# RELATIVE CONVEXITY AND TOTAL CONCAVITY

BY

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## I. Relative convexity

The purpose of this paper is to generalize the idea of convexity in Euclidean space and study the properties of those sets, which we call totally concave, which are in some sense as non-convex as possible.

If $p_1, p_2, \cdots, p_s$ are points of $E^n$ (Euclidean $n$-space), $1 \leqq n$, we will denote the closed convex cell spanned by these points by $p_1 p_2 \cdots p_s$. The dimension of this cell may of course be less than $s - 1$. The complement of a set $T$ in $E^n$ will be denoted by $cT$. If $S$ and $T$ are sets in $E^n$, $S \circ T = \bigcup_{p \in S, q \in T} (pq)$.

DEFINITION I.1. If $S$ and $T$ are sets in $E^n$, $S$ will be said to be convex with respect to $T$ if and only if, given $p_1, p_2, \cdots, p_s$; $s \leqq n + 1$; $p_i \in S \cap T$, the convex cell $p_1 p_2 \cdots p_s \subset S \cup cT$.

This is a generalization of convexity in the sense that if $p$ and $q$ are points of $S \cap T$, those points of the line segment $pq$ which are in $T$ are also in $S$. A set $S \subset E^n$ is convex if and only if it is convex with respect to $E^n$. Any set is convex with respect to itself. The empty set is convex with respect to any set. A convex set is convex with respect to any set. If $T$ is convex, $\mathrm{Bd}(T)$ is convex with respect to $c\,\mathrm{Int}(T)$.

PROPOSITION I.2. *If $T$ is convex and $S$ is convex with respect to $T$, then $S \cap T$ is convex.*

*Proof.* If $p_1, p_2 \in S \cap T$, then $p_1 p_2 \subset S \cup cT$ and by convexity $p_1 p_2 \subset T$. Hence $p_1 p_2 \subset S \cap T$.

*Example* I.3. In Figure 1a, $S$ is convex with respect to $T$, $R \subset S$, but $R$ is not convex with respect to $T$. In Figure 1b, $S$ is convex with respect to $T$, $T$ is convex with respect to $R$, but $S$ is not convex with respect to $R$.

PROPOSITION I.4. *If $\Lambda$ is an index set and $S_\alpha$ is convex with respect to $T$ for all $\alpha \in \Lambda$, then $\bigcap_{\alpha \in \Lambda} S_\alpha$ is convex with respect to $T$.*

*Proof.* If $p_1, p_2, \cdots, p_s \in (\bigcap_{\alpha\in\Lambda} S_\alpha) \cap T$ the cell

$$p_1 p_2 \cdots p_s \subset \textstyle\bigcap_{\alpha\in\Lambda} (S_\alpha \cup cT) = (\bigcap_{\alpha\in\Lambda} S_\alpha) \cup cT.$$

PROPOSITION I.5. *If $S$ is convex with respect to $R$, $T \subset R$, then $S$ is convex with respect to $T$.*

*Proof.* Let $p_1, p_2, \cdots, p_s \in S \cap T \subset S \cap R$. Then

$$p_1 p_2 \cdots p_s \subset S \cup cR \subset S \cup cT.$$

Received October 10, 1966.

COROLLARY I.6. *If $\Lambda$ is an index set and $S$ is convex with respect to $T_\alpha$ for all $\alpha \in \Lambda$ then $S$ is convex with respect to $\bigcap_{\alpha\in\Lambda} T$.*

COROLLARY I.7. *$\bar{T}$ is convex with respect to $T$.*

*Example* I.8. In Figure 2a, $S$ is convex with respect to both $T_1$ and $T_2$ but not with respect to $T_1 \cup T_2$. In Figure 2b, both $S_1$ and $S_2$ are convex with respect to $T$ but $S_1 \cup S_2$ is not.

PROPOSITION I.9. *$S$ is convex with respect to $T$ if and only if $S \cap T$ is convex with respect to $T$.*

*Proof.* If $p_1, p_2, \cdots, p_s \in (S \cap T) \cap T = S \cap T$ then

$$p_1 p_2 \cdots p_s \subset S \cup cT = (S \cap T) \cup cT.$$

PROPOSITION I.10. *$S$ is convex with respect to $T$ if and only if*

$$S \cap T = H(S \cap T) \cap T.$$

*($H(T)$ is the convex cover of $T$.)*

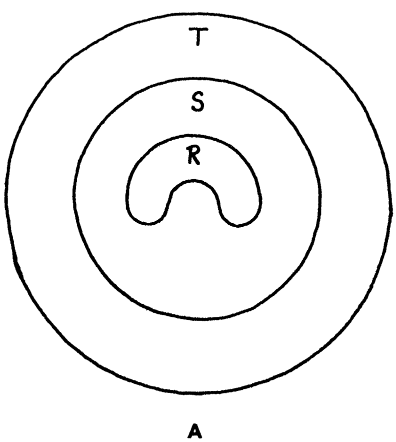



FIGURE 1a

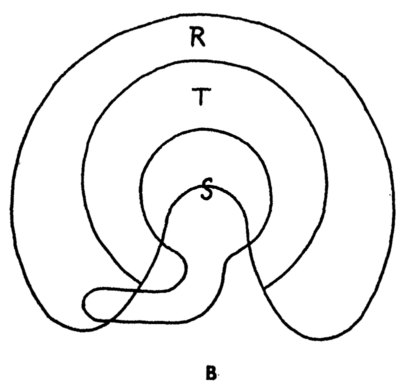



FIGURE 1b

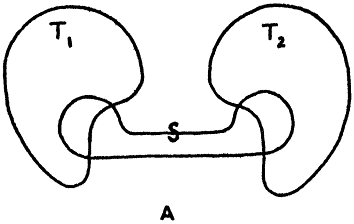



FIGURE 2a

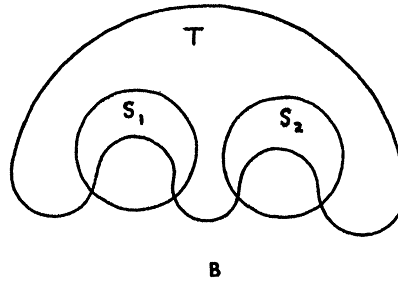



FIGURE 2b

*Proof.* $S \cap T \subset H(S \cap T) \cap T$. If $p \in [H(S \cap T) \cap T] - [S \cap T]$, by Caratheodory's theorem [2] there exist $p_1, p_2, \cdots, p_s$, $s \leqq n + 1$, $p_i \in S \cap T$, and $p \in p_1 p_2 \cdots p_s$. If $S$ is convex with respect to $T$, $p_1 p_2 \cdots p_s \subset S \cup cT$. Hence

$$\begin{aligned} p &\in [H(S \cap T) \cap T] \cap c[S \cap T] \cap [S \cup cT] \\ &= [H(S \cap T) \cap T] \cap \{[cS \cup cT] \cap [S \cup cT]\} \\ &= H(S \cap T) \cap T \cap cT = \emptyset. \end{aligned}$$

Since $H(S \cap T)$ is convex it is convex with respect to $T$. By Proposition I.9, $H(S \cap T) \cap T = S \cap T$ is convex with respect to $T$, and by the same proposition $S$ is convex with respect to $T$.

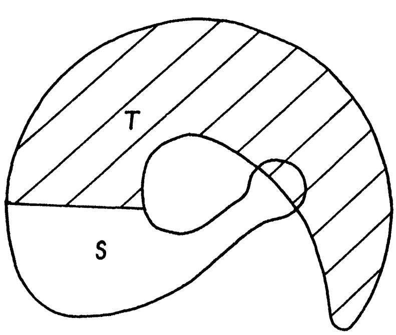
*Example* I.11. If Figure 3a below $S$ is open, $T$ closed and $S$ is convex with respect to $T$, but $S \cap T \neq H(S) \cap T$. This example also shows that neither $\bar{S}$ nor $\bar{S} \cap T$ need be convex with respect to $T$. In Figure 3b, $S$ is open, $S \cap T$ is open, but the closed segment $pq \subset T$. $S$ is convex with respect to $T$, but none of $\bar{S}, \overline{S \cap T}, \overline{S \cap T} \cap T, \overline{S \cap T} \cap \bar{T}, N(S, \delta)$ (a $\delta$-neighborhood of $S$) is convex with respect to $T$. Hence none is convex with respect to $\bar{T}$.

In fact, about all that can be said about closures is that if $S$ is convex with respect to $T$, then so is $S \cap \bar{T}$.

*Example* I.12. $S$ and $T$ are open, $S$ is convex with respect to $T$, but $S \cap \bar{T}$ is not convex with respect to $\bar{T}$. Also note that Int $(\bar{S})$ is not convex with respect to $\bar{T}$.

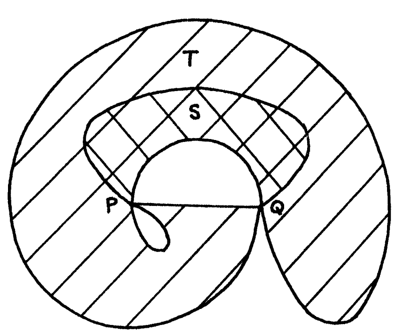
PROPOSITION I.13. *If $S$ is convex with respect to $T$, then* Int $(S)$ *is convex with respect to* Int $(T)$.

*Proof.* Let $p_1, p_2, \cdots, p_{n+1} \in$ Int $(S) \cap$ Int $(T)$, and for each $i$, pick a neighborhood $N(p_i) \subset$ Int $(S) \cap$ Int $(T)$. Let $R = H(\bigcup_{i=1}^{n+1} N(P_i))$. If $x \in R$, we can, by Caratheodory's theorem, pick $q_{i0}, q_{i1}, \cdots, q_{is}$; $q_{ij} \in N(p_{ij})$,




A

FIGURE 3a




B

FIGURE 3b

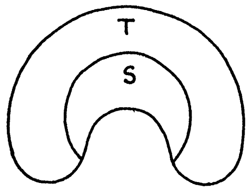



FIGURE 4

where each $p_{ij}$ is some $p_k$, so that $q_{i0} q_{i1} \cdots q_{is}$ is a convex cell containing $x$. Each $q_{ij} \in S \cap \text{Int}(T)$, so that

$$x \in q_{i0} q_1 \cdots q_{is} \subset S \cup c \, \text{Int}(T)$$

It follows that

$$R \subset S \cup c \, \text{Int}(T) \quad \text{and} \quad p_1 p_2 \cdots p_{n+1} \subset R \subset S \cup c \, \text{Int}(T).$$

Now let $p \in p_1 p_2 \cdots p_{n+1}$. We wish to show that

$$p \in \text{Int}(S) \cup c \, \text{Int}(T).$$

If $p \in \text{Int}(T)$, $p \in \text{Int}(R) \cap \text{Int}(T)$ and there is a neighborhood

$$N(p) \subset \text{Int}(R) \cap \text{Int}(T) \subset (S \cup c \, \text{Int}(T)) \cap \text{Int}(T) = S \cap \text{Int}(T) \subset S.$$

Hence $p \in \text{Int}(S)$ and the theorem is proved.

DEFINITION I.14. The relative convex cover of a set $S$ with respect to another set $T$ is denoted by $H_T(S)$, and defined to be $H(S \cap T) \cap T$. Note that Propositon I.10 states that $S$ is convex with respect to $T$ if and only if $H_T(S) = S \cap T$.

PROPOSITION I.15. *If $R$ is convex with respect to $T$ and $S \subset R$, then*

$$H_T(S) \subset R \cap T = H_T(R).$$

*Proof.* $S \cap T \subset R \cap T$ implies $H(S \cap T) \subset H(R \cap T)$. Hence

$$H_T(S) = H(S \cap T) \cap T \subset H(R \cap T) \cap T = R \cap T = H_T(R).$$

PROPOSITION I.16. *If $S \subset R$ then $H_S(T) \subset H_R(T)$.*

PROPOSITION I.17. *If $S$ is compact and $T$ closed then $H_T(S)$ is compact.*

DEFINITION I.18. *A* point $p$ of *a* convex set $S$ is called an extreme point of $S$ provided that $p$ does not lie on the interior of any interval in $S$. The set of extreme points of $S$ will be denoted by $E(S)$. If $S$ is convex with respect to $T$ and $p \in S \cap T$, $p$ is extreme in $S$ with respect to $T$ provided $p \in E(H(S \cap T))$. The set of extreme points of $S$ with respect to $T$ is denoted by $E_T(S)$. $E_T(S) = E(H(S \cap T)) \cap (S \cap T)$.

LEMMA I.19. *If $S$ is compact, then*

$$H(S) = H(E(H(S))) = H(E(H(S)) \cap S).$$

*Proof.* The first equality is well known and the inclusion

$$H(E(H(S)) \cap S) \subset H(E(H(S)))$$

is clear. If $p \in H(S) - S$, there exist points $q_1, q_2, \cdots, q_s$ ; $q_i \in S$; $s \leqq n+1$ ; with $p \in q_1 q_2 \cdots q_s \subset H(S)$. Since $p \neq q_i$ for any $i$, $p$ is on the interior of some face of $q_1 q_2 \cdots q_s$ and cannot be an extreme point. Hence

$$E(H(S)) \subset E(H(S)) \cap S,$$

and the lemma is established by taking convex covers.

PROPOSITION I.20. *If $S$ is convex with respect to $T$, and $S \cap T$ is compact then $H_T(S) = H_T(E_T(S))$.*

*Proof.*

$$\begin{aligned} H_T(S) &= H(S \cap T) \cap T \\ &= H(E(H(S \cap T)) \cap (S \cap T)) \cap T \qquad \text{by the previous lemma} \\ &= H(E_T(S)) \cap T = H(E_T(S) \cap T) \cap T = H_T(E_T(S)). \end{aligned}$$

THEOREM I.21 (Helly's theorem for relatively convex sets). *If $S_1$, $S_2, \cdots, S_m$ are sets, relatively convex with respect to $T \subset E^n$, and if every $n+1$ of them intersect in a point of $T$, and $S_1 \cap T$ is convex, then $\bigcap_{i=1}^m (S_i \cap T)$ is not empty.*

*Proof.* Let $S_{i1}, S_{i2}, \cdots, S_{i,n+1}$ be $n+1$ of the sets $S_i$. Since $S_{ij}$ is convex wih respect to $T$, $S_{ij} \cap T = H(S_{ij} \cap T) \cap T$. By hypothesis there is a point

$$p \in (\textstyle\bigcap_{j=1}^{n+1} S_{ij}) \cap T = \bigcap_{j=1}^{n+1} (S_{ij} \cap T) \subset \bigcap_{j=1}^{n+1} H(S_{ij} \cap T).$$

By Helly's theorem, there is a point $x \in \bigcap_{i=1}^m H(S_i \cap T)$. Since $S_1 \cap T$ is convex, $x \in H(S_1 \cap T) = S_1 \cap T \subset T$. Hence

$$x \in (\textstyle\bigcap_{i=1}^m H(S_i \cap T)) \cap T = \bigcap_{i=1}^m (H(S_i \cap \mathrm{T}) \cap T) = \bigcap_{i=1}^m (S_i \cap T).$$

COROLLARY I.22. *If $S_1, S_2, \cdots, S_m$ are subsets of $T \subset E^n$, which are convex with respect to $T$ and if every $n+1$ of them have a non-empty intersection, and if $S_1$ is convex, then $\bigcap_{i=1}^m S_i$ is not empty.*

*Example* I.23. The condition that $S_1$ be convex cannot be removed in the above theorem. In Figure 5, $S_1, S_2, S_3$, and $S_4$ are all convex with respect to the annulus $T$, and intersect in threes, but have an empty intersection.

It is easy to generalize Caratheodory's theorem to relative covers and relatively convex cells.

PROPOSITION I.24. *If $p \in H_T(S)$, $S \cup T \subset E^n$, then there exist points $p_1, p_2, \cdots, p_s$ ; $s \leqq n+1$; $p_i \in S \cap T$ such that $p \in p_1 p_2 \cdots p_s \subset S \cup cT$.*

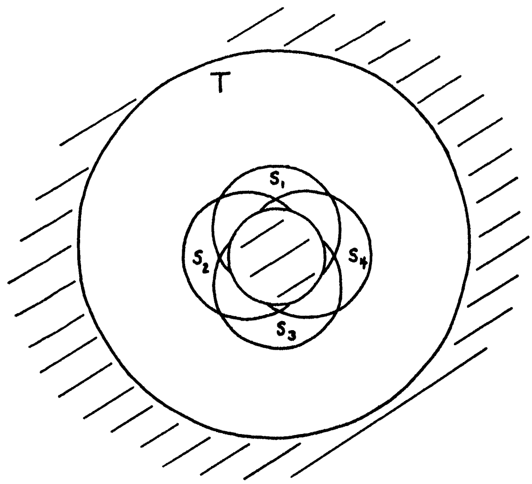



FIGURE 5

## II. Totally concave sets

DEFINITION II.1. A set $T$ in $E^n$ is said to be totally concave if and only if $S$ is convex with respect to $T$ for any subset $S$ of $T$ and $T$ is not convex.

There are many examples of totally concave sets; the vertices of a simplex, the unit sphere, the rational points of a circle, and the vertices of a cube are a few.

PROPOSITION II.2. *A totally concave set has no non-degenerate convex subsets.*

*Proof.* If $S$, convex, is contained in $T$, totally concave, and $S$ contains more than one point, it contains a segment whose endpoints form a subset of $T$ which is not convex with respect to $T$.

The converse of this proposition is clearly false. The unit sphere plus its center has no non-degenerate convex subsets and is not totally concave.

PROPOSITION II.3. *Any non-degenerate subset of a totally oncave set is totally concave.*

*Proof.* If $T$ is totally concave and $S \subset R \subset T$ and

$$p_1, p_2, \cdots, p_s \in S \cap R \subset S \cap T,$$

then

$$p_1 p_2 \cdots p_s \subset S \cup cT \subset S \cup cR$$

and $R$ is totally concave.

PROPOSITION II.4. *$T$ is totally concave if and only if it is convex with respect to any set of $k$ points, $k \leqq n + 1$, contained in it.*

*Proof.* Necessity is trivial. If $T$ is not totally concave it has a subset $S$ which is not convex with respect to $T$. Hence, in $S$, there are $k$ points $p_1, p_2, \cdots, p_k$ ; $k \leqq n + 1$; such that $p_1 p_2 \cdots p_k$ is not contained in $S \cup cT$. It now follows that the points $p_1, p_2, \cdots, p_k$ form a subset of $T$ which is not convex with respect to $T$.

PROPOSITION II.5. *The only totally concave sets on the line are sets consisting of exactly two points.*

*Proof.* If $T$ is totally concave and contains three points, the two farthest apart form a subset of $T$ which is not convex with respect to $T$.

PROPOSITION II.6. *The extreme points of a convex set form a totally concave set. (Provided, of course, there is more than one.)*

*Proof.* If $E(T)$ is convex and $p_1$ and $p_2 \in E(T)$, then the segment $p_1 p_2 \subset E(T)$, and is an interior point of the segment. Since $E(T) \subset T$, this cannot happen.

Let $S \subset E(T)$, $S = \{p_1, p_2, \cdots, p_k\}$; $k \leqq n + 1$. We must show that

$$p_1 p_2 \cdots p_k \subset S \cup cE(T).$$

$p_1 p_2 \cdots p_k \subset T$ by convexity. Hence if $p$ is a point of $p_1 p_2 \cdots p_k$ which is not in $S$, it is sufficient to prove that $p$ is not an extreme point of $T$. Since $p \notin S$, $p \neq p_i$ and hence is on the interior of some face of $p_1 p_2 \cdots p_k$. Thus $p$ is on the interior of an interval in $T$ and not extreme in T.

PROPOSITION II.7. *A totally concave set in the plane contains no triod.*

*Proof.* It is sufficient to show that a triod cannot be totally concave in the plane. We may assume, without loss of generality, that the triod in question lies in a circular disk, has its center at the center of the disk, and has its three endpoints on the circumference of the disk. Let $v$ be the center of the triod. Let $R$ be the triangle formed by $p_1$, $p_2$, and $p_3$, the endpoints of the triod $T$. If $v \in R$, the set $S = \{p_1, p_2, p_3\}$ is not convex with respect to $T$. If $v \notin R$, $p_1$, $p_2$, and $p_3$ lie on a semi-circle and may be numbered so that $p_2$ lies between $p_1$ and $p_3$. The arc $vp_2$ must intersect the segment $p_1 p_3$ in a point of $T$. Thus the set $S = \{p_1, p_3\}$ is not convex with respect to $T$.

Note that in $E^3$ we can find collections of arbitrarily many arcs having a common endpoint which lie on a sphere and hence form a totally concave set.

Since any compact convex set is the convex cover of its extreme points, it is of interest to ask whether there is any intrinsic property of the set of extreme-points which characterizes it. The following theorem, together with Proposition II.6, gives a complete answer to this question.

THEOREM II.8. *If $T$ is totally concave, $T = E(H(T))$.*

*Proof.* $E(H(T)) \subset T$ for any set $T$. If $T \subset E^n$ is totally concave and $p \in T - E(H(T))$, $p$ is in the relative interior of some cell $H(S)$ where $S \subset T$

and $S$ has at most $n + 1$ points. This means that $S$ is not convex with respect to $T$. This contradiction establishes the theorem.

Corollary II.9. *Any totally concave set lies on the boundary of a convex set.*

Corollary II.10. *If $T$ is totally concave in $E^n$*, $\dim T \leqq n - 1$.

*Proof.* The boundary of a convex set in $E^n$ is topologically a $k$-sphere $S^k$, a $k$-hyperplane $E^k$, or a product $S^r \times E^{k-r}$, $1 \leqq k \leqq n - 1$.

Note that for each $k$, $0 \leqq k \leqq n - 1$, the unit $k$-sphere is a totally concave set of dimension $k$.

Corollary II.11. *If $T$ is totally concave, it is convex with respect to $H(T)$.*

We can now generalize Proposition II.7.

Proposition II.12. *If $T$ is a totally concave set in $E^n$ and if $A$ is an $n - 2$ disk on $T$, then $A$ is on at most two $n - 1$ disks in $T$.*

*Proof.* There is nothing to prove if $\dim T < n - 1$. If $\dim T = n - 1$, we need only note that the desired property holds on the boundary of any convex set of dimension $n - 1$ in $E^n$.

## III. Totally convex sets

Definition III.1. A convex set $S$ is called totally convex if and only if $\mathrm{Bd}\,(S)$ is totally concave.

Proposition III.2. *A non-degenerate convex set $S$ in $E^n$ is totally convex if and only if* $\mathrm{Bd}\,(S) = E(\bar{S})$.

*Proof.* If $\mathrm{Bd}\,(S)$ is totally concave, $\mathrm{Bd}\,(S) = E(H(\mathrm{Bd}\,(S))) = E(\bar{S})$. If $\mathrm{Bd}\,(S) = E(\bar{S})$, $\mathrm{Bd}\,(S)$ is totally concave by Proposition II.6.

It is not difficult to see that all sets of constant width are totally convex. Recall that a compact convex set has constant width if and only if the distance between parallel support planes is constant. Since $S = \bar{S}$ and $E(S) \subset \mathrm{Bd}\,(S)$, it is only necessary to see that $E(S) \supset \mathrm{Bd}\,(S)$. If $p \in \mathrm{Bd}\,(S) - E(S)$, there is a segment $p_1 p_2$ in $S$ with $p \in \mathrm{Int}\,(p_1 p_2)$. Let $\pi_p$ be a support plane at $p$. Since $S$ lies on one side of $\pi_p$, $p_1 p_2 \subset \pi_p \cap S \subset \mathrm{Bd}\,(S)$. Parallel to $\pi_p$ there is another support plane $\pi_q$ with $\pi_q \cap \mathrm{Bd}\,(S) = q$ and $\rho(\pi_p, \pi_q) = \alpha$ (the width of the set $S$). Hence $\rho(p, q) \geqq \alpha$. If $\rho(p, q) > \alpha$, the diameter of $S$, $\delta(S) > \alpha$, which is impossible. If $\rho(p, q) = \alpha$, either $\rho(p_1, q) > \alpha$ or $\rho(p_2, q) > \alpha$ and again $\delta(S) > \alpha$.

Proposition III.3. *If $T_1$ and $T_2$ are totally convex and $T_1 \cap T_2$ is non-degenerate, then $T_1 \cap T_2$ is totally convex.*

*Proof.* Let $p \in \mathrm{Bd}\,(T_1 \cap T_2) \subset \mathrm{Bd}\,(T_1) \cup \mathrm{Bd}\,(T_2)$. If $p \in \mathrm{Bd}\,(T_1)$, $p$ is an extreme point of $\bar{T}_1$. Since $p \in \overline{T_1 \cap T_2}$, $p \in E(\overline{T_1 \cap T_2})$.

If we let $T_1$ be a circle and inscribe in it an equilateral triangle, we may construct a new totally convex set $T_2$ by replacing each arc of the circle, cut off by two vertices of the triangle, by a new arc, with the same end points, but closer to the side of the triangle in question. We will in particular pick $T_2$ so that the perpendicular distance from a point of the triangle to $T_2$ is exactly half the distance from the point to $T_1$. It is now easy to see how to construct a sequence of totally convex sets whose intersection is not totally convex.

DEFINITION III.4. If $S$ is convex and $p \in \mathrm{Bd}(S)$, there is an $n-1$ support plane to $S$ at $p$. If there is only one such plane, $p$ is called a regular point of $S$, and $\mathrm{Bd}(S)$ is said to be differentiable at $p$. Otherwise, $p$ is called a corner of $S$. The set of corners of $S$ is denoted by $C(S)$.

Before considering the relation of the set $C(S)$ to the set $S$ itself in the case where $S$ is totally convex, it is worthwhile to note a few things about corners in general. An interior point of a 1-face of a 3-simplex is a corner but not an extreme point; any point on the boundary of a circular disk in $E^2$ is an extreme point but not a corner. If $S \subset E^2$, $C(S) \subset E(S)$.

If $S$ is a convex set in $E^n$, $C(S)$ is the set of points where $\mathrm{Bd}(S)$ is not differentiable. It is known that the measure of such a set is always zero [1]. Since $C(S)$ is contained in $\mathrm{Bd}(S)$, which is an $n-1$ sphere or a cylinder, $\dim C(S) \leqq n-1$. If $\dim C(S) = n-1$, $C(S)$ contains an open set in $\mathrm{Bd}(S)$, [3], and $C(S)$ has positive measure. Hence $\dim C(S) \leqq n-2$.

If $S$ is a compact convex set in $E^2$, we may construct two horizontal and two vertical support lines to $S$. These lines intersect $\mathrm{Bd}(S)$ in single points or intervals; we will suppose the intersections are points for simplicity, although this is not necessary. The four points so defined divide $\mathrm{Bd}(S)$ into four arcs. Suppose one of these arcs, $p_1 p_2$, has uncountably many corners on it; and suppose $p_1$ lies on the lower support line and $p_2$ on the right hand support line. Let the intersection of these lines be $q$. If $x$ is a corner of $S$ lying on $p_1 p_2$, the two limiting support lines to $S$ at $x$ form a positive angle which intersects the segment $p_2 q$ in an interval of positive length. Moreover, for different corners $x_1$ and $x_2$, these intervals are disjoint. This is impossible, and the set of corners of $S$ is therefore at most countable.

*Example* III.5. Let $S$ be the unit disk in $E^2$ and consider the sequence of points $\{x_n\}$, where $x_n$ has polar coordinates $(1, \pi/2^n)$. $x_n \to (1, 0)$. For each $n$, we replace the arc $x_n x_{n+1}$ by a straight line interval. The resulting set $T$ has a corner at each point $x_n$, but no corner at $(1, 0)$. If we do the same thing on a semi-circle we get a corner at $(1, 0)$. It is not hard to see that by replacing the arcs by other arcs, we can get the same sort of examples with the sets involved being totally convex. Hence $C(T)$ may be infinite, and it may or may not be closed.

THEOREM III.6. *If $S$ is a compact totally convex set in $E^n$, there is a convex set $K(S) \subset S$ with $C(S) = E(K(S))$.*

*Proof.* Since $S$ is closed $C(S) \subset S$. We define $K(S) = H(C(S))$. That $K(S) \subset S$ is clear. If $p \in K(S) - E(K(S))$ then $p \notin E(S)$ and $E(S) \cap K(S) \subset E(K(S))$.

If $p \in K(S) - E(S)$ there are, in $C(S)$, points $p_1, p_2, \cdots, p_k$; $k \leqq n + 1$; with $p \in p_1 p_2 \cdots p_k$. By the total convexity $C(S) \subset E(S)$ and $p \neq p_i$ for any $i$. Hence $p$ is on the interior of a face of $p_1 p_2 \cdots p_k \subset K(S)$ and $p \notin E(K(S))$. Hence $E(K(S)) \subset E(S) \cap K(S)$ and

$$(1) \qquad E(K(S)) = E(S) \cap K(S).$$

Now $C(S) \subset K(S) \cap E(S)$, and if $p \in [K(S) \cap E(S)] - C(S)$, there are, in $C(S)$, points $p_1, p_2, \cdots, p_k$; $k \leqq n + 1$; with $p \in p_1 p_2 \cdots p_k$. As before $p \neq p_i$ and $p$ is not extreme in $K(S)$ and therefore not extreme in $S$. Hence $K(S) \cap E(S) \subset C(S)$ and

$$(2) \qquad E(S) \cap K(S) = C(S).$$

Equations (1) and (2) give the desired result.

We now concern ourselves with the converse problem. If $T$ is a given closed convex set, can we find a totally convex set $S$, containing $T$, with $C(S) = E(T)$. If $T$ is compact, we will have

$$K(S) = H(C(S)) = H(E(T)) = T.$$

*Example* III.7. Let $T$ be the set in the plane consisting of a square with semicircular disks on two opposite sides. In this case it is not possible to find a set $S$ as described above. $T$ has no corners but a corner $p$ of the rectangle is an extreme point of $T$. Hence, the desired set $S$ must have $p$ as a corner. However, each support plane to $S$ at $p$ must contain the unique support line to $T$ at $p$. This support line contains an interval on Bd $(T)$, which must then be an interval on Bd $(S)$, contradicting the total convexity of $S$.

THEOREM III.8. *If $T$ is a compact convex set in $E^2$ with* Bd $(E(T)) \subset C(T)$, *then there is a totally convex set $S$ in $E^2$, $S \supset T$, with*

$$C(S) = C(T) \quad \textit{and} \quad \mathrm{Bd}\,(S) \cap \mathrm{Bd}\,(T) = E(T).$$

*Proof.* If $T$ consists of a single point a teardrop shape with the point as vertex will suffice. Now assume $T$ is non-degenerate and let $p \in \mathrm{Bd}\,(T) - E(T)$ so that $p$ is on the interior of an interval $x_1 x_2$ on Bd $(T)$, with $x_1$ and $x_2 \in \mathrm{Bd}\,(E(T)) \subset C(T)$; $x_1 x_2$ is on a support line to $T$.

Since $x_1$ and $x_2$ are corners we may construct a circle $C_1$ through $x_1$ and $x_2$ and containing $T$. We let $T_1$ be the circular segment bounded by the interval $x_1 x_2$ and that arc of $C_1$ which is on the opposite side of $x_1 x_2$ from $T$. Let $S_1 = T \cup T_1$. $S_1$ is convex and since $x_1$ and $x_2$ are corners the construction may be carried out in such a way as to leave $x_1$ and $x_2$ corners of $S_1$. The interior of $x_1 x_2$, which consisted entirely of non-extreme points, has been replaced by an arc consisting of extreme points which are not corners.

We now let $p'$ be a point of $\mathrm{Bd}\,(S_1) - E(S_1)$. Since no intervals have been added in the construction of $S_1$, $p'$ is on the interior of an interval which must be on $\mathrm{Bd}\,(T)$. The endpoints of this interval are corners of $T$ and therefore corners of $S_1$. We repeat the above construction, defining $T_2$ as a segment of a circle and $S_2 = S_1 \cup T_2$. In this manner we get a sequence $\{S_i\}$, (possibly infinite), of sets with $C(S_i) = C(T)$; $S_i = S_{i-1} \cup T_i$. $S = \bigcup_i S_i$.

If $p_1$ and $p_2 \in S$, let $k$ be the first integer for which $p_1$ and $p_2$ are both in $S_k$; $p_1 p_2 \subset S_k \subset S$. It is now easy to see that $S$ is totally convex. Each corner of $T$ is the endpoint of at most two intervals of non-extreme points. The support lines, after adjustments corresponding to these intervals, remain support lines at each succeeding stage and are support lines to $S$. Hence, $C(T) \subset C(S)$. Since $S \supset T$, each support line to $S$, which intersects $T$, is a support line to $T$. No corners are added in constructing $T$, each corner of $S$ is in $T$, so that $C(S) \subset C(T)$. Hence $C(S) = C(T)$. That $\mathrm{Bd}\,(S) \cap \mathrm{Bd}\,(T) = E(T)$ is clear from the construction.

If we insist only that $T$ be bounded, we can perform the same construction on $\bar{T}$, replacing intervals by open pieces $T_i$. We will not, of course, get the last condition in this case.

COROLLARY III.9. *If $T$ is compact and convex in $E^2$, and $E(T) \subset C(T)$, there is a totally convex set $S$ in $E^2$, $S \supset T$, with $C(S) = E(T)$. (In fact,* $\mathrm{Bd}\,(S) \cap \mathrm{Bd}\,(T) = E(T) = C(S) = C(T)$.*)*

*Proof.* Since $E(T)$ is closed in $E^2$, we can pick $S$ by the theorem with $C(S) = C(T)$. Since $C(T) \subset \mathrm{E}(T)$ in $E^2$, $C(S) = C(T) = E(T)$.

It is clear that the set $S$ in Theorem III.8 can be chosen so that $\rho(\mathrm{Bd}\,(S), T)$ is arbitrarily small simply by taking the circles $C_i$ large enough.

THEOREM III.10. *If $T$ is compact and convex in $E^2$, in order that there shall exist a totally convex set $S \supset T$, $S \subset E^2$, and $K(S) = T$, it is necessary and sufficient that $E(T) = C(\mathrm{T})$.*

*Proof.* Corollary III.9 proves sufficiency. If the set $S$ exists, $E(T) = C(S)$. If $p \in C(S) \subset T$, there are distinct support lines to $S$ at $p$ and, since $S \supset T$ these are support lines to $T$ as well. Hence, $p \in C(T)$ and $C(S) \subset C(T)$. However, in $E^2$, $C(T) \subset E(T)$. Hence $C(T) \subset E(T) = C(S) \subset C(T)$ and $E(T) = C(\mathrm{T})$.

It may be possible, given the set $T$ in $E^2$, to find the set $S$ in some higher dimensional Euclidean space, without the condition of the above theorem. If $T$ is a circular disk, we can take a lozenge shaped set in $E^3$ to be the set $S$; $K(S) = T$ although $T$ has no corners at all.

One is tempted to parallel Theorem III.10 with a necessary and sufficient condition for the existence of a set $S \subset E^2$, with $C(S) = C(T)$. If this can be done the condition is not that of Theorem III.8. In Example III.5 the point $(1, 0) \in \mathrm{Bd}\,(E(T)) - C(T)$. It is clear, however, that the construction of Theorem III.8 works in this case also.

We have used the facts that $C(S) \subset E(S)$ and that $E(S)$ is closed in Bd $(S)$ in an essential way in these theorems. Being peculiar to $E^2$, these properties make generalizations to higher dimensions appear quite difficult at this time.

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