RISES OF NONNEGATIVE SEMIMARTINGALES[1]

BY
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A real-valued stochastic process $f_0, f_1, \cdots$ has a *rise* of size y if $\exists i, j$ with $i < j$ such that $f_j - f_i \geq y$. This note obtains sharp upper bounds to the probability of a rise of size y for certain natural classes of stochastic processes.

Let Θ be a class of probability measures on the real line. If, for every n, given any partial history $f_0, \cdots, f_n$, the conditional distribution θ of the increment $f_{n+1} - f_n$ is in Θ, then $\{f_j\}$ is a *Θ-process*. If, in addition, $f_0 \equiv x$, then $\{f_j\}$ is an *(x, Θ)-process*. One can think of an (x, Θ)-process as the successive fortunes of a gambler whose initial fortune is x, and who chooses his successive lotteries from Θ.

Let $\rho(x, y) = \rho(x, y, \Theta)$ be the least upper bound over all nonnegative (x, Θ)-processes (including not necessarily countably additive processes) to the probability that the process experiences a rise of size y. The determination of ρ can sometimes be reduced to solving a simpler problem, namely that of determining U, where $U(x, y) = U(x, y, \Theta)$ is the least upper bound over all nonnegative (x, Θ)-processes $\{f_j\}$ to the probability that there is j with $f_j \geq y$.

As will soon be evident, there are interesting Θ for which

$$(1) \qquad U(x - m, y - m) = \frac{U(x, y) - U(m, y)}{1 - U(m, y)},$$

whenever $0 < m < x$, and $m < y$.

Incidentally, for every Θ, the left side of (1) is majorized by the right side. This inequality is quite simple to establish and is analogous to Theorem 4.2.1, p. 64 in [2].

I do not investigate the regularity conditions that U perhaps automatically satisfies once it satisfies (1), but, at least in interesting examples,

(2) $U(x, y)$ is convex in x for $0 \leq x \leq y$,

and

(3) $U(x, y)$ is continuously differentiable in x and y for $0 \leq x \leq y$.

Let

$$(4) \qquad \lambda = \lambda(y) = \frac{\partial U}{\partial x}(0, y).$$

Theorem 1. *If U satisfies* (1), (2) *and* (3), *then*

$$\rho(x, y) = 1 - e^{-\lambda x}.$$

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For an interesting example of a Θ whose ρ can be calculated with the help of Theorem 1, let μ and σ^2 be the mean and variance of θ, let $\alpha > 0$, and let Θ_α consist of all θ such that $\mu \leq -\alpha\sigma^2$. Then, as is shown in (Theorem 9.4.1, p. 182 in [2]),

$$U_\alpha(x, y) = \frac{x}{y} \cdot \frac{1}{1 + \alpha(y - x)}. \tag{5}$$

In view of Theorem 1,

$$\rho_\alpha(x, y) = 1 - e^{-x/y(1+\alpha y)}. \tag{6}$$

The instance of (6) in which $\alpha = 0$ was established in [1]. Interest in evaluating the left side of (6) for general α led me to Theorem 1.

As a second example, for each $\beta > 0$, let Θ_β be the set of all θ such that $\int e^{\beta z} d\theta(z) \leq 1$. As in (8.7.8) p. 166 in [2], the U associated with Θ_β—there will be no confusion if it is here designated by U_β—satisfies

$$U_\beta(x, y) = \frac{e^{-\beta(y-x)} - e^{-\beta y}}{1 - e^{-\beta y}}. \tag{7}$$

Again, the hypotheses of Theorem 1 apply to U_β, as is verified by an easy calculation, so

$$\rho_\beta(x, y) = 1 - e^{-\lambda x}, \tag{8}$$

where $\lambda = \beta/(e^{\beta y} - 1)$.

For a third example, see Chap. 9, Sec. 3 in [2].

Incidentally, if Θ is a Borel set of probability measures which are countably additive on the Borel subsets of the line, then U and ρ would not change if the suprema were taken over countably additive processes only, as follows from [3].

Since finitely additive stochastic processes or, more precisely, strategies, as defined in [2], are not familiar objects, the essential ideas of the proof of Theorem 1 will be given in a countably additive setting.

Proof of Theorem 1. The proof that $\rho(x, y) \leq 1 - e^{-\lambda x}$ is basically an application of [2, Theorem 2.12.1] and will use two lemmas.

LEMMA 1. *Let $u_0, u_1, \cdots$ and $\alpha_0, \alpha_1, \cdots$ be two real-valued stochastic processes and $\mathfrak{F}_0, \mathfrak{F}_1, \cdots$ be an increasing sequence of sigma fields which satisfy*

(i) $u_n = 0$ or 1;

(ii) $0 \leq \alpha_n \leq 1$;

(iii) *if* $u_n = 0$ *and* $u_{n+1} = 1$, *then* $\alpha_{n+1} = 1$;

(iv) u_n *and* α_n *are* $\mathfrak{F}_n$-*measurable.*

Then, if $\alpha_0, \alpha_1, \cdots$ is an expectation-decreasing semimartingale relative to $\mathfrak{F}_0, \mathfrak{F}_1, \cdots$, so is $u_n + (1 - u_n)\alpha_n$.

Proof of Lemma 1. Let $Q_n = u_n + (1 - u_n)\alpha_n$. If $Q_n = 1$, then $Q_n \geq E[Q_{n+1} \mid \mathfrak{F}_n]$, since Q_{n+1} is everywhere majorized by 1.

If $Q_n < 1$, then $u_n = 0$. Verify that whenever $u_n = 0$, $Q_n = \alpha_n$ and $Q_{n+1} = \alpha_{n+1}$. Consequently, on the event $\{u_n = 0\}$,

$$Q_n = \alpha_n \geq E[\alpha_{n+1} \mid \mathfrak{F}_n] = E[Q_{n+1} \mid \mathfrak{F}_n]. \tag{9}$$

The final equality holds because the event $\{u_n = 0\}$ is in $\mathfrak{F}_n$.

As a preliminary to the next lemma, a definition is needed.

Let α be a (measurable) real-valued function defined on the cartesian product of two sets M and F (endowed with σ-fields), let $f_0, f_1, \cdots$ be a stochastic process with values in F, and let γ_n be the conditional distribution of f_{n+1} given $f_0, \cdots, f_n$, which is here assumed to exist. If, for all n,

$$\alpha(m, f_n) \geq \int \alpha(m, z)\, d\gamma_n(z), \tag{10}$$

except possibly for an event of probability zero which does not vary with m, then $\alpha(m, f_0), \alpha(m, f_1), \cdots$ is an expectation-decreasing, *semimartingale family*.

LEMMA 2. *Suppose that $\alpha(m, f_0), \alpha(m, f_1), \cdots$ is an expectation-decreasing semimartingale family and that m_n is measurable with respect to $f_0, \cdots, f_n$. Then*

$$\alpha(m_n, f_n) \geq E[\alpha(m_n, f_{n+1}) \mid f_0, \cdots, f_n]. \tag{11}$$

If, in addition, $\alpha(m_n, f_{n+1})$ majorizes $\alpha(m_{n+1}, f_{n+1})$ almost certainly, then $\alpha(m_0, f_0), \alpha(m_1, f_1), \cdots$ is an expectation-decreasing semimartingale.

Proof of Lemma 2. Outside the null event where (10) fails to hold, $\alpha(m_n, f_n)$ majorizes $\int \alpha(m_n, z)\, d\gamma_n(z)$. Since m_n is measurable with respect to $f_0, \cdots, f_n$, the latter is easily seen to be a version of the right side of (11).

(It is important in Lemma 2 that $\alpha(m, f_n)$ be a semimartingale family; if this assumption is replaced by the weaker one that for each m, $\alpha(m, f_n)$ is an expectation-decreasing semimartingale then (11) can fail to hold.)

Proof that $\rho(x, y) \leq 1 - e^{-\lambda x}$. Let

$$q(f) = q(f, y) = 1 - e^{-\lambda f}. \tag{12}$$

As in (12), the functional dependence on y will often not be indicated.

Let $U(m, x, y)$ be the right-hand side of (1), which is meaningful even for $0 < x < m$, and define

$$\alpha(m, f) = q(m) + (1 - q(m))U(m, f, m + y).$$

Let $f_0, f_1, \cdots$ be a nonnegative (x, Θ)-process and let m_n be the minimum of $(f_0, \cdots, f_n)$.

The immediate goal is to indicate that the hypotheses, and hence the conclusion, of Lemma 2 are satisfied. That $\alpha(m, f_0), \alpha(m, f_1), \cdots$ is an expectation-decreasing semimartingale family can be verified directly, or with the help of (Theorem 2.14.1, p. 32 in [2]). To check that

$$\alpha(m_n, f_{n+1}) \geq \alpha(m_{n+1}, f_{n+1})$$

almost surely, it certainly suffices that $\alpha(m, f) \geq \alpha(m \wedge f, f)$ where $m \wedge f$ is the minimum of m and f. So suppose $m \wedge f = f < m$. To be verified is that $\alpha(m, f) \geq \alpha(f, f)$, or

$$q(m) + (1 - q(m))U(m, f, m + y) \geq q(f) \tag{13}$$

for $0 \leq f \leq m$. In this region, the left side of (13) is convex in f, the right side concave in f, and both sides equal at $f = m$. So for (13), to hold, it suffices that

$$\frac{\partial U}{\partial f}(m, m, m + y) \leq \frac{\dot{q}(m)}{1 - q(m)} = \lambda. \tag{14}$$

In evaluating the left side of (14), it is most convenient to consider the derivative on the right at $f = m$, and hence to shift attention to the interval $m \leq f$; for there, according to (1),

$$U(m, f, m + y) = U(f - m, y). \tag{15}$$

Hence the left side of (14) also is λ, according to (4). So the conclusion of Lemma 2 holds.

Now let $u_n = 1$ or 0 according as there is or there is not an i, j with $0 \leq i < j \leq n$ such that $f_j - f_i \geq y$, and let $Q_n = u_n + (1 - u_n)\alpha_n$. Then

$$\begin{aligned} Eu_n \leq EQ_n \leq EQ_0 = E\alpha_0 = \alpha(m_0, f_0) \\ = q(x) = 1 - e^{-\lambda x} \qquad \text{for all } n, \end{aligned} \tag{16}$$

where the second inequality is justified by Lemma 1. Plainly, $\lim E\mu_n$ is the probability, P, that the process $\{f_n\}$ experience a rise of size y. So in view of (16), $P \leq 1 - e^{-\lambda x}$. Except for the need to attend to processes $\{f_n\}$ that are not countably additive, the proof that $\rho(x, y) \leq 1 - e^{-\lambda x}$ would be complete. But the above proof does apply in the general finitely additive case, which is easily checked with the help of [2, Chap. 2].

That the bound in Theorem 1 cannot be improved does not require hypotheses (1) and (2); only (3) will be used. Consider a gambler who divides his fortune x into N equal parts. He constructs an (x, Θ)-process which gains y before losing x/N with probability $U(x/N, y + x/N) + o(1/N)$. By N repetitions, he constructs an (x, Θ)-process which fails to have a rise of size y with a probability of at most

$$[1 - U(x/N, y + x/N) + o(1/N)]^N = [(1 - \lambda x/N) + o(1/N)]^N. \tag{17}$$

The equality in (17) holds because

$$\frac{\partial U}{\partial x}(0, y) = \lambda, \qquad \frac{\partial U}{\partial y}(0, y) = 0$$

and U has a differential at $(0, y)$. Take the limit as $N \to \infty$ to see that there is a nonnegative (x, Θ)-process for which the probability of a rise of size y is arbitrarily close to $1 - e^{-\lambda x}$. This completes the proof of Theorem 1.

If the U associated with Θ does not satisfy (1), I do not see how to calculate ρ, nor even interesting lower bounds for ρ. On the other hand, nontrivail upper bounds, perhaps not very sharp, can typically be found with the help of any of the three examples above.

For instance, let w be a fixed positive number less than $\frac{1}{2}$, let θ gain 1 with probability w and lose 1 with probability $1 - w$, and let Θ consist of all positive multiples of θ. The U associated with this Θ is essentially the U of the red-and-black casino in [2], and certainly does not satisfy (1). But setting α equal to $(1 - 2w)/y$ or even $(1 - 2w)/4w(1 - w)y$, the right-hand side of (6) majorizes $\rho(x, y)$.

References

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