EIGENVALUES AND BOUNDARY SPECTRA

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1. Introduction

Let T be a bounded operator on a Hilbert space of elements x with $\| T \| = \sup \| T \|$, where $\| x \| = 1$, and let sp (T) denote the spectrum of T. It is well known that if $\lambda \notin \text{sp}\,(T)$ then

$$\|(T - \lambda I)^{-1}\| \geqq 1/d_T(\lambda),$$

where $d_T(\lambda)$ denotes the distance from λ to the (compact) set sp (T). This paper will deal with bounded operators for which the equality sign holds, so that

$$\|(T - \lambda I)^{-1}\| = 1/d_T(\lambda), \qquad \lambda \notin \text{sp}\,(T). \tag{1}$$

The condition (1) is known to be valid for normal operators ($TT^* = T^*T$) as well as for semi-normal ones ($TT^* - T^*T$ semi-definite); cf. Stampfli [5]. Also, as is noted there, if T satisfies (1) and if sp (T) is real then T is self-adjoint (Nieminen [3]), and if T satisfies (1) and sp (T) lies on the unit circle then T is unitary (Donoghue [1]). Further, Stampfli has shown that if T satisfies (1) and if λ is an isolated point of sp (T) then necessarily λ is in the point spectrum of T, so that $Tx = \lambda x$ holds for some $x \neq 0$. In addition, $T^*x = \bar{\lambda}x$, so that the eigenvectors of T belonging to λ determine a reducing space on which T is normal. Thus, if T satisfies (1) on a finite-dimensional Hilbert space, then T must be normal.

Concerning other implications of (1) and of related growth restrictions on the resolvent operator see, in addition to the references cited above, also Meng [2], Orland [4].

It will be convenient to have the following

DEFINITION. Let λ belong to the point spectrum of an operator T and suppose that the eigenvectors of T belonging to λ determine a reducing space of T on which T is normal. Then λ will be called a normal eigenvalue of T.

Thus, by Stampfli's result, isolated points of the spectrum of an operator satisfying (1) must be normal eigenvalues. It will be shown below that an analogous assertion holds for certain other eigenvalues lying in the boundary of the spectrum of an operator T satisfying (1).

2. The main theorem

THEOREM. *Let T satisfy* (1) *and suppose that λ_0 is in the point spectrum of T and also in the boundary of* sp (T). *Suppose that there exist $\lambda_n \notin$* sp (T) *for*

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which $\lambda_n \rightarrow \lambda_0$ *and*

$$(2) \qquad |\lambda_n - \lambda_0|/d_T(\lambda_n) \rightarrow 1 \quad as \quad n \rightarrow \infty.$$

Then necessarily λ_0 *is a normal eigenvalue of* T.

The above result will be proved in section 3. Several consequences follow readily:

COROLLARY 1. *Let* T *satisfy* (1) *and let the entire boundary of* sp (T) *be a convex simple closed curve* G. *Then all eigenvalues of* T *on* C *must be normal eigenvalues.*

For, in this case, one can choose λ_n outside C and tending to λ_0 along the normal to a support line of C, so that even $|\lambda_n - \lambda_0|/d_T(\lambda_n) \equiv 1$ in (2).

COROLLARY 2. *Let* T *satisfy* (1) *and suppose that the boundary of* sp (T) *contains a rectifiable simple closed curve or simple arc* C *lying at a positive distance from the rest of the boundary of* sp (T). *Then the set of eigenvalues of* T *on* C *which are not normal eigenvalues must be of one-dimensional Lebesgue measure zero, the measure being that determined by ordinary arc length on* C.

In order to see this, note that, since C is rectifiable, its arc length s can be used in a parametrization $x = x(s)$, $y = y(s)$ by absolutely continuous functions satisfying $x'^2(s) + y'^2(s) = 1$ almost everywhere. Hence, at almost every point, C has a tangent and C pierces (that is, in a neighborhood of such a point, lies on both sides of) the corresponding normal line. (Thus, for instance, the point cannot be a cusp.) It follows that the theorem is applicable to almost all eigenvalues λ_0 on C by allowing the approach to λ_0 along (at least one side of) the normal line at λ_0.

COROLLARY 3. *Let* T *satisfy* (1) *on a separable Hilbert space and let the boundary of* sp (T) *consist of a finite number of mutually disjoint simple closed curves, simple arcs, and isolated points. Then the set of all eigenvalues of* T *lying in the boundary of* sp (T) *has (one-dimensional) measure zero.*

For, since the space is separable, the set of (all) normal eigenvalues is countable and hence the set of those in the boundary of sp (T) has one-dimensional measure zero. Since, by Corollary 2, the set of non-normal eigenvalues in the boundary of sp (T) also has measure zero, the result follows.

As already noted, the proof of the theorem will be given in Section 3. In Section 4, there will be given an example showing that the boundary of sp (T) in Corollary 2 may consist of a single rectifiable simple closed curve C, while the set mentioned there, of non-normal eigenvalues on C, is non-empty.

3. Proof of theorem

It will be useful to establish first the following

LEMMA. *Let* T *be arbitrary and let* λ_0 *belong to the point spectrum of* T *and* *also to the boundary of* sp (T). *In addition, suppose that there exist* $\lambda_n \notin$ sp (T) *for which* $\lambda_n \to \lambda_0$ *and*

$$(3) \qquad |\lambda_n - \lambda_0| \, \|(T - \lambda_n I)^{-1}\| \to 1 \quad \textit{as} \quad n \to \infty.$$

Then λ_0 *is a normal eigenvalue of* T.

In order to prove this result, let x be a unit vector satisfying $Tx = \lambda_0 x$ and, for $\lambda \notin$ sp (T), let $R_\lambda = (T - \lambda I)^{-1}$. Then

$$R_\lambda x = (\lambda_0 - \lambda)^{-1} x$$

and hence

$$0 \leqq \|((\bar{\lambda} - \bar{\lambda}_0)R_\lambda^* + I)x\|^2 = |\lambda - \lambda_0|^2 \, \|R_\lambda^* x\|^2 - 1.$$

Since $\|R_\lambda\| = \|R_\lambda^*\|$, then, on letting $\lambda = \lambda_n$ and using (3), one obtains

$$(\bar{\lambda}_n - \bar{\lambda}_0)R_{\lambda_n}^* x + x \to 0 \quad \text{as} \quad n \to \infty.$$

On applying the (uniformly bounded sequence of) operators $(T - \lambda_n I)^*$, one obtains $(T^* - \bar{\lambda}_0)x = 0$, as was to be shown.

In order to prove the theorem, let λ_0, λ_n be defined as in the hypothesis. Since relations (1) and (2) imply (3), the assertion of the theorem follows from the lemma.

4. An example

In this section there will be constructed an operator T satisfying (1) and such that sp (T) has as its boundary a simple closed curve C, smooth at all points except one (the origin), at which point the curve has a corner. Further, this point will be an eigenvalue of T but not a normal eigenvalue.

The operator T will be defined as the direct sum of the operator

$$(4) \qquad A = \begin{pmatrix} 1 & t \\ 0 & 0 \end{pmatrix},$$

for some $t \neq 0$, on a two-dimensional Hilbert space, and a suitably chosen normal operator N on an infinite-dimensional Hilbert space. (The example in this regard is similar to one given by Stampfli [5] to show that condition (1) and complete continuity of T are, on an infinite-dimensional space, not sufficient to imply the normality of T.)

It is seen that for all $t \neq 0$, A is not normal and that sp (A) consists of the two numbers 0 and 1. The resolvent $(A - \lambda I)^{-1}$ is given by

$$(5) \qquad (A - \lambda I)^{-1} = \begin{pmatrix} 1/(1-\lambda) & t/\lambda(1-\lambda) \\ 0 & -1/\lambda \end{pmatrix}, \qquad \lambda \neq 0 \text{ and } 1,$$

and it is easily verified that

$$(6) \qquad \|(A - \lambda I)^{-1}\| \leqq \max\{1/|\lambda|, 1/|1-\lambda|\} + |t|/|\lambda(1-\lambda)|.$$

Let C be a simple closed curve to be constructed so that, in some neighborhood of the origin, C is given by $y = \pm x \sin \phi$, $x \geqq 0$, where ϕ is any fixed angle satisfying $0 < \phi < \pi/2$. If $\lambda > 0$ and if λ is sufficiently small, then the distance $D(\lambda)$ from λ to C is given by $D(\lambda) = \lambda \sin \phi$. It is seen from (6) that (for $0 < \lambda < \frac{1}{2}$)

$$\| (A - \lambda I)^{-1} \| \leqq 1/\lambda + |t|/\lambda(1 - \lambda),$$

and so

$$D(\lambda)\| (A - \lambda I)^{-1} \| \leqq \sin \phi(1 + |t|/(1 - \lambda)).$$

Hence, one can choose $t = t_\phi > 0$ so small that $D(\lambda)\| (A - \lambda I)^{-1} \| \leqq 1$, for λ positive and sufficiently small.

Next, define a normal operator N so that its spectrum is a region which, in a neighborhood of the origin, lies on the left side of, and has as its boundary, the curve C as already defined and, further, is such that 0 is not in the point spectrum of N. Then, if

$$T = A \oplus N, \tag{7}$$

it is clear that for all *complex* λ satisfying $\lambda \notin \mathrm{sp}\,(T)$ and $|\lambda|$ sufficiently small one has $d_T(\lambda)\| (T - \lambda I)^{-1} \| = 1$. (Note that λ is in the wedge formed by C at 0, that the distance from λ to C is not greater than the distance from $|\lambda|$ to C, and that $d_N(\lambda)\| (N - \lambda I)^{-1} \| = 1$.)

The operator T of (7) satisfies (1) if λ lies in some neighborhood of 0 and, in addition, 0 is an eigenvalue of T but is not a normal eigenvalue. The completion of the construction of C and T with the properties described at the beginning of this section can be carried out easily. In particular, C should be further defined so as to become a simple closed curve smooth (except at 0) and so as to include the point 1 in its interior, while N should be defined so that its spectrum is precisely C together with its interior and so that 0 is not an eigenvalue of N. Since $\mathrm{sp}\,(N) = \mathrm{sp}\,(T)$ and $d_N(\lambda) = d_T(\lambda)$ in this case, and since

$$d_N(\lambda)\| (N - \lambda I)^{-1} \| = 1 \quad \text{for } \lambda \notin \mathrm{sp}\,(N),$$

it is only necessary to insure that

$$d_N(\lambda)\| (A - \lambda I)^{-1} \| \leqq 1 \quad \text{for } \lambda \notin \mathrm{sp}\,(N).$$

This can be readily managed in view of the estimate (6). The details will be omitted.

The operator T of (7) constructed in this way has of course a (non-trivial) reducing subspace on which T is normal. It could be arranged however that T has no such reducing space by choosing, instead of N, any operator M with the same spectrum and for which 0 is not in the point spectrum of M, and such that M satisfies (1), that is,

$$d_M(\lambda)\| (M - \lambda I)^{-1} \| = 1 \quad \text{for } \lambda \notin \mathrm{sp}\,(M),$$

and is such that M has no non-trivial reducing space on which M is normal.

In order to show that this can be done, let V denote the non-unitary, isometric operator on the sequential Hilbert space of vectors $x = (x_1, x_2, \cdots)$, $\sum |x_j|^2 < \infty$, defined by the matrix $V = (a_{ij})$, where $i, j = 1, 2, \cdots$, with $a_{ij} = 1$ if $i = j + 1$ and $a_{ij} = 0$ otherwise. Then $V^*V - VV^* \geqq 0\ (\neq 0)$, thus V is semi-normal and hence V satisfies (1) with $T = V$. It is known that V has no non-trivial normal reducing space, that its point spectrum is empty, and that sp (V) is the closed unit disk $|\lambda| \leqq 1$. Next, choose constants a_n, b_n $(n = 1, 2, \cdots)$ so that the closure of the union of the disks sp $(a_n V + b_n)$ is the set sp (N). If M is defined as the direct sum

$$M = \sum_{n=1}^{\infty} \oplus (a_n V + b_n),$$

then M is bounded and semi-normal, and sp (M) = sp (N). It is clear that $T = A \oplus M$ has the desired properties.

Remarks. The above constructed operators T satisfy (1) but are not semi-normal, due to the presence of the term A in the direct sum. It seems to be an open question whether a semi-normal operator can even have an eigenvalue, other than a normal one, in the boundary of sp (T), at least if the boundary is a simple closed curve. Such eigenvalues can exist in the interior of the spectrum of a semi-normal operator however. For instance, all points of $|\lambda| < 1$ are eigenvalues of the operator V^* considered above, while the boundary of sp (V^*) is the circle $|\lambda| = 1$.

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