MATERIAL SYMMETRY RESTRICTIONS FOR CERTAIN LOCALLY COMPACT SYMMETRY GROUPS[1]

BY

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Part I: Preliminaries

1. Introduction. The response functions or functionals which appear in constitutive equations may all be symbolized by $y = \Phi(x)$, where x and y are tensors, or aggregates of tensors, and may thus be considered vectors in abstract real vector spaces X and Y respectively. The material symmetries of such a system may then all be described by a group G, which has representations S and T by invertible linear operators in X and Y respectively. A central problem in the formulation of constitutive equations is to find the canonical forms of form-invariant, and thus physically admissible, functions Φ—that is, functions Φ which satisfy $T(g)\Phi(x) = \Phi(S(g)x)$ for all g in G.

The standard techniques for this problem have been developed largely by Rivlin and his co-workers, [7], [11], and ref.'s in [14], but often require the assumption that Φ is a polynomial. All assumptions on the form of Φ have been removed by Wineman and Pipkin, first for finite symmetry groups [9] and then for compact symmetry groups [14]. This paper extends the conclusions of Wineman and Pipkin to a large class of locally compact groups, namely Lindelof mean-ergodic groups, through the use of a more general concept of the group average. That is, the now possible non-compactness of G requires that we first prove a topological ergodic theorem, based on the mean ergodic theorem of Calderón [2], and then use this to find as before, the restrictions on form-invariant functions.

In the arguments of Wineman and Pipkin, the invariant Hurwitz integral on a compact group [12] is used repeatedly to take averages over the group G. In our more general discussion, the left (or right) invariant Haar integral [6] is required for this purpose; but this integral cannot be used uncritically to compute group averages, for a non-compact group has infinite Haar measure. To obtain group averages, we require a theorem stating the convergence of averages on larger and larger compact open subsets of G—that is, a topological ergodic theorem in the style of the ergodic theorems for various norms [7]. Our first task is therefore to prove Theorem 4, which will be the principal tool in this paper, but readers who wish to avoid such details may skip to Part II, and read only the statement of this theorem.

2. Integration of vector-valued functions. Over the group G we shall wish to integrate not only functions $\varphi(g)$ on G alone, but also functions $\Phi(g, x)$ on

Received January 15, 1966.

[1] The work described here was carried out under an Office of Naval Research contract with Brown University.

both G and X. For each fixed x in X such expressions, of course, are functions on G alone, and can be integrated accordingly, but then any continuity in x of the resulting integral must be derived separately. A less concrete but more elegant approach is to regard $\Phi(g, x)$ for each g as a vector in a suitable real function space, and to integrate at once for all values of x. However, this approach requires a brief recapitulation of the integration theory for vector-valued functions.

A topological vector space $(Z, \mathcal{V})$ is a vector space Z with a vector topology $\mathcal{V}$—that is, a topology in which the standard vector compositions $(y, z) \to y + z$ and $(\alpha, z) \to \alpha z$ are jointly continuous for all scalars α and all vectors y, z. Such a topology need not be describable by a norm, but may often be characterized by a family of pseudo-norms—that is, non-negative functions ν on Z such that $\nu(y + z) \leq \nu(y) + \nu(z)$ and $\nu(\alpha z) = |\alpha| \nu(z)$ but $\nu(z)$ may be zero for non-zero vectors z.

In a topological vector space $(Z, \mathcal{V})$ the subset V is a neighborhood of 0 if and only if the subset $z + V$ is a neighborhood of z; so that the family $\mathcal{V}_0$ of all neighborhoods of 0 characterizes $\mathcal{V}$. The subfamily $\mathcal{B}$ of $\mathcal{V}_0$ is a local base for $\mathcal{V}$ if for each V in $\mathcal{V}_0$ there exists U in $\mathcal{B}$ such that $U \subset V$; a topological vector space $(Z, \mathcal{V})$ is locally convex if it has a local base consisting of convex sets. Local convexity is a convenient restriction, for it permits the choice of convex neighborhoods in all proofs, and is a plausible restriction, for it includes nearly all of the well-known function spaces [5].

To integrate the vector-valued functions which will appear in this paper, we shall need the following theorem, a slight specialization of the results of Bourbaki [1, Ch. III §4]. Let A be a compact topological space with elements h, and let π be a Borel probability measure on the space A. Let Z and Z' be real locally convex Hausdorff topological vector spaces with the property EC: the closed convex extension of each compact subset is itself a compact subset; and let $C(A, Z)$ be the real vector space of all continuous functions from A into Z.

THEOREM 1 (vector integral theorem). *For each Φ in $C(A, Z)$ there exists in A a unique $\int \Phi(h)\, d\pi(h)$, which lies in the closed convex extension of $\Phi(A)$, such that*

(1) *the integral is a uniformly continuous real linear functional on $C(A, Z)$*

(2) $\nu(\int \Phi(h)\, d\pi(h)) \leq \int \nu(\Phi(h))\, d\pi(h)$ *for each continuous pseudonorm ν on Z*

(3) $T(\int \Phi(h)\, d\pi(h)) = \int T\Phi(h)\, d\pi(h)$ *for each continuous linear transformation T from Z into Z'.*

Proof. See Chap. II §4 of ref. 1.

If we take $Z' = R$ then by Theorem 1.3

$$z^*\left(\int \Phi(h)\, d\pi(h)\right) = \int z^*\Phi(h)\, d\pi(h)$$

for each continuous linear functional z^* on Z. Conversely, this equation for all z^* defines the integral uniquely as a linear functional on Z^*, the continuous dual space of Z.

3. Auxiliary fixed-point theorem. Let G be a locally compact topological group, and let μ be the left-invariant Haar measure on G. Let $(\mathfrak{A}, <)$ be a directed family of compact open subsets of G, let $\lim_A$ denote the limit, in the topology of the context, as A runs through $(\mathfrak{A}, <)$, and let $S_\triangle(\mathfrak{A})$ be the subset, clearly a semigroup, of all h in G such that

$$\lim_A \mu(hA \triangle A)/\mu(A) = \lim_A \mu(Ah \triangle A)/\mu(A) = 0,$$

where $\triangle$ denotes the symmetric difference. Call the family $(\mathfrak{A}, <)$ ergodic if $\bigcup \mathfrak{A} \subset S_\triangle(\mathfrak{A})$, and call the group G mean-ergodic if it contains an ergodic family $(\mathfrak{A}, <)$ for which $S_\triangle(\mathfrak{A}) = G$. In Part II we shall assume that G is mean-ergodic but now we assume only that G contains an ergodic family $(\mathfrak{A}, <)$.

Let $(Z, \mathcal{V})$ be a real locally convex Hausdorff topological vector space with the property EC, and let U be a representation of G by invertible linear operators on Z with the properties

(U1) $U(g)$ is continuous for each g in G
(U2) $g \to U(g)z$ is a continuous map for each z in Z
(U3) the closure of $\{U(g)z : g \in S_\triangle(\mathfrak{A})\}$ is compact for each z in Z.

Let K_z be the closed convex extension of $\{U(g)z : g \in S_\triangle(\mathfrak{A})\}$; then K_z is also compact for each z in Z by the property EC. We shall now introduce the means M_A over subsets A in $\mathfrak{A}$, note that cluster points of $\{M_A z : A \in \mathfrak{A}, <\}$ exist, and show following Calderón that these are fixed points of the $U(g)$.

COROLLARY 1.1. *For each compact open subset A of G and each vector z in Z there exists in Z a unique $M_A z = \int_A U(g)z \, d\mu(g)/\mu(A)$, which lies in K_z. The map M_A thus defined is a linear operator on Z.*

Proof. If we take $\Phi(g) = U(g)z$ and $\pi = \mu/\mu(A)$ then by Theorem 1 there exists a unique $M_A z$ in Z, which lies in the closed convex extension of $\{U(g)z : g \in A\}$, a subset of K_z; and by Theorem 1.1 in particular $M_A z$ depends linearly on $U(g)z$, hence linearly on z.

LEMMA 1. *In the vector topology $\mathcal{V}$, for each z in A and each h in $S_\triangle(\mathfrak{A})$,*

$$\lim_A [U(h)M_A z - M_A z] = \lim_A [M_A U(h)z - M_A z] = 0.$$

Proof. In effect we have made the assumption that K_z is compact, but at this point we shall use only the consequence that K_z is bounded; that is, if V is any convex circled neighborhood of 0 then there exists a positive real α such that $K_z \subset \alpha V$. We shall prove only the first statement, since the second follows similarly by permuting a few terms. Now if $B(g)$ denotes the characteristic function of a subset B of G, then

$$\begin{aligned}\mu(A)[U(h)M_A z - M_A z] &= \int [U(h)U(g)z - U(g)z]A(g)\, d\mu(g)\\ &= \int U(g)z[hA(g) - A(g)]\, d\mu(g)\\ &= \mu(hA - A)M_{hA-A}\, z - \mu(A - hA)M_{A-hA}\, z.\end{aligned}$$

Thus

$$\mu(A)[U(h)M_A z - M_A z] \in \mu(hA - A)K_z - \mu(A - hA)K_z \subset \mu(hA \triangle A)\cdot \alpha V,$$

and since h lies in $S_\triangle(\mathfrak{A})$, the first statement follows on division by $\mu(A)$.

THEOREM 2 (fixed-point theorem). *For each z in Z there exists z_0 in K_z which is a fixed point of $U(h)$ for all h in $S_\triangle(\mathfrak{A})$.*

Proof. The net $\{M_A z : A \in \mathfrak{A}, <\}$ lies in the compact set K_z, and thus has a cluster point z_0 in K_z; hence we may restrict A to a cofinal subfamily of $\mathfrak{A}$ on which $\lim_A M_A z = z_0$. Then on the right side of

$$U(h)z_0 - z_0 = [U(h) - I][z_0 - M_A z] + [U(h)M_A z - M_A z]$$

with h in $S_\triangle(\mathfrak{A})$, the first term has limit 0 by the continuity of $U(h)$ and the second term has limit 0 by Lemma 1; so that $U(h)z_0 - z_0$, which is independent of A, must be 0 by the uniqueness of limits.

4. Topological ergodic theorem. Let the topological group G, the topological vector space Z, and the representation U retain the properties stated in section 3. Now let $\mathcal{V}'$ be another vector topology, for which $(Z, \mathcal{V}')$ is also a real locally convex Hausdorff topological vector space with the properties

(U4) $(Z, \mathcal{V})$ and $(Z, \mathcal{V}')$ have the same continuous dual space Z^*
(U5) the $U(g)$ are equicontinuous at 0 for all g in $S_\triangle(\mathfrak{A})$.

Condition U4 is satisfied trivially when $\mathcal{V} = \mathcal{V}'$, as in our planned application, and is satisfied generally if and only if $\mathcal{V}$ and $\mathcal{V}'$ lie between the weak topology $w(Z, Z^*)$ and the Mackey topology $m(Z, Z^*)$, [5, 18.8].

We shall now show following Calderón that fixed points of $\{U(g) : g \in S_\triangle(\mathfrak{A})\}$ are unique $\mathcal{V}'$-limits of $\{M_A z : A \in \mathfrak{A}, <\}$, and thus introduce the mean M over the whole subset $S_\triangle(\mathfrak{A})$. We shall soon find that M is a continuous linear operator on Z, and in fact show that M commutes with $U(h)$ for all h in $S_\triangle(\mathfrak{A})$.

LEMMA 2. *In the vector topology $\mathcal{V}'$, for each z in Z,*
(1) *K_z is a bounded subset of Z*

(2) *K_z is the closed convex extension of* $\{U(g)z : g \in S_\triangle(\mathfrak{A})\}$
(3) *the* M_A *are equicontinuous at* 0 *for all* A *in* $\mathfrak{A}$.

Proof. K_z is $\mathcal{V}$-compact and thus $\mathcal{V}$-bounded; by condition U4, $\mathcal{V}$ and $\mathcal{V}'$ yield the same bounded sets and the same closed convex sets [5, 17.1 and 17.5]. By condition U5, for each closed convex $\mathcal{V}'$-neighborhood W of 0 there exists a convex $\mathcal{V}'$-neighborhood V of 0 such that $U(g)V \subset W$ for all g in $S_\triangle(\mathfrak{A})$. If z is in V, then $U(g)z$ is in W for all g in $S_\triangle(\mathfrak{A})$, and thus K_z is in W, so that $M_A z$ is in W; that is, $M_A V \subset W$.

THEOREM 3 (topological ergodic theorem) *For each z in Z let there exist z_0 in K_z which is a fixed point of $T(h)$ for all h in $S_\triangle(\mathfrak{A})$. Then z_0 is the unique $\mathcal{V}'$-limit of $\{M_A z : A \in \mathfrak{A}, <\}$; and the map M defined by $Mz = z_0$ is a $\mathcal{V}'$-continuous linear operator on Z such that*
(1) *$M^2 = M$, i.e., M is a projection operator*
(2) *$U(h)M = MU(h) = M$ for each h in $S_\triangle(\mathfrak{A})$.*

Proof. For each convex $\mathcal{V}'$-neighborhood W of 0, there exists by Lemma 2.3 a convex $\mathcal{V}'$-neighborhood V of 0 such that $M_A V \subset W$ for all A in $\mathfrak{A}$. Thus there exist by Lemma 2.2 elements v in V, $h_1, \cdots, h_m$ in $S_\triangle(\mathfrak{A})$, and $\alpha_1, \cdots, \alpha_m > 0$ with sum unity such that

$$z_0 - z = v + \textstyle\sum_{i=1}^m \alpha_i[U(h_i)z - z].$$

But by hypothesis z_0 is clearly a fixed point of M_A for all a in $\mathfrak{A}$, so that

$$z_0 - M_A z \in W + \textstyle\sum_{i=1}^m \alpha_i[M_A U(h_i)z - M_A z].$$

Moreover, Lemma 1 holds for $\mathcal{V}'$ as well as $\mathcal{V}$, by Lemma 2.1; so that $M_A U(h_i)z - M_A z \in W$ for all i, and thus $z_0 - M_A z \in W + W$, whenever $A >$ some A_W in $\mathfrak{A}$.

Now on Z, M is the pointwise limit of an equicontinuous family $\{M_A\}$ of linear operators, and is thus itself a continuous linear operator. Also, by the fixed point property, $U(h)Mz = Mz$ for each z in Z and each h in $S_\triangle(\mathfrak{A})$, so that by definition $M_A Mz = Mz$, and in the limit $M^2 z = Mz$. Finally, by definition $U(h)z$ is in K_z, and by Lemma 1,

$$MU(h)z = \mathcal{V}'\text{-}\lim_A M_A U(h)z = \mathcal{V}'\text{-}\lim_A M_A z = Mz.$$

Part II. Group mean on $C_{\mathcal{K}}(X', Y)$

5. Definitions. Let X and Y be finite-dimensional real Banach spaces with elements x and y respectively, let $\mathcal{K}$ be an arbitrary family of compact subsets of X with $\bigcup \mathcal{K} = X$, and let X' be an arbitrary union of subsets K in $\mathcal{K}$. We need not specify $\mathcal{K}$, but in all sections following we shall take its sets to be each S-invariant, and in Section 9 we shall take them all to be S-orbits, these concepts being defined below. Also we need not specify X', but in certain arguments we shall take X' as a single K in $\mathcal{K}$, as $X_\rho = \{x : \| x \| \leq \rho\}$, or as the whole of X.

Let $F(X', Y)$ be the set of all functions from X' into Y; and let $B_{\mathcal{K}}(X', Y)$, $C_{\mathcal{K}}(X', Y)$, and $C(X', Y)$ respectively be the subsets of all functions bounded on each K in $\mathcal{K}$, all functions continuous on each K in $\mathcal{K}$, and all functions continuous on X'. Then clearly

$$C(X', Y) \subset C_{\mathcal{K}}(X', Y) \subset B_{\mathcal{K}}(X', Y) \subset F(X', Y),$$

and these sets of functions, with elements $\Phi, \Psi, \cdots$, are all real vector spaces.

Let $\mathfrak{I}_{\mathcal{K}}$ be the topology of uniform convergence in norm on the sets K in $\mathcal{K} \cap X'$. That is, for a single K in $\mathcal{K}$ the topology $\mathfrak{I}_{\mathcal{K}}$ on $F(K, Y)$ is that of convergence in the norm

$$\| \Phi \|_K = \sup \{ \| \Phi(x) \| : x \in K \},$$

and for an arbitrary X', the topology $\mathfrak{I}_{\mathcal{K}}$ on $F(X', Y)$ is that of convergence individually in the pseudo-norms $\| \Phi \|_K$ for all K in $\mathcal{K} \cap X'$. Now convergence individually on subsets K in $\mathcal{K}$ is equivalent to convergence simultaneously on finite numbers of such subsets; so that we may, and henceforth shall, assume $\mathcal{K}$ closed under finite unions.

With the topology $\mathfrak{I}_{\mathcal{K}}$ it can be shown that $B_{\mathcal{K}}(X', Y)$ is a topological vector space, and that $C_{\mathcal{K}}(X', Y)$ is a closed subspace. It can further be shown that $B_{\mathcal{K}}(X', Y)$, and therefore all its closed subspaces, inherit from Y the properties of being complete, locally convex, and Hausdorff; and that any such topological vector space has in addition the property EC needed for integration of vector-valued functions [5, 7.6 and 13.4].

Let G be a locally compact topological group with left-invariant Haar measure μ; indeed let G be a mean-ergodic such group with ergodic family $(\mathfrak{A}, <)$. Let S and T be representations of G by invertible linear operators on X and Y respectively; and for each Φ in $F(X, Y)$ let

$$U(g)\Phi(x) = T(g)\Phi(S(g)^{-1}x).$$

Then $U(e) = I$, where e is the identity of G, and

$$U(gh)\Phi(x) = T(g)T(h)\Phi(S(h)^{-1}S(g)^{-1}x) = U(g)U(h)\Phi(x);$$

so that U is a representation of G by invertible linear operators on $F(X, Y)$.

An S-invariant subset of X is a subset invariant under $S(g)$ for all g in G; the S-orbit of an element x is the set $S(G)x = \{S(g)x;\, g \in G\}$, which is clearly S-invariant. Similarly we define a T- invariant subset of Y and a U-invariant subset of $F(X, Y)$, and we define the T-orbit of y and the U-orbit of Φ, which are T-invariant and U-invariant respectively. On any U-invariant subspace of $F(X, Y)$, note that U is still a representation of G by invertible linear operators on this space. In particular, if X' is S-invariant then $F(X', Y)$ is U-invariant, for functions in $F(X', Y)$ may be regarded as vanishing outside X'.

Let X' be S-invariant in X, and let U be restricted to $F(X', Y)$. Then for each Φ in $F(X', Y)$ we call Φ form-invariant if $\{\Phi\}$ is U-invariant, that is, if

$U(g)\Phi = \Phi$ for all g, and we call Φ invariant if Φ is form-invariant, where T is specifically the identity representation. We shall now put continuity assumptions on S and T which imply the U-invariance of $B_{\mathcal{K}}(X', Y)$ and $C_{\mathcal{K}}(X', Y)$ and which yield the properties U1–U5 for the representation U on $C_{\mathcal{K}}(X', Y)$. We shall then apply the topological ergodic theorem on this space, and use the resulting group mean M to obtain our conclusions.

6. Boundedness property. Since X and Y are Banach spaces, we may define norms on their linear operators L in the usual way; that is,

$$\| L \| = \sup \{\| Lx \| / \| x \| : x \neq 0\}.$$

Now let S and T be representations of G with the properties

(ST1) $\| S(g) \| \leq 1$ and $\| T(g) \| \leq 1$ for all g in G;
(ST2) $S(g)$ and $T(g)$ are continuous in norm.

In this section, we shall consider only the boundedness property ST1, from which we can obtain conditions U1, U3, and U5 by suitably restricting $\mathcal{K}$; in the next section we shall consider the continuity property ST2, from which we can obtain the remaining hypotheses for the topological ergodic theorem.

By condition ST1, the sets

$$X_\rho = \{x : \| x \| \leq \rho\} \quad \text{and} \quad Y_\sigma = \{y : \| y \| \leq \sigma\}$$

for all ρ and σ in $[0, \infty)$ are respectively S- and T-invariant; and by the finite-dimensionality of X and Y, the sets X_ρ and Y_σ, and all their closed subsets, are compact. Thus in X there exist families of S-invariant compact subsets which cover X, such as: the sets X_ρ with $\rho = 1, 2, \cdots$ by our preceding remarks; the sets $\{x : \| x \| = \rho\}$ with $0 \leq \rho < \infty$ by Lemma 3.1; and the closures $S(G)x^-$ of all S —orbits in X, since $S(G)x \subset X_\rho$ whenever $\| x \| \leq \rho$. If we enlarge these families to include finite unions of the given sets, then the resulting systems are all candidates for the family $\mathcal{K}$, which we assumed closed under finite unions. This enlargement does not change $\mathfrak{I}_{\mathcal{K}}$ or the first system, but adds further sets, all S-invariant and compact, to the second and third families

Henceforth let $\mathcal{K}$ be an arbitrary family of S-invariant compact subsets of X, closed under finite unions, with $\bigcup \mathcal{K} = X$, and let X' be an arbitrary union of subsets K in $\mathcal{K}$, so that X' is also S-invariant. By condition ST1 and this restriction on $\mathcal{K}$ the spaces $C(X', Y)$, $C_{\mathcal{K}}(X' \cdot Y)$, and $B_{\mathcal{K}}(X', Y)$, as well as $F(X', Y)$, are all U-invariant; so that on any of these spaces the representation U is well defined, and may be studied further.

Lemma 3. *If $\| S(g) \| \leq 1$ and $\| T(g) \| \leq 1$ for all g in G, then*
(1) *$S(g)$ and $T(g)$ are isometries on X and Y respectively for each g in G.*

(2) *$U(g)$ is an isometry on $B_{\mathcal{K}}(K, Y)$ for each g in G and each K in $\mathcal{K}$.*
(3) *the $U(g)$ are equicontinuous on $B_{\mathcal{K}}(X', Y)$ for all g in G.*

Proof. For any K in $\mathcal{K}$ take any Φ in $B_{\mathcal{K}}(K, Y)$; then for each g in G,

$$\| U(g)\Phi \|_K = \sup \| T(g)\Phi(S(g)^{-1}x) \| \leq \sup_K \| T(g)\Phi(x) \|$$
$$\leq \sup_K \| \Phi(x) \| = \| \Phi \|_K .$$

Thus $\| U(g) \| \leq 1$ on $B_{\mathcal{K}}(K, Y)$; and the $U(g)$ are equicontinuous on $B_{\mathcal{K}}(X' \cdot Y)$ by definition of $\mathfrak{J}_{\mathcal{K}}$. Suppose $\| U(g)\Phi \|_K = \alpha \| \Phi \|_K$ with $0 \leq \alpha < 1$ for some g in G and some $\Phi \neq 0$; then

$$\| \Phi \|_K = \| U(g^{-1})U(g)\Phi \|_K \leq \| U(g^{-1}) \| \cdot \| U(g)\Phi \|_K = \alpha \| U(g^{-1}) \| \cdot \| \Phi \|_K$$

Thus $1 \leq \alpha \| U(g^{-1}) \| \leq \alpha$ on $B_{\mathcal{K}}(K, Y)$, a contradiction; and $U(g)$ is an isometry on $B_{\mathcal{K}}(K, Y)$. The argument for (2) holds in any Banach space, so that $S(g)$ and $T(g)$ are likewise isometries.

LEMMA 4. *If $\| S(g) \| \leq 1$ and $\| T(g) \| \leq 1$ for all g in G, then for each Φ in $C_{\mathcal{K}}(X', Y)$, the closure of $\{U(g)\Phi : g \in G\}$ is compact for the topology $\mathfrak{J}_{\mathcal{K}}$.*

Proof. We need a slight modification of the proof of Ascoli's theorem˙ Let $\mathcal{P}$ be the family of all finite subsets of X, and let $\mathfrak{J}_{\mathcal{P}}$ be the topology of convergence in (uniform) norm on the sets P in $\mathcal{P} \cap X'$. Let $J_\Phi = \{U(g)\Phi : g \in G\}$ and let $\bar{J_\Phi}$ be the $\mathfrak{J}_{\mathcal{P}}$-closure of J_Φ, that is, the pointwise closure of J_Φ. Now each x in X lies in some K in $\mathcal{K}$, and $\Phi(K)$ is compact since K is compact, so that $\Phi(K) \subset Y_\sigma$ for some $\sigma < \infty$; thus

$$J_\Phi[x] \subset \{T(g)\Phi(S(h)x) : g, h \in G\} \subset \{T(g)\Phi(K) : g \in G\} \subset Y_\sigma ,$$

whence $\bar{J_\Phi}[x] \subset Y_\sigma$. That is, $\bar{J_\Phi}[x]$ has compact closure by the compactness of Y_σ, and thus $\bar{J_\Phi}$ is $\mathfrak{J}_{\mathcal{P}}$-compact by the Tychonoff theorem.

Also Φ is uniformly continuous on each K in $\mathcal{K}$, so that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\| x - x' \| < \delta$ implies $\| \Phi(x) - \Phi(x') \| < \varepsilon$ on this K. But then $\| x - x' \| < \delta$ implies $\| S(g)^{-1}x - S(g)^{-1}x' \| < \delta$, and thus

$$\| T(g)\Phi(S(g)^{-1}x) - T(g)\Phi(S(g)^{-1}x') \|$$
$$\leq \| \Phi(S(g)^{-1}x - \Phi(S(g)^{-1}x') \| < \epsilon$$

for each g in G. That is, J_Φ is equicontinuous on each K in $\mathcal{K}$, and thus, by a standard theorem [4, 7. 14], $\bar{J_\Phi}$ is equicontinuous on each K in $\mathcal{K}$. Therefore in $\bar{J_\Phi}$, by a standard theorem [4, 7.15], for each K, $\mathfrak{J}_{\mathcal{P}}$-convergence implies $\mathfrak{J}_{\mathcal{K}}$-convergence, and in $C_{\mathcal{K}}(X', Y)$, since $\bigcup \mathcal{K} = X$, $\mathfrak{J}_{\mathcal{K}}$-convergence implies $\mathfrak{J}_{\mathcal{P}}$-convergence. That is, $\mathfrak{J}_{\mathcal{P}} = \mathfrak{T}_{\mathcal{K}}$ on $\bar{J_\Phi}$, and thus the $\mathfrak{J}_{\mathcal{K}}$-closure of J_Φ is $\mathfrak{J}_{\mathcal{K}}$-compact.

7. Existence of group mean. We have just studied the boundedness property ST1 and obtained from it conditions U3 and U5; we shall now study the continuity property ST2 and obtain from it condition U2. But if we choose $\mathcal{V} = \mathcal{V}' = \mathfrak{I}_{\mathcal{K}}$ in the space $C_{\mathcal{K}}(X', Y)$, then U1 follows from U5, and U4 becomes trivial; and since G was assumed a mean-ergodic group, the topological ergodic theorem will now yield a group mean on $C_{\mathcal{K}}(X', Y)$.

LEMMA 5. *If $S(g)$ and $T(g)$ are continuous in norm on G, then $g \to U(g)\Phi$ is continuous on G for each fixed Φ in $C_{\mathcal{K}}(X', Y)$.*

Proof. For each K in $\mathcal{K}$, $\sup_K \| x \|$ and $\| \Phi \|_K$ are finite, since K and $\Phi(K)$ are compact. Also

$$\begin{aligned} \| U(g)\Phi - \Phi \| &\leq \sup_K \| [T(g) - I]\Phi(S(g)^{-1}x) \| \\ &\quad + \sup_K \| \Phi(S(g)^{-1}x) - \Phi(x) \| \\ &\leq \| T(g) - I \| \cdot \| \Phi \|_K + \sup_K \|\Phi(S(g^{-1})x) - \Phi(x) \|. \end{aligned}$$

But Φ is uniformly continuous on K, and

$$\| S(g^{-1})x - x \| \leq \| S(g^{-1}) - I \| \cdot \sup_K \| x \|;$$

so that, on the right side of the preceding inequality, both terms approach 0 as $g \to e$.

THEOREM 4 ($\mathfrak{I}_{\mathcal{K}}$-ergodic theorem). *Let S and T satisfy the conditions $ST1$ and $ST2$, and let G be a mean-ergodic group. Then there exists on $C_{\mathcal{K}}(X', Y)$ with the topology $\mathfrak{I}_{\mathcal{K}}$ a continuous linear operator M such that, for each Φ in $C_{\mathcal{K}}(X', Y)$,*

(1) $M\Phi =$ *unique* $\mathfrak{I}_{\mathcal{K}}$-$\lim_A M_A \Phi$
(2) $M^2 = M$, i.e., *M is a projection operator*
(3) *$U(g)M = MU(g) = M$ for each g in G*
(4) *Φ is form-invariant if and only if $M\Phi = \Phi$.*

Proof. We have noted that $C_{\mathcal{K}}(X', Y)$ with the topology $\mathfrak{I}_{\mathcal{K}}$ is a real locally convex Hausdorff topological vector space which is complete, and therefore has the property EC, and that U is a representation of G by invertible linear operators on $C_{\mathcal{K}}(X', Y)$. If we put $\mathcal{V} = \mathcal{V}' = \mathfrak{I}_{\mathcal{K}}$ then this topological vector space with this representation satisfies the conditions for Theorems 2 and 3 by Lemmas 3.3, 4, and 5; so that the existence of M, the continuity of M, and parts (1), (2), and (3) follow from Theorem 3.

If Φ is form-invariant then $U(g)\Phi = \Phi$ for all g in G, so that $M_A \Phi = \Phi$ for all A in $\mathfrak{A}$, and thus $M\Phi = \Phi$ by part (1). If $M\Phi = \Phi$ then $U(g)\Phi = U(g)M\Phi = M\Phi = \Phi$ for all g in G by part (3), so that Φ is form-invariant.

COROLLARY 4.1. $\| M \| = 1$ *on $C_{\mathcal{K}}(K, Y)$ with the topology $\mathfrak{I}_{\mathcal{K}}$, for each K in $\mathcal{K}$.*

Proof. $\mathfrak{I}_{\mathcal{K}}$ on $C_{\mathcal{K}}(K, Y)$ is described by the norm $\| \Phi \|_K$; so that by Theorems 1.2 and 4.1,

$$\| M\Phi \|_K \leq \limsup_A \| M_A \Phi \|_K \leq \sup_G \| T(g)\Phi \|_K \leq \| \Phi \|_K$$

for all Φ in $C_{\mathcal{K}}(K, Y)$. Thus $\| M \| \leq 1$; but by Theorem 4.3, $\| M \| = \| M^2 \| \leq \| M \|^2$ on $C_{\mathcal{K}}(K, Y)$, and $1 \leq \| M \|$.

Finite-dimensionality. If we let $\mathcal{K}$ be a family of S-invariant compact subsets of X, but drop the requirement that $\bigcup \mathcal{K} = X$, then Lemmas 3 and 5 hold for any Banach spaces X and Y. However, Lemma 4 used explicitly the conditions that $\bigcup \mathcal{K} = X$ and that Y_σ is compact, and thus presupposes X and Y to be finite-dimensional.

To see that Lemma 4 can fail otherwise, let X and Y both be a Hilbert space H with denumerable basis $\{e_i\}$, and let G_P be the (denumerable) group of unitary operators on H induced by the group of permutations on finite subsets of $\{e_i\}$. Also let $G = G_P \times G_P$ with the discrete topology, let $S(g_1, g_2) = g_1$ and $T(g_1, g_2) = g_2$, and let $\Phi_j(x) = e_j$ for all j and all x in X. If e_j is in K for some K in $\mathcal{K}$, then $\{e_i\} = S(G)e_j$ is in K; if $U(G)\Phi_j^-$ is $\mathfrak{I}_{\mathcal{K}}$-compact and x is in K, then $\{e_i\} = [U(g)\Phi_j^-][x]$ has compact closure. But $\{e_i\}$ is contained in no compact subset of H.

Part III. Form-invariant functions

8. Polynomial functions. Let $L_r(X, Y)$ be the set of all r-linear functions $\Phi(x_1, \cdots, x_r)$ from X into Y which are separately continuous in each variable x_i, and thus, by a standard argument, are jointly continuous in these variables. Let $P_r(X, Y)$ be the set of all functions $\Phi(x, \cdots, x)$ with Φ in $L_r(X, Y)$, and let $P_r(X', Y)$ be the restriction to X' of $P_r(X, Y)$. Then X and Y, hence $L_r(X, Y)$, hence $P_r(X, Y)$, and hence $P_r(X', Y)$ are finite-dimensional real vector spaces; so that $P_r(X', Y)$ is a closed subspace of $C_{\mathcal{K}}(X', Y)$, [5, 7.3].

Let G be a mean-ergodic group, so that Theorem 4 holds; then $P_r(X', Y)$ is M-invariant, since it is closed and U-invariant. Let $P(X', Y)$ be the set of all finite sums of functions in the spaces $P_r(X', Y)$, that is, the set of all polynomial functions from X' into Y; then $P(X', Y)$ is an M-invariant real linear manifold in $C_{\mathcal{K}}(X', Y)$ by the preceding arguments.

When $Y = R$ we shall always understand T to be the identity representation. Then by Theorem 4, $MC_{\mathcal{K}}(X', Y)$ is the subspace of all form-invariant functions in $C_{\mathcal{K}}(X', Y)$, and $MC_{\mathcal{K}}(X', R)$ is the subspace of all invariant real-valued functions in $C_{\mathcal{K}}(X', R)$. Similarly $MP(X', Y)$ is the subspace of all form-invariant polynomial functions in $P(X', Y)$, and $MP(X', R)$ is the subspace of all invariant real-valued polynomial functions in $P(X', R)$. We shall now use these concepts to prove the results of Wineman and Pipkin.

LEMMA 6. *$P(X', Y)$ is $\mathfrak{I}_{\mathcal{K}}$-dense in $C_{\mathcal{K}}(X', Y)$.*

Proof. Since $\mathcal{K}$ is closed under finite unions, we need only show $P(K, Y)$ uniformly dense in $C(K, Y)$ for each K in $\mathcal{K}$. Let $\{e_i\}$ be a basis of Y, and $\{e_i^*\}$ a dual basis of Y^*, such that $\| e_i \| = \| e_i^* \| = 1$ for $i = 1, \cdots, \dim(Y)$. Then $e_i^* P(K, Y) \subset P(K, R)$ for each i, and if φ is in $P(K, R)$ then $e_i\varphi$ is in $P(K, Y)$ with $e_i^*(e_i\varphi) = \varphi$; so that $e_i^* P(K, Y) = P(K, R)$ Now take $\varepsilon > 0$, take any Φ in $C(K, Y)$, and recall the Stone-Weierstrass theorem, by which there exists φ_i in $P(K, R)$ for each i such that $\| e_i^*\Phi - \varphi_i \|_K < \varepsilon$. Thus

$$\| \Phi - \textstyle\sum_i e_i\varphi_i \|_K \leq \sum_i \| e_i e_i^*\Phi - e_i\varphi_i \|_K < \varepsilon \dim(Y)$$

and $\dim(Y)$ is finite.

THEOREM 5 (density theorem). *$MP(X', Y)$ is $\mathfrak{I}_{\mathcal{K}}$-dense in $MC_{\mathcal{K}}(X', Y)$.*

Proof. Since $\mathcal{K}$ is closed under finite unions, we need only show $MP(K, Y)$ uniformly dense in $MC(K, Y)$ for each K in $\mathcal{K}$. Take $\varepsilon > 0$, take any Φ in $MC(K, Y)$, hence in $C(K, Y)$, and note by Lemma 6 that there exists Ψ in $P(K, Y)$ such that $\| \Psi - \Phi \|_K \leq \varepsilon$. Then $M\Phi = \Phi$ and $M\Psi$ is in $MP(K, Y)$, so that by Corollary 4.1,

$$\| \Phi - M\Psi \|_K = \| M\Phi - M\Psi \|_K \leq \| \Phi - \Psi \|_K \leq \varepsilon.$$

THEOREM 6 (separation theorem). *Let K_1 and K_2 be disjoint S-invariant compact subsets of X; then $MP(X, R)$ separates K_1 and K_2.*

Proof. Since we have not specified $\mathcal{K}$, and since $MP(X, R)$ is independent of $\mathcal{K}$, we may assume here that K_1, K_2, and $K_1 \cup K_2$ are in $\mathcal{K}$. Now the space X is Hausdorff, so that K_1 and K_2 are closed; and the space X is normal, so that there exists φ in $C(X, R)$ such that $\varphi(x) = 0$ on K_1, $\varphi(x) = 1$ on K_2, and $0 \leq \varphi(x) \leq 1$ on X. Then $M\varphi$ is defined and in $C_{\mathcal{K}}(X, R)$, such that $M\varphi(x) = 0$ on K_1, $M\varphi(x) = 1$ on K_2, and $0 \leq M\varphi(x) \leq 1$ on X; and by Theorem 5 there exists ψ in $MP(X, R)$ such that $\| M\varphi - \psi \|_{K_1 \cup K_2} < \frac{1}{2}$, so that ψ separates K_1 and K_2.

9. Closed S-orbits. To exploit Theorems 5 and 6 for the desired conclusions, we must now assume not only that G is mean-ergodic but also that G has closed S-orbits—that for each x in X the orbit $S(G)x$ is closed in X. Now the orbit $S(G)x$ is contained in the compact subset X_ρ for each $\rho \geq \| x \|$, and is thus itself compact, so that by Theorem 6, $MP(X, R)$ separates distinct orbits. Moreover, the family of all orbits $S(G)x$ clearly covers X, so that we may, and henceforth shall, choose the family $\mathcal{K}$ to consist of all finite unions of S-orbits.

We have previously noted that $MP(X', Y)$ and $MC_{\mathcal{K}}(X', Y)$ are real vector spaces for any S-invariant subset X' of X and any finite-dimensional Banach spaces X and Y. We now further observe that $MP(X', R)$ and

$MC_{\mathcal{K}}(X', R)$ are real algebras, by the convention that T is the identity representation, and that $MP(X', Y)$ and $MC_{\mathcal{K}}(X', Y)$ are respectively modules over $MP(X', R)$and $MC_{\mathcal{K}}(X', R)$. Indeed, the former pair are closed under the obvious multiplication of real-valued functions within them, the latter pair closed under the obvious multiplication by real-valued functions in their respective algebras [3].

A generating set for an algebra (resp. a module) is a subset Γ such that any given element can be expressed in terms of the elements of Γ as a finite polynomial with scalar coefficients (resp. a finite linear combination with coefficients in the multiplier algebra); and an integrity basis (resp. module basis) is simply a minimal generating set. In particular, for a given generating set Γ of $MP(X', Y)$ and for each x in X let $\hat{x}$ be the function on all γ in Γ defined by $\hat{x}(\gamma) = \gamma(x)$, and let $\hat{\Gamma}(x) = \hat{x}$, so that $\hat{\Gamma}$ is a map from X onto a subset $\hat{X}$ of R^{Γ}. Then we shall characterize $MC_{\mathcal{K}}(X', R)$ in terms of these concepts, and $MC_{\mathcal{K}}(X', Y)$ in terms of a generating set for $MP(X', Y)$.

LEMMA 7. *$\hat{\Gamma}$ is a function constant on each S-orbit such that, for each x_1 and x_2 in X, the following are equivalent:*
(1) $\hat{\Gamma}(x_1) = \hat{\Gamma}(x_2)$
(2) $S(G)x_1 \cap S(G)x_2 \neq \emptyset$
(3) $S(G)x_1 = S(G)x_2$.

Proof. Now $\{S(g) : g \in G\}$ is a transformation group on X, and thus defines an equivalence relation on X by: $x_1 \sim x_2$ if and only if $x_2 = S(g)x_1$ for some g in G. Therefore parts (2) and (3) each hold if and only if $x_1 \sim x_2$, since $S(G)x$ is the equivalence class of x. If $x_1 \sim x_2$ then $\varphi(x_1) = \varphi(S(g)^{-1}x_2) = \varphi(x_2)$ for any φ in $MC_{\mathcal{K}}(X', R)$; so that φ is constant on $S(G)x_1$, and in particular $\hat{\Gamma}$ is constant on $S(G)x_1$. Conversely if $\hat{\Gamma}(x_1) = \hat{\Gamma}(x_2)$ then $\varphi(x_1) = \varphi(x_2)$ for any φ in $MC_{\mathcal{K}}(X', R)$; so that $S(G)x_1 \cap S(G)x_2 \neq \emptyset$ by Theorem 6, since by assumption $S(G)x_i$ is compact for $i = 1, 2$.

THEOREM 7 (characterization of invariant real-valued functions). *For any φ in $F(X', R)$, the following are equivalent:*
(1) *φ is an invariant function*
(2) *φ is in $MC_{\mathcal{K}}(X', R)$*
(3) *$\varphi = \Phi \circ \hat{\Gamma}$ for some map $\Phi : \hat{X} \to R$*

Proof. A function φ in $F(X' R)$ is invariant if and only if it is constant on S-orbits, and thus by Lemma 7 if and only if it has the form $\Phi \circ \hat{\Gamma}$ of part (3). Such a function is clearly continuous on finite unions of S-orbits, and thus by Theorem 4.4 is an element of $MC_{\mathcal{K}}(X', R)$, all of whose elements, conversely, are invariant.

LEMMA 8. *If $\{\Theta_1, \cdots, \Theta_n\}$ with $n < \infty$ is a generating set for $MP(X, Y)$, then it is a generating set for $MC_{\mathcal{K}}(X', Y)$.*

Proof. If Ψ_x is an element of $MP(S(G)x, Y)$ for any x in X, then by definition there exists an element Ψ of $P(X, Y)$ such that $\Psi_x = \Psi \mid S(G)x$. But by hypothesis there exist elements $\psi_1, \cdots, \psi_n$ in $MP(X, R)$ such that that $M\Psi = \sum_{i=1}^n \psi_i \Theta_i$, and therefore

$$\Psi_x = M\Psi \mid S(G)x = \sum_{i=1}^n (\psi_i \mid S(G)x)(\Theta_i \mid S(G)x).$$

However $\psi \mid S(G)x$ is constant, so that $MP(S(G)x, Y)$ considered as a real vector space is spanned by the functions $\Theta_i \mid S(G)x$, and is thus finite-dimensional.

Clearly $MP(S(G)x, Y)$ is then uniformly closed, and is thus $MC_{\mathfrak{K}}(S(G)x, Y)$ by Theorem 5. Now take any x in X' and any Φ in $MC_{\mathfrak{K}}(X', Y)$; then $\Phi \mid S(G)x$ is in $MC_{\mathfrak{K}}(S(G)x, Y)$, so that there exist scalars $\alpha_i(S(G)x)$ with $i = 1, \cdots, n$ such that

$$\Phi \mid S(G)x = \sum_{i=1}^n \alpha_i(S(G)x)(\Theta_i \mid S(G)x).$$

If we let $\varphi_i(x) = \alpha_i(S(G)x)$ for each i and all x then φ_i is clearly invariant, so that φ_i is in $MC_{\mathfrak{K}}(X', Y)$, and $\Phi = \sum_{i=1}^n \varphi_i \Theta_i$ by construction.

10. Form-invariant functions for Lindelof groups. To obtain the Wineman-Pipkin canonical form of a form-invariant function in $F(X', Y)$ we must finally assume that $MP(X, Y)$ has a finite generating set, hence a finite module basis, and that the topological group G is also a Lindelof space—that each open cover of G has a countable subcover. We shall need this hypothesis for a category argument [4, 6.34] but, before proceeding, we shall study its effect on the group G; clearly every compact group is a Lindelof space, but so, we shall note, are much less restricted groups.

The subset C of G is called σ-connected (resp. σ-compact) if it is the union of a countable family of connected (resp. compact) subsets. We shall see that the Lindelof property and these two are all equivalent, and imply that all form-invariant functions lie in $MC_{\mathfrak{K}}(X', Y)$, so that by lemma 8 they are all representable via a module basis for $MP(X, Y)$.

LEMMA 9. *Let G be a locally compact group; then the following are equivalent:*

(1) *G is a Lindelof space*
(2) *G is σ-connected*
(3) *G is σ-compact.*

Proof. Let G_0 be the identity component of G, which is well known to be an open and closed normal subgroup [4, 3T] so that $G = \{hG_0 : h \in G\}$ contains a disjoint open cover of G. If G is a Lindelof space, then the distinct subsets hG_0 can form at most a countable family, in which every subset hG_0 is connected by definition. If G is σ-connected, then the distinct subsets hG_0 can again form a countable family, and $G_0 \subset \bigcup_{r=1}^{\infty} N^r$ for any neighborhood N of e, [4, 3T]. But we can choose N compact, and thus N^r compact, so that G_0 is σ-compact, and thus G is σ-compact. If $G = \bigcup_r^{\infty} C_r$ with each C_r compact, and if $\{A_j : j \in J\}$ is an open cover of G, then we can select from the A_j a finite cover of each C_r, and thus a countable cover of G.

LEMMA 10. *Let G be a Lindelof group and $S(G)x$ a closed orbit in X; then $g \to S(g)x$ is an open map.*

Proof. With distances measured as in X, the closed subset $S(G)x$ is a complete metric space, and thus a set of second category in itself [4, 6.34]. If N is any neighborhood of e, then $\{gN : g \in G\}$ is an open cover of G and there exists a countable subcover $\{g_i N : i = 1, 2, \cdots\}$ since G is a Lindelof group. But $S(G)x \subset \bigcup_{i=1}^{\infty} S(g_i N)x$, so that some $S(g_k N)x$ has non-void interior in $S(G)x$, and $S(gN)x = S(gg_K^{-1})S(g_k N)x$ has non-void interior for each g in G. Thus $S(g)x \in$ interior of $S(N)x$ for some g in N, and $x \in$ interior of $S(N^{-1}N)x$.

THEOREM 8 (characterization of form-invariant functions). *For any Φ in $F(X' \cdot Y)$ and any generating set $\{\Theta_1, \cdots, \Theta_n\}$ of $MP(X, Y)$, the following are equivalent;*

(1) *Φ is a form-invariant function*

(2) *Φ is in $MC_{\mathcal{K}}(X', Y)$*

(3) *$\Phi = \sum_{i=1}^{n} \varphi_i \Theta_i$ for some $\varphi_1, \cdots, \varphi_n$ in $MC_{\mathcal{K}}(X', R)$.*

Proof. Clearly (3) $\Rightarrow$ (2) $\Rightarrow$ (1), so that we need only prove the reverse implications. Take any form-invariant Φ and any open subset V of Y, and recall that $g \to T(g)y$ is continuous for each fixed y in Y by condition ST2. But $T(g)\Phi(x) = \Phi(S(g)x)$ for each x in X, so that the set

$$N = \{g : T(g)\Phi(x) \in V\} = \{g : \Phi(S(g)x) \in V\}$$

is open in G, and by Lemma 10,

$$S(N) = \{x' \in S(G)x : \Phi(x') \in V\}$$

is open in $S(G)x$. Thus $\Phi \mid S(G)x$ is continuous for each x in X, so that Φ is in $MC_{\mathcal{K}}(X', Y)$ by definition, and $\Phi = \sum_{i=1}^{n} \varphi_i \Theta_i$ by Lemma 8.

11. Summary. We have assumed that X and Y are finite-dimensional real Banach spaces, that G is a mean-ergodic locally compact group, and that S and T are representations of G by invertible linear operators on X and Y respectively, such that $\| S(g) \| \leq 1$ and $\| T(g) \| \leq 1$ for all g in G, $S(g)$ and $T(g)$ are continuous in their operator norms, and that all S-orbits are closed sets. We have then proved in Theorem 7 that an integrity basis of $MP(X, R)$ is also a functional basis for all invariant functions in $F(X', R)$.

We have further assumed that G is a Lindelof group and that $MP(X, Y)$ has a finite set of generators $\Theta_1, \cdots, \Theta_n$. We have finally proved in Theorem 8 that the Θ_i are a set of generators, over the algebra of all invariant functions in $F(X', R)$, for all form-invariant functions in $F(X', Y)$.

We have noted that G is a Lindelof group if and only if it is either σ-connected or σ-compact; we can offer only somewhat less exact conditions that G be mean-ergodic. Again, each compact group is mean-ergodic, but so is each locally compact group G in which

$$G = \bigcup_{r=1}^{\infty} N^r \quad \text{and} \quad \lim_{r\to\infty} \mu(N^{r+1})/\mu(N^r) = 1$$

for some compact symmetric neighborhood N of e. To see this we take $(\mathfrak{A}, <)$ to be $\{N^r : r = 1, 2, \cdots\}$ with set inclusion, and for any h in G we let s be an integer such that h lies in N^s; then

$$hN^r - N^r \subset N^{r+s} - N^r \quad \text{and} \quad N^r - hN^r \subset h(N^{r+s} - N^r),$$

so that $\lim_{r\to\infty} \mu(hN^r \triangle N^r)/\mu(N^r) = 0$.

Without the condition that G have closed S-orbits, the only asymmetric requirement on the two representations, there exist counterexamples to Theorems 7 and 8 for such simple G as the additive groups of the integers and of the real numbers. However, we may conjecture that a restriction to measurable invariants and form-invariant functions might again yield the conclusions of these theorems, by suitable methods of topological dynamics.

Wineman and Pipkin also obtain canonical forms of arbitrary form-invariant functionals of vector histories, that is, of functions $x(\tau)$ with values in X. However, given Theorems 7 and 8, the discussion of functionals becomes entirely algebraic, requiring not even continuity in τ, and may be carried through exactly as before to the same conclusion.

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