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IMAGINARIES IN BEAUTIFUL PAIRS

ANAND PILLAY AND EVGUENI VASSILIEV

ABSTRACT. We prove that if $T = T^{eq}$ is a stable theory (without the finite cover property), then the theory T_P of "beautiful pairs" eliminates imaginaries if and only if no infinite group is definable in a model of T.

1. Introduction

An imaginary in a first order structure M is something of the form a/E where a is a finite tuple from M and E is an $\emptyset$-definable equivalence relation on M^n. It is nowadays recognized that a model-theoretic understanding of a structure M involves not only classifying/describing the definable sets in M (via some kind of quantifier elimination) but also classifying/describing the imaginaries in M, up to interdefinability.

In this paper we are concerned with the question of what, if any, new imaginaries arise when we pass from a stable theory T (without the finite cover property) to the theory T_P of "beautiful pairs" of T. We were led to the questions dealt with in the current paper by our work [1] on lovely pairs of models of a simple theory, and some of the preliminary work in the current paper will be at that level of generality. It was natural to ask what, if any, new *hyperimaginaries* appear in lovely pairs. But we realized that we did not even know what new *imaginaries* arise, even in the most straightforward examples, such as pairs of algebraically closed fields. Lovely pairs are the "simple" generalization of Poizat's "belles paires" or beautiful pairs of models of a stable theory, which we will now describe.

Let T be a complete first order theory in the language L. Let L_P be the language obtained by augmenting L with a new unary predicate symbol P. An *(elementary) pair of models of* T, or a T*-pair*, is a structure (M, P) where $M \models T$, and P defines an elementary substructure of M. A T-pair (M, P) is *proper*, if $P(M) \neq M$. To be a T-pair is clearly a first order property.

Pairs of stable structures were first studied in [5], where the central notion is that of a *beautiful pair*. A T-pair (M, P) is beautiful (belle), if $P(M)$ is

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$|T|^+$-saturated and any L-type over $P(M)$ together with a finite tuple from M is realized in M. If T is stable, then any two beautiful T-pairs are elementarily equivalent (in L_P). It was shown in [5] that if T is stable and does not have the finite cover property, then the (complete) theory of all beautiful T-pairs is again stable.

We will prove:

THEOREM. *Let T be stable, without the finite cover property. Then the following are equivalent:*

(i) *T_P has elimination of imaginaries relative to T (that is any imaginary in a model (M, P) of T_P is interdefinable with an imaginary of $M \models T$).*

(ii) *T_P has geometric elimination of imaginaries relative to T (namely any imaginary in $(M, P) \models T_P$ is interalgebraic with an imaginary of M).*

(iii) *No infinite group is definable in any model of T.*

Our general approach to understanding imaginaries will be via canonical bases. Our proof will make heavy use of the work [1] on lovely pairs (M, P) of models of a simple theory, which of course is valid in the stable case too where lovely pairs and Poizat's beautiful pairs coincide. In particular Proposition 7.5 in that paper, which gives a description of canonical bases in T_P up to interdefinability, will be crucial. It says, roughly speaking, that any canonical base $b \in (M, P)^{eq}$ can be assumed to be the canonical base of a type $tp(a/b)$ say which is internal to P. The general theory of internality and definable automorphism groups then yields an L_P-definable group G internal to P, which is infinite if and only if $a \notin acl(bP)$. Then we prove that if $a \in acl(bP)$ then b is interdefinable with an element of M^{eq}. This is carried out in Section 3. We present some background and preliminary material in Section 2.

The work here does not yield an explicit description of the imaginaries in T_P (even up to interalgebraicity), although these imaginaries are clearly closely related to "codes" for L_P-definable principal homogeneous spaces for definable groups in P. In the best of all possible worlds, the new imaginaries will be accounted for by elements of $G(M)/G(P)$ for G a group that is L-definable over P. We expect this to be more or less the case when T is the theory of algebraically closed fields.

2. Preliminaries

We will now recall the definition and some properties of "lovely pairs" of simple structures from [1], prove some technical lemmas, and also recall some facts from stability theory. Lovely pairs give a common generalization of Poizat's beautiful pairs of stable structures [5] and the "generic" pairs of supersimple SU-rank 1 structures [6]. We assume that our base theory T is complete, simple and has quantifier elimination. Many of the results below have simpler proofs in the stable or strongly minimal case (or the case of ACF), but we will try to state and/or prove them in the most general setting.

For any set A in (M, P), $P(A)$ denotes $A \cap P$, the "P-part" of A. When dealing with a group G which is definable in L with parameters from P, we will use notation $G(P)$ to denote the P-part of G. For a simple T and $\kappa \geq |T|^+$, a T-pair (M, P) is called *κ-lovely*, if it satisfies

(i) (*κ-extension property*) for any L-type $p \in S(A)$, where $|A| < \kappa$, some non-forking extension of p to $A \cup P(M)$ is realized in M;
(ii) (*κ-coheir property*) for any L-type $p \in S(A)$, where $|A| < \kappa$, if p does not fork over $P(A)$, then it is realized in $P(M)$.

A pair is called *lovely*, if it is $|T|^+$-lovely. In the stable case, κ-lovely pairs are exactly the "κ-beautiful" pairs, with the obvious definition of κ-beautiful. Any two lovely T-pairs are elementarily equivalent. Any T-pair can be embedded in a lovely one. If T has finite and definable $D(-, \phi)$-ranks (equivalent to non-finite cover property in the stable case), then any sufficiently saturated model of the (complete) theory T_P of all lovely T-pairs is again a lovely pair, and T_P is simple. We assume that $T = T^{eq}$ and has finite and definable $D(-, \phi)$-ranks, and that we are working in a saturated model (M, P) of T_P (which will be a lovely pair), and we will assume elimination of hyperimaginaries (which holds in the stable case). Although some of the facts below hold in a more general setting, this assumption makes some arguments and notation simpler. One can still work with hyperimaginaries, and in this case, *acl* should be changed to *bdd* and *stp* to *Lstp*.

We should also point out that when working in a pair (M, P), acl_L and dcl_L are taken in $M = M^{eq}$, while acl_{L_P} and dcl_{L_P} are taken in $(M, P)^{eq}$, which may contain new sorts.

Letters $a, b, c, \ldots$ may denote tuples (possibly infinite, but of small length) of imaginaries. We may also view sets as tuples, and use set or tuple notation interchangeably. Given $a \in M$, we let $c_P(a) = Cb(tp_L(a/P(M)))$ (in [1] it was denoted by a^c). Then $c_P(a)$ is a definably closed small subset of $P(M)$ (since we assume that $T = T^{eq}$). By $\widehat{a}$ we denote $(a, c_P(a))$. We call a set A *P-independent*, if $A \mathop{\smile\!\!\!\!\!\!\;|}^{L}_{P(A)} P(M)$. Clearly, $\widehat{a}$ is always P-independent.

Notice also that $c_P(a) \subset dcl_{L_P}(a)$ (any L_P-automorphism fixes $P(M)$ setwise, and if it also fixes a pointwise, then it fixes $tp_L(a/P(M))$). Clearly, we also have $\widehat{a} \subset dcl_{L_P}(\ a)$.

FACT 2.1 (see [1, Lemma 3.8]).

(i) *If A and B are P-independent and $qftp_{L_P}(A) = qftp_{L_P}(B)$, then $tp_{L_P}(A) = tp_{L_P}(B)$.*
(ii) *$tp_{L_P}(a) = tp_{L_P}(a')$ iff $tp_L(a, c_P(a)) = tp_L(a', c_P(a'))$.*

Notice that since any subset of P is P-independent, (i) implies that any relation on $P(M)$ definable in a lovely pair (M, P) with parameters in $P(M)$ is actually definable in the model $P(M)$ of T over the same parameters. In particular the imaginaries coming from quotients of tuples from $P(M)$ by equivalence relations L_P-definable over $P(M)$ are interdefinable with "old" imaginaries from $P(M)^{eq}$. So, $P(M)^{eq}$ computed in L_P and in L will be identified.

We have the following characterization of independence (non-forking) in T_P.

FACT 2.2 (see [1, Proposition 7.3]). *The following are equivalent:*

(i) $A \mathop{\smile\!\!\!\!\!\mid}^{L_P}_{C} B$;
(ii) $A \mathop{\smile\!\!\!\!\!\mid}^{L}_{C\cup P(M)} B$ *and* $\widehat{AC} \mathop{\smile\!\!\!\!\!\mid}^{L}_{\widehat{C}} \widehat{BC}$;
(iii) $A \mathop{\smile\!\!\!\!\!\mid}^{L}_{C\cup P(M)} B$ *and* $c_P(AC) \mathop{\smile\!\!\!\!\!\mid}^{L}_{c_P(C)} c_P(BC)$.

The next fact reduces canonical bases in T_P to canonical bases of some special kind of types.

FACT 2.3 (see [1, Proposition 7.5]). *Let B be an elementary substructure of (M, P), $a \in M$. Let $d = Cb(stp_L(a/B \cup P(M))$ (so $d \in M$), $e = Cb(tp_{L_P}(a/B))$ and $e' = Cb(tp_{L_P}(d/B))$. Then*

(i) $e' \in dcl_{L_P}(e)$;
(ii) $e \in bdd_{L_P}(e')$;
(iii) *if T is stable, then* $e \in dcl_{L_P}(e')$.

By [3], for any (hyper)imaginary e in T_P, there is a real tuple a and a model (M, P), such that if $c = Cb(tp_{L_P}(a/M))$, we have $e \in dcl_{L_P}(c)$ and $c \in acl_{L_P}(e)$ (resp. $c \in bdd_{L_P}(e)$). Since the tuple d above is in $acl_L(B \cup P(M))$, we get the following characterization of imaginaries in T_P.

FACT 2.4. *Assume T is stable. Then for any imaginary e in T_P there is a real tuple d and a model B such that $d \in acl_L(B \cup P(M)))$, $e \in dcl_{L_P}(Cb(tp_{L_P}(d/B)))$ and $Cb(tp_{L_P}(d/B)) \subset acl_{L_P}(e)$.*

The next three lemmas deal with some properties of algebraic/definable closure in T_P.

LEMMA 2.5.

(i) $acl_{L_P}(a) \cap M = acl_L(\widehat{a})$;
(ii) $dcl_{L_P}(a) \cap M = dcl_L(\widehat{a})$.

Proof. (i) First we will show that $acl_{L_P}(a) \cap P(M) = acl_L(c_P(a))$. Let $b \in acl_{L_P}(a) \cap P(M)$. Then $b \mathop{\smile\!\!\!\!\!\mid}^{L}_{c_P(a)} \widehat{a}$. Assume $b \notin acl_L(c_P(a))$. Then $b \notin$

$acl_L(\widehat{a})$. So there is an infinite sequence $(b_i|i \in \omega)$ of realizations of $tp_L(b/\widehat{a})$, such that $(b_i|i \in \omega)\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{L}_{c_P(a)}\widehat{a}$. By the coheir property, we may assume that $b_i \in P(M)$ for any $i \in \omega$. Then for any $i \in \omega$, $\widehat{a}b_i$ is P-independent, and realizes the same quantifier free L_P-type as $\widehat{a}b$ (which is also P-independent). Then by Fact 2.1 (i), $tp_{L_P}(b/a)$ has infinitely many realizations, a contradiction.

Now, take any $e \in M\backslash P(M)$ such that $e \in acl_{L_P}(a)$, but $e \notin acl_L(\widehat{a})$. Then $c_P(e\widehat{a}) \subset dcl_{L_P}(e\widehat{a}) \cap P(M) \subset acl_{L_P}(\widehat{a}) \cap P(M) = acl_L(c_P(a))$. So, $e\widehat{a}$ is P-independent, namely $e\widehat{a}\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{L}_{c_P(a)}P(M)$. Take a Morley sequence $(e_i|i \in \omega)$ in $tp_L(e/acl_L(\widehat{a})$. Since $e \notin acl_L(\widehat{a})$, this sequence is infinite. By the extension property, we may assume that $(e_i|i \in \omega)\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{L}_{\widehat{a}}P(M)$. Then $e_i\widehat{a}$ are P-independent, with the same quantifier free L_P-type as $e\widehat{a}$. So, again by Fact 2.1 (i), $tp_{L_P}(e/a)$ has infinitely many realizations, a contradiction.

(ii) First we show that $dcl_{L_P}(a) \cap P(M) = dcl_L(\widehat{a}) \cap P(M)$. Let $b \in dcl_{L_P}(a) \cap P(M)$. By (i), $b \in acl_L(c_P(a))$. If $b \notin dcl_L(\widehat{a})$, then there is $b' \equiv^L_{\widehat{a}} b'$, $b' \neq b$. Then $b' \in acl_L(c_P(a))$ (a subset of $P(M)$). But then $b\widehat{a}$ and $b'\widehat{a}$ have the same quantifier free L_P-type and are P-independent, so by Fact 2.1(i) they have the same L_P-type, contradicting $b \in dcl_{L_P}(a)$.

Now, let $e \in M\backslash P(M)$, $e \in dcl_{L_P}(a)$ but $e \notin dcl_L(\widehat{a})$. Let $e' \neq e$ be such that $e \equiv^L_{\widehat{a}} e'$. By (i), both e and e' are in $acl_L(\widehat{a})$, so $e\widehat{a}$ and $e'\widehat{a}$ are both P-independent. Note that since $e \notin P(M)$, $e \notin acl_L(c_P(a))$. Then also $e' \notin acl_L(c_P(a))$. But $acl_L(\widehat{a}) \cap P(M) = acl_{L_P}(a) \cap P(M) = acl_L(c_P(a))$. So, $e' \notin P(M)$. Thus, by Fact 2.1(i) again, e and e' realize the same L_P-type over a, contradicting $e \in dcl_{L_P}(a)$. $\square$

Lemma 2.6.

(i) *Any L_P-algebraically closed subset of M ($= M^{eq}$) is $P-$independent.*
(ii) *If $A \subset M$ is P-independent, then $acl_{L_P}(A) \cap M = acl_L(A)$.*

(Note that it follows that any L_P-algebraically closed subset of M^{eq} is P-independent.)

Proof. (i) Follows from $c_P(A) \subset acl_{L_P}(A) \cap P(M)$.
(ii) Follows from Lemma 2.5, since $c_P(A) \subset acl_L(P(A))$. $\square$

Lemma 2.7. *For any $A \subset M$,*

(i) $acl_{L_P}(A \cup P(M)) \cap M = acl_L(A \cup P(M))$;
(ii) $dcl_{L_P}(A \cup P(M)) \cap M = dcl_L(A \cup P(M))$.

Proof. The right to left inclusions are clear. Now, by Lemma 2.5, for any small $B \subset P(M)$ we have

$$dcl_{L_P}(AB) \cap M = dcl_L(\widehat{AB}) = dcl_L(ABc_P(AB)) \subset dcl_L(AP(M)),$$

and

$$acl_{L_P}(AB) \cap M = acl_L(\widehat{AB}) = acl_L(ABc_P(AB)) \subset acl_L(AP(M)). \quad \square$$

The following notion plays an important role in our proofs. Let $p(x, A)$ be a complete type over a set A in a simple theory T (with elimination of hyperimaginaries), and $\Sigma(y, A)$ a partial type over A. We say that p is (almost) Σ-*internal*, if there is $B \supset A$ and $a \models p(x, A)$ such that $a \mathop{\smile\!\!\!\!\!\!\;\;|}_A B$ and there is a tuple e of realizations of $\Sigma(y, A)$ such that $a \in dcl(eB)$ (resp. $a \in acl(eB)$).

The following is a well-known fact from stability theory, due to Hrushovski, which we will apply when the theory is T_P and Σ is P.

FACT 2.8 (see [4, Theorem 7.4.8]). *Let T be stable, $p(x, A)$ a $\Sigma(y, A)$-internal stationary type. Let $G = Aut(p/A \cup \Sigma)$ be the group of all of automorphisms of the monster model fixing A and all realizations of Σ pointwise, restricted to the realizations of p. Then*

(i) *G is finite iff $a \in acl(Ae)$ for some tuple e of realizations of Σ.*

(ii) *There is an A-type-definable group (in T^{eq}) acting A-type-definably on p, so that this group together with its action on p are isomorphic to the group G acting on p.*

(iii) *Let Q be the set of all realizations of $\Sigma(y, A)$, and Q^{eq} be the set of all imaginaries coming from quotients of tuples in Q. Then the A-type-definable group in* (ii) *is definably isomorphic to a group type-definable in Q^{eq} over parameters in Q.*

Since type-definable groups are intersections of definable groups, the existence of an infinite type-definable group implies the existence of an infinite definable group (in the same sort, over the same parameters).

The following fact, proved in [2, Theorem 1.2], allows one to "transform" an almost Σ-internal type into a Σ-internal type.

FACT 2.9. *Let T be simple (with elimination of hyperimaginaries), A algebraically closed in T^{eq}, and $p = tp(a/A)$ almost $\Sigma(y, A)$-internal. Then there is an imaginary $e \in dcl(aA)$ such that $a \in acl(e)$ and $tp(e/A)$ is Σ-internal. More precisely, e is a finite set of realizations of p.*

3. Definability of groups and elimination of imaginaries

In this section we prove the main theorem. But we try to work at as general a level as possible. So the blanket assumption for this section is that $T = T^{eq}$ is a simple theory with finiteness and definability of all D_ϕ-ranks. In particular the assumption holds for $T = T^{eq}$ stable with non-fcp. We work in a large saturated model (M, P) of T_P, which we know is a lovely (beautiful) pair.

LEMMA 3.1. *Suppose T to be stable and that no infinite group is definable in a model of T. Then for any set B and a possibly infinite tuple a in (M, P), $a \in acl_L(B \cup P(M))$ implies $a \in acl_{L_P}(Cb(stp_{L_P}(a/B)) \cup P(M))$.*

Proof. We may assume that the tuple a is finite, since in any stable theory $Cb(stp(a/A))$ is interdefinable with $\bigcup_{a'\subset a \text{ finite}} Cb(stp(a'/A))$. Now assume that $a \in acl_L(B \cup P(M))$, let $B_0 = Cb(stp_{L_P}(a/B))$, and assume that $a \notin acl_{L_P}(B_0 \cup P(M))$. Clearly, $tp_{L_P}(a/B_0)$ is stationary and almost P-internal (i.e., $\Sigma(y) = P(y)$). Then by Fact 2.9, there is an L_P-imaginary e such that $e \in dcl_{L_P}(aB_0)$, $a \in acl_{L_P}(eB_0)$, and $tp_{L_P}(e/B_0)$ is P-internal. Clearly, $e \notin acl_{L_P}(B_0 \cup P(M))$. Then, since T_P is stable, by Fact 2.8, there is an infinite group G L_P-definable in $P(M)^{eq}$ over parameters in $P(M)$. But then G is actually L-definable in $P(M)^{eq}$ (a model of T^{eq}), a contradiction. □

LEMMA 3.2. *Suppose T eliminates hyperimaginaries. Let a be a (possibly infinite) tuple in M and $b = Cb(stp_{L_P}(a/b))$, and $a \in acl_{L_P}(bP(M))$. Let $d = Cb_{L_P}(ab/P(M))$. Let B be an L_P-algebraically closed subset of the (real part of) M such that $b \in dcl_{L_P}(B)$ and $a \mathop{\smile\!\!\!\!\!\!\mid}^{L_P}_b B$.*

(i) *d is (L_P-)interdefinable with a (possibly infinite) tuple in $P(M)$. So we may assume $d \in P(M)$.*

(ii) *Let $c = Cb(tp_L(ad/B))$. Then $c \in dcl_{L_P}(b)$ and $b \in acl_{L_P}(c)$. If T is stable, then $b \in dcl_{L_P}(c)$.*

Proof. (i) Follows from $d \in P(M)^{eq}$.

(ii) First note that since B is L_P-algebraically closed in $M = M^{eq}$, it is P-independent, and so is Bd. Now, $a \in acl_{L_P}(bP(M))$, so $a \in acl_{L_P}(bd) \subset acl_{L_P}(Bd)$. So, $a \in acl_{L_P}(Bd) \cap M = acl_L(Bd)$. Let $p_b = tp_{L_P}(a/b)$, $r_B = tp_{L_P}(a/B)$ (so implies a non-forking extension of p_b to bB), $q_B = tp_L(ad/B)$. So, $c = Cb(q_B)$.

Claim 1. $c \in dcl_{L_P}(b)$.

Proof. We need to show that whenever $B' \equiv^{L_P}_b B$, we have $Cb(q_B) = Cb(q_{B'})$ (with the obvious meaning of $q_{B'}$). Since we can always find $B'' \equiv^{L_P}_b B$ such that $B'' \mathop{\smile\!\!\!\!\!\!\mid}^{L_P}_b BB'$, we may assume that $B \mathop{\smile\!\!\!\!\!\!\mid}^{L_P}_b B'$. Then, since p_b is an amalgamation base we can find $a' \models p_b$ with $a' \mathop{\smile\!\!\!\!\!\!\mid}^{L_P}_b BB'$ and $a'B \equiv^{L_P}_b a'B' \equiv^{L_P}_b aB$. Take $d' \in P(M)$ such that $a'd'B' \equiv^{L_P}_b adB$. Note that $d \in dcl_{L_P}(ab)$, so

$$a'd'B \equiv^{L_P}_b a'd'B' \equiv^{L_P}_b adB.$$

Now, $a'd' \mathop{\smile\!\!\!\!\!\!\mid}^{L_P}_b BB'$, so since b is in the L_P-definable closure of each of B and B', we have $a'd' \mathop{\smile\!\!\!\!\!\!\mid}^{L_P}_B BB'$ and $a'd' \mathop{\smile\!\!\!\!\!\!\mid}^{L_P}_{B'} BB'$. But since B and B' are both P-independent and L-algebraically closed, $\widehat{B} = B$ and $\widehat{B'} = B'$. So, by Fact 2.2, $a'd' \mathop{\smile\!\!\!\!\!\!\mid}^{L}_B BB'$ and $a'd' \mathop{\smile\!\!\!\!\!\!\mid}^{L}_{B'} BB'$. Hence $a'd'$ realize a common non-forking extension of q_B and $q_{B'}$. Thus $Cb(q_B) = Cb(q_{B'})$, as needed. □

Claim 2. $b \in acl_{L_P}(c)$. If T is stable, then $b \in dcl_{L_P}(c)$.

Proof. Note that $b = Cb(p_b) = Cb(r_B)$. Also recall that $\widehat{c} \subset acl_{L_P}(c)$. So, it suffices to prove that if $B' \equiv^{L_P}_{acl_L(\widehat{c})} B$ and $B' \mathop{\downarrow}^{L_P}_{acl_L(\widehat{c})} B$, then r_B and $r_{B'}$ have a common non-forking extension. Now, $B' \mathop{\downarrow}^{L_P}_{acl_L(\widehat{c})} B$ implies that $B' \mathop{\downarrow}^{L}_{acl_L(\widehat{c})} B$, so, since $q_B|_{acl_L(\widehat{c})}$ is an amalgamation base (in the sense of L), q_B and $q_{B'}$ have a common non-forking extension. Extend it non-forkingly (in the sense of L) to a type q' over $\widehat{BB'}$. Let $a'd'$ realize q'. Since $d \in P(M)$ and both B and B' are P-independent, we have (by transitivity) $d' \mathop{\downarrow}^{L}_{P(B)} \widehat{BB'}$ and $d' \mathop{\downarrow}^{L}_{P(B')} \widehat{BB'}$. So, $d' \mathop{\downarrow}^{L}_{P(\widehat{BB'})} \widehat{BB'}$, and by the coheir property, we may assume that $d' \in P(M)$. By Fact 2.2, d' is then L_P-independent from $\widehat{BB'}$ over each of $B = \widehat{B}$ and $B' = \widehat{B'}$. Since $a' \in acl_L(Bd')$ and $a' \in acl_L(B'd')$, the same is true for a'. Also, each of $a'd'B$ and $a'd'B'$ is P-independent and has the same quantifier free L_P-type as adB (which is also P-independent). So, by Fact 2.1, $a'B \equiv^{L_P} a'B' \equiv^{L_P} aB$. Thus a' realizes a common non-forking extension of r_B and $r_{B'}$. To show the second statement of the claim, it suffices to show that (assuming T is stable) if $B' \equiv^{L_P}_c B$ then r_B and $r_{B'}$ have a common non-forking extension. By stability of T, q_B and $q_{B'}$ have a common non-forking extension, and then exactly as above, we find a common non-forking extension of r_B and $r_{B'}$. $\square$

LEMMA 3.3. *If there is an infinite group G definable in T, then T_P does not have geometric elimination of imaginaries. That is, there is some element of $(M,P)^{eq}$ which is not interalgebraic with any element of $M = M^{eq}$.*

Proof. Assume G is an infinite group defined over some c (in a sufficiently saturated model of T). We may assume that $c \in P(M)$. Take a generic element $g \in G(M)$ such that $g \mathop{\downarrow}^{L}_{c} P(M)$. Now, if $g_0 \in G(P)$, then $g \mathop{\downarrow}^{L}_{c} g_0$, so by genericity of g, we have $g \cdot g_0 \mathop{\downarrow}^{L}_{c} g_0$. On the other hand, $g \cdot g_0 \mathop{\downarrow}^{L}_{c,g_0} P(M)$, and hence $g \cdot g_0 \mathop{\downarrow}^{L}_{c} P(M)$. Thus for any $g' \in g \cdot G(P)$, $g' \mathop{\downarrow}^{L}_{c} P(M)$. Consider an L_P-imaginary g_P representing the coset $g \cdot G(P)$ (its canonical parameter). Then $g_P \in dcl_{L_P}(g, c)$. Note that since $c \in P(M)$, $g \mathop{\downarrow}^{L}_{c} P(M)$ implies $g \mathop{\downarrow}^{L_P}_{c} P(M)$. So also $g_P \mathop{\downarrow}^{L_P}_{c} P(M)$. Since $g \cdot G(P)$ is a large set, we may also assume that g is not L_P-algebraic over g_P, c (we will only need $g \mathop{\downarrow}^{L}_{c} P(M)$, which holds for any element of $g \cdot G(P)$).

Let a be any real element (i.e., imaginary in $M^{eq} = M$). We need to show that a is not interalgebraic with g_P. Assume a is interalgebraic with g_P. In particular, $a \in acl_{L_P}(g, c)$. But by Lemma 2.6, since $g \mathop{\downarrow}^{L}_{c} P(M)$, $acl_{L_P}(g, c) \cap M = acl_L(g, c)$. So, $a \in acl_L(g, c)$.

Case 1. If g is not in $acl_L(\{a\} \cup P(M))$, then $g \notin acl_L(a, c_P(g, a, c))$. Take a Morley sequence $(g_i | i \in \omega)$ in $tp_L(g/a, c_P(g, a, c))$. Choose it L-independent from $P(M)$ over $a, c_P(g, a, c)$ (by the extension property). Then for any $i \in \omega$ $c_P(g_i, a, c) = c_P(g, a, c)$, so by Lemma 2.1, $tp_{L_P}(g_i/a, c) = tp_{L_P}(g/a, c)$. Hence also $tp_{L_P}(g_{iP}/a, c) = tp_{L_P}(g_P/a, c)$, where g_{iP} is the canonical parameter of the coset $g_i \cdot G(P)$. Now, since $g \notin acl_L(a, c_P(g, a, c))$, the g_i's are algebraically independent over P, i.e., $g_i \notin acl_L(\{g_j | j \neq i\} \cup P(M))$ for any $i \in \omega$. Thus they are in different $G(P)$-cosets, and hence all g_{iP}'s are distinct. This shows that g_P is not in $acl_{L_P}(a)$, a contradiction.

Case 2. Assume $g \in acl_L(\{a\} \cup P(M))$. We know that $a \in acl_L(g, c)$ and g is L-independent from $P(M)$ over c. Thus $g \in acl_L(a, c)$. So, $g \in acl_{L_P}(g_P, c)$, a contradiction. □

We can now deduce the main result of this paper:

THEOREM 3.4. *Let $T = T^{eq}$ be stable without the finite cover property, and let T_P be the theory of beautiful (lovely) T-pairs. Then the following are equivalent:*

(i) *T_P has elimination of imaginaries.*
(ii) *T_P has geometric elimination of imaginaries.*
(iii) *No infinite group is definable in a model of T.*

Proof. (i) implies (ii) is immediate. (ii) implies (iii) is Lemma 3.3.
(iii) implies (i): Let $e \in (M, P)^{eq}$. By Fact 2.4, Lemma 3.1 and Lemma 3.2, there is some real tuple c such that $e \in dcl(c)$ and $c \in acl(e)$ (in (M, P)). Let c' be a code (in $M = M^{eq}$) for the (finite, so definable in M) set of conjugates of c over e. Then e is interdefinable with c'. □

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Anand Pillay, Department of Mathematics, University of Illinois, Urbana, IL 61801, USA

E-mail address: pillay@math.uiuc.edu

Evgueni Vassiliev, Department of Mathematics, University of Illinois, Urbana, IL 61801, USA

Current address: Wilkes Honors College, Florida Atlantic University, 5353 Parkside Drive, Jupiter, FL 33458, USA

E-mail address: evassili@fau.edu