LOCAL SYMMETRY IN THE MOMENT PROBLEM

BY

GORDON G. JOHNSON

If one studies the moment problem for [0, 1], one is led to consider certain step functions on [0, 1]. In this paper a symmetry condition, at a point of (0, 1), of the step functions is shown to be enough to give convergence of the step function sequence at that point.

DEFINITION. Suppose $x \in [0, 1]$ and n is a positive integer. Let Q_n^x denote the restriction to [0, 1] of $\sum_{t=0}^{k} C(n, t)j^t(1 - j)^{n-t}$ where $x \in [k/n, (k + 1)/n)$ and j is the function such that $j(a) = a$ for each number a.

A straight forward computation shows that

$$(Q_n^x)' = -nC(n - 1, k)j^k(1 - j)^{n-k-1}$$

where $x \in [k/n, (k + 1/n)$.

The following are well known facts concerning Bernstein polynomials and are stated here without proof.

LEMMA 1. *If $x \in (0, 1)$ then $\lim_{n\to\infty} Q_n^x(x) = \frac{1}{2}$.*

LEMMA 2[2]. *If $\varepsilon > 0$ and $0 < d \leq \frac{1}{2}$, then there is a positive number N such that if $n > N$, and $x \in [d, 1 - d]$ then*

$$1 - Q_n^x(a) < \varepsilon \quad \text{if } 0 \leq a \leq x - d$$

and

$$Q_n^x(a) < \varepsilon \quad \text{if } x + d \leq a \leq 1.$$

LEMMA 3. *Suppose $x \in (0, 1)$, n is an integer greater than 2, a is a non-negative integer such that $x \in [a/n,(a+1)/n)$ and*

$$f_n^x = (2x - j)^a(1 - 2x + j)^{n-a-1} - j^a(1 - j)^{n-a-1}.$$

(1) *If $x \in (0, \frac{1}{2})$ and $x \leq a/(n - 1)$ then $f_n^x \geq 0$ on $[2x/3, x]$, and if $x > a/(n - 1)$ then $f_n^x \leq 0$ on $[2x/3, x]$.*
(2) *If $x \in [\frac{1}{2}, 1)$ and $x \leq a/(n - 1)$ then $f_n^x \geq 0$ on $[x, (1 + 2x)/3]$ and if $x > a/(n - 1)$ then $f_n^x \leq 0$ on $[X, (1 + 2x)/3]$.*

Proof. Suppose $x \in (0, \frac{1}{2}]$ and $x \leq a/(n - 1)$. Let $y = x/(1 - x)$ and $c \in [1, 2]$; then $(1 - y) - (1 - y)c^{-y} \geq 0$. Since $0 < y \leq 1$, it follows that

$$c^y[1 - y(1 - c^{-1})] \geq 1,$$

i.e., $c^{x/(1-x)}[1 - (x/(1 - x))(1 - c^{-1})] \geq 1$. This last expression can be written as

$$c^{x/(1-x)}[(1 - 2x)(1 - x)c + (1 + c - 2cx)x]/c(1 - x) \geq 1,$$

Received September 13, 1965.

from which it follows that

$$(1 - 2x)c^{x/(1-x)} + (1 + c - 2cx)(x/(1 - x))c^{[x/(1-x)]-1} \geq 1.$$

Hence if $F_x(c) = (1 + c - 2cx)c^{x/(1-x)} - (1 + c - 2x)$ then F_x is non negative if $x \in (0, \frac{1}{2})$ and $c \in [1, 2]$ since $F_x(1) = 0$ and $F_x'(c) \geq 0$.

Since $0 < [(1 + c - 2cx)/(1 + c - 2x)] \leq 1$ it follows that

$$c^{x/(1-x)}[(1 + c - 2cx)/(1 + c - 2x)] \geq 1$$

and hence

$$c^a[(1 + c - 2cx)/(1 + c - 2x)]^{a(1-x)/x} \geq 1.$$

Recall that $x \leq a/(n - 1)$ and therefore $a(1 - x)/x \geq n - a - 1 > 0$. Combining this and the previous result we have that

$$c^a[(1 + c - 2cx)/(1 + c - 2x)]^{n-a-1} \geq 1.$$

Let $b = 2/(c + 1)$; then $c = (2 - b)/b$ and $\frac{2}{3} \leq b \leq 1$. The above then becomes

$$x^a(2 - b)^a[1 - (2 - b)x]^{n-a-1} \geq (xb)^a(1 - bx)^{n-a-1}.$$

If $z = bx$ then $2x/3 \leq z \leq x$ and finally

$$f_n^x(z) = (2x - z)^a(1 - 2x + z)^{n-a-1} - z^a(1 - z)^{n-a-1} \geq 0.$$

Suppose now that $x > a/(n - 1)$. If $d \in [\frac{1}{2}, 1]$ then

$$2(n - a - 1)[(1 + d - 2x)/(1 + d - 2x\,d)]^{n-2-a}(1 - d^2)/(1 + d - 2x\,d) \geq 0,$$

from which it follows that

$$-(n - a - 1)[(1 + d - 2x)/(1 + d - 2x\,d)]^{n-2-a}$$
$$\cdot[-2(1 + d - 2x\,d) + 2d(1 + d - 2x)]/(1 + d - 2x\,d)^2 \geq 0$$

and hence if

$$F_d(x) = d^a - [(1 + d - 2x)/(1 + d - 2x\,d)]^{n-a-1}$$

then $F_d' \geq 0$ on $[0, \frac{1}{2}]$.

Therefore $F_d \geq 0$ if $d \in [\frac{1}{2}, 1]$ since $\lim_{x\to 0} F_d(x) = 0$. Let $b = 2d/(1 + d)$; then $d = b/(2 - b)$, $\frac{2}{3} \leq b \leq 1$ and

$$[b/2 - b)]^a \geq ([1 - (2 - b)x]/[1 - bx])^{n-a-1}.$$

Hence if $z = bx$ then $2x/3 \leq z \leq x$ and

$$f_n^x(z) = z^a(1 - z)^{n-a-1} - (2x - z)^a(1 - 2x - z)^{n-a-1} \geq 0.$$

Similarly for part two.

LEMMA 4. *If $x \in (0, 1)$ and $\varepsilon > 0$ then there is a number $d > 0$ and a number N such that*

$$V_{x-d}^x(Q_n^x + Q_n^x[2x - j]) < \varepsilon \quad \text{if } n > N.$$

Proof. Suppose $x \in (0, 1)$; let $d = \min [x/3, (1 - x)/3]$.

$$V_{x-d}^{x}(Q_n^x + Q_n^x[2x - j])$$
$$= \int_{x-d}^{x} | dQ_n^x + Q_n^x[2x - j] |$$
$$= \int_{x-d}^{x} | (Q_n^x + Q_n^x[2x - j])' | \, dj$$
$$= \int_{x-d}^{x} nC(n - 1, a) \, | j^a(1 - j)^{n-a-1} - (2x - j)^a(1 - 2x + j)^{n-a-1} | \, dj$$

where $x \in [a/n, (a + 1)/n)$.

Suppose $x \in (0, \frac{1}{2}]$; then by Lemma 3

$$V_{x-d}^{x}(Q_n^x + Q_n^x[2x - j])$$
$$= \left| \int_{x-d}^{x} nC(n - 1, a)[j^a(1 - j)^{n-a-1} - (2x - j)(1 - 2x + j)^{n-a-1}] \, dj \right|$$
$$= | (Q_n^x + Q_n^x[2x - j] |_{x-d}^{x} |$$
$$= | 2Q_n^x(x) - Q_n^x(x - d) - Q_n^x(x + d) |$$
$$\leq 2 | Q_n^x(x) - \tfrac{1}{2} | + | 1 - Q_n^x(x - d) | + | Q_n^x(x + d) |.$$

For this d, there is by Lemmas 1 and 2 a number $N > 0$ such that if $n > N$ then

$$| Q_n^x(x) - \tfrac{1}{2} | < \varepsilon/6, \quad Q_n^x(x + d) < \varepsilon/3 \quad \text{and} \quad 1 - Q_n^x(x - d) < \varepsilon/3.$$

Hence $V_{x-d}^{x}(Q_n^x + Q_n^x[2x - j]) < 2\varepsilon/6 + \varepsilon/3 + \varepsilon/3 = \varepsilon$ if $n > N$.

Note that $V_{x-d}^{x}(Q_n^x + Q_n^x[2x - j]) = V_x^{x+d}(Q_n^x + Q_n^x[2x - j])$. The case $x \in [\frac{1}{2}, 1)$ follows by an entirely similar argument.

DEFINITION. Suppose $\{c_i\}_{i=0}^{\infty}$ is a real number sequence and n is a positive integer. Let $\varphi_n(x) = 0$ if $x \geq 0$, $\varphi_n(x) = c_0$ if $x \geq 1$ and

$$\varphi_n(x) = \textstyle\sum_{t=0}^{k} C(n, t) \sum_{i=0}^{n-t} C(n - t, i)(-1)^i c_{i+t}$$

if $x \in (0, 1) \cap [k/n, (k + 1)/n)$.

LEMMA 5[3]. *If $\{c_i\}_{i=0}^{\infty}$ is a real number sequence and $\{\varphi_i\}_{i=1}^{\infty}$ is the associated step function sequence then*

$$\varphi_n(x) = \lim_{a\to\infty} \int_0^1 Q_n^x \, d\varphi_a$$

for each x in $[0, 1]$.

THEOREM. *Suppose $\{c_i\}_{i=0}^{\infty}$ is a real number sequence such that the associated step function sequence $\{\varphi_i\}_{i=1}^{\infty}$ is uniformly bounded on* $[0, 1]$. *If $x \in (0, 1)$ and*

for each $\varepsilon > 0$ *there is a number* $d > 0$ *and a number* $N > 0$ *such that*

$$V_x^{x+d}(\varphi_n - \varphi_n[2x - j]) < \varepsilon$$

then $\lim_{n\to\infty} \varphi_n(x)$ *exists.*

Proof. If each of m and n is a positive integer then by Lemma 5

$$|\varphi_m(x) - \varphi_n(x)| = \lim_{a\to\infty} \left| \int_0^1 (Q_m^x - Q_n^x)\, d\varphi_a \right|.$$

If t is a positive integer and d is a number such that $0 < d$, $0 < x - d$ and $x + d < 1$ then

$$\begin{aligned}
&\left| \int_0^1 (Q_m^x - Q_n^x)\, d\varphi_t \right| \\
&\leq \left| \int_0^{x-d} (Q_m^x - Q_n^x)\, d\varphi_t \right| + \left| \int_{x-d}^{x+d} (Q_m^x - Q_n^x)\, d\varphi_t \right| + \left| \int_{x+d}^1 (Q_m^x - Q_n^x)\, d\varphi_t \right| \\
&\leq B\{|Q_m^x(x-d) - Q_n^x(x-d)| + V_0^{x-d}(Q_m^x - Q_n^x) \\
&\qquad + |Q_m^x(x+d) - Q_n^x(x+d)| + V_{x+d}^1(Q_m^x - Q_n^x)\} \\
&\qquad + \left| \int_{x-d}^{x+d} (Q_m^x - Q_n^x)\, d\varphi_t \right|
\end{aligned}$$

where B is bound for the sequence $\{\varphi_i\}_{i=1}^{\infty}$ on $[0, 1]$.

Suppose that $x \in (0, \frac{1}{2}]$.

$$\begin{aligned}
&\left| \int_{x-d}^{x} (Q_m^x - Q_n^x)\, d\varphi_t + \int_x^{x+d} (Q_m^x - Q_n^x)\, d\varphi_t \right| \\
&\leq \left| \int_x^{x+d} (Q_m^x - Q_n^x)\, d(\varphi_t - \varphi_t[2x - j]) \right| \\
&\qquad + \left| \int_x^{x+d} (Q_m^x - Q_n^x)\, d\varphi_t[2x - j] + \int_{x-d}^{x} (Q_m^x - Q_n^x)\, d\varphi_t \right| \\
&\leq 2V_x^{x+d}(\varphi_t - \varphi_t[2x - j]) \\
&\qquad + \left| \int_{x-d}^{x} (Q_m^x[2x - j] + Q_m^x - Q_n^x[2x - j] - Q_n^x)\, d\varphi_t \right| \\
&\leq 2V_x^{x+d}(\varphi_t - \varphi_t[2x - j]) + B\{|Q_m^x(x-d) - Q_n^x(x-d) \\
&\qquad + Q_m^x(x+d) - Q_n^x(x+d)| + V_{x-d}^x(Q_m^x + Q_m^x[2x - j]) \\
&\qquad + V_{x-d}^x(Q_n^x + Q_n^x[2x - j])]\}.
\end{aligned}$$

Combining the above results

$$\begin{aligned}
|\varphi_m(x) - \varphi_n(x)| &\leq B\{2|Q_m^x(x-d) - Q_n^x(x-d)| \\
&\qquad + 2|Q_m^x(x+d) - Q_n^x(x+d)| + V_0^{x-d}(Q_m^x - Q_n^x)
\end{aligned}$$

$$+ V_{x+d}^{1}(Q_m^x - Q_n^x) + V_{x-d}^{x}(Q_m^x + Q_m^x[2x - j])$$
$$+ V_{x-d}^{x}(Q_n^x + Q_n^x[2x - j])\} + 2V_x^{x+d}(\varphi_t - \varphi_t[2x - j]).$$

If $\varepsilon > 0$ there is a number d, $0 < d < x/3$ and a number $N > 0$ such that if $t > N$, $V_x^{x+d}(\varphi_t - \varphi_t[2x - j]) < \epsilon/4$.

For this number d there is, by Lemmas 2 and 4, a number $N > 0$ such that if each of m and n is an integer greater than N then $|\varphi_m(x) - \varphi_n(x)| < \varepsilon$. There is an entirely similar argument for $x \in [\frac{1}{2}, 1)$.

References

1. G. G. Lorentz, *Bernstein polynomials*, Toronto, University of Toronto Press, 1953.
2. J. H. Wells, *Concerning the Hausdorff inclusion problem*, Duke Math. J., vol. 26 (1959), pp. 669–645.
3. G. G. Johnson, *Concerning local variations in the moment problem*, J. London Math. Soc., to appear.

University of Georgia
Athens, Georgia