.2. Thus

$$\begin{aligned} \|z\| &= \|P(\lambda_p x^\sharp + z) - P\lambda_p x^\sharp - P\lambda_p x + P\lambda_p x + z - Pz\| \\ &\leq \|P\| \cdot \|\lambda_p x^\sharp + z\| + |\lambda_p| \cdot \|P\| \cdot \|x^\sharp - x\| + |\lambda_p| \|Px\| + \|(I - P)z\| \\ &\leq \|P\| \cdot \gamma_p \|z\| + \|P\| |\lambda_p| \cdot \frac{3}{2} \delta_p + |\lambda_p| \varepsilon + \varepsilon \\ &\leq \|z\| \cdot \|P\| \left(\gamma_p + \frac{3\delta_p |\lambda_p|}{6 + 9\delta_p} \right) + \varepsilon (|\lambda_p| + 1) \\ &\leq \|z\| \cdot \|P\| \left(\gamma_p + \frac{1}{2} \delta_p |\lambda_p| \right) + \varepsilon (|\lambda_p| + 1). \end{aligned}$$

Since ε was arbitrary we obtain

$$\|P\| \geq \left(\gamma_p + \frac{1}{2} \delta_p |\lambda_p| \right)^{-1} \stackrel{\text{def}}{=} \varrho_p.$$

By Proposition 3.2, $\varrho_p > 1$. □

REMARK 3.4. Note that the same proof will demonstrate that whenever T is a narrow operator on $L_{p,w}$, $p > 2$, such that 1 is an eigenvalue of T , i.e., such that there exists a nonzero element $x \in L_{p,w}$ with $Tx = x$, then

$$\|I - T\| \geq \varrho_p > 1.$$

(Simply replace P in the proof by $I - T$, and note that Propositions 3.2 and 3.3 do not depend on the operator at all.)

Proof of Proposition 3.2. Let $\delta \in (0, 1/8)$ and let $x = \sum_{k=1}^m a_k \chi_{A_k}$ be a simple function such that $1 \leq \|x\|_{p,w} \leq 1 + (3/2)\delta$ and

$$(3.2) \quad \frac{|a_i|}{|a_j|} \notin (3 - \delta, 3),$$

for all $i, j = 1, \dots, m$. We assume without loss of generality that $|a_1| \geq |a_2| \geq \dots \geq |a_m|$. Let B_k, C_k for $k = 1, \dots, m$ be subsets as described in the statement of the proposition. Denote

$$y = \sum_{k=1}^m 3a_k \chi_{B_k} - a_k \chi_{C_k},$$

and set $b_k = 3a_k, c_k = -a_k$ for $k = 1, \dots, m$. Thus

$$\begin{aligned} y &= \sum_{k=1}^m b_k \chi_{B_k} + c_k \chi_{C_k}, \\ \lambda x + y &= \sum_{k=1}^m b_k \left(1 + \frac{\lambda}{3} \right) \chi_{B_k} + c_k (1 - \lambda) \chi_{C_k}. \end{aligned}$$

We first notice that if $0 \geq \lambda > \delta/(\delta - 4) > -1$, then for all $i, j = 1, \dots, m$ the following hold:

$$(3.3) \quad |b_i| \left(1 + \frac{\lambda}{3}\right) \leq |b_j| \left(1 + \frac{\lambda}{3}\right) \iff |b_i| \leq |b_j|;$$

$$(3.4) \quad |c_i|(1 - \lambda) \leq |c_j|(1 - \lambda) \iff |c_i| \leq |c_j|;$$

$$(3.5) \quad |c_i|(1 - \lambda) < |b_i| \left(1 + \frac{\lambda}{3}\right) \quad \text{for all } i;$$

$$(3.6) \quad |b_i| \left(1 + \frac{\lambda}{3}\right) \leq |c_j|(1 - \lambda) \iff |b_i| \leq |c_j|;$$

$$(3.7) \quad |b_i| \left(1 + \frac{\lambda}{3}\right) \neq |c_j|(1 - \lambda) \quad \text{for all } i, j.$$

Indeed, (3.3), (3.4) and (3.5) are obvious since $1 + \lambda/3 > 0$, $1 - \lambda > 0$ and $1 - \lambda < 3(1 + \lambda/3)$. To see “ \Rightarrow ” of (3.6), suppose for a contradiction that there exist i, j such that

$$|b_i| \left(1 + \frac{\lambda}{3}\right) \leq |c_j|(1 - \lambda), \quad \text{and} \quad |b_i| > |c_j|.$$

Then

$$1 > \frac{|c_j|}{|b_i|} \geq \frac{1 - \lambda}{1 + \lambda/3} = \frac{1 - \lambda}{3(3 + \lambda)}.$$

Thus, since $\lambda \in (\delta/(\delta - 4), 0)$,

$$3 > \frac{|a_j|}{|a_i|} \geq \frac{1 - \lambda}{3 + \lambda} > \frac{3 - \frac{\delta}{4 - \delta}}{1 + \frac{\delta}{4 - \delta}} = 3 - \delta,$$

which contradicts (3.2), and “ \Rightarrow ” of (3.6) is proved.

Next, suppose $|b_i| \leq |c_j|$. Since $\lambda < 0$ we get

$$|b_i| \left(1 + \frac{\lambda}{3}\right) < |b_i| \leq |c_j| < |c_j|(1 - \lambda).$$

Thus “ \Leftarrow ” of (3.6) and (3.7) are proved.

Now define numbers t_{C_k}, t_{B_k} for $k = 1, \dots, m$ as follows:

$$t_{C_i} = \sum_{k < i} \mu(C_k) + \sum_{l: |b_l| > |c_i|} \mu(B_l),$$

$$t_{B_j} = \sum_{k < j} \mu(B_k) + \sum_{i: |c_i| \geq |b_j|} \mu(C_i).$$

It follows from (3.3)–(3.7) that for all $i, j = 1, \dots, m$

$$\begin{aligned}
 (3.8) \quad & t_{C_i} > t_{C_j} \Rightarrow \left(|c_i| \leq |c_j| \text{ and } |c_i|(1 - \lambda) \leq |c_j|(1 - \lambda) \right), \\
 & \left(|c_i| < |c_j|, \text{ or, equivalently, } |c_i|(1 - \lambda) < |c_j|(1 - \lambda) \right) \Rightarrow t_{C_i} > t_{C_j}, \\
 & t_{B_i} > t_{B_j} \Rightarrow \left(|b_i| \leq |b_j| \text{ and } |b_i| \left(1 + \frac{\lambda}{3} \right) \leq |b_j| \left(1 + \frac{\lambda}{3} \right) \right), \\
 & \left(|b_i| < |b_j| \text{ or, equivalently, } |b_i| \left(1 + \frac{\lambda}{3} \right) < |b_j| \left(1 + \frac{\lambda}{3} \right) \right) \Rightarrow t_{B_i} > t_{B_j}, \\
 & t_{B_i} > t_{C_i} \Leftrightarrow |b_i| \left(1 + \frac{\lambda}{3} \right) < |c_i|(1 - \lambda) \Leftrightarrow |b_i| \leq |c_i|.
 \end{aligned}$$

Now define the weights

$$w_{B_k} = \int_{t_{B_k}}^{t_{B_k} + \mu(B_k)} w d\mu, \quad w_{C_k} = \int_{t_{C_k}}^{t_{C_k} + \mu(C_k)} w d\mu.$$

By (3.8) we obtain

$$\begin{aligned}
 \|y\|_{p,w}^p &= \sum_{k=1}^m [|b_k|^p w_{B_k} + |c_k|^p w_{C_k}], \\
 \|\lambda x + y\|_{p,w}^p &= \sum_{k=1}^m \left[|b_k|^p \left(1 + \frac{\lambda}{3} \right)^p w_{B_k} + |c_k|^p (1 - \lambda)^p w_{C_k} \right].
 \end{aligned}$$

(Heuristically, the non-increasing order of the moduli of the coefficients of y is the same as the non-increasing order of the moduli of the coefficients of $\lambda x + y$.) Thus if we set

$$\psi(\lambda) = \sum_{k=1}^m \left[|b_k|^p \left(1 + \frac{\lambda}{3} \right)^p w_{B_k} + |c_k|^p (1 - \lambda)^p w_{C_k} \right],$$

for $\lambda \in (-3, 1)$, then

$$(3.9) \quad \psi(0) = \|y\|_{p,w}^p, \quad \psi(\lambda) = \|\lambda x + y\|_{p,w}^p \text{ for } \lambda \in (\delta/(\delta - 4), 0).$$

Clearly ψ is differentiable for all $\lambda \in (-3, 1)$ and

$$\psi'(\lambda) = \sum_{k=1}^m \left[p|b_k|^p \left(1 + \frac{\lambda}{3} \right)^{p-1} \frac{1}{3} w_{B_k} - p|c_k|^p (1 - \lambda)^{p-1} w_{C_k} \right].$$

Thus

$$(3.10) \quad \begin{aligned} \psi'(0) &= p \sum_{k=1}^m \left[\frac{1}{3} |b_k|^p w_{B_k} - |c_k|^p w_{C_k} \right] \\ &= p \sum_{k=1}^m |a_k|^p [3^{p-1} w_{B_k} - w_{C_k}]. \end{aligned}$$

We now need to compare the quantities w_{B_k} and w_{C_k} for a given k , $1 \leq k \leq m$. It follows from (3.5) that $t_{C_k} > t_{B_k}$. Moreover, by the definition of B_k and C_k we have $\mu(C_k) = 3\mu(B_k)$. Thus, since w is non-increasing, we get

$$\begin{aligned} w_{C_k} &= \int_{t_{C_k}}^{t_{C_k} + \mu(C_k)} w d\mu = \int_{t_{C_k}}^{t_{C_k} + 3\mu(B_k)} w d\mu \\ &\leq \int_{t_{B_k}}^{t_{B_k} + \mu(B_k)} w d\mu + \int_{t_{B_k} + \mu(B_k)}^{t_{B_k} + 2\mu(B_k)} w d\mu + \int_{t_{B_k} + 2\mu(B_k)}^{t_{B_k} + 3\mu(B_k)} w d\mu \\ &\leq 3 \int_{t_{B_k}}^{t_{B_k} + \mu(B_k)} w d\mu \\ &= 3w_{B_k}. \end{aligned}$$

Thus for all $k = 1, \dots, m$,

$$(3.11) \quad 3^{p-1} w_{B_k} - w_{C_k} \geq 3^{p-1} w_{B_k} - 3w_{B_k} = w_{B_k} (3^{p-1} - 3).$$

In analogy to the numbers $t_{C_k}, t_{B_k}, w_{C_k}, w_{B_k}$ we define

$$\begin{aligned} t_{A_k} &= \sum_{l < k} \mu(A_l), \\ w_{A_k} &= \int_{t_{A_k}}^{t_{A_k} + \mu(A_k)} w d\mu. \end{aligned}$$

Since we assumed that $|a_1| \geq |a_2| \geq \dots \geq |a_m|$ we have

$$\|x\|_{p,w}^p = \sum_{k=1}^m |a_k|^p w_{A_k}.$$

Further, since for all k , $\mu(A_k) = \mu(B_k) + \mu(C_k)$ and since $|c_l| \geq |b_j| \Rightarrow |c_l| > |c_j| \Rightarrow l < j$, we obtain

$$\begin{aligned} t_{B_j} &= \sum_{k < j} \mu(B_k) + \sum_{l: |c_l| \geq |b_j|} \mu(C_l) \leq \sum_{k < j} \mu(B_k) + \sum_{l < j} \mu(C_l) \\ &= \sum_{k < j} \mu(A_k) = t_{A_j}. \end{aligned}$$

Therefore, since for all $j = 1, \dots, m$, $\mu(B_j) = \frac{1}{4}\mu(A_j)$, we obtain

$$\begin{aligned} w_{B_j} &= \int_{t_{B_j}}^{t_{B_j} + \mu(B_j)} w d\mu \geq \int_{t_{A_j}}^{t_{A_j} + \frac{1}{4}\mu(A_j)} w d\mu \\ &\geq \frac{1}{4} \int_{t_{A_j}}^{t_{A_j} + \mu(A_j)} w d\mu = \frac{1}{4} w_{A_j}. \end{aligned}$$

Thus we can continue the estimate from (3.11) as follows:

$$3^{p-1}w_{B_k} - w_{C_k} \geq w_{B_k}(3^{p-1} - 3) \geq \frac{1}{4}(3^{p-1} - 3)w_{A_k}.$$

Plugging this into (3.10) we get

$$\begin{aligned} (3.12) \quad \psi'(0) &= p \sum_{k=1}^m |a_k|^p [3^{p-1}w_{B_k} - w_{C_k}] \\ &\geq \frac{1}{4}p(3^{p-1} - 3) \sum_{k=1}^m |a_k|^p w_{A_k} \\ &= \frac{1}{4}p(3^{p-1} - 3) \|x\|_{w,p}^p \\ &\geq \frac{1}{4}p(3^{p-1} - 3) \stackrel{\text{def}}{=} C_p. \end{aligned}$$

Note that our assumption that $p > 2$ guarantees that $C_p > 0$.

Our next step is to estimate from above the value of $|\psi''(\lambda)|$ when $\lambda \in (\delta/(\delta - 4), \delta/(4 - \delta))$. We have, since $\delta \in (0, 1/8)$,

$$\begin{aligned} (3.13) \quad |\psi''(\lambda)| &= \left| p(p-1) \sum_{k=1}^m \left[\frac{1}{9} |b_k|^p \left(1 + \frac{\lambda}{3}\right)^{p-2} w_{B_k} + |c_k|^p (1-\lambda)^{p-2} w_{C_k} \right] \right| \\ &\leq p(p-1) \left(1 + \frac{\delta}{4-\delta}\right)^{p-2} \sum_{k=1}^m \left[\frac{1}{9} |b_k|^p w_{B_k} + |c_k|^p w_{C_k} \right] \\ &\leq p(p-1) \left(1 + \frac{\delta}{4-\delta}\right)^{p-2} \|y\|_{w,p}^p \\ &\leq p(p-1) \left(\frac{4}{4-\delta}\right)^{p-2} 3^p \|x\|_{w,p}^p \\ &\leq p(p-1) \left(\frac{4}{4-\delta}\right)^{p-2} 3^p \left(1 + \frac{3}{2}\delta\right)^p \\ &\leq p(p-1)4^p \stackrel{\text{def}}{=} M_p. \end{aligned}$$

By Taylor's Theorem we get for $\lambda \in (\delta/(\delta - 4), 0)$

$$\psi(\lambda) = \psi(0) + \lambda\psi'(0) + \frac{1}{2}\lambda^2\psi''(\theta),$$

where $\theta \in (\lambda, 0)$. Note that

$$\psi(0) = \|y\|_{p,w}^p \leq 3^p \|x\|_{p,w}^p \leq \left[3 \left(1 + \frac{3}{2}\delta\right)\right]^p = \left(3 + \frac{9}{2}\delta\right)^p.$$

Hence by (3.12) and (3.13) we have

$$\begin{aligned} \psi(\lambda) &\leq \psi(0) + \lambda C_p + \frac{1}{2}\lambda^2 M_p \\ &\leq \psi(0) \left[1 + \frac{\lambda}{\left(3 + \frac{9}{2}\delta\right)^p} \left(C_p + \frac{1}{2}\lambda M_p\right)\right] \\ &\leq \psi(0) \left[1 + \lambda 3^{-p} \left(C_p + \frac{1}{2}\lambda M_p\right)\right]. \end{aligned}$$

Thus when $|\lambda| \leq C_p/M_p$ we get

$$\psi(\lambda) \leq \psi(0) \left[1 + \lambda \frac{C_p}{2 \cdot 3^p}\right].$$

If $\delta \leq \min\{1/8, 4C_p/M_p\}$ we set $\lambda = -\delta/4$ and

$$\gamma(\delta) \stackrel{\text{def}}{=} \left(1 - \delta \cdot \frac{C_p}{8 \cdot 3^p}\right)^{1/p}.$$

Then $\gamma(\delta) < 1$ and by the definition of the function ψ we have

$$\left\| -\frac{\delta}{4} \cdot x + y \right\| \leq \gamma(\delta) \|y\|.$$

To finish the proof of the proposition we notice that, by the Bernoulli inequality,

$$\gamma(\delta) < 1 - \delta \cdot \frac{C_p}{8 \cdot 3^p \cdot p}.$$

Set

$$D_p \stackrel{\text{def}}{=} \frac{C_p}{8 \cdot 3^p \cdot p} = \frac{3^{p-2} - 1}{32 \cdot 3^{p-1}}.$$

Since $p > 2$ we have $D_p > 0$ and we obtain

$$\gamma(\delta) + \frac{1}{2}|\lambda|\delta = \gamma(\delta) + \frac{1}{8}\delta^2 < 1 - \delta D_p + \frac{1}{8}\delta^2 = 1 - \delta(D_p - \frac{1}{8}\delta).$$

Thus if $\delta \leq 8D_p$ then

$$\gamma(\delta) + \frac{1}{2}|\lambda|\delta < 1.$$

Hence we can take $\delta_p = \min\{1/8, 4C_p/M_p, 8D_p\}$, $\lambda_p = -\delta_p/4$, $\gamma_p = \gamma(\delta_p) = (1 - \delta_p p D_p)^{1/p}$, and the proposition is proved. \square

REMARK 3.5. It is clear that the above proof does not work for $p < 2$. Indeed, when $p < 2$ then the estimate (3.11) becomes meaningless and both constants C_p and D_p are negative. Moreover, for every p , $1 \leq p < 2$, it is not difficult to construct weights w_p such that when $x = \chi_{[0,1]}$ is partitioned into any disjoint sets $[0, 1] = B \sqcup C$ with $\mu(B) = 1/4$ then for any $\lambda \in \mathbb{R}$

$$(3.14) \quad \|\lambda x + (3\chi_B - \chi_C)\|_{p,w_p} \geq \|3\chi_B - \chi_C\|_{p,w_p}.$$

In fact, one can take, e.g.,

$$w_p = \frac{4}{3^{p-1} + 1} \chi_{[0,1/4]} + \frac{4 \cdot 3^{p-2}}{3^{p-1} + 1} \chi_{[1/4,1]}.$$

This is a well defined weight when $1 \leq p < 2$. It is routine, although tedious, to check that $L_{w_p,p}$ satisfies (3.14) for all p with $1 \leq p < 2$. We leave the details to the interested reader.

We suspect that if $1 \leq p < 2$ Proposition 3.2 fails in $L_{w,p}$ for any nonconstant weight w , but we have not checked it carefully. This, of course, does not mean that we believe that Theorem 3.1 fails for $p < 2$. For some comments on duality please see Section 5.

4. Proof of Proposition 3.3

The first step of the proof of Proposition 3.3 is the following lemma.

LEMMA 4.1. *Let X be an r.i. space and $x \in X$ be a simple function $x = \sum_{k=1}^m a_k \chi_{A_k}$. For any $\eta > 0$ there exists $\bar{x} = \bar{x}(\eta) = \sum_{k=1}^m \bar{a}_k \chi_{A_k}$ such that for all $i, j = 1, \dots, m$,*

$$\frac{|\bar{a}_j|}{|\bar{a}_i|} \notin (1, 1 + \eta)$$

and $\|x - \bar{x}\| < \eta \|x\|$, $\|\bar{x}\| \geq \|x\|$.

Proof of Lemma 4.1. Without loss of generality we assume that $|a_1| \geq |a_2| \geq \dots \geq |a_m|$.

Let $r_0 = 1 < r_1 < r_2 \dots < r_n = m$ be such that

$$\frac{|a_j|}{|a_i|} \begin{cases} < 1 + \eta & \text{if there exists } k \text{ with } r_k \leq j < i < r_{k+1}, \\ \geq 1 + \eta & \text{if there exists } k \text{ with } j < r_k \leq i. \end{cases}$$

Define

$$\bar{a}_j = \text{sgn}(a_j) |a_{r_{k(j)}}|,$$

where $k(j)$ is such that $r_{k(j)} \leq j < r_{k(j)+1}$. Then for any $j < i$ we have $k(j) \leq k(i)$ and

$$\frac{|\bar{a}_j|}{|\bar{a}_i|} = \frac{|a_{r_{k(j)}}|}{|a_{r_{k(i)}}|} \begin{cases} = 1 & \text{if } k(j) = k(i), \\ \geq 1 + \eta & \text{if } k(j) < k(i). \end{cases}$$

Thus

$$\frac{|\bar{a}_j|}{|\bar{a}_i|} \notin (1, 1 + \eta)$$

as required.

Moreover, for all $j = 1, \dots, m$,

$$\frac{\bar{a}_j}{a_j} = \frac{|a_{r_k(j)}|}{|a_j|} \in [1, 1 + \eta).$$

Thus $\|\bar{x}\| \geq \|x\|$ and

$$\begin{aligned} \|\bar{x} - x\| &= \left\| \sum_{j=1}^m (\bar{a}_j - a_j) \chi_{A_k} \right\| = \left\| \sum_{j=1}^m a_j \left(\frac{\bar{a}_j}{a_j} - 1 \right) \chi_{A_k} \right\| \\ &< \eta \left\| \sum_{j=1}^m a_j \chi_{A_j} \right\| = \eta \|x\|, \end{aligned}$$

which completes the proof of the lemma. □

In the next lemma we gather, for easy reference, a few simple arithmetic inequalities which will be useful in the proof of Proposition 3.3.

LEMMA 4.2. *Let $\delta \in (0, 3)$ and $t_i, i = 1, 2, 3, 4$, be positive real numbers such that*

$$\frac{\tilde{t}_i}{t_i} \in \left[1, \frac{3}{3 - \delta} \right) \text{ for } i = 1, 2, 3, 4.$$

Then we have:

- (i) *If $t_i = t_2, \tilde{t}_i/\tilde{t}_3 \in (3 - \delta, 3), \tilde{t}_2/\tilde{t}_4 \in (3 - \delta, 3)$, then $\tilde{t}_3/\tilde{t}_4 \in ((3 - \delta)^3/27, 27/(3 - \delta)^3)$.*
- (ii) *If $t_1/t_2 \in (3 - \delta, 3), t_3/t_2 = 3$, then $t_3/t_1 \in (1, 3/(3 - \delta))$.*
- (iii) *If $t_1 < t_2$, then $\tilde{t}_1/\tilde{t}_2 < 9/(3 - \delta)^2$.*
- (iv) *If $t_1/\tilde{t}_2 = 3, t_1/\tilde{t}_3 \in (3 - \delta, 3)$, then $t_2/t_3 \in ((3 - \delta)^2/9, 3/(3 - \delta))$.*

Proof of Lemma 4.2. The proofs of Lemma 4.2(i)–(iv) are very simple and very similar to each other. As an illustration we prove the implication (i).

We have

$$\frac{t_3}{t_4} = \frac{t_3}{\tilde{t}_3} \cdot \frac{\tilde{t}_3}{\tilde{t}_1} \cdot \frac{\tilde{t}_1}{t_1} \cdot \frac{t_1}{t_2} \cdot \frac{t_2}{\tilde{t}_2} \cdot \frac{\tilde{t}_2}{\tilde{t}_4} \cdot \frac{\tilde{t}_4}{t_4}$$

Since

$$\begin{aligned} \frac{t_3}{\tilde{t}_3} &\in \left(\frac{3 - \delta}{3}, 1 \right], \quad \frac{\tilde{t}_3}{\tilde{t}_1} \in \left(\frac{1}{3}, \frac{1}{3 - \delta} \right), \quad \frac{\tilde{t}_1}{t_1} \in \left[1, \frac{3}{3 - \delta} \right), \quad \frac{t_1}{t_2} = 1, \\ \frac{t_2}{\tilde{t}_2} &\in \left(\frac{3 - \delta}{3}, 1 \right], \quad \frac{\tilde{t}_2}{\tilde{t}_4} \in (3 - \delta, 3), \quad \frac{\tilde{t}_4}{t_4} \in \left[1, \frac{3}{3 - \delta} \right), \end{aligned}$$

we obtain

$$\begin{aligned} \frac{t_3}{t_4} &\in \left(\frac{3-\delta}{3} \cdot \frac{1}{3} \cdot 1 \cdot 1 \cdot \frac{3-\delta}{3} \cdot (3-\delta) \cdot 1, 1 \cdot \frac{1}{3-\delta} \cdot \frac{3}{3-\delta} \cdot 1 \cdot 1 \cdot 3 \cdot \frac{3}{3-\delta} \right) \\ &= \left(\frac{(3-\delta)^3}{27}, \frac{27}{(3-\delta)^3} \right). \end{aligned}$$

Implications (ii)–(iv) are proved in a very similar way. □

In the proof of Proposition 3.3 we will also need the following definition:

DEFINITION 4.3. For any simple function $y = \sum_{k=1}^m d_k \chi_{D_k}$ define the sets

$$S(y, k) = \left\{ j \in \{1, \dots, m\} : \frac{|d_j|}{|d_k|} \in (3-\delta, 3) \right\}.$$

Proof of Proposition 3.3. Let $\eta > 0$ be such that

$$1 + \eta = \left(\frac{3}{3-\delta} \right)^3.$$

By Lemma 4.1 there exists $\bar{x} = \bar{x}(\eta) = \sum_{k=1}^m \bar{a}_k \chi_{A_k}$ with

$$(4.1) \quad \|x - \bar{x}\| < \eta \|x\|,$$

such that for all $i, j = 1, \dots, m$,

$$\frac{|\bar{a}_i|}{|\bar{a}_j|} \notin (1, 1 + \eta).$$

By symmetry this means that for all $i, j = 1, \dots, m$,

$$(4.2) \quad \frac{|\bar{a}_i|}{|\bar{a}_j|} \in \left(\frac{1}{1+\eta}, 1 + \eta \right) \implies |\bar{a}_i| = |\bar{a}_j|.$$

To prove the lemma we need to construct a simple function x^\sharp such that $\|x - x^\sharp\| < (3/2)\delta$, $\|x\| \leq \|x^\sharp\| < (1 + (3/2)\delta)\|x\|$ and

$$(4.3) \quad S(x^\sharp, k) = \emptyset \text{ for } k = 1, \dots, m.$$

We will construct x^\sharp satisfying (4.3) inductively. To start the induction we set

$$\begin{aligned} \bar{a}_k^{(0)} &= \bar{a}_k \text{ for } k = 1, \dots, m, \\ \bar{x}^{(0)} &= \sum_{k=1}^m \bar{a}_k^{(0)} \chi_{A_k} = \bar{x}, \\ k_0 &= \max(\{k : S(\bar{x}^{(0)}, k) \neq \emptyset\} \cup \{0\}). \end{aligned}$$

If $k_0 = 0$ then $\bar{x}^{(0)}$ satisfies (4.3) and we are done. If $k_0 > 0$ then

$$S(\bar{x}^{(0)}, k) = \emptyset \text{ for } k > k_0.$$

In the inductive process we will describe a way of defining a sequence of non-negative integers $k_0 > k_1 > k_2 > \dots$ and a sequence of simple functions $(\bar{x}^{(0)}, \bar{x}^{(1)}, \bar{x}^{(2)}, \dots)$ such that for all n

$$S(\bar{x}^{(n)}, k) = \emptyset \text{ for } k > k_n.$$

Once these sequences are defined we observe that since $k_0 \leq m$ and the sequence $(k_n)_n$ is a strictly decreasing sequence of non-negative integers, there exists $N \leq m + 1$, such that $k_N = 0$ and $\bar{x}^{(N)}$ satisfies (4.3).

To describe the inductive process, suppose that $(\bar{x}^{(\nu)})_{\nu=0}^n$ and $k_0 > k_1 > \dots > k_n > 0$ have been defined such that for all $\nu \leq n$

$$\begin{aligned} \bar{x}^{(\nu)} &= \sum_{k=1}^m \bar{a}_k^{(\nu)} \chi_{A_k}, \\ (4.4) \quad S(\bar{x}^{(\nu)}, k_\nu) &\neq \emptyset && \text{if } k_\nu > 0, \end{aligned}$$

$$(4.5) \quad S(\bar{x}^{(\nu)}, k) = \emptyset \quad \text{for } k > k_\nu,$$

$$(4.6) \quad \frac{\bar{a}_k^{(\nu)}}{\bar{a}_k} \in \left[1, \frac{3}{3-\delta} \right) \quad \text{for } k = 1, \dots, m,$$

$$(4.7) \quad \bar{a}_k^{(\nu)} = \bar{a}_k \quad \text{for } k \notin \bigcup_{\alpha=0}^{\nu-1} S(\bar{x}^{(\alpha)}, k_\alpha),$$

$$(4.8) \quad |\bar{a}_k| = |\bar{a}_l| \implies |\bar{a}_k^{(\nu)}| = |\bar{a}_l^{(\nu)}|,$$

$$(4.9) \quad |\bar{a}_{k_\nu}| > |\bar{a}_{k_{\nu-1}}| \quad \text{if } k_\nu > 0.$$

Now define

$$\bar{a}_j^{(n+1)} = \begin{cases} \operatorname{sgn}(\bar{a}_j) \cdot 3 \cdot |\bar{a}_{k_n}^{(n)}| & \text{if } j \in S(\bar{x}^{(n)}, k_n), \\ \bar{a}_j^{(n)} & \text{if } j \notin S(\bar{x}^{(n)}, k_n), \end{cases}$$

$$\bar{x}^{(n+1)} = \sum_{k=1}^m \bar{a}_k^{(n+1)} \chi_{A_k},$$

$$k_{n+1} = \max(\{k : S(\bar{x}^{(n+1)}, k) \neq \emptyset\} \cup \{0\}).$$

To prove the induction step we need to show that (4.4)–(4.9) are satisfied for $\nu = n + 1$.

Clearly, if $k_{n+1} \neq 0$ then $S(\bar{x}^{(n+1)}, k_{n+1}) \neq \emptyset$, so (4.4) holds for $\nu = n + 1$. Similarly, (4.5) holds for $\nu = n + 1$ by the definition of k_{n+1} . To verify (4.6) for $\nu = n + 1$ we first observe that if $j \notin S(\bar{x}^{(n)}, k_n)$ then, by definition, $\bar{a}_j^{(n+1)} = \bar{a}_j^{(n)}$, and by (4.6) we get

$$(4.10) \quad \frac{\bar{a}_j^{(n+1)}}{\bar{a}_j} = \frac{\bar{a}_j^{(n)}}{\bar{a}_j} \in \left[1, \frac{3}{3-\delta} \right) \text{ for } j \notin S(\bar{x}^{(n)}, k_n).$$

Thus it only remains to check that (4.6) is valid for $\nu = n + 1$ and $j \in S(\bar{x}^{(n)}, k_n)$. For this we first establish that

$$(4.11) \quad \bar{a}_j^{(n)} = \bar{a}_j \text{ for } j \in S(\bar{x}^{(n)}, k_n).$$

To prove (4.11), by (4.7), it is enough to show that for all α , $0 \leq \alpha < n$,

$$S(\bar{x}^{(n)}, k_n) \cap S(\bar{x}^{(\alpha)}, k_\alpha) = \emptyset.$$

Suppose, for a contradiction, that there exists α , $0 \leq \alpha < n$, and i , $1 \leq i \leq m$, such that

$$i \in S(\bar{x}^{(n)}, k_n) \cap S(\bar{x}^{(\alpha)}, k_\alpha).$$

Set $t_1 = t_2 = |\bar{a}_i|$, $\tilde{t}_1 = |\bar{a}_i^{(n)}|$, $\tilde{t}_2 = |\bar{a}_i^{(\alpha)}|$, $t_3 = |\bar{a}_{k_n}|$, $\tilde{t}_3 = |\bar{a}_{k_n}^{(n)}|$, $t_4 = |\bar{a}_{k_\alpha}|$, $\tilde{t}_4 = |\bar{a}_{k_\alpha}^{(\alpha)}|$. Then, by (4.6) and Lemma 4.2(i),

$$\frac{|\bar{a}_{k_n}|}{|\bar{a}_{k_\alpha}|} = \frac{t_3}{t_4} \in \left(\frac{(3 - \delta)^3}{3^3}, \frac{3^3}{(3 - \delta)^3} \right) = \left(\frac{1}{1 + \eta}, 1 + \eta \right).$$

Thus, by (4.2), $|\bar{a}_{k_n}| = |\bar{a}_{k_\alpha}|$, which contradicts (4.9). Hence (4.11) is proven.

Next, by (4.11) and by the definition of $S(\bar{x}^{(n)}, k_n)$, we see that for $j \in S(\bar{x}^{(n)}, k_n)$

$$\frac{|\bar{a}_j|}{|\bar{a}_{k_n}^{(n)}|} = \frac{|\bar{a}_j^{(n)}|}{|\bar{a}_{k_n}^{(n)}|} \in (3 - \delta, 3).$$

Thus, by Lemma 4.2(ii) with $t_1 = |\bar{a}_j|$, $t_2 = |\bar{a}_{k_n}^{(n)}|$, $t_3 = |\bar{a}_j^{(n+1)}|$, we obtain

$$\frac{\bar{a}_j^{(n+1)}}{\bar{a}_j} = \frac{|\bar{a}_j^{(n+1)}|}{|\bar{a}_j|} = \frac{t_3}{t_1} \in \left(1, \frac{3}{3 - \delta} \right) \text{ for } j \in S(\bar{x}^{(n)}, k_n).$$

Together with (4.10) this ends the proof that (4.6) is satisfied for $\nu = n + 1$.

Next we check that (4.7) is valid for $\nu = n + 1$, i.e.,

$$\bar{a}_k^{(n+1)} = \bar{a}_k \text{ for } k \notin \bigcup_{\alpha=0}^n S(\bar{x}^{(\alpha)}, k_\alpha).$$

Let $k \notin \bigcup_{\alpha=0}^n S(\bar{x}^{(\alpha)}, k_\alpha)$. Then $k \notin S(\bar{x}^{(n)}, k_n)$ and, by definition, $\bar{a}_k^{(n+1)} = \bar{a}_k^{(n)}$. And since $k \notin \bigcup_{\alpha=0}^{n-1} S(\bar{x}^{(\alpha)}, k_\alpha)$, by (4.7), $\bar{a}_k^{(n)} = \bar{a}_k$. Thus (4.7) holds for $\nu = n + 1$.

Our next step is to check (4.8) for $\nu = n + 1$. We know, by (4.8), that if $|\bar{a}_k| = |\bar{a}_l|$ then $|\bar{a}_k^{(n)}| = |\bar{a}_l^{(n)}|$. Thus $k \in S(\bar{x}^{(n)}, k_n)$ if and only if $l \in S(\bar{x}^{(n)}, k_n)$. In either case it follows directly from the definition that $|\bar{a}_k^{(n+1)}| = |\bar{a}_l^{(n+1)}|$, i.e., (4.8) holds for $\nu = n + 1$.

Our final step is to verify (4.9) for $\nu = n + 1$, i.e., to show that if $k_{n+1} > 0$ then

$$|\bar{a}_{k_{n+1}}| > |\bar{a}_{k_n}|.$$

Since the numbers $(|\bar{a}_k|)_{k=1}^m$ are arranged in a non-increasing order and $k_{n+1} = \max(\{k : S(\bar{x}^{(n+1)}, k) \neq \emptyset\} \cup \{0\}) > 0$, it is enough to prove that

$$(4.12) \quad |\bar{a}_k| \leq |\bar{a}_{k_n}| \implies S(\bar{x}^{(n+1)}, k) = \emptyset.$$

If $|\bar{a}_k| = |\bar{a}_{k_n}|$ then by (4.8) for $\nu = n + 1$ we get $|\bar{a}_k^{(n+1)}| = |\bar{a}_{k_n}^{(n+1)}|$ and thus

$$(4.13) \quad S(\bar{x}^{(n+1)}, k) = S(\bar{x}^{(n+1)}, k_n).$$

Notice that $k_n \notin S(\bar{x}^n, k_n)$, so $\bar{a}_{k_n}^{(n+1)} = \bar{a}_{k_n}^{(n)}$. Hence, when $j \notin S(\bar{x}^n, k_n)$ we get $\bar{a}_j^{(n+1)} = \bar{a}_j^{(n)}$ and

$$\frac{|\bar{a}_j^{(n+1)}|}{|\bar{a}_{k_n}^{(n+1)}|} = \frac{|\bar{a}_j^{(n)}|}{|\bar{a}_{k_n}^{(n)}|} \notin (3 - \delta, 3).$$

Thus $j \notin S(\bar{x}^{(n+1)}, k_n)$.

If $j \in S(\bar{x}^n, k_n)$ then, by definition,

$$\frac{|\bar{a}_j^{(n+1)}|}{|\bar{a}_{k_n}^{(n+1)}|} = \frac{|\bar{a}_j^{(n+1)}|}{|\bar{a}_{k_n}^{(n)}|} = 3 \notin (3 - \delta, 3).$$

Hence $S(\bar{x}^{(n+1)}, k_n) = \emptyset$ and by (4.13) we see that

$$(4.14) \quad |\bar{a}_k| = |\bar{a}_{k_n}| \implies S(\bar{x}^{(n+1)}, k) = \emptyset.$$

Next we consider the case

$$|\bar{a}_k| < |\bar{a}_{k_n}|.$$

In this case $k > k_n$ and by the definition of k_n , $S(\bar{x}^n, k) = \emptyset$. By (4.6)

$$\frac{|\bar{a}_k^{(n)}|}{|\bar{a}_k|} \in \left[1, \frac{3}{3 - \delta}\right), \quad \frac{|\bar{a}_{k_n}^{(n)}|}{|\bar{a}_{k_n}|} \in \left[1, \frac{3}{3 - \delta}\right).$$

Hence, if we set $t_1 = |\bar{a}_k|$, $\tilde{t}_1 = |\bar{a}_k^{(n)}|$, $t_2 = |\bar{a}_{k_n}|$, $\tilde{t}_2 = |\bar{a}_{k_n}^{(n)}|$, we obtain by Lemma 4.2(iii) that

$$\frac{|\bar{a}_k^{(n)}|}{|\bar{a}_{k_n}^{(n)}|} = \frac{\tilde{t}_1}{\tilde{t}_2} < \frac{9}{(3 - \delta)^2} < 3 - \delta.$$

Thus $k \notin S(\bar{x}^n, k_n)$ and, by definition, $\bar{a}_k^{(n+1)} = \bar{a}_k^{(n)}$. Further, for all $j \notin S(\bar{x}^n, k_n)$ we have $\bar{a}_j^{(n+1)} = \bar{a}_j^{(n)}$, and therefore

$$\frac{|\bar{a}_j^{(n+1)}|}{|\bar{a}_k^{(n+1)}|} = \frac{|\bar{a}_j^{(n)}|}{|\bar{a}_k^{(n)}|} \notin (3 - \delta, 3),$$

since $S(\bar{x}^n, k) = \emptyset$. Hence

$$S(\bar{x}^{(n+1)}, k) \subset S(\bar{x}^n, k_n).$$

But if $j \in S(\bar{x}^{(n+1)}, k) \cap S(\bar{x}^{(n)}, k_n)$ then

$$\frac{|\bar{a}_j^{(n+1)}|}{|\bar{a}_{k_n}^{(n)}|} = 3, \quad \frac{|\bar{a}_j^{(n+1)}|}{|\bar{a}_k^{(n)}|} = \frac{|\bar{a}_j^{(n+1)}|}{|\bar{a}_k^{(n+1)}|} \in (3 - \delta, 3),$$

and if we set $t_1 = |\bar{a}_j^{(n+1)}|$, $t_2 = |\bar{a}_{k_n}|$, $\tilde{t}_2 = |\bar{a}_{k_n}^{(n)}|$, $t_3 = |\bar{a}_k|$, $\tilde{t}_3 = |\bar{a}_k^{(n)}|$ then by (4.6) and by Lemma 4.2(iv) we get

$$\frac{|\bar{a}_{k_n}|}{|\bar{a}_k|} = \frac{t_2}{t_3} \in \left(\frac{(3 - \delta)^2}{9}, \frac{3}{3 - \delta} \right) \subset \left(\frac{1}{1 + \eta}, 1 + \eta \right).$$

Thus by (4.2), $|\bar{a}_{k_n}| = |\bar{a}_k|$, which contradicts our assumption that $|\bar{a}_{k_n}| > |\bar{a}_k|$.

Thus $S(\bar{x}^{(n+1)}, k) = \emptyset$ if $|\bar{a}_k| < |\bar{a}_{k_n}|$, which together with (4.14) concludes the proof of (4.12) and (4.9) for $\nu = n + 1$.

Note that (4.9) for $\nu = n + 1$ implies that $k_{n+1} < k_n$.

This ends the proof of the inductive process.

To finish the proof of the proposition we notice, as indicated above, that since $(k_n)_{n \geq 0}$ is a strictly decreasing sequence of nonnegative integers, it must be finite, i.e., there exists $N \leq m + 1$, such that $k_N = 0$. Set $x^\# = \bar{x}^{(N)}$. Then, by (4.5),

$$S(\bar{x}^{(N)}, k) = \emptyset, \quad \text{for all } k > k_N = 0,$$

and, by (4.6),

$$\begin{aligned} \|x^\# - \bar{x}\| &= \left\| \sum_{k=1}^m (\bar{a}_k^{(N)} - \bar{a}_k) \chi_{A_k} \right\| = \left\| \sum_{k=1}^m \bar{a}_k \left(\frac{\bar{a}_k^{(N)}}{\bar{a}_k} - 1 \right) \chi_{A_k} \right\| \\ &\leq \left(\frac{3}{3 - \delta} - 1 \right) \cdot \left\| \sum_{k=1}^m \bar{a}_k \chi_{A_k} \right\| = \frac{\delta}{3 - \delta} \|\bar{x}\|. \end{aligned}$$

Thus, by (4.1), when $\delta < 1/8$ we have

$$\begin{aligned} \|x^\# - x\| &\leq \|x^\# - \bar{x}\| + \|\bar{x} - x\| \leq \frac{\delta}{3 - \delta} \|\bar{x}\| + \eta \|x\| \\ &\leq \frac{\delta}{3 - \delta} \cdot (1 + \eta) \|x\| + \eta \|x\| \\ &= \left[\frac{\delta}{3 - \delta} \cdot \left(\frac{3}{3 - \delta} \right)^3 + \left(\frac{3}{3 - \delta} \right)^3 - 1 \right] \|x\| \\ &= \frac{81 - (3 - \delta)^4}{(3 - \delta)^4} \|x\| \\ &< \frac{3}{2} \delta \|x\|. \end{aligned}$$

Finally note that, by (4.6), for all $j = 1, \dots, m$,

$$\frac{\bar{a}_j^{(N)}}{\bar{a}_j} \geq 1.$$

Combining the above two inequalities and Lemma 4.1 we get

$$\left(1 + \frac{3}{2}\delta\right) \|x\| > \|x^\sharp\| = \|\bar{x}^{(N)}\| \geq \|\bar{x}\| \geq \|x\|.$$

This ends the proof of the proposition. \square

5. Remarks about duality and narrow operators

As mentioned above (see Remark 3.5), our proof does not work in the case $1 \leq p < 2$. Unfortunately this case cannot be handled by duality arguments either. Indeed the dual of $L_{w,p}$ is never isometric to a Lorentz space, even if we restrict ourselves to the more classical spaces $L_{p,q}$ (cf., e.g., [4]). The isometric structure of the duals of Lorentz spaces $L_{w,p}$ has been described in [9]. However, this description is so complicated that there are later papers describing the isomorphic structure of duals of Lorentz spaces $L_{w,p}$; see [1], [25], [26], [15], [16] and the references in these papers. These results require too much technical notation to be stated here, so we just mention the result for spaces $L_{p,q}$.

Recall that the space $L_{p,q}$ for $p, q \in (1, \infty)$ is defined as $L_{w,q}$, where $w(t) = qt^{p/q-1}/p$. It is easily seen that $\|\cdot\|_{w,q}$ with this weight is not a norm when $q > p$, since then $w(t)$ is increasing rather than decreasing. Nevertheless, $L_{p,q}$ is a linear space also for $q > p$ and it can be shown that $L_{p,q}$ can be made into a Banach space when $p > 1$ by introducing an actual norm $\|\cdot\|_{p,q}$ which satisfies $\|f\|_{p,q} \leq \|f\|_{w,q} \leq C(p,q)\|f\|_{p,q}$.

For the spaces $L_{p,q}$ we have the following isomorphic characterization of the dual space

$$(L_{p,q})^* \cong L_{p',q'},$$

where $1/p + 1/p' = 1/q + 1/q' = 1$.

Another difficulty with the duality approach comes from the fact that in general the conjugate operator T^* to a narrow operator $T : E \rightarrow E$ need not be narrow (for any r.i. space E with E^* having an absolutely continuous norm to consider the notion of narrow operators); see [22, p. 60]. In general we can only conclude that if P is a projection onto a finite-codimensional subspace $X \subseteq E$ then P^* is also a projection onto a finite-codimensional subspace in E^* . Since $\|P\| = \|P^*\|$, we have the following obvious result.

PROPOSITION 5.1. *Suppose that for a reflexive r.i. space E there is a constant k_E such that $\|P\| \geq k_E$ for any finite-codimensional projection P in E . Then the same is true for finite-codimensional projections in E^* .*

We obtain the following immediate corollary.

COROLLARY 5.2.

- (1) For any $p > 2$ there exists a constant $k_p > 1$ such that for every projection P from $(L_{w,p})^*$ onto a finite-codimensional subspace we have $\|P\| \geq k_p$.
- (2) For any $q, 1 < q < 2$, and any $p \in (1, \infty)$ there exists an equivalent norm $\|\cdot\|$ on $L_{p,q}$, such that there exists a constant $k_q > 1$ such that for every projection P from $L_{p,q}$ onto a finite-codimensional subspace we have $\|P\| \geq k_q$.

We recall here that it follows from [24, Theorem 4] (Theorem 1.3 above) that in $L_{w,p}$ with the usual norm, for all $p, 1 \leq p < \infty$, and for all weights w , we have $\|P\| > 1$ for all finite-codimensional projections.

In the remainder of this section we study operators which are conjugate to narrow. We start with the following definition:

DEFINITION 5.3. We say that an operator $T \in \mathcal{L}(E)$ on an r.i. space E with absolutely continuous norm is **-narrow* provided $T^* \in \mathcal{L}(E^*)$ is narrow.

Next we will need the following notion.

DEFINITION 5.4. Let E be an r.i. space with absolutely continuous norm. We call a subset $M \subseteq E$ *poor* if for each measurable subset $A \subseteq [0, 1]$ and each $\varepsilon > 0$ there exists a decomposition $A = A^+ \sqcup A^-$ into measurable subsets such that

$$\left| \int_{A^+} f d\mu - \int_{A^-} f d\mu \right| < \varepsilon$$

for every $f \in M$.

Let $B(X)$ be the closed unit ball of a Banach space X . Here and in the sequel we denote by $\mathcal{L}(X, Y)$ the space of all continuous linear operators acting from X to Y and write $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$.

PROPOSITION 5.5. Let E be a reflexive r.i. space. An operator $T \in \mathcal{L}(E)$ is narrow if and only if $T^*(B(E^*))$ is a poor subset of E^* .

Proof. For each $x \in E$ we have

$$\|Tx\| = \sup_{f \in B(E^*)} |f(Tx)| = \sup_{f \in B(E^*)} |(T^*f)(x)|.$$

Now suppose that $x^2 = \chi_A$ and put $A^+ = x^{-1}(1)$ and $A^- = x^{-1}(-1)$. Then

$$\|Tx\| = \sup_{f \in B(E^*)} \left| \int_{A^+} T^*f d\mu - \int_{A^-} T^*f d\mu \right|.$$

This observation completes the proof. \square

As an immediate corollary we obtain the following result:

COROLLARY 5.6. *Let E be a reflexive r.i. space with absolutely continuous norm. An operator $T \in \mathcal{L}(E)$ is $*$ -narrow if and only if $T(B(E))$ is a poor subset of E .*

Proof. Indeed, $T \in \mathcal{L}(E)$ is $*$ -narrow if and only if $T^* \in \mathcal{L}(E^*)$ is narrow, and since $T^{**} = T$ for operators T on a reflexive space, by Proposition 5.5, we have that $T^* \in \mathcal{L}(E^*)$ is $*$ -narrow if and only if $T^{**}(B(E^{**})) = T(B(E))$ is poor. \square

Denote by $\text{Narr}(E)$, resp. $*$ - $\text{Narr}(E)$, the set of all narrow, resp. $*$ -narrow, operators on E .

The existence of a narrow operator whose conjugate is not narrow shows that both inclusions $\text{Narr}(E) \subseteq *$ - $\text{Narr}(E)$ and $\text{Narr}(E) \supseteq *$ - $\text{Narr}(E)$ are false. Indeed, let T be narrow on E and T^* not be narrow on E^* . If T were $*$ -narrow, then T^* would be narrow. Thus, first inclusion fails. Let T^* be narrow in E^* and T not be narrow in E . Then T is $*$ -narrow in E and the second inclusion fails.

The intersection $\text{Narr}(E) \cap *$ - $\text{Narr}(E)$ contains the set $\mathcal{K}(E)$ of all compact operators on E . The inclusion $\mathcal{K}(E) \subseteq \text{Narr}(E)$ is very simple and is proved in [22, p. 55]. The second inclusion is also simple and follows from the following result.

PROPOSITION 5.7. *Let E be an r.i. space with absolutely continuous norm. Then every relatively compact set $K \subset E$ is poor.*

Proof. Given $\varepsilon > 0$ and a measurable subset $A \subseteq [0, 1]$, pick any $\varepsilon/2$ -net in K , say $\{f_1, \dots, f_m\}$. Let $\{r_n(A)\}$ be any “Rademacher” type sequence on A , i.e., $(r_n(A))^2 = \chi_A$ and $r_n(A) \xrightarrow{w} 0$. Now choose n such that $|\int f_i r_n(A) d\mu| < \varepsilon/2$ for each $i = 1, \dots, m$. Then for each $f \in K$ we obtain

$$\left| \int f r_n(A) d\mu \right| \leq \left| \int f_i r_n(A) d\mu \right| + \|f_i - f\| < \varepsilon$$

for suitable i . The proposition is proved. \square

A principal difference between the notions of narrow and $*$ -narrow operators is that $*$ -narrowness of an operator is a property of the image under it of the unit ball, while the notion of narrow operators cannot be formulated in terms of the image. This fact is a consequence of the following result.

THEOREM 5.8. *Let E be an r.i. space with an absolutely continuous norm. Then there exists a complemented subspace E_0 of E isomorphic to E and for*

which there are two projections onto E_0 , one of which is narrow and second is not narrow at any measurable subset $B \subseteq [0, 1]$ of positive measure.

(An operator T on an r.i. space E is said to be narrow at a measurable subset $B \subseteq [0, 1]$ if the operator $T(\bullet \cdot \chi_B)$ is narrow.)

In other words, there are two decompositions $E = E_0 \oplus E_1 = E_0 \oplus E_2$ with rich E_1 , while E_2 is not rich at any measurable subset $B \in [0, 1]$.

Proof. First we construct two subspaces E_0 and E_1 . For convenience, we consider E at $[0, 1] \times [0, 1]$ instead of $[0, 1]$ (cf. [22, Example 1, p. 57]). We consider elements of E as equivalence classes of functions of two variables $x(t_1, t_2)$. Put

$$E_0 = \left\{ x \in E : x(t_1, t_2) = \int_{[0,1]} x(t_1, s) ds : x \in E \text{ (a.e.)} \right\},$$

$$E_1 = \left\{ x \in E : \int_{[0,1]} x(t_1, t_2) dt_2 = 0 \text{ (a.e.)} \right\}.$$

Thus, $E = E_0 \oplus E_1$. In [22, p. 57] it was proved that E_1 is rich and therefore the projection P of E onto E_0 with $\ker P = E_1$ is narrow.

Now we construct another complement E_2 to E_0 . We use the following simple argument. Suppose $Z = X \oplus Y$, where X, Y, Z are Banach spaces and $T \in \mathcal{L}(Y, X)$. Then $Z = X \oplus \{y + Ty : y \in Y\}$ is a decomposition into closed subspaces. (This is easily verified.) For $T \neq 0$ we somewhat “bend” the subspace Y . Our purpose is to “bend” the subspace E_1 by a suitable T .

Put $D_1 = [0, 1/2) \times [0, 1]$, $D_2 = [1/2, 1] \times [0, 1]$, $E'_i = \chi_{D_1} E_i$, $E''_i = \chi_{D_2} E_i$ for $i = 0, 1$. So we have $E_i = E'_i \oplus E''_i$, $i = 0, 1$. Let T be an isomorphic embedding of E_1 into E_0 with $TE'_1 \subseteq E'_0$ and $TE''_1 \subseteq E''_0$. (Such an operator exists since both E'_0 and E''_0 are evidently isomorphic to E .)

Next we show that the subspace $E_2 = \{y + Ty : y \in E_1\}$ (which is a complement to E_0) is not rich at any subset $B \subseteq [0, 1] \times [0, 1]$. Fix any such B of positive measure. Then at least one of the two subsets $B \cap D_1$ or $B \cap D_2$ is of positive measure, say the first and denote it by A . Let x be any element of E with $x^2 = \chi_A$ and $y + Ty$ be any element of E_2 ($y \in E_1$). Since $y\chi_{D_1}$ and $y\chi_{D_2}$ belong to E_1 and since the restriction of a function to a subset of the domain is a projection of norm one we may write

$$\begin{aligned} \|x - y - Ty\| &= \|x - y\chi_{D_1} - y\chi_{D_2} - T(y\chi_{D_1}) - T(y\chi_{D_2})\| \\ &\geq \max\{\|x - y\chi_{D_1} - T(y\chi_{D_2})\|, \|y\chi_{D_2} + T(y\chi_{D_1})\|\} \\ &= \max\left\{ \left\| \int x dt_2 - T(y\chi_{D_2}) + \left(x - \int x dt_2\right) - y\chi_{D_1} \right\|, \right. \\ &\quad \left. \|y\chi_{D_2} + T(y\chi_{D_1})\| \right\} \end{aligned}$$

$$\geq \max \left\{ \left\| \int x dt_2 - T(y\chi_{D_2}) \right\|, \alpha^{-1} \left\| x - \int x dt_2 - y\chi_{D_1} \right\|, \right. \\ \left. \alpha^{-1} \|y\chi_{D_2}\|, \|T(y\chi_{D_1})\| \right\},$$

where α is the norm of the projection of E onto E_1 having the kernel E_0 . (Note that the projection of E onto E_0 with kernel E_1 is a conditional expectation operator and therefore is of norm one.)

Now suppose to the contrary that E_2 is rich at B . Then for each $\varepsilon > 0$ there are $x_\varepsilon \in E$ with $x_\varepsilon^2 = \chi_A$ and $y_\varepsilon \in E_1$ ($y_\varepsilon + Ty_\varepsilon \in E_2$) such that $\|x_\varepsilon - y_\varepsilon - Ty_\varepsilon\| < \varepsilon$. By the above estimates we have

$$\left\| \int x_\varepsilon dt_2 - T(y_\varepsilon\chi_{D_2}) \right\| < \varepsilon, \quad \left\| x_\varepsilon - \int x_\varepsilon dt_2 - y_\varepsilon\chi_{D_1} \right\| < \alpha\varepsilon, \\ \|y_\varepsilon\chi_{D_2}\| < \alpha\varepsilon, \quad \|T(y_\varepsilon\chi_{D_1})\| < \varepsilon,$$

whence

$$\| \chi_A \| = \| x_\varepsilon \| = \left\| \int x_\varepsilon dt_2 + (x_\varepsilon - \int x_\varepsilon dt_2) \right\| \\ \leq \left\| \int x_\varepsilon dt_2 - T(y_\varepsilon\chi_{D_2}) \right\| + \|T\| \|y_\varepsilon\chi_{D_2}\| \\ + \left\| x_\varepsilon - \int x_\varepsilon dt_2 - y_\varepsilon\chi_{D_1} \right\| + \|y_\varepsilon\chi_{D_1}\| \\ \leq \varepsilon(1 + \alpha\|T\| + \alpha + \|T^{-1}\|).$$

The contradiction completes the proof. □

Theorem 5.8 implies a negative answer to the following question of A. Plichko and M. Popov [22, Question 2, p. 71]:

Given an r.i. space E , $E \neq L_2$, such that E^ has absolutely continuous norm. Suppose that $E = X \oplus Y$ and $E^* = Y^\perp \oplus X^\perp$, where X^\perp (Y^\perp) consists of all functionals vanishing on X (on Y). Is $X \in E$ rich if and only if $Y^\perp \in E^*$ is?*

Indeed, we have constructed decompositions $E = E_0 \oplus E_1 = E_0 \oplus E_2$ with E_1 rich and E_2 not rich, which imply the dual decompositions $E^* = E_1^\perp \oplus E_0^\perp = E_2^\perp \oplus E_0^\perp$. If we assume an affirmative answer to the above question then E_0^\perp should be rich and therefore E_2 also should be rich.

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