THE REFLECTED DIRICHLET SPACE[1]

BY
MARTIN L. SILVERSTEIN

Introduction

We consider a regular Dirichlet space $(\mathbf{F}, E)$ with corresponding symmetric resolvent $\{G_u, u > 0\}$ and construct a second Dirichlet space $(\mathbf{F}^r, E^r)$ which we call the "reflected Dirichlet space." This Dirichlet space is interesting for its own sake and also because it classifies symmetric resolvents $\{G_u^*, u > 0\}$ with the property that $G_u^* f - G_u f$ is nonnegative and u-harmonic for $u > 0$ and for square integrable $f \geq 0$.

In Section 1 we introduce what seems to be the appropriate notion of "irreducibility" and we distinguish the transient and recurrent cases. In the remainder of the paper we assume that the given Dirichlet space is transient and irreducible.

In Section 2 the completion $\mathbf{F}_{(e)}$ of $\mathbf{F}$ relative to the form E alone (that is, without a piece of the standard inner product) is shown to be a Hilbert space. It is also pointed out that the corresponding completion in the recurrent case cannot be a Hilbert space. This generalizes the well known result that the classical Dirichlet space is a Hilbert space only for dimension three or more. (See [1].)

Various results on exit distributions and time reversal are collected in Section 3. These play an important role in the remainder of the paper. In particular some of the results are used in Section 4 to construct an "approximate Markov process" as introduced in a slightly different context in [9] by G. A. Hunt.

Section 5 is a streamlined version of a special case of Section 5 in [15]. The Dirichlet space associated with the process time changed by the 1-balayage of the given speed measure m onto a closed set M is identified and an important estimate is established which involves N^M, the "universal Dirichlet norm on M".

Sections 6 and 7 extend the techniques introduced in Section 7 of [15]. "Probabilistic interpretations" are given for various Dirichlet norms and in particular for N^M. The results of these sections are useful but dispensable for the general theory. However they play a crucial role in the treatment of examples. We refer to Section 9 of [15] and to [19] and [20] for specific instances.

The reflected Dirichlet space $(\mathbf{F}^r, E^r)$ is introduced in Section 9. This decomposes into the extended Dirichlet space $\mathbf{F}_{(e)}$ of Section 2 and a global "universal Dirichlet space" $\mathbf{N}$. The latter is defined in Section 8 as a space of

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random variables on the appropriate sample space. It is clear that if additional hypotheses are introduced so that the "Martin boundary" can be introduced then $\mathbf{N}$ can be identified with a function space on the Martin boundary. We refer to [19] where this is verified in a special case.

Finally the classification is obtained in Section 10. This explicitly involves only the "active" part $\mathbf{N}_a$ of $\mathbf{N}$ which is closely related to the "active boundary" of W. Feller [4].

We do not consider here the problem of explicitly constructing the processes which are associated with the classified Dirichlet spaces. For results in special cases we refer to [5] and [13]. For other related work we refer to [2], [3], [4], [8], [10] and [14]. We also refer to [19] where a more or less self contained presentation is given for a special case.

As in [15] I acknowledge my debt to M. Fukushima. As a general rule we refer to [15] for background results. However many of these results were first established by Fukashima in [6], [7] and [8].

Added in proof. The author is now preparing a monograph in which the main results of this paper are established in a broader context using different techniques.

Notations

Throughout the paper $\mathbf{X}$ is a separable locally compact Hausdorff space, m is a Radon measure on $\mathbf{X}$ and $(\mathbf{F}, E)$ is a regular Dirichlet space on $L^2(\mathbf{X}, m)$.

The indicator of a set will be denoted both by 1_A and by $I(A)$. The integral of a function ξ over the set determined by a condition such as "X_t is in Γ" will be denoted both by $\mathcal{E}[X_t \text{ is in } \Gamma; \xi]$ and by $\mathcal{E}I(X_t \text{ is in } \Gamma)\xi$. All functions are real valued. In particular $L^2(m)$ or $L^2(\mathbf{X}, m)$ is the real Hilbert space of square integrable functions on the measure space $(\mathbf{X}, m)$ and $C_{\text{com}}(\mathbf{X})$ is the collection of real valued continuous functions on $\mathbf{X}$ with compact support. Questions of measurability are generally taken for granted. In particular functions are always understood to be measurable with respect to the obvious σ — algebra.

Notations and results in Sections 1 through 3 of [15] will be used throughout the paper. To help orient the reader we collect some of them here. However we will continue to refer to [15] in specific instances.

A *dead point* ∂ is adjoined to $\mathbf{X}$ with the usual conventions. The standard sample space Ω is the collection of maps ω from the time axis $[0, \infty)$ into the augmented phase space $\mathbf{X} \cup \{\partial\}$ which satisfy the following two conditions.

0.1.1. $\omega(\cdot)$ is right continuous and has one sided limits everywhere.

0.1.2. There exists a life time $\zeta(\omega)$ with $0 \leq \zeta(\omega) \leq +\infty$ such that $\omega(t) = \partial$ if and only if $t \geq \zeta$ and such that $X_{t-0} \neq \partial$ for $t < \zeta$.

Trajectory variables are defined by $X_t(\omega) = \omega(t)$ and first entrance times are defined by

$$\sigma(E) = \inf\{t \geq 0 : X_t \text{ is in } E\}$$

with the understanding that $\sigma(E) = +\infty$ when not otherwise defined. In addition we introduce here the last exit times

$$\sigma^*(E) = \sup\{t > 0 : X_{t-0} \text{ is in } E\}$$

with the understanding that $\sigma^*(E) = -\infty$ when not otherwise defined. There is an exceptional polar set N and a family of probabilities $\mathcal{P}_x$ indexed by x in $\mathbf{X} - N$ which form a quasi-left-continuous Markov process on $\mathbf{X} - N$. In particular $\mathcal{P}_x(\sigma(N) < +\infty) = 0$ for all x in $\mathbf{X} - N$. The transition operators P_t and the resolvent operators G_u, first regarded as operators on $L^2(\mathbf{X}, m)$ are refined so that

$$P_t f(x) = \mathcal{E}_x f(X_t), \qquad G_u f(x) = \mathcal{E}_x \int_0^\zeta dt e^{-ut} f(X_t)$$

for $f \geq 0$ on $\mathbf{X}$ and for x in $\mathbf{X} - N$. These operators act on measures which do not charge N according to

$$\int (\nu P_t)(dy) f(y) = \int \nu(dx) P_t f(x), \qquad \int (\nu G_u)(dy) f(y) = \int \nu(dx) G_u f(x).$$

If ν charges no polar set then νG_u is absolutely continuous and therefore

$$G_u \nu = (d/dm)(\nu G_u) \tag{0.1}$$

is well defined almost everywhere. Indeed $G_u \nu$ is well defined up to quasi-equivalence by the conventions in [15]. The phrase "almost everywhere" is understood to mean almost everywhere with respect to m. The prefix "quasi" means that the exceptional set is polar. Often identities will be stated as if valid everywhere when they are valid only quasi-everywhere.

Hitting operators are defined by

$$H^E f(x) = \mathcal{E}_x[\sigma(E) < +\infty; f(X_{\sigma(E)})],$$
$$H_u^E f(x) = \mathcal{E}_x\, e^{-u\sigma(E)} f(X_{\sigma(E)}).$$

Unless otherwise specified, functions in $\mathbf{F}$ are understood to be represented by their quasi-continuous refinement, which is unique up to quasi-equivalence.

Strictly speaking our definition of the sample space Ω is inconsistent with [15]. The newer version is technically more convenient and the difference causes no difficulty in applying results from [15].

1. Preliminaries

A subset A of $\mathbf{X}$ is properly invariant if $m(A) > 0$, if $m(X - A) > 0$ and if for $t > 0$, $P_t 1_A < 1_A$ almost everywhere. It follows from symmetry that A is properly invariant if and only if its complement $A^c = \mathbf{X} - A$ is. Moreover if f belongs to $L^2(m)$ then

$$\begin{aligned}\int m(dx) f(x)\{f - P_t f(x)\} &= \int m(dx) 1_A f(x)\{1_A f(x) - P_t 1_A f(x)\} \\ &\quad + \int m(dx) 1_{A^c} f(x)\{1_{A^c} f(x) - P_t 1_{A^c} f(x)\} \\ &\geq \int m(dx) 1_A f(x)\{1_A f(x) - P_t 1_A f(x)\}.\end{aligned}$$

By Theorem 4.1 in [19], if f belongs to $\mathbf{F}$ then modulo refinements also $1_A f$ belongs to $\mathbf{F}$ and therefore we can obtain a new Dirichlet space by restricting everything to A. This suggests that an appropriate analogue of irreducibility for Markov chains is

1.1. Condition of Irreducibility. There exist no properly invariant sets.

We assume from now on that this condition is satisfied. Our feeling is that this restriction will be harmless in practice.

In the remainder of this section we apply the techniques associated with the Hopf decomposition to distinguish the transient and recurrent cases. Our source for these techniques is the book of Foguel [23].

Lemma 1.1. *Let u be in $L^1(m)$ and let*

$$E = \left\{x \text{ in } \mathbf{X} : \sup \int_0^t ds P_s u(x) > 0\right\}.$$

Then $\int_E m(dx) u(x) \geq 0$.

This can be proved for example by adapting Garsia's well known argument in the discrete time case.

Both in Lemma 1.1 and in the proof of Corollary 1.2 below the supremum is understood to be taken only as t runs over the dyadic fractions $k2^{-n}$.

The potential operator G is defined by

$$Gu(x) = \mathcal{E}_x \int_0^\zeta dt u(X_t) \tag{1.2}$$

when it converges.

Corollary 1.2. *Let u, $v > 0$ almost everywhere be in $L^1(m)$. Then*

$$[Gu < +\infty] = [Gv < +\infty] \text{ almost everywhere.}$$

Proof. Let $A = [Gu = +\infty, Gv + < \infty]$. Then for $a > 0$ clearly

$$B = \left\{x : \sup_{t>0} \int_0^t ds P^s (u - av)(x) > 0\right\}$$

contains A and therefore by Lemma 1.1,

$$\int_B m(dx)\{u(x) - av(x)\} \geq 0$$

and 'a fortiori,'

$$a \int_A m(dx) v(x) \leq \int m(dx) u(x).$$

The corollary follows upon letting $a \uparrow \infty$.

For $u \geq 0$ and nontrivial in $L^1(m)$ the set $[Gu = 0]$ is clearly invariant and therefore must be m-null by our condition of irreducibility. Similarly

$[Gu < +\infty]$ is invariant and therefore either this set or its complement $[Gu = +\infty]$ is m-null. By Corollary 1.2 the choice is independent of u and so the following definition makes sense.

1.2. Definition. The Dirichlet space (F, E) is *transient* if $Gu < +\infty$ almost everywhere and *recurrent* if $Gu = +\infty$ almost everywhere for $u > 0$ in $L^1(m)$.

Theorem 1.3. *If the Dirichlet space* $(\mathbf{F}, E)$ *is recurrent, then for quasi-every* x,

$$\mathcal{P}_x[\zeta = +\infty] = 1.$$

Proof. For $h > 0$,

$$\int_0^t dsP_0(1 - P_h 1) = \int_0^h dsP_s 1 - \int_t^{t+h} dsP_s 1$$

stays bounded as $t \uparrow 0$ and it follows that $P_h 1 = 1$ almost everywhere and therefore quasi-everywhere.

In the remainder of the paper we assume that $(\mathbf{F}, E)$ is transient. If $u \geq 0$ in $L^1(m)$ and if $\Gamma = [Gu = +\infty]$ then

$$\mathcal{P}_x(\sigma(\Gamma) < +\infty) = 0$$

for x in $\mathbf{X} - N - \Gamma$. This follows from the supermartingale property of $Gu(X_t)$ and since Gu is the limit of an increasing sequence of quasi-continuous functions. But then it follows from Theorem 3.6 in [15] that every compact subset of Γ is polar and therefore by the Choquet extension theorem Γ itself is polar. Thus Gu is *defined and finite quasi-everywhere* for any u in $L^1(m)$.

2. The extended Dirichlet space

In this section we give a direct construction of the extended Dirichlet space introduced in Section 8 of [15]. This Dirichlet space is analogous to the discrete time minimal Dirichlet space in [19].

Lemma 2.1. *Let* $\varphi \geq 0$ *on* $\mathbf{X}$. *If* $G\varphi \leq M$ *almost everywhere on the set* $[\varphi > 0]$ *then* $G\varphi \leq M$ *quasi-everywhere on* $\mathbf{X}$.

Proof. It suffices to consider $G_u \varphi$ with φ in $L^2(m)$ and with $u > 0$, to apply the maximum principle Theorem 1.13 in [15] and to pass to the limit in u and φ.

2.1. *Notation.* $\mathcal{S}$ is the collection of $\varphi \geq 0$ in $L^1(m)$ such that $G\varphi$ is bounded.

Lemma 2.2. *Assume that m is bounded. If φ is in $\mathcal{S}$, then $G\varphi$ belongs to* $\mathbf{F}$ *and*

$$E(g, G\varphi) = \int m(dx)\varphi(x)g(x)$$

for g in $\mathbf{F}$.

Proof. Clearly $G\varphi$ is in $L^2(m)$ and since

$$(1/t)\int m(dx)g(x(\{G\varphi(x) - P^tG\varphi(x)\} = (1/t)\int m(dx)g(x)\int_0^t ds\, P^s\varphi(x)$$

the lemma follows from Theorem 4.1. in [19].

Let $\{g_n\}$ be a sequence in $\mathbf{F}$ which is Cauchy relative to the Dirichlet form E. If m is bounded, then Lemma 2.2 applies and by the contraction property of E also $\{g_n\}$ is Cauchy in $L^1(\varphi\cdot m)$ for any φ in $\mathcal{S}$. This result then follows for general m with the help of a simple special case of Proposition 5.4 in [15]. Also with the help of Lemma 2.1 it is easy to see that any finite nonnegative function can be approximated from below by functions in $\mathcal{S}$ and in particular there exists φ in $\mathcal{S}$ which is >0 almost everywhere. Thus there exists g defined almost everywhere on $\mathbf{X}$ such that for a subsequence, $g_n \to g$ almost everywhere. This suggests

2.2. Definition. g belongs to the *extended Dirichlet space* $\mathbf{F}_{(e)}$ if there exists a sequence g_n in $\mathbf{F}$ such that

2.2.1. $\{g_n\}$ is Cauchy relative to E and

2.2.2. $g_n \to g$ almost-everywhere on $\mathbf{X}$.

The Dirichlet form E extends to $\mathbf{F}_{(e)}$ by continuity and the paragraph preceding the definition shows that $\mathbf{F}_{(e)}$ is a Hilbert space relative to E. Therefore the arguments of Section 1 in [15] can be applied directly to the pair $(\mathbf{F}_{(e)}, E)$.

Lemma 2.3. *The following are equivalent for h in $\mathbf{F}_{(e)}$.*

(i) *$P^th \le h$ almost everywhere for $t > 0$.*

(ii) *$E(h, g) \ge 0$ whenever g in $\mathbf{F}_{(e)}$ satisfies $g \ge 0$.*

(iii) *$uG_u h \le h$ almost everywhere for $u > 0$.*

(iv) *There exists a Radon measure ν on $\mathbf{X}$ which charges no polar set such that*

$$E(h, g) = \int \nu(dx)g(x) \tag{2.1}$$

for g in $\mathbf{F}_{(e)} \cap C_{\mathrm{com}}(\mathbf{X})$. Also

$$h = G\nu \tag{2.2}$$

in the sense of (0.1).

Proof. For the equivalence of (i) and (ii) we refer to [24, p. 72]. That (i) implies (iii) follows from Theorem 4.1 in [19]. That (iii) implies (ii) and that (iii) is equivalent to (2.1) follows from the proof of Proposition 1.2 in [15]. Finally (2.1) implies (2.2) by Lemma 2.2 and an obvious symmetry argument. (Note that potentials $G\varphi$ with φ in S are dense in $\mathbf{F}_{(e)}$ because of Lemma 2.1.)

Any function h which can be represented (2.2) with ν a Radon measure charging no polar set will be called a potential. A measure ν is said to have *finite energy* if there exists a potential h in $\mathbf{F}_{(e)}$ such that $h = G\nu$. Clearly this is so if and only if

$$\int \nu(dx)f(x) \leq c\{E(f, f)\}^{1/2}$$

for f in $\mathbf{F}_{(e)} \cap C_{\mathrm{com}}(\mathbf{X})$. The collection of such ν will be denoted by $\mathfrak{M}$. For ν in $\mathfrak{M}$ the *energy* of ν will be denoted and defined by

$$E(\nu) = \int \nu(dx)G\nu(x).$$

The proof of Proposition 1.4 in [15] shows that $\mathfrak{M}$ is complete relative to the energy metric $E^{1/2}(\mu - \nu)$.

Remark. Our use of $\mathfrak{M}$ here is inconsistent with [15]. In general $\mathfrak{M}$ defined here is a proper subset of $\mathfrak{M}$ defined in [15].

For G open we define the *capacity*

$$\mathrm{Cap}\,(G) = \inf E(f, f) \tag{2.3}$$

as f runs over functions in $\mathbf{F}_{(e)}$ such that $f \geq 1$ almost everywhere on G, with the understanding that $\mathrm{Cap}\,(G) = +\infty$ if no such G exists. For A Borel we define

$$\mathrm{Cap}\,(A) = \inf \mathrm{Cap}\,(G)$$

as G runs over the open supersets of A. It is obvious that

$$\mathrm{Cap}\,(A) \leq \mathrm{Cap}_1(A)$$

where Cap_1 is defined in the same way as Cap except that E is replaced by E_1. In particular if $\mathrm{Cap}_1(A) = 0$ then also $\mathrm{Cap}\,(A) = 0$. Conversely if $\mathrm{Cap}\,(A) = 0$ and if A has compact closure, then there exist relatively compact open sets $G_n \downarrow$ and containing A such that $\mathrm{Cap}\,(G_n) \downarrow 0$ and therefore there exist $f_n \geq 1$ almost everywhere on G_n and belonging to $\mathbf{F}$ such that $E(f_n, f_n) \to 0$. After possibly arguing as in the proof of Lemma 8.2 in [15] and passing to a subsequence, we can assume also that $E_1(f_n, f_n) \to 0$ and therefore $\mathrm{Cap}_1(A) = 0$. Thus the notion of polar set and therefore quasi-equivalence is the same for the Dirichlet forms E and E_1. In particular it follows from the proof of Theorem 1.1 in [15] that every f in $F_{(e)}$ has a well defined refinement which is specified and finite quasi-everywhere.

For A a nonpolar Borel subset of $\mathbf{X}$ let $\mathfrak{M}(A)$ be the closure in $\mathfrak{M}$ of measures concentrated on A. For μ in $\mathfrak{M}$ let $\pi^A\mu$ be the measure in $\mathfrak{M}(A)$ determined by the condition that $E(\mu - \pi^A\mu)$ is minimal. This makes sense because $\mathfrak{M}(A)$ is a closed convex subset of $\mathfrak{M}$ and because of the quadratic nature of the energy metric. We call $\pi^A\mu$ the *balayage* of μ onto A. Let $\mathbf{F}_{(e)}(A)$ be the closure in $\mathbf{F}_{(e)}$ of the linear span of $G\mathfrak{M}(A)$. The proof of Lemma 3.5 in [15] shows that $G\pi^A\mu$ is the orthogonal projection of $G\mu$ onto $\mathbf{F}_{(e)}(A)$. Also the proof of Theorem 3.6 in [15] shows that H^A implements orthogonal projection onto $\mathbf{F}_{(e)}(A)$ and so in particular

$$H^A G\mu = G\pi^A \mu.$$

Finally consider again h a nontrivial potential in $\mathbf{F}_{(e)}$ and let $A = [h = 0]$. If A is nonpolar, then by the Choquet extension theorem it contains a nonpolar compact set and therefore by an obvious compactness argument, after approximating from above by open sets, there exists a nontrivial measure μ in $\mathfrak{M}$ which is concentrated on A. But then

$$\int \mu(dx)P^t h(x) = 0$$

for $t > 0$ and therefore

$$\int \mu(dx)G1_{A^c} = 0$$

which is impossible since A is m-null. Therefore $h > 0$ quasi-everywhere and the argument at the end of Section 1 shows that also $h < +\infty$ quasi-everywhere.

Remark. If $(\mathbf{F}, E)$ is recurrent and if m is bounded, then by Theorem 1.3 the function 1 is in $\mathbf{F}$ and $E(1, 1) = 0$. Thus $\mathbf{F}$ cannot be completed relative to E alone to get a Hilbert space. Again the restriction on m is easily removed with the help of random time change.

3. Exit distributions and time reversal

3.1. *Convention.* D_k, $k \geq 1$ is an increasing sequence of open subsets of $\mathbf{X}$ such that $D_k \uparrow \mathbf{X}$ and such that each D_k has compact closure. The complements $\mathbf{X} - D_k$ are denoted by M_k. Often k and $\sim k$ will be used in place of D_k and M_k for subscripts and superscripts.

Let

$$p(x) = \mathcal{P}_x(\zeta = +\infty), \qquad r(x) = \mathcal{P}_x(\zeta < +\infty; X_{\zeta-0} \neq \partial),$$

$$h_0(x) = \mathcal{P}_x(\zeta < +\infty; X_{\zeta-0} = \partial)$$

and note the decomposition

$$1 = p + r + h_0.$$

Our first result is

LEMMA 3.1. *There exists a unique Radon measure κ charging no polar set such that $r = G\kappa$.*

Proof. For each k clearly $H^k r$ is a potential in $\mathbf{F}^{(e)}$ and therefore there exist measure ν_k in M such that $H^k r = G\nu_k$. If $\nu_k^{(l)}$ is the restriction of ν_k to D_l and if $k \geq l$ then by the proof of Theorem 3.11 in [15],

$$E(\nu_k^{(l)}) \leq E(H^{\mathrm{cl}(D_l)} r, H^{\mathrm{cl}(D_l)} r)$$

which is bounded independent of k. After applying the appropriate analogue of Proposition 1.4 in [15] and after selecting a subsequence we can assume that $\nu_k \to \kappa$ vaguely and that for each l there exists a measure $\kappa^{(l)}$ in M such that $\kappa^{(l)}$ dominates the restriction of κ to D_l and is dominated by the restriction of κ to $\mathrm{cl}(D_l)$ and such that $G\nu_k^{(l)} \to G\kappa^{(l)}$ weakly in $\mathbf{F}_{(e)}$. For $f \geq 0$ in $\mathcal{S} \cap B_{\mathrm{com}}(\mathbf{X})$,

$$\int m(dx)r(x)f(x) = \mathrm{Lim}_k \int m(dx)G\nu_k(x)f(x)$$

$$= \mathrm{Lim}_k \left\{ \int_{\mathrm{cl}(D_l)} \nu_k(dx)Gf(x) + \int_{\mathbf{X}-\mathrm{cl}(D_l)} \nu_k(dx)Gf(x) \right\}$$

and for fixed l the first term inside the brackets converges to

$$\int \kappa^{(l)}(dx)Gf(x)$$

and the second is dominated by

$$\int \pi^{\sim l}\nu_k(dx)Gf(x) = \int m(dx)f(x)H^{\sim l}G\nu_k(x)$$

$$\leq \int m(dx)f(x)H^{\sim l}r(x).$$

But it is easy to check that $H^{\sim l}r \downarrow 0$ quasi-everywhere (and therefore almost everywhere) as $l \uparrow \infty$ and we conclude that $r = G\kappa$. Uniqueness of κ follows directly from uniqueness of ν in Lemma 2.3 (iv).

We identify κ as the "killing measure" in

THEOREM 3.2. *For $f \geq 0$ and for quasi-every x,*

$$\mathcal{E}_x[\zeta < +\infty;\, X_{\zeta-0} \neq \partial;\, f(X_{\zeta-0})] = G(f\cdot\kappa)(x). \tag{3.1}$$

Also for $u > 0$,

$$\mathcal{E}_x[X_{\zeta-0} \neq \partial;\, e^{-u\zeta}f(X_{\zeta-0})] = G_u(f\cdot\kappa)(x). \tag{3.1$'$}$$

Proof. It suffices to prove (3.1)′ since (3.1) then follows by passage to the limit $u \downarrow 0$. For this purpose let R_u be the usual terminal time which is independent of the trajectory variables X_t and which is exponentially distributed with density ue^{-ul} and let $\zeta_u = \min(R_u, \zeta)$. Since for quasi-every x,

$$\mathcal{E}_x[R_u < \zeta;\, f(X_{R_u-0})] = uG_u f(x),$$

it suffices to establish

$$\mathcal{E}_x[X_{\zeta_u-0} \neq \partial;\, f(X_{\zeta_u-0})] = G_u(f\cdot\kappa_u)(x) \tag{3.2}$$

where $\kappa_u(dx) = \kappa(dx) + um(dx)$. Indeed it suffices to consider f in $C_{\mathrm{com}}(\mathbf{X})$ and to verify that for $g \geq 0$ in $C_{\mathrm{com}}(\mathbf{X})$,

$$\int m(dx)g(x)\mathcal{E}_x[X_{\zeta_u-0} \neq \partial;\, f(X_{\zeta_u-0})] = \int \kappa_u(dy)f(y)G_u g(y). \tag{3.2$'$}$$

Let

$$h_u(x) = \mathcal{E}_x[X_{\zeta-0} = \partial;\, e^{-u\zeta}], \qquad r_u(x) = \mathcal{P}_x[X_{\zeta_u-0} \neq \partial]$$

and note that

$$h_0 = h_u + uG_u h_0$$

and therefore

$$\begin{aligned} r_u &= 1 - h_u \\ &= 1 - h_0 + uG_u h_0 \\ &= r + p + uG_u h_0 \qquad (3.3)\\ &= G_u\{\kappa + ur\cdot m\} + uG_u p + uG_u h_0 \\ &= G_u \kappa_u. \end{aligned}$$

Thus the left side of (3.2′)

$$= \operatorname{Lim} \textstyle\sum_{k=0}^{\infty} \displaystyle\int m(dx)g(x)\mathcal{E}_x\,[X_{\zeta_u-0} \neq \partial;\, f(X_{k/2^n});\, k/2^n < \zeta_u \leq (k+1)/2^n]$$

$$= \operatorname{Lim} \textstyle\sum_{k=0}^{\infty} \displaystyle\int m(dx)g(x)e^{-uk/2^n}P_{k/2^n}\, f\{r_u - e^{-u/2^n}P_{1/2^n}\, r_u\}(x)$$

$$= \operatorname{Lim} \textstyle\sum_{k=0}^{\infty} \displaystyle\int \kappa_u(dy)G_u\{1 - e^{-u/2^n}P_{1/2^n}\}fe^{-uk/2^n}P_{k/2^n}\, g(y)$$

$$= \operatorname{Lim} \textstyle\sum_{k=0}^{\infty} \displaystyle\int \kappa_u(dy) \int_0^{1/2^n} dt\; e^{-ut}P_t\, fe^{-uk/2^n}P_{k/2^n}\, g(y)$$

and the theorem follows since

$$P_s f(P_{t-s}\, g(y) = \mathcal{E}_y f(X_s)g(X_{t-s})$$

converges to $f(y)P_t\, g(y)$ as $s \downarrow 0$ for $t > 0$ and for quasi-every y.

Now fix D an arbitrary open subset of $\mathbf{X}$ and denote the complement $\mathbf{X} - D$ by M. For x in D let

$$\begin{aligned} p^D(x) &= \mathcal{P}_x[\zeta = +\infty;\, \sigma(M) = +\infty], \\ r^D(x) &= \mathcal{P}_x[\zeta < +\infty;\, \sigma(M) = +\infty;\, X_{\zeta-0} \neq \partial], \\ s^D(x) &= \mathcal{P}_x[\zeta < +\infty;\, \sigma(M) = +\infty;\, X_{\zeta-0} = \partial]. \end{aligned}$$

(This notation is inconsistent with Section 5 in [15].) Clearly

$$1 = H^M 1 + r^D + s^D + p^D$$

quasi-everywhere on D and $s^D = 0$ whenever D has compact closure. Let ν^M be the unique Radon measure on M such that

$$\int \nu^M(dy)\varphi(y) = \int m(dx)H_1^M\varphi(x) \qquad (3.4)$$

for $\varphi \geq 0$ on M. For $f \geq 0$ on $\mathbf{X}$ and for $u \geq 0$ let $\pi_u^M f$ be the unique function specified up to ν^M equivalence on M and satisfying

$$\int \nu^M(dy)\pi_u^M f(y)\varphi(y) = \int m(dx)f(x)H_u^M\varphi(x) \tag{3.5}$$

for $\varphi \geq 0$ on M. More generally if μ is a Radon measure concentrated on R and charging no polar set, let $\pi_u^M\mu$ be the function on M determined by

$$\int \nu^M(dy)\pi_u^M\mu(y)\varphi(y) = \int \mu(dx)H_u^M\varphi(x).$$

That is, $\nu^M(dy)\pi_u^M\mu(y)$ is the u-balayage of μ onto M. This makes sense because of Theorem 3.6(iv) in [15].

Routine computations establish the following identities for $u > 0$ and for quasi-every x in D:

$$p^D(x) = uG_u^D p^D(x) \tag{3.6}$$

$$r^D(x) = uG_u^D r^D(x) + \mathcal{E}_x[X_{\zeta-0} \neq \partial; \sigma(M) = +\infty; e^{-u\zeta}] \tag{3.7}$$

$$s^D(x) = uG_u^D s^D(x) + \mathcal{E}_x[X_{\zeta-0} = \partial; \sigma(M) = +\infty; e^{-u\zeta}]. \tag{3.8}$$

From (3.6) and the identity

$$\pi_u^M - \pi_v^M = (v-u)\pi_v^M G_u^D \tag{3.9}$$

which is dual to the familiar and easily verified

$$H_u^M - H_v^M = (v-u)G_u^D H_v^M \tag{3.10}$$

follows

$$v\pi_v^M p^D = u\pi_u^M p^D, \qquad 0 < u < v. \tag{3.11}$$

The identities (3.7) and (3.8) lead in the same way to inequalities

$$v\pi_v^M r^D \geq u\pi_u^M r^D, \qquad 0 < u < v, \tag{3.12}$$

$$v\pi_v^M s^D \geq u\pi_u^M s^D, \qquad 0 < u < v. \tag{3.13}$$

Indeed (3.12) can be improved to

$$v\pi_v^M r^D + \pi_v^M(1_D\cdot\kappa) = u\pi_u^M r^D + \pi_u^M(1_D\cdot\kappa), \qquad u, v > 0. \tag{3.14}$$

To see this note first that

$$r_u = G_u^D\kappa_u + H_u^M r_u$$

which follows from (3.3) and the familiar identity

$$G_u = G_u^D + H_u^M G_u. \tag{3.15}$$

Since

$$uG_u^D 1(x) = \mathcal{P}_x[R_u \leq \min(\sigma(M), \zeta)]$$

it follows that

$$\mathcal{P}_x[X_{\zeta_u-0} \neq \partial] = G_u^D \kappa(x) + \mathcal{P}_x[R_u \leq \min(\sigma(M), \zeta)] + \mathcal{P}_x[\sigma(M) < R_u; X_{\zeta_u-0} \neq \partial]$$

and therefore

$$\mathcal{P}_x[\zeta < R_u; \sigma(M) = +\infty; X_{\zeta-0} \neq \partial] = G_u^D \kappa(x) \tag{3.16}$$

for quasi-every x in D. Then (3.7) can be written

$$r^D = uG_u^D r^D + G_u^D \kappa \tag{3.7'}$$

and therefore

$$\begin{aligned} u\pi_u^M r^D + \pi_u^M \kappa &= u\pi_u^M \{vG_v^D r^D + G_v^D \kappa\} + \pi_u^M \kappa \\ &= v\pi_v^M \{uG_u^D r^D + G_u^D \kappa\} + (u - v)\pi_v^M G_u^D \kappa + \pi_u^M \kappa \\ &= v\pi_v^M r^D + \pi_v^M \kappa \end{aligned}$$

which proves (3.14). Also (3.16) and the proof of Theorem 3.2 suffice to establish

THEOREM 3.3. *For $f \geq 0$ on D and for quasi-every x in D,*

$$\mathcal{E}_x[\sigma(M) = +\infty; X_{\zeta-0} \neq \partial; f(X_{\zeta-0})] = G^D(f \cdot \kappa)(x). \tag{3.17}$$

Also for $u > 0$,

$$\mathcal{E}_x[\zeta < R_u; \sigma(M) = +\infty; X_{\zeta-0} \neq \partial; f(X_{\zeta-0})] = G_u^D(f \cdot \kappa)(x). \tag{3.17'}$$

Finally we establish a local time reversal result which is essential for our construction of Hunt's "approximate Markov process" in Section 4. For each k let L_k be the unique measure in M such that

$$H^k 1 = GL_k.$$

The time reversal operator ρ_k and the truncation operator τ_k are defined on $\Omega \cap [\sigma(D_k) < +\infty]$ by

$$\begin{aligned} \rho_k \omega(t) &= \omega(\sigma^*(D_k) - t - 0), && 0 \leq t < \sigma^*(D_k) \\ &= \partial, && t \geq \sigma^*(D_k) \\ \tau_k \omega(t) &= \omega(t), && 0 \leq t < \sigma^*(D_k) \\ &= \partial && t \geq \sigma^*(D_k) \end{aligned}$$

Note that by transience $\sigma^*(D_k) < +\infty$ [a.e. $\mathcal{P}_x$] whenever $\sigma(D_k) < +\infty$.

THEOREM 3.4. *For $\xi \geq 0$ on Ω and for $k, l \geq 1$,*

$$\int L_k(dx)\mathcal{E}_x\, \xi \circ \rho_l = \int L_l(dy)\mathcal{E}_y\, \xi \circ \tau_k. \tag{3.18}$$

Proof. We begin by establishing

$$\mathcal{E}_x[t_n < \sigma^*(D_l); f_0(X_{\sigma^*(D_l)-0}) \cdots f_n(X_{\sigma^*(D_l)-t_n-0})] = G\{f_n P_{t_n-t_{n-1}} \cdots P_{t_1} f_0 \cdot L_l\}(x) \tag{3.19}$$

for $0 < t_1 < \cdots < t_n$ for $f_0, \cdots, f_n \geq 0$ on $\mathbf{X}$ and for quasi-every x. It suffices to establish (3.19) for almost every x and therefore we can replace (3.19) by

$$\int m(dx)\varphi(x)\mathcal{E}_x[t_n < \sigma^*(D_l); f_0(X_{\sigma^*(D_l)-0}) \cdots f_n(X_{\sigma^*(D_l)-t_n-0})] = \int L_l(dy)f_0(y)P_{t_1} \cdots P_{t_n-t_{n-1}} f_n G\varphi(y) \tag{3.19$'$}$$

with φ in $\mathcal{S}$. Also we can assume that $f_0, \cdots, f_n$ are in $\mathbf{F} \cap C_{\text{com}}(\mathbf{X})$. The left side of (3.19′)

$$= \operatorname{Lim} \textstyle\sum_{k=0}^{\infty} \displaystyle\int m(dx)\varphi(x)P_{k/p} f_n \cdots P_{t_1} f_0(1 - P_{1/p})H^{D_l}1(x)$$

$$= \operatorname{Lim} \int L_l(dy) \int_0^{1/p} ds\, P_s f_0 \cdots f_n \textstyle\sum_{k=0}^{\infty} P_{k/p} \cdot \varphi(y).$$

Clearly $p \int_0^{1/p} ds P_s$ converges strongly to the identity as an operator on $\mathbf{F}_{(e)}$ and with the help of the spectral theorem it is easy to check (after first considering m bounded) that

$$(1/p) \textstyle\sum_{k=0}^{\infty} P_{k/p}\, \varphi \to G\varphi$$

in $\mathbf{F}_{(e)}$ as $p \uparrow \infty$ and (3.19′) is proved. In (3.18) it suffices to consider

$$\xi = f_0(X_0) \cdots f_n(X_{t_n})$$

and then (3.18) follows from

$$\int L_k(dx)\mathcal{E}_x\, \xi \circ \rho_l$$

$$= \int L_k(dx)\mathcal{E}_x[t_n < \sigma^*(D_l); f_0(X_{\sigma^*(D_l)-0}) \cdots f_n(X_{\sigma^*(D_l)-t_n-0})]$$

$$= \int L_k(dx)G\{f_n P_{t_n-t_{n-1}} \cdots f_0 \cdot L_l\}(x)$$

$$= \int L_l(dx)f_0(x)P_{t_1} f_1 \cdots P_{t_n-t_{n-1}} f_n H^{D_k}1(x).$$

Remark. The proof of Theorem 4.2 in [15] is incorrect and indeed the last lines on page 32 do not make sense. However a correct proof can easily be supplied using the techniques of this section.

4. An approximate Markov process

For each k let Ω_k be the subcollection of Ω satisfying

4.1. $\omega(0) = \partial$ or $\omega(0)$ is in the closure $\mathrm{cl}(D_k)$.

There is a unique trajectory ω in Ω_k such that $\omega(t) = \partial$ for $0 \le t \le +\infty$. We refer to this trajectory as the *dead trajectory* and denote it by δ_k. We consider Ω_k with the Skorohod metric as defined for a special case and for compact time intervals in [12, Chap VII]. A simple extension of the results in [12] shows that relative to the Skorohod topology Ω_k is a separable metric space and an absolute Borel set. That is, Ω_k is a Borel subset of one (and therefore any) of its completions. The mapping J_k from Ω_{k+1} to Ω_k is defined by

$$J_k\,\omega(t) = \begin{cases} \partial \text{ for all } t & \text{if } \sigma(D_k) = +\infty \\ \omega(\sigma(\mathrm{R}_k) + t) & \text{if } \sigma(D_k) < +\infty. \end{cases}$$

Clearly each J_k is Borel measurable and surjective. The *inverse limit* of the Ω_k is the collection Ω_∞^0 of sequences $\{\omega_k\}_{k=1}^{\infty}$ with each ω_k in Ω_k and such that $J_k\,\omega_{k+1} = \omega_k$ for all k. The *extended sample space* is the reduced inverse limit $\Omega_\infty = \Omega_\infty^0 - \{\delta\}$ where δ is the dead sequence in Ω_∞^0 whose components are the dead trajectories δ_k. We denote by $J_{\infty,k}$ the natural projection of Ω_∞ onto Ω_k. It follows from [12, Chap. V] that Ω_∞ is a separable metric space and an absolute Borel set in the product Skorohod topology and that the projections $J_{\infty,k}$ are Borel measurable. The point of this is

THEOREM 4.1. *There exists a unique countably additive measure $\mathcal{P}$ on the extended sample space Ω_∞ such that*

$$\mathcal{E}\xi \circ J_{\infty,k} = \int L_k(da)\mathcal{E}_x\,\xi \tag{4.1}$$

for each k and for $\xi \ge 0$ on Ω_k and vanishing on δ_k.

Proof. We note first that for $\varphi \ge 0$ on $\mathbf{X}$,

$$\begin{aligned}\int L_{k+1}(dx)\mathcal{E}_x[\sigma(D_k) < +\infty;\, \varphi(X_{\sigma(D_k)})]. &= \int L_k(dy)\mathcal{E}_y\,\varphi(X_{\sigma^*(D_k)}) \\ &= \int L_k(dx)\mathcal{E}_x\,\varphi(X_0) \\ &= \int L_k(dx)\varphi(x)\end{aligned}$$

where we have used Theorem 3.4 twice, and from this it follows that

$$\int L_{k+1}(dx)\mathcal{E}_x\,\xi \circ J_k = \int L_k(dx)\mathcal{E}_x\,\xi.$$

This is the consistency condition which is necessary for the existence of a finitely additive measure satisfying (4.1). To establish countable additivity it suffices to show that if A_k is a Borel subset of $\Omega_k - \{\delta_k\}$ and if the inverse images $J_{\infty,k}^{-1}(A_k)$ in Ω_∞ decrease with k and if

$$\mathcal{P}^{(k)}(A_k) = \int L_k(dx)\mathcal{P}_x(A_k) \geq a > 0$$

for all k, then

$$\bigcap_{k=1}^{\infty} J_{\infty,k}^{-1}(A_k) \quad \text{is nonempty.} \tag{4.2}$$

According to [12, Theorem 3.2, p. 139] there exists a sequence of compact metric spaces Ω_k^* and for each k a surjective continuous map $J_k^* : \Omega_{k+1} \to \Omega_k$ and an injective Borel map $\Phi_k : \Omega_k \to \Omega_k^*$ such that $J_k^* \Phi_{k+1} = \Phi_k J_k$. The inverse limit Ω_∞^* of the Ω_k^* is a compact metric space and the projections $J_{\infty,k}^* : \Omega_\infty^* \to \Omega_k^*$ are continuous. Let $\mathcal{P}^{(k)*}$ be the unique Borel measure on Ω_k^* such that $\mathcal{P}^{(k)*}(B^*) = \mathcal{P}^{(k)}(\Phi_k^{-1}B^*)$ for B^* a Borel subset of Ω_k^*. By Kuratowski's theorem [12, Theorem 3.9, p. 21] the images $A_k^* = \Phi(A_k^*)$ are Borel subsets of Ω_k^* and of course $\mathcal{P}^{(k)*}(A_k^*) = \mathcal{P}^{(k)}(A_k)$. Choose compact subsets B_k^* of A_k^* such that

$$\mathcal{P}^{(k)*}(B_k^*) \geq \mathcal{P}^{(k)*}(A_k^*) - a2^{-k}$$

and define

$$C_2^* = B_2^* \cap J_1^{*-1}(B_1^*), \qquad C_{k+1}^* = B_{k+1}^* \cap J_k^{*-1}(C_k^*). \qquad k \geq 2.$$

Each C_k^* is compact in Ω_k^* and the inverse images $J_{\infty,k}^{*-1}(C_k^*)$ are compact and decreasing in the inverse limit Ω_∞^*. Clearly $(J_1^*)^{-1}(A_1^*)$ contains A_2^* and so

$$\begin{aligned}\mathcal{P}^{(2)*}(C_2^*) &\geq \mathcal{P}^{(2)*}(B_2^*) - \mathcal{P}^{(1)*}(A_1^* - B_1^*)\\ &\geq a - \frac{a}{4} - \frac{a}{2},\end{aligned}$$

and similarly for $k \geq 2$,

$$\begin{aligned}\mathcal{P}^{(k+1)*}(C_{k+1}^*) &\geq \mathcal{P}^{(k+1)*}(B_{k+1}^*) - \mathcal{P}^{(k)*}(A_k^* - C_k^*)\\ &\geq a - a/2^{k+1} - (a/2 + \cdots + a/2^k).\end{aligned}$$

Thus C_k^* is nonempty for each k and therefore $\bigcap_{k=r}^{\infty} J_{\infty,k}^{*-1}(C_k^*)$ is nonempty. Now (4.2) follows and the theorem is proved.

Remark. Theorem 4.1 is an adaption of Hunt's construction of "approximate Markov chains" as outlined in [9]. This was first done in continuous time by M. Weil in [25].

As in [15] we introduce trajectory variables parametrized by an artificial two sided time side for $\omega = \{\omega_k\}$ in Ω_∞. Let k_0 be the first integer k such that $\omega_k \neq \delta_k$ and for $t \geq 0$ define

$$X_t(\omega) = \omega_{k_0}(t).$$

For $t < 0$ there is at most one integer $k_t > k_0$ such that

$$\sigma(D_{k_0}, \omega_{k_t}) \geq |t|, \qquad \sigma(D_{k_0}, \omega_{k_t-1}) < |t|.$$

Define

$$X_t(\omega) = \omega_{k_t}(\sigma(D_{k_0}) + t)$$

if k_t exists and otherwise define $X_t(\omega) = \partial$. The coordinates X_t are Borel measurable on Ω_∞ and in addition ω in Ω_∞ is determined by its coordinates $X_t(\omega)$. However the X_t *do not in general* form a Markov process relative to the measure $\mathcal{P}$. First hitting times $\sigma(A)$, last exit times $\sigma^*(A)$, the death time ζ and the birth time ζ^* are defined in the usual manner. The time reversal operator ρ is defined so that

$$X_t(\rho\omega) = X_{\sigma^*(D_{k_0})-t-0}$$

with k_0 as above. Clearly ρ is bijective and $\rho = \rho^{-1}$ is Borel measurable. Our general result on time reversal is

THEOREM 4.2. *For $\xi > 0$ on Ω,*

$$\mathcal{E}\xi \circ \rho = \mathcal{E}\xi. \tag{4.3}$$

Proof. It suffices to consider

$$\xi = f_0(X_{\sigma(D_p)})f_1(X_{\sigma(D_p)+t_1}) \cdots f_n(X_{\sigma(D_p)} + t_n)$$

with $0 < t_1 < \cdots < t_n$ and with $f_i \geq 0$ and in $\mathrm{C}_{\mathrm{com}}(\mathbf{X})$. Then for k sufficiently large,

$$\mathcal{E}\xi \circ \rho = \mathcal{E}\xi \circ \rho_k \circ J_{\infty,k} = \int L_k(dx)\mathcal{E}_x\, \xi \circ \rho_k$$

which by Theorem 3.4

$$= \int L_k(dy)\mathcal{E}_y\, \xi \circ \tau_k = \mathcal{E}\xi \circ \tau_k \circ J_{\infty,k} = \mathcal{E}\xi.$$

From the very definition of $\mathcal{E}$,

$$\mathcal{E} \int_{\zeta^*}^{\zeta} dt\varphi(X_t) = \int m(dx)\varphi(x) \tag{4.4}$$

for $\varphi \geq 0$ on $\mathbf{X}$. This will play an important role in later sections.

Finally we return to the capacity Cap introduced in Section 2. An elementary compactness argument shows that a Borel subset E of $\mathbf{X}$ has *finite capacity* if and only if

$$H^E 1 = GL_E$$

with L_E a measure in M and then

$$\text{Cap }(E) = \int L_E(dx). \tag{4.5}$$

For each k,

$$\int L_k(dx)\mathcal{P}_x(\sigma(E) < +\infty) = \int L_k(dx)GL_E(x) = \int L_E(dx)GL_k(x)$$

and after passing to the limit $k \uparrow \infty$,

$$\text{(4.6)} \qquad \text{Cap}(E) = \mathcal{P}(\sigma(E) < +\infty).$$

This also will play an important role in later sections.

5. A time changed process

We consider again a general open set D with notations as in Section 3. For $0 \leq u < v$ we define the symmetric kernel $U^M_{u,v}(y, z)$ by

$$\text{(5.1)} \qquad (v - u) \int_D m(dx) H^M_u(x, dy) H^M_v(x, dz) = U^M_{u,v}(y, z) \nu^M(dy) \nu^M(dz)$$

and also we define the bilinear forms

$$\text{(5.2)} \qquad U^M_{u,v}(\varphi, \varphi) = \int \nu^M(dy) \int \nu^M(dz) U^M_{u,v}(y, z) \varphi(y) \varphi(z)$$

$$\text{(5.3)} \qquad U^M_{u,v}\langle \varphi, \varphi \rangle = \int \nu^M(dy) \int \nu^M(dz) U^M_{u,v}(y, z) \{\varphi(z) - \varphi(y)\}^2.$$

This notation is more or less consistent with Section 5 in [19] but not with Section 5 in [15]. It is easy to check that

$$\text{(5.4)} \qquad U^M_{u,v}(1, \varphi^2) = \tfrac{1}{2} U^M_{u,v} \langle \varphi, \varphi \rangle + U^M_{u,v}(\varphi, \varphi)$$

when the left side converges and that

$$\text{(5.5)} \qquad \begin{aligned} (v - u) \pi^M_v H^M_u \varphi &= (v - u) \pi^M_u H^M_v \varphi \\ &= (v - u) dm/d\nu^M \varphi + \int U^M_{u,v}(\cdot, z) \varphi(z) \nu^M(dz) \end{aligned}$$

for $\varphi \geq 0$ on M. Also

$$\text{(5.6)} \qquad U^M_{u,w} = U^M_{u,v} + U^M_{v,w}$$

for $0 \leq u < v < w$ and then the estimate

$$\text{(5.7)} \qquad U^M_{0,1}(\varphi, \varphi) \leq U^M_{0,1}(1, \varphi^2) \leq \int \nu^M(dy) \varphi^2(y)$$

guarantees that

$$\text{(5.8)} \qquad U^M_{0,u}(\varphi, \varphi) \to 0$$

as $u \downarrow 0$ for φ in $L^2(\nu^M)$.

LEMMA 5.1. *For $u > 0$ the operator H^M_u is bounded from $L^2(\nu^M)$ to $L^2(m)$ and the operator π^M_u is bounded from $L^2(m)$ to $L^2(\nu^M)$.*

Proof. It suffices to consider H^M_u, $u > 0$ because of (3.5) and it suffices to consider $u = 1$ because of (3.10). But then the lemma follows since

$$\int m(dx) \{H^M_1 \varphi(x)\}^2 \leq \int m(dx) H^M_1 \varphi^2(x) = \int \nu^M(dy) \varphi^2(y).$$

We consider now the time changed process

$$X_t^M = X_{B(\nu^M;t)}$$

as defined in Section 5 of [15]. Our attention will focus on the corresponding resolvent operators

$$(5.9)\qquad \begin{aligned} R_a^M \varphi(y) &= \mathcal{E}_y \int_0^\infty dt e^{-at} \varphi(X_t^M) \\ &= \mathcal{E}_y \int_0^\infty A(\nu^M; dt) \exp\{-aA(\nu^M; t)\}\varphi(X_t) \end{aligned}$$

and the modified resolvent operators

$$(5.10)\qquad R_{(u)a}^M \varphi(y) = \mathcal{E}_y \int_0^\infty A(\nu^M; dt) e^{-ut} \exp\{-aA(\nu^M; t)\}\varphi(X_t).$$

It is shown in Lemma 5.1 of [15] that $\{R_a^M, a > 0\}$ and $\{R_{(u)a}^M, a > 0\}$ are symmetric submarkovian resolvents on $L^2(M, \nu^M)$. We proceed to identify the Dirichlet spaces which correspond in the sense of Proposition 1.1 in [15]. We refer to Section 6 in [19] where this is done in the context of Markov chains.

The first step is

LEMMA 5.2. (i) *F is contained in $L^2(\nu^M)$.*

(ii) *Let $\mathbf{H}^M$ be the restriction of $\mathbf{F}$ to M and for $u > 0$ and φ in $\mathbf{H}^M$ define*

$$(5.11)\qquad Q_{(u)}^M(\varphi, \varphi) = E_u(H_u^M \varphi, H_u^M \varphi).$$

Then $(\mathbf{H}^M, Q^M)$ is the Dirichlet space on $L^2(\nu^M)$ which corresponds to the modified resolvent $\{R_{(u)a}^M, a > 0\}$.

We begin with (ii). This is a special case of Lemma 5.2 in [15] but we repeat the proof here with appropriate simplifications.

For $u > 0$ let $R_{(u)}^M$ be defined by (5.10) with $a = 0$. By Theorem 3.3 in [15]

$$(5.12)\qquad R_{(u)}^M \psi = G_u(\psi \cdot \nu^M).$$

Since

$$\begin{aligned} R_{(u)}^M 1 &= G_u \nu^M = G_u \pi_1^M 1 = G_u\{\pi_u^M 1 + (u-1)\pi_u^M G_1^D 1\} \\ &= H_u^M G_u\{1 + (u-1)G_1^D 1\} \end{aligned}$$

is bounded, the operator $R_{(u)}^M$ is bounded on $L^\infty(\nu^M)$ and therefore by symmetry, on $L^2(\nu^M)$. Thus $R_{(u)}^M$ is exactly the inverse to the generator $B_{(u)}^M$ for the modified resolvent (see Section 1 in [15]) and φ in $L^2(\nu^M)$ belongs to the domain of $B_{(u)}^M$ if and only if $\varphi = G_u(\psi \cdot \nu^M)$ with ψ in $L^2(\nu^M)$. In this case since for $\psi \geq 0$,

$$\int \nu^M(dy)\varphi(y)\psi(y) < +\infty$$

and since H_u^M projects $\mathbf{F}$ onto $\mathbf{F}_u(M)$ (see Theorem 3.6(i) in [15]), the function φ belongs to $\mathbf{H}^M$ and

$$Q_{(u)}^M(\varphi, \varphi) = \int \nu^M(dy)\varphi(y)\psi(y).$$

Clearly such φ are dense in the Dirichlet space which corresponds to the $R_{(u)a}^M$ and (ii) will follow if we show that such φ are dense in $\mathbf{H}^M$. But as $v \uparrow \infty$ clearly $vH_u^M G_v H_u^M \varphi \to \varphi$ in $\mathbf{F}$ and by the resolvent identity we can assume that $\varphi = H_u^M G_u g$ with g in $L^2(m)$. But then $\varphi = G_u \pi_u^M g = R_{(u)}^M \pi_u^M g$ and we are done because of Lemma 5.1. Finally (i) follows because $R_{(u)}^M$ is bounded on $L^2(\nu^M)$ and therefore $Q_{(u)}^M$ dominates a multiple of the standard inner product on $L^2(\nu^M)$.

Among other things the time changed Dirichlet space itself is identified in

THEOREM 5.3. (i) *Let $\mathbf{F}_{(e)}^D$ be the closure in $\mathbf{F}_{(e)}$ of $\mathbf{F}^D$. Then the operator H^M implements orthogonal projection of $\mathbf{F}_{(e)}$ onto the E-orthogonal complement of $\mathbf{F}_{(e)}^D$.*

(ii) *Let $\mathbf{H}_{(e)}^M$ be the restriction of $\mathbf{F}_{(e)}$ to M and for φ in $\mathbf{H}_{(e)}$ define*

$$Q^M(\varphi, \varphi) = E(H^M\varphi, H^M\varphi). \tag{5.13}$$

Then $(\mathbf{H}^M, Q^M)$ is the Dirichlet space on $L^2(\nu^M)$ which corresponds to the time changed resolvent $\{R_a^M, a > 0\}$. Moreover this is a regular Dirichlet space and $(\mathbf{H}_{(e)}^M, Q^M)$ is the extended Dirichlet space as defined in Section 2.

(iii) *For $u > 0$ and for φ in $\mathbf{H}^M$,*

$$Q_{(u)}^M(\varphi, \varphi) = Q^M(\varphi, \varphi) + U_{0,u}^M(\varphi, \varphi) + u\int_M m(dy)\varphi^2(y). \tag{5.14}$$

Proof. H^M can be identified as an orthogonal projector either by directly adapting the proof of Theorem 3.6(i) in [15] or by passing to the limit $u \downarrow 0$ as in the proof of Lemma 8.3 in [15] and then (5.14) follows by direct computation. To identity $(\mathbf{H}^M, Q)$ as the time changed Dirichlet space we must proceed indirectly since the operator R^M is in general not bounded on $L^2(\nu^M)$. Again the needed argument is a special case of one given in Section 5 of [15] and is repeated here with appropriate simplifications.

It follows in particular from (5.14) that for $0 < u < 1$,

$$Q_{(1)}^M(\varphi, \varphi) = Q_{(u)}^M(\varphi, \varphi) + U_{u,1}^M(\varphi, \varphi) + (1-u)\int_M m(dy)\varphi^2(y) \tag{5.15}$$

and therefore for $a > 0$ and for φ, ψ in $L^2(\nu^M)$,

$$\begin{aligned}
&\int \nu^M(dy)\psi(y)R_{(u)a}^M\varphi(y)\\
&= \int \nu^M(dy)\varphi(y)R_{(u)a}^M\psi(y)\\
&= Q_{(1)a}^M(R_{(1)a}^M\varphi, R_{(u)a}^M\psi)\\
&= Q_{(u)a}^M(R_{(1)a}^M\varphi, R_{(u)a}^M\psi) + U_{u,1}^M(R_{(1)a}^M\varphi, R_{(u)a}^M\psi)\\
&\qquad + (1-u)\int m(dy)R_{(1)a}^M\varphi(y)R_{(u)a}^M\psi(y).
\end{aligned} \tag{5.16}$$

Thus by (5.5),

$$R^M_{(u)a} = R^M_{(1)a} + (1 - u)R^M_{(1)a}\,\pi^M_1\, H^M_u\, R^M_{(u)a}$$

and after passing to the limit $u \downarrow 0$,

$$R^M_a = R^M_{(1)a} + R^M_{(1)a}\,\pi^M_1\, H^M R^M_a. \tag{5.17}$$

If $\varphi = R^M_a\,\psi$ with $\psi \geq 0$ in $L^2(\nu^M)$ then also

$$\varphi = R^M_{(1)a}\{\psi + \pi^M_1\, H^M\varphi\}$$

and since

$$\int \nu^M(dy)\varphi(y)\{\psi(y) + \pi^M_1 H^M\varphi(y)\} < +\infty$$

it follows from Lemma 5.2 that φ belongs to $\mathbf{H}^M$ and that

$$Q^M_a(\varphi, \varphi) = \int \nu^M(dy)\varphi(y)\psi(y).$$

It follows that the time changed Dirichlet space is contained in $\mathbf{H}^M$ and that the Dirichlet form is Q^M. But since $R^M_{(1)a} \leq R^M_a$ it follows either by Proposition 1.1 in [15] or by Theorem 4.1 in [19] that the time changed Dirichlet space is precisely $\mathbf{H}^N$. Finally, it is routine to check that $(\mathbf{H}^M, Q^M)$ is a regular Dirichlet space and that $(\mathbf{H}^M_{(e)}, Q^M)$ is the extended Dirichlet space as defined in Section 2.

Remark. Potential theory for $(\mathbf{H}^M, Q^M)$ as developed in Section 2 and in Section 1 of [15] is entirely consistent with the potential theory for $(\mathbf{F}, E)$. In particular the definitions of capacity and of quasi-equivalence on M are identical

Note that for $u \geq 0$ and $f \geq 0$ on $\mathbf{X}$,

$$G_u f = G^D_u f + H^M_u\, G_u f = G^D_u f + H^M_u\, G_u\, \pi^M_u\, f$$

and therefore

$$G_u = G^D_u + H^M_u\, R^M_{(u)}\, \pi^M_u. \tag{5.18}$$

This is a special case of (5.21) in [15].
It follows from (3.13) that

$$\pi^M_* s^D(y) = \mathrm{Lim}_{v\uparrow\infty}\, v\pi^M_v\, s^D(y) \tag{5.19}$$

and from (5.6) that for $u \geq 0$,

$$U^M_{u,\infty}(y, z) = \mathrm{Lim}_{v\uparrow\infty}\, U^M_{u,v}(y, z) \tag{5.20}$$

are well defined [a.e. ν^M] and [a.e. $\nu^M \times \nu^M$] respectively.

5.1. Definition. The *universal Dirichlet form* on M is given by

$$N^M(\varphi, \varphi) = \tfrac{1}{2}U^M_{0,\infty}\langle\varphi, \varphi\rangle + \int \nu^M(dy)\{\pi_1^M p^D(y) + \pi_1^M r^D(y) + \pi_1^M \kappa 1_D(y) + \pi_*^M s_D(y)\}\varphi^2(y).$$

The *extended universal Dirichlet space* on M is the collection $\mathbf{N}^M_{(e)}$ of functions φ specified up to ν^M equivalence on M such that $N^M(\varphi, \varphi) < +\infty$. The universal Dirichlet space on M is the intersection $\mathbf{N}^M = \mathbf{N}^M_{(e)} \cap L^2(\nu^M)$.

The above definition is consistent which Definition 6.4 in [15]. We prove

THEOREM 5.4. *$\mathbf{H}^M_{(e)}$ is contained in $\mathbf{N}^M_{(e)}$ and*

$$Q^M(\varphi, \varphi) - N^M(\varphi, \varphi)$$

is contractive on $\mathbf{H}^M_{(e)}$.

The meaning of Theorem 5.4 is this. If φ belongs to $\mathbf{H}^M_{(e)}$ and if ψ is a normalized contraction of φ then

$$Q^M(\varphi, \varphi) - N^M(\varphi, \varphi) \geq Q^M(\psi, \psi) - N^M(\psi, \psi).$$

In particular we can take $\psi = 0$ and therefore

$$Q^M(\varphi, \varphi) \geq N^M(\varphi, \varphi).$$

Theorem 5.4 is a special case of Theorem 5.8 in [15]. Because of its importance we repeat the proof here with appropriate simplifications, at the same time providing details which were taken for granted in [15].

It suffices to consider φ in $\mathbf{H}^M$ bounded and we assume this throughout the argument. For $0 < u < v$,

$$\begin{aligned}
E_u([v - u]G_v H^M_u\varphi, H^M_u\varphi)
&= E_u([v - u]H^M_v G_v H^M_u\varphi, H^M_u\varphi) \\
&= Q^M_{(u)}([v - u]R^M_{(v)}\pi^M_v H^M_u\varphi, \varphi) \\
&= Q^M_{(v)}([v - u]R^M_{(v)}\pi^M_v H^M_u\varphi, \varphi) \\
&\qquad - (v - u)^2 \int \nu^M(dy)\pi^M_v H^M_u R^M_{(v)}\pi^M_v H^M_u\varphi(y)\varphi(y) \\
&= (v - u)\int \nu^M(dy)(1 - (v - u)\pi^M_v H^M_u R^M_{(v)})\pi^M_v H^M_u\varphi(y)\varphi(y)
\end{aligned}$$

and therefore

$$\begin{aligned}
Q^M_{(u)}(\varphi, \varphi)
&= \mathrm{Lim}_{v\uparrow\infty}\,(v - u)\int \nu^M(dy)(1 - (v - u)\pi^M_v H^M_u R^M_{(v)})\pi^M_v H^M_u\varphi(y)\varphi(y).
\end{aligned}$$

Let $\lambda_{u,v}(dy, dz)$ be the unique symmetric measure on $M \times M$ such that

$$\iint \lambda_{u,v}(dy, dz)\varphi(y)\varphi(z) = (v - u)^2\int \nu^M(dy)\pi^M_v H^M_u R^M_{(v)}\pi^M_v H^M_u\varphi(y)\varphi(y).$$

Then also

$$Q^M_{(u)}(\varphi, \varphi)$$
$$= \operatorname{Lim}_{v\uparrow\infty}\Big[\int \nu^M(dy)\{1 - (v-u)R^M_{(v)}\pi^M_v H^M_u 1(y)\}(v-u)\pi^M_v H^M_u \varphi^2(y)$$
$$- \frac{1}{2}U^M_{u,v}\langle\varphi,\varphi\rangle + \frac{1}{2}\int\int \lambda_{u,v}(dy,dz)\{\varphi(y)-\varphi(z)^2\}\Big]$$

and therefore if ψ is a normalized contraction of φ,

$$Q^M_{(u)}(\varphi, \varphi) + \tfrac{1}{2}U^M_{u,\infty}\langle\varphi,\varphi\rangle - Q^M_{(u)}(\psi,\psi) - \tfrac{1}{2}U^M_{u,\infty}\langle\psi,\psi\rangle$$
$$\geq \operatorname{Lim\,inf}_{v\uparrow\infty}\int \nu^M(dy)\{1-(v-u)R^M_{(v)}\pi^M_v H^M_u 1(y)\}$$
$$\cdot(v-u)\pi^M_v H^M_u(\varphi^2-\psi^2)(y). \tag{5.21}$$

But

$$vR^M_{(v)}\pi^M_v 1 = vG_v 1 \leq 1 \quad [\text{a.e. } \nu^M]$$

and therefore (5.21) is valid with the right side replaced by

$$\operatorname{Lim\,inf}_{v\uparrow\infty}\int \nu^M(dy)\pi^M_v\{v1 - (v-u)H^M_u 1\}$$
$$\cdot(y)(v-u)R^M_{(v)}\pi^M_v H^M_u(\varphi^2-\psi^2)(y).$$

On the one hand

$$(v-u)R^M_{(v)}\pi^M_v H^M_u(\varphi^2-\psi^2)(y) = (v-u)G_v H^M_u(\varphi^2-\psi^2)(y)$$

converges [a.e. ν^M] to $\varphi^2(y) - \psi^2(y)$ as $v \uparrow \infty$ through a subsequence and on the other hand

$$\pi^M_v\{v1 - (v-u)H^M_u 1\} = \pi^M_v\{vp^D + vr^D + vs^D + vH^M 1 - (v-u)H^M 1$$
$$+ (v-u)uG^D_u H^M 1\}$$
$$= v\pi^M_v\{p^D + r^D + s^D\} + u\pi^M_u H^M 1$$
$$= \pi^M_1\{p^D + r^D + 1_D\kappa\} - \pi^M_v 1_D\kappa$$
$$+ v\pi^M_v s^D + u\pi^M_u H^M 1$$

and it follows that

$$Q^M_{(u)}(\varphi,\varphi) + \tfrac{1}{2}U^M_{u,\infty}\langle\varphi,\varphi\rangle - U^M_{0,u}(1,\varphi^2)$$
$$- \int \nu^M(dy)\{\pi^M_1(p^D + r^D + 1_D\kappa)(y) + \pi^M_* s^D(y)\}\varphi^2(y)$$

dominates the corresponding expression with φ replaced by ψ. The theorem follows upon letting $u \uparrow \infty$ since

$$Q^M_{(u)}(\varphi,\varphi) + \tfrac{1}{2}U^M_{u,\infty}\langle\varphi,\varphi\rangle - U^M_{0,u}(1,\varphi^2)$$
$$= Q^M(\varphi,\varphi) - \tfrac{1}{2}U^M_{0,u}\langle\varphi,\varphi\rangle + \tfrac{1}{2}U^M_{u,\infty}\langle\varphi,\varphi\rangle.$$

Remark. The proof of Theorem 6.2 in [19] would establish the cruder result

$$(5.22) \qquad Q^M_{(u)}(\varphi, \varphi) - \tfrac{1}{2}U^M_{u,v}\langle\varphi, \varphi\rangle \geq Q^M_{(u)}(\psi, \psi) - \tfrac{1}{2}U^M_{u,v}\langle\psi, \psi\rangle.$$

This has a simple heuristic interpretation which we repeat here. The process corresponding to $Q^M_{(v)}$ is in part obtained from the process corresponding to $Q^M_{(u)}$ by replacing "jumps at the rate $U^M_{u,v}(\cdot, z)\nu^M(dz)$" by "killing at the rate $(v - u)\pi^M_v H^M_u 1_D$". Therefore $Q^M_{(u)}$ must already contain "jumping at the rate $U^M_{u,v}(\cdot, z)\nu^M(dz)$". That is, (5.22) must be valid.

Added in proof. In fact our proof of Theorem 5.4 is not valid unless we first establish the cruder estimate (5.22).

6. Associated functionals

For t a constant time on Ω or Ω_∞ the *past* $\mathfrak{F}_t$ is the σ-algebra generated by the trajectory variables X_s, $s \leq t$. For τ a variable time on Ω or Ω_∞ the *past* $\mathfrak{F}_\tau$ is the σ-algebra of Borel subsets A of Ω such that

$$A \cap [\tau \leq t] \quad \text{is in} \quad \mathfrak{F}_t$$

for all constant times t. This terminology is consistent with standard usage.

Meyer's decomposition theory for supermartingales and for square integrable martingales plays an important role in this and the next section. We cite [21] as a general reference.

6.1. DEFINITION. An *additive functional* on the extended sample space Ω_∞ is a real valued function $\alpha(t, \omega)$ defined and jointly measurable for t real and for ω in Ω_∞ and satisfying the following conditions

6.1.1. $\alpha(t, \cdot)$ is $\mathfrak{F}_t$ measurable,
6.1.2. $\alpha(t, \omega) = \alpha(\zeta^*, \omega) = 0$ for $t \leq \zeta^*$,
6.1.3. $\alpha(t, \omega) = \alpha(\zeta, \omega)$ for $t \geq \zeta$,
6.1.4. $\alpha(t + h, \omega) - \alpha(t, \omega) = \alpha(t' + h, \omega') - \alpha(t', \omega')$,

whenever $X_{t+s}(\omega) = X_{t'+s}(\omega')$ for $0 \leq s \leq h$.

6.2. DEFINITION. An *additive functional* on the standard sample space Ω is a real valued function $\alpha(t, \omega)$ defined and jointly measurable for $t \geq 0$ and for ω in Ω and satisfying the following conditions

6.2.1. $\alpha(t, \cdot)$ is $\mathfrak{F}_t$ measurable,
6.2.2. $\alpha(t, \omega) = \alpha(\zeta, \omega)$ for $t \geq \zeta$,
6.2.3. $\alpha(t + h, \omega) - \alpha(t, \omega) = \alpha(t' + h, \omega') - \alpha(t', \omega')$,

whenever $X_{t+s}(\omega) = X_{t'+s}(\omega')$ for $0 \leq s \leq h$.

Properties 6.1.4 and 6.2.3 guarantee that an additive functional is always perfect in the sense of [24]. In the constructions given below this property can always be obtained by first selecting a sequence of approximating functionals for which the appropriate limits exist almost everywhere and then defining the limiting functional by an explicit limiting procedure involving this sequence which makes sense for all sample paths. We take this for granted throughout the section.

For $\varphi \geq 0$ on $\mathbf{X}$ define

$$A(\varphi; t) = \int_{\zeta^*}^{t} \varphi(X_s)\, ds$$

on the extended sample space Ω_∞ with the understanding that $A(\varphi; t) = 0$ for $t < \zeta^*$. It is easy to check that

$$\mathcal{E}A(\varphi; \zeta) = \int m(dx)\varphi(x) \tag{6.1}$$

$$\frac{1}{2}\,\mathcal{E}\{A(\varphi, \zeta)\}^2 = \int m(dx)\varphi(x)G\varphi(x). \tag{6.2}$$

For μ in $\mathfrak{M}$ the functions $uG_u\, G\mu \to G\mu$ in $\mathbf{F}_{(e)}$ and also increase to $G\mu$ quasi-everywhere as $u \uparrow \infty$. For typographical convenience put

$$\varphi_u = uG_u\, \mu.$$

Then for $0 < u < v$,

$$\begin{aligned} &\tfrac{1}{2}\mathcal{E}\{A(\varphi_v; \zeta) - A(\varphi_u, \zeta)\}^2 \\ &\qquad = \int m(dx)\{\varphi_v(x) - \varphi_u(x)\}\{G\varphi_v(x) - G\varphi_u(x)\} \\ &\qquad = E(G\varphi_v - G\varphi_u, G\varphi_v - G\varphi_u) \end{aligned}$$

which $\to 0$ as $u, v \uparrow \infty$. Thus as $u \uparrow \infty$,

$$A(\mu, \zeta) = \operatorname{Lim} A(\varphi_u; \zeta)$$

exists in the L^2 sense relative to $\mathcal{P}$ and therefore relative to $\mathcal{P}^{(k)} = \int L_k(dx)P_x$. For $t \geq 0$ clearly

$$\mathcal{E}^{(k)}(A(\varphi_u; \zeta) \mid \mathfrak{F}_t) = A(\varphi_u; t) + G\varphi_u(X_t).$$

By the maximal inequality for martingales

$$\sup \mid \mathcal{E}^{(k)}(A(\varphi_v; \zeta) \mid \mathfrak{F}_t) - \mathcal{E}^{(k)}(A(\varphi_u; \zeta) \mid \mathfrak{F}_t) \mid \to 0$$

in probability as $u, v \uparrow \infty$ and by (4.6),

$$\sup \mid G\varphi_v(X_t) - G\varphi_u(X_t) \mid \to 0$$

in probability relative to $\mathcal{E}$ and therefore relative to $\mathcal{E}^{(k)}$ as $u, v \uparrow \infty$. (The above suprema are taken for $t \geq 0$ and rational.) Thus after passing to the limit $k \uparrow \infty$ and taking into account Meyer's uniqueness results for the decomposition of supermartingales, we deduce

THEOREM 6.1. *For μ in $\mathfrak{M}$ there is a nonnegative additive functional $A(\mu; t)$ on Ω_∞ which is unique up to $\mathcal{P}$ equivalence and satisfies the following conditions.*

(i) *Except for a $\mathcal{P}$ null set of sample paths, $A(\mu;\cdot)$ is continuous and nondecreasing.*

(ii) *For each k and for $t \ge 0$,*

$$\mathcal{E}(A(\mu;\zeta) \mid \mathcal{F}_{\sigma(D_k)+t},\ \sigma(D_k) < +\infty) = A(\mu;\sigma(D_k)+t) + G\mu(X_{\sigma(D_k)}+t). \tag{6.3}$$

The identities

$$\mathcal{E}A(\mu;\zeta) = \int \mu(dx) \tag{6.1'}$$

$$\tfrac{1}{2}\mathcal{E}\{A(\mu;\zeta)\}^2 = E(\mu) \tag{6.2'}$$

can be established either by applying (6.3) or by passing to the limit in (6.1) and (6.2). Also the proof of Theorem 3.3 in [15] is easily adapted to establish

$$A(\varphi\cdot\mu;t) = \int_{\zeta^*}^{t} A(\mu;ds)\varphi(X_s) \tag{6.4}$$

for almost every sample path whenever μ and $\varphi\cdot\mu$ both belong to $\mathfrak{M}$. This leads to a *property of universality* for the measure $\mathcal{P}$.

THEOREM 6.2. *Let $\xi \ge 0$ on Ω and let μ be in $\mathfrak{M}$. Then*

$$\int \mu(dx)\mathcal{E}_x\xi = \mathcal{E}\int_{\zeta^*}^{\zeta} A(\mu;dt)\xi\circ\theta_t. \tag{6.5}$$

Of course the *shift* θ_t is interpreted as a mapping from Ω_∞ to Ω and is defined by

$$\theta_t\,\omega(s) = \omega(t+s).$$

For the proof it suffices to observe that if $\varphi(x) = \mathcal{E}_x\xi$ then the right side of (6.5)

$$= \mathcal{E}\int_{\zeta^*}^{\zeta} A(\mu,dt)\varphi(X_t) = \mathcal{E}A(\varphi\cdot\mu,\zeta) = \int \mu(dx)\varphi(x).$$

For $f = G\varphi$ with φ in $\mathcal{S}$ define

$$Mf(t) = f(X_t) + \int_{\zeta^*}^{t} ds\varphi(X_s). \tag{6.6}$$

It is easy to check that

$$Mf(\zeta) = \int_{\zeta^*}^{\zeta} ds\,\varphi(X_s)$$

and therefore

$$E(f,f) = \tfrac{1}{2}\mathcal{E}\{Mf(\zeta)\}^2. \tag{6.7}$$

Passing to the limit in f and arguing as for Theorem 6.1, we prove

THEOREM 6.3. *For f in $\mathbf{F}_{(e)}$ there is an additive functional $Mf(t)$ on Ω_∞ which is unique up to $\mathcal{P}$ equivalence and sztisfies the following conditions.*

(i) *Except for a $\mathcal{P}$ null set of sample paths, $Mf(t) - f(X_t)$ is continuous.*

(ii) *Conditioned on the set $[\sigma(D_k) < +\infty]$, the process $Mf(\sigma(D_k) + t)$ is a martingale relative to the σ-algebras $\mathfrak{F}_{\sigma(D_k)+t}$.*

(iii) *Mf is given by (6.6) whenever $f = G\varphi$ with φ in S and (6.7) is valid for all $\mathbf{F}_{(e)}$.*

It is easy to see that

$$Mf(t) = f(X_t) + A(\mu; t) \tag{6.8}$$

whenever $f = G\mu$ with μ in M.

It will also be convenient to define the processes $A(\mu, t)$ and $Mf(t)$ directly on the standard sample space Ω. Our main tool for doing this is the "property of universality" (6.5). To see how this works fix f in $\mathbf{F}_{(e)}$ and choose f_n such that each $f_n = G\varphi_n$ with φ_n the difference of two functions in $\mathcal{S}$, such that $f_n \to f$ in $\mathbf{F}_{(e)}$ and such that except for a $\mathcal{P}$ null subset of Ω_∞ the functionals $Mf_n(t)$ converge uniformly in t as $n \uparrow \infty$. If ξ is the indicator of the set where Mf_n does not converge uniformly, then by (6.5),

$$\int \mu(dx)\mathcal{E}_x\, \xi = 0$$

for all μ in $\mathfrak{M}$ and it follows that for quasi-every x in $\mathbf{X}$ also

$$Mf_n(t) = f_n(X_t) + \int_0^t \varphi_n(X_s)\, ds$$

converges uniformly except for a $\mathcal{P}_x$ null subset of Ω. Another application of (6.5) shows that for quasi-every x the limiting process $Mf(t)$ is a square integrable martingale. This plus a similar argument for the functionals $A(\mu; t)$ suffices to establish the following two theorems.

THEOREM 6.4. *For f in $\mathbf{F}_{(e)}$ there is an additive functional $Mf(t)$ defined on the standard sample space Ω and satisfying the following conditions.*

(i) *If $f = G\varphi$ with φ the difference of two functions in $\mathcal{S}$, then*

$$Mf(t) = f(X_t) + \int_0^t \varphi(X_s)\, ds.$$

(ii) *For general f in $\mathbf{F}_{(e)}$ there exists a sequence f_n as in (i) such that $f_n \to f$ in $\mathbf{F}_{(e)}$ and such that for quasi-every x in $\mathbf{X}$,*

$$Mf_n(t) \to Mf(t)$$

uniformly in t as $n \uparrow \infty$ except for a $\mathcal{P}_x$ null set.

(iii) *For quasi-every x the process $Mf(t)$ is a square integrable martingale* *relative to $\mathcal{P}_x$ and the difference $Mf(t) - f(X_t)$ is continuous in t except for a $\mathcal{P}_x$ null set; also $Mf(0) = f(X_0)$.*

Moreover if $M'f(t)$ is another additive functional satisfying (i) *and* (ii), *then for quasi-every x the functionals $Mf(t)$ and $M'f(t)$ are $\mathcal{P}_x$ equivalent.*

THEOREM 6.5. *For ν in M there is a nonnegative additive functional $A(\nu; t)$ defined on the standard sample space Ω and satisfying the following conditions.*

(i) *For quasi-every x the functional is nondecreasing and continuous in t except for a $\mathcal{P}_x$ null set; also $A(\mu, 0) = 0$.*

(ii) *For quasi-every x and for $t \geq 0$,*

$$\mathcal{E}_x(A(\mu, \zeta) \mid \mathfrak{F}_t) = G\mu(X_t) + A(\mu; t).$$

Moreover if $A'(\mu; t)$ is another additive functional satisfying (i) *and* (ii) *then for quasi-every x the functionals $A(\mu; t)$ and $A'(\mu; t)$ are $\mathcal{P}_x$ equivalent.*

By considering first f in $\mathbf{F}_{(e)} \cap C_{\mathrm{com}}(\mathbf{X})$ and passing to the limit as above, we prove also

THEOREM 6.6. *Let f be in $F_{(e)}$. For quasi-every x and except for a $\mathcal{P}_x$ null subset of Ω the functional $f(X_t)$ is right continuous with one sided limits everywhere and with discontinuities only at the discontinuities of the trajectory X_t; also $f(X_t) \to 0$ as $t \uparrow \infty$. The same is true except for a $\mathcal{P}$ null subset of Ω_∞ and in addition $f(X_t) \to 0$ as $t \downarrow -\infty$.*

Remark. Some of the results established above are paralleled by results established in Sections 3 and 7 of [15]. The main difference is that here we make systematic use of the measure $\mathcal{P}$ and the identity (6.5).

For f in $\mathbf{F}_{(e)}$ the process $Mf(t)$ is a square integrable martingale adapted to the pasts $\mathfrak{F}_t$ relative to the measures $\mathcal{P}_x$ for quasi-every x. For such x let $\langle Mf\rangle(t)$ be the unique nondecreasing continuous process adapted to the $\mathfrak{F}_t$ such that $\langle Mf\rangle(0) = 0$ and such that $\{Mf(t)\}^2 - \langle Mf\rangle(t)$ is a martingale. Also let $M_c f(t)$ be the continuous part of $Mf(t)$ and let $\langle M_c f\rangle(t)$ be the unique nondecreasing continuous process adapted to the $\mathfrak{F}_t$ such that

$$\{M_c f(t)\}^2 - \langle M_c f\rangle(t)$$

is a martingale. Transferring structure to Ω_∞ in the obvious way we see that also $\langle Mf\rangle(t)$ and $\langle M_c f\rangle(t)$ are well defined on Ω_∞ up to a $\mathcal{P}$ null set by the conventions

$$\langle Mf\rangle(\zeta^* - 0) = \langle M_c f\rangle(\zeta^*) = 0, \qquad \langle Mf\rangle(\zeta^*) = I(X_{\zeta^*} \neq \partial) f^2(X_{\zeta^*}).$$

Both $\langle Mf\rangle$ and $\langle M_c f\rangle$ will be used below. The process $\langle Mf\rangle$ is convenient for calculations but $\langle M_c f\rangle$ is often better for stating results since its increments are invariant under time reversal.

We remark that the preceding can be refined to define $\langle Mf\rangle$, $M_c f$ and $\langle M_c f\rangle$ as additive functions.

7. Formulae for Dirichlet norms

For D an open subset of $\mathbf{X}$ and for f defined on $\mathbf{X}$ we introduce the special notation

$$\sum_D \Delta f^2 = \sum_t \{f(X_t) - f(X_{t-0})\}^2$$

where t runs over all times such that either X_t or X_{t-0} belongs to D. We also introduce entrance and return times for excursions into D as in Section 7 of [15]. The random set

$$\{t > \sigma(M) : X_t \text{ and } X_{t-0} \text{ are in } D\}$$

is a finite or countable union of disjoint intervals $I = (e, r)$. We index the intervals $I_i = (e(i), r(i))$ so that the entrance times $e(i)$ and the return times $r(i)$ are Borel measurable. In general this indexing does not respect the "natural ordering" and neither the $e(i)$ nor the $r(i)$ are stopping times. (See however Section 1 and 9 in [19] and Sections 3 and 5 in [20] where the natural ordering is respected and where the entrance and return times are stopping times.) We also introduce for τ a random time the special notation

$$f_a(X_\tau) = I(\tau < \zeta) f(X_\tau).$$

Our first result is

THEOREM 7.1. *Let D be an open subset of $\mathbf{X}$, let $M = \mathbf{X} - D$ and let f be in the extended Dirichlet space $\mathbf{F}_{(e)}$. Then*

$$\begin{aligned} \tfrac{1}{2}\mathcal{E} \int_{\zeta^*}^{\zeta} \langle M_c f\rangle(dt) 1_D(X_t) + \tfrac{1}{2}\mathcal{E} \textstyle\sum_D f^2 &= E(f - H^M f, f - H^M f) \\ \text{(7.1)} \qquad &+ \tfrac{1}{2}\mathcal{E} \textstyle\sum_i \{f_a(X_{r(i)}) - f(X_{e(i)-0})\}^2 \\ &+ \tfrac{1}{2}\mathcal{E} I(\zeta^* < \sigma(M) < \zeta)\{f(X_{\sigma(M)})\}^2. \end{aligned}$$

Theorem 7.1 parallels Theorem 7.3 in [15]. However our notations are not entirely consistent with [15] and care must be taken in making direct comparisions. Also the proof given in [15] is incorrect, the main problem being that the approximating times are poorly chosen.

For a correct proof of Theorem 7.1 we consider first the standard sample Ω and fix an open subset D' having compact closure in D. Approximating entrance and return times are defined by

$$\begin{aligned} \varepsilon(1) &= \inf\{t > \sigma(M) : X_t \text{ is in } D'\} \\ \rho(1) &= \inf\{t > \varepsilon(1) : X_t \text{ or } X_{t-0} \text{ is in } M\} \\ \varepsilon(2) &= \inf\{t > \rho(1) : X_t \text{ is in } D'\} \\ &\text{etc.} \end{aligned}$$

with the usual understanding that these times are $+\infty$ when not otherwise defined. We begin with the special case $f = G\varphi$ with φ in $\mathcal{S}$ and compute

$$
\begin{aligned}
\tfrac{1}{2}\mathcal{E}_x \int_0^{\sigma(M)} \langle Mf\rangle(dt) &= \tfrac{1}{2}\mathcal{E}_x\{Mf(\sigma(M)) - Mf(0)\}^2 \\
&= \tfrac{1}{2}\mathcal{E}_x\left\{f(X_{\sigma(M)}) - f(X_0) + \int_0^{\sigma(M)} dt\, \varphi(X_t)\right\}^2 \\
&= \mathcal{E}_x \int_0^{\sigma(M)} dt\, \varphi(X_t) \int_t^{\sigma(M)} ds\, \varphi(X_s) \\
&\quad + \tfrac{1}{2}\mathcal{E}_x\{f(X_{\sigma(M)}) - f(X_0)\}^2 \\
&\quad + \mathcal{E}_x\{f(X_{\sigma(M)}) - f(X_0)\} \int_0^{\sigma(M)} dt\, \varphi(X_t). \\
&= \mathcal{E}_x \int_0^{\sigma(M)} dt\, \varphi(X_t) G^D\varphi(X_t) \\
&\quad + \tfrac{1}{2}\mathcal{E}_x f^2(X_{\sigma(M)}) - \tfrac{1}{2}\mathcal{E}_x f^2(X_0) \\
&\quad - \mathcal{E}_x f(X_0)\{f(X_{\sigma(M)}) - f(X_0)\} \\
&\quad + \mathcal{E}_x f(X_{\sigma(M)}) \int_0^{\sigma(M)} dt\, \varphi(X_t) \\
&\quad - \mathcal{E}_x f(X_0) \int_0^{\sigma(M)} dt\, \varphi(X_t)
\end{aligned}
\tag{7.2}
$$

and since

$$
\mathcal{E}_x f(X_0) \left\{f(X_{\sigma(M)}) - f(X_0) + \int_0^{\sigma(M)} dt\, \varphi(X_t)\right\} = 0
$$

we get

$$
\begin{aligned}
\tfrac{1}{2}\mathcal{E}_x &\int_0^{\sigma(M)} \langle Mf\rangle(dt) + \tfrac{1}{2}\mathcal{E}_x f^2(X_0) \\
&= \mathcal{E}_x \int_0^{\sigma(M)} dt\, \varphi(X_t) G^D\varphi(X_t) + \tfrac{1}{2}\mathcal{E}_x f^2(X_{\sigma(M)}) \\
&\quad + \mathcal{E}_x f(X_{\sigma(M)}) \int_0^{\sigma(M)} dt\, \varphi(X_t).
\end{aligned}
\tag{7.3}
$$

Next a computation exactly analagous to (7.2) establishes for each i,

$$
\begin{aligned}
\tfrac{1}{2}\mathcal{E}_x \int_{\varepsilon(i)}^{\rho(i)} \langle Mf\rangle(dt) &= \mathcal{E}_x \int_{\varepsilon(i)}^{\rho(i)} dt\, \varphi(X_t) G^D\varphi(X_t) \\
&\quad + \tfrac{1}{2}\mathcal{E}_x\{f_a(X_{\rho(i)}) - f(X_{\varepsilon(i)})\}^2 \\
&\quad + \mathcal{E}_x\{f_a(X_{\rho(i)}) - f(X_{\varepsilon(i)})\} \int_{\varepsilon(i)}^{\rho(i)} dt\, \varphi(X_t).
\end{aligned}
\tag{7.4}
$$

To transform this we assume the intervals (e_i, r_i) labeled so that

$$
e_i = \sup\{t < \varepsilon_i : X_t \text{ or } X_{t-0} \text{ is in } M\}
$$

whenever $\varepsilon_i < +\infty$. Then for such i clearly

$$\mathcal{E}_x\{f(X_{\varepsilon(i)}) - f(X_{e(i)-0})\}\left\{f_a(X_{r(i)}) - f(X_{\varepsilon(i)}) + \int_{\varepsilon(i)}^{r(i)} dt\, \varphi(X_t)\right\} = 0$$

and so the same algebraic manipulations as above lead to

$$\begin{aligned} &\tfrac{1}{2}\mathcal{E}_x \int_{\varepsilon(i)}^{\rho(i)} \langle Mf\rangle(dt) + \tfrac{1}{2}\mathcal{E}_x\{f(X_{\varepsilon(i)}) - f(X_{e(i)-0})\}^2 \\ &= \mathcal{E}_x \int_{\varepsilon(i)}^{\rho(i)} dt\, \varphi(X_t) G^D\varphi(X_t) + \tfrac{1}{2}\mathcal{E}_x\{f_a(X_{r(i)}) - f(X_{e(i)-0})\}^2 \\ &\qquad + \mathcal{E}_x\{f_a(X_{r(i)}) - f(X_{e(i)-0})\}\int_{\varepsilon(i)}^{r(i)} dt\, \varphi(X_t). \end{aligned} \tag{7.5}$$

Next we sum (7.5) over i, combine with (7.3) and pass to the limit $D' \uparrow D$ to establish

$$\begin{aligned} &\tfrac{1}{2}\mathcal{E}_x \int_0^\zeta \langle Mf\rangle(dt) 1_D(X_t) + \tfrac{1}{2}\mathcal{E}_x f^2(X_0) + \tfrac{1}{2}\mathcal{E}_x \textstyle\sum_i \{f(X_{e(i)}) - f(X_{e(i)-0})\}^2 \\ &\quad = \mathcal{E}_x \int_0^\zeta dt \varphi(X_t) G^D\varphi(X_t) + \tfrac{1}{2}\mathcal{E}_x f^2(X_{\sigma(M)}) \\ &\qquad + \tfrac{1}{2}\mathcal{E}_x \textstyle\sum_i \{f_a(X_{r(i)-0})\}^2 \\ &\qquad + \mathcal{E}_x f(X_{\sigma(M)}) \int_0^{\sigma(M)} dt \varphi(X_t) \\ &\qquad + \mathcal{E}_x \textstyle\sum_i \{f_a(X_{r(i)}) - f(X_{e(i)-0})\} \int_{e(i)}^{r(i)} dt \varphi(X_t). \end{aligned} \tag{7.6}$$

The difficulty in the last passage to the limit is controlling the terms

$$\tfrac{1}{2}\mathcal{E}_x \{f(X_{e(i)}) - f(X_{e(i)-0})\}^2.$$

But this is easily done with the help of the maximal inequality for martingales. Integrating (7.6) with respect to $L_k(dx)$ and then passing to the limit $k \uparrow \infty$ establishes

$$\begin{aligned} &\tfrac{1}{2}\mathcal{E} \int_{\zeta^*}^\zeta \langle Mf\rangle(dt) 1_D(X_t) + \tfrac{1}{2}\mathcal{E}_x I(\zeta^* > -\infty, \zeta^* = \sigma(D)) \{f(X_{\zeta^*})\}^2 \\ &\qquad + \tfrac{1}{2}\mathcal{E} \textstyle\sum_i \{f(X_{e(i)}) - f(X_{e(i)-0})\}^2 \\ &\quad = \mathcal{E} \int_{\zeta^*}^\zeta dt \varphi(X_t) G^D\varphi(X_t) \\ &\qquad + \tfrac{1}{2}\mathcal{E} \textstyle\sum_i \{f_\alpha(X_{r(i)}) - f(X_{e(i)-0})\}^2 \\ &\qquad + \tfrac{1}{2}\mathcal{E} I(\zeta^* < \sigma(M) < +\infty) \{f(X_{\sigma(M)})\}^2 \\ &\qquad + \mathcal{E} f(X_{\sigma(M)}) \int_{\zeta^*}^{\sigma(M]} dt \varphi(X_t) \\ &\qquad + \mathcal{E} \textstyle\sum_i \{f(X_{r(i)}) - f(X_{e(i)-0})\} \int_{e(i)}^{r(i)} dt \varphi(X_t). \end{aligned} \tag{7.7}$$

The theorem follows after passage to the limit in f since

$$\mathcal{E}f(_{\sigma(M)}) \int_{\zeta^*}^{\sigma[M)} dt\varphi(X_t) + \mathcal{E} \sum_i \{f(X_{r(i)}) - f(_{e(i)-0})\} \int_{e(i)}^{r(i)} dt\varphi(X_t)$$

changes sign under time reversal and therefore must vanish, since the left side of (7.7) equals

$$\frac{1}{2}\,\mathcal{E} \int_{\zeta^*}^{\zeta} \langle M_c f\rangle(dt) 1_D(X_t) + \frac{1}{2} E \sum_D \Delta f^2$$

and since

$$\mathcal{E} \int_{\zeta^*}^{\zeta} dt\varphi(X_t) G^D\varphi(X_t) = \int m(dx)\varphi(x) G^D\varphi(x) = E(f - H^M f, f - H^M f).$$

THEOREM 7.2. *Let D be an open subset of $\mathbf{X}$ and let $M = \mathbf{X} - D$.*

(i) *If φ belongs to the extended universal Dirichlet space $\mathbf{N}^M_{(e)}$, then*

$$\begin{aligned} N^M(\varphi, \varphi) = {}& \tfrac{1}{2}\mathcal{E} \sum_i \{\varphi_\alpha(X_{r(i)}) - \varphi(X_{e(i)-0})\}^2 \\ & + \tfrac{1}{2}\mathcal{E} I(\zeta^* < \sigma(M) < +\infty) \{\varphi(X_{\sigma(M)})\}^2. \end{aligned} \tag{7.8}$$

Moreover φ specified up to ν^M equivalence on M belongs to $\mathbf{N}^M_{(e)}$ if and only if the right side of (7.8) *converges.*

(ii) *If φ belongs to $\mathbf{N}^M_{(e)}$ then $H^M\varphi(x)$ and therefore $H^M_u\varphi(x)$ for $u > 0$ converges for quasi-every x in D.*

(iii) *If f belongs to the extended Dirichlet space $\mathbf{F}_{(e)}$ then*

$$E(f - H^M f, f - H^M f) + N^M(f, f) = \frac{1}{2}\,\mathcal{E} \int_{\zeta^*}^{\zeta} \langle M_c f\rangle(dt) 1_D(X_t) + \frac{1}{2}\,\mathcal{E} \sum_D \Delta f^2.$$

Theorem 7.2 parallels Theorem 7.4 and Theorem 8.6 in [15]. However Theorem 7.2 is a cleaner result and its proof is simpler since we avoid the auxiliary times R_u.

We begin by considering φ specified on M up to ν^M equivalence and we compute

$$\begin{aligned} & U^M_{0,u}\langle\varphi, \varphi\rangle \\ & = \mathcal{E} \int_{\zeta^*}^{\zeta} dt 1_D(X_t) \int u H^M_u(X_t, dy) \int H^M(X_t, dz) \{\varphi(y) - \varphi(z)\}^2 \\ & = \mathcal{E} I(\sigma(M) < \zeta) \int_{\zeta^*}^{\sigma(M)} dt \int u H^M_u(X_t, dy) \{\varphi(y) - \varphi(X_{\sigma(M)})\}^2 \\ & \quad + \mathcal{E} \sum_i I(r(i) < \zeta) \int_{e(i)}^{r(i)} dt \int u H^M_u(X_t, dy) \{\varphi(y) - \varphi(X_{r(i)})\}^2 \end{aligned} \tag{7.9}$$

which after application of time reversal

$$= \mathcal{E} \sum_i \int_{e(i)}^{r(i)} dt \int uH_u^M(X_t, dy)\{\varphi(y) - \varphi(X_{e(i)-0})\}^2$$
$$+ \mathcal{E} \sum_i I(r(i) < \zeta)\{\varphi(X_{r(i)}) - \varphi(X_{e(i)-0})\}^2 \int_{e(i)}^{r(i)} dt\, ue^{-u[r(i)-t]}$$
$$= \mathcal{E} \sum_i I(r(i) < \zeta)\{\varphi(X_{r(i)}) - \varphi(X_{e(i)-0})\}^2\{1 - e^{-u[r(i)-e(i)]}\}.$$

Thus after passage to the limit $u \uparrow \infty$,

$$U_{0,\infty}^M\langle\varphi, \varphi\rangle = \mathcal{E}\sum_i I(r(i) < \zeta)\,\{\varphi(X_{r(i)} - \varphi(X_{e(i)-0})\}^2. \tag{7.10}$$

Similarly

$$\begin{aligned} &\int \nu^M(dy)\{u\pi_u^M(p^D + r^D + s^D)(y)\}\varphi^2(y) \\ &\quad= \int m(dx)\{1 - H^M1(x)\}uH_u^M\varphi^2(x) \\ &\quad= \mathcal{E}\int_{\zeta^*}^{\zeta} dt\{1 - H^M1(X_t)\}uH_u^M\varphi^2(X_t) \\ &\quad= \mathcal{E}I(\sigma(M) < \zeta)\int_{\zeta^*}^{\sigma(M)} dt\{1 - H^M1(X_t)\}\varphi^2(X_{\sigma(M)})ue^{-u[\sigma(M)-t]} \\ &\qquad+ \mathcal{E}\sum_i I(r(i) < \zeta)\int_{e(i)}^{r(i)} dt\{1 - H^M1(X_t)\}\varphi^2(X_{r(i)})ue^{-u[r(i)-t]} \end{aligned} \tag{7.11}$$

which after application of time reversal

$$\begin{aligned} &= \mathcal{E}\sum_i \varphi^2(X_{e(i)-0})\int_{e(i)}^{r(i)} dt\{1 - H^M1(X_t)\}ue^{-u[t-e(i)]} \\ &= \mathcal{E}I(\zeta^* < \sigma^*(M) < \zeta)\varphi^2(X_{\sigma^*(M)-0})\int_{\sigma^*(M)}^{\zeta} dtue^{-u[t-\sigma^*(M)]} \\ &= \mathcal{E}I(\zeta^* < \sigma^*(M) < \zeta)\varphi^2(X_{\sigma^*(M)-0})\,\{1 - e^{-u[\zeta-\sigma^*(M)]}\}. \end{aligned}$$

After passage to the limit $u \uparrow \infty$,

$$\begin{aligned} &\int \nu^M(dy)\{\pi_1^M(p^D + r^D + \kappa 1_D)(y) + \pi_*^M s^D(y)\}\varphi^2(y) \\ &\qquad\qquad= \mathcal{E}I(\zeta^* < \sigma^*(M) < \zeta)\varphi^2(X_{\sigma^*(M)-0}) \end{aligned} \tag{7.12}$$

and after application of time reversal

$$\begin{aligned} &\int \nu^M(dy)\{\pi_1^M(p^D + r^D + \kappa 1_D)(y) + \pi_*^M s^D(y)\}\varphi^2(y) \\ &\qquad\qquad= \mathcal{E}I(\zeta^* < \sigma(M) < \zeta)\varphi^2(X_{\sigma(M)}). \end{aligned} \tag{7.12'}$$

Now (7.8) follows upon combining (7.10), (7.12) and (7.12′) and multiplying through by $\frac{1}{2}$. This proves (i) and (iii) follows directly with the help of

Theorem 7.1. Finally (ii) follows upon combining (7.8) with the principle of universality (6.5).

8. The universal Dirichlet space

We recall that by Convention 3.1,

$$H^{\sim k} = H^{M_k}. \tag{8.1}$$

where $M_k = \mathbf{X} - D_k$.

8.1. DEFINITION. A function h on $\mathbf{X}$, specified and finite up to quasi-equivalence is *u-harmonic*, $u \geq 0$ if for each k,

$$H_u^{\sim k} h = h \quad \text{quasi-everywhere on } D_k.$$

In practice we omit the prefix u- for $u = 0$.

8.2. DEFINITION. A *terminal variable* is a Borel function $h^{\#}$ defined on $\Omega \cap [X_{\zeta-0} = \partial]$ and such that for quasi-every x the function $h^{\#}$ is $\mathcal{P}_x$ integrable and for all k,

$$h^{\#} = h^{\#} \circ \theta_{\sigma(M_k)} \tag{8.2}$$

except for a $\mathcal{P}_x$ null set.

Of course for τ a random time θ_τ is the usual shift transformation defined on $[\tau < +\infty]$ by

$$X_t(\theta_t\, \omega) = X_{\tau(\omega)+t}(\omega).$$

It is easy to see that if $h^{\#}$ is a terminal variable then on $\Omega_\infty \cap [X_{\zeta-0} = \partial]$,

$$h^{\#} \cdot J_{\infty,k} \tag{8.3}$$

is independent of k except for a $\mathcal{P}$ null set. To simplify the notation we continue to use $h^{\#}$ for the random variable defined on Ω_∞ by (8.3). Since for quasi-every x every Borel function on $[X_{\zeta-0} = \partial]$ agrees up to a $\mathcal{P}_x$ null set with a function measurable with respect to the σ-algebra generated by the $\mathcal{F}_{\sigma(M_k)}$, routine arguments establish

LEMMA 8.1. *If $h^{\#}$ is a terminal variable, then*

$$h(x) = \mathcal{E}_x\, h^{\#} \tag{8.4}$$

is harmonic. Conversely h harmonic can be represented by (8.4) with $h^{\#}$ a terminal variable if and only if for quasi-every x the random variables $\{h(X_{\sigma(M_k)})\}$ are uniformly integrable with respect to $\mathcal{P}_x$. In this case

$$h^{\#} = \operatorname{Lim} h(X_{\sigma(M_k)}) \tag{8.4$'$}$$

both almost everywhere and in L^1 relative to $\mathcal{P}_x$.

8.3. *Terminology.* A harmonic function h will be called *resolutive* if it satisfies the condition of Lemma 8.1. In this case the corresponding terminal variable will always be denoted by $h^{\#}$ and we will write

$$h = Hh^{\#} \tag{8.5}$$

8.4. DEFINITION. The *universal Dirichlet form* N is defined on terminal variables $h^{\#}$ by

$$N(h^{\#}, h^{\#}) = \tfrac{1}{2}E\{h^{\#} - h^{\#}\cdot\rho\}^2$$

The *universal Dirichlet space* is the collection $\mathbf{N}$ of terminal variables $h^{\#}$ such that

$$N(h^{\#}, h^{\#}) < +\infty.$$

For h harmonic let $M_c h(t)$, $\langle M_c h\rangle(t)$ and $\langle Mh\rangle(t)$ be defined, when they make sense, as in Section 6 for f in $\mathbf{F}_{(e)}$. Notice in particular that on Ω_∞,

$$\langle Mh\rangle(\zeta^*) = I(X_{\zeta^*} \neq \partial)h^2(X_{\zeta^*}). \tag{8.6}$$

THEOREM 8.2. *Let $h^{\#}$ be a terminal variable and let $h = Hh^{\#}$. Then $h^{\#}$ belongs to the universal Dirichlet space* $\mathbf{N}$ *if and only if*

$$\mathcal{E}\langle Mh\rangle(\zeta) < +\infty \tag{8.7}$$

and in this case

$$N(h^{\#}, h^{\#}) = \tfrac{1}{2}\mathcal{E}\langle Mh\rangle(\zeta).$$

Proof. For each k

$$\begin{aligned} \mathcal{E}\{\langle Mh\rangle(\zeta) - \langle Mh\rangle(\sigma(D_k))\} &= \mathcal{E}\{h^{\#} - h(X_{\sigma(D_k)})\}^2 \\ &= \mathcal{E}\{h(X_{\sigma^*(D_k)-0}) - h^{\#}\circ\rho\}^2. \end{aligned} \tag{8.8}$$

As $k \uparrow \infty$ clearly

$$h(X_{\sigma^*(D_k)-0}) \to I(X_\zeta \neq \partial)h(X_{\zeta-0}) + h^{\#} \quad \text{[a.e. } P]$$

and Fatou's lemma plus the maximal inequality for square integrable martingales guarantees that

$$\mathcal{E}\{h(X_{\sigma^*(D_k)-0}) - h^{\#}\circ\rho\}^2 \to E\{h^{\#} + I(X_\zeta \neq \partial)h(X_{\zeta-0}) - h^{\#}\circ\rho\}^2.$$

Also

$$\mathcal{E}\{\langle Mh\rangle(\zeta) - \langle Mh\rangle(\sigma(D_k))\} \to \mathcal{E}\{\langle Mh\rangle(\zeta) - \langle Mh\rangle(\zeta^*)\}$$

and the theorem follows with the help of (8.6) and time reversal since

$$\begin{aligned} &\mathcal{E}\{h^{\#} + I(X_{\zeta-0} \neq \partial)h(X_{\zeta-0}) - h^{\#}\circ\rho\}^2 \\ &\quad = \mathcal{E}\{h^{\#} - h^{\#}\circ\rho\}^2 I(X_{\zeta-0} = X_{\zeta^*} = \partial) + \mathcal{E}\{h^{\#}\}^2 I(X_{\zeta^*} \neq \partial) \\ &\qquad + \mathcal{E}\{h(X_{\zeta-0}) - h^{\#}\circ\rho\}^2 I(X_{\zeta-0} \neq \partial) \\ &\quad = \mathcal{E}\{h^{\#} - h^{\#}\circ\rho\}^2 I(X_{\zeta-0} = X_{\zeta^*} = \partial) + 2\int \kappa(dx)\mathcal{E}_x\{h^{\#}\}^2 \\ &\qquad - \mathcal{E}I(X_{\zeta^*} \neq \partial)h^2(X_{\zeta^*}). \end{aligned}$$

For h harmonic and resolutive and for τ a random time we introduce the special notation

$$h_r(X_\tau) = I(\tau < \zeta)h(X_\tau) + I(\tau \le \zeta, X_{\zeta-0} = \partial)h^\#.$$

The techniques of Section 7 are easily adapted to prove

THEOREM 8.3. (i) *If D is an arbitrary open set and if h is harmonic and resolutive then*

$$\begin{aligned}\frac{1}{2}\,\mathcal{E}\int_{\zeta^*}^{\zeta}\langle M_c h\rangle(dt)1_D(X_t) + \frac{1}{2}\,\mathcal{E}\textstyle\sum_D h^2 \\ &= \tfrac{1}{2}\mathcal{E}\textstyle\sum_i\{h_r(X_{r(i)}) - h(X_{e(i)-0})\}^2 \\ &\quad + \tfrac{1}{2}\mathcal{E}I(\zeta^* < \sigma(M) < \zeta)\,\{h(X_{\sigma(M)}) - I(X_{\zeta^*} = \partial)h^\# \circ \rho\}^2\end{aligned}$$

(ii) *If D is an open set with compact closure and if h is an arbitrary harmonic function, then*

$$\frac{1}{2}\,\mathcal{E}\int_{\zeta^*}^{\zeta}\langle M_c h\rangle(dt)1_D(X_t) + \frac{1}{2}\,\mathcal{E}\textstyle\sum_D h^2 = N^M(h, h).$$

After taking $D = D_k$ in Theorem 8.3-(ii) and applying Theorem 8.2 we get

COROLLARY 8.4. *If h is harmonic then $N^{\sim k}(h, h)$ increases with k. Also $h = Hh^\#$ with $h^\#$ in the universal Dirichlet space if and only if*

$$\sup_k N^{\sim k}(h, h) < +\infty$$

and in this case

$$N^{\sim k}(h, h) \uparrow N(h^\#, h^\#)$$

as $k \uparrow \infty$.

9. The reflected Dirichlet space

9.1. DEFINITION. A function f belongs to the *reflected Dirichlet space* $\mathbf{F}^r$ if it can be represented by

$$f = Hh^\# + g \tag{9.1}$$

with $h^\#$ in the universal Dirichlet space $\mathbf{N}$ and with g in the extended Dirichlet space $\mathbf{F}_{(e)}$.

From the identification of the operators $H^{\sim k}$ as orthogonal projections on $\mathbf{F}_{(e)}$ it follows that $\mathbf{F}_{(e)}$ contains no harmonic functions and in particular the representation (9.1) is unique for f in $\mathbf{F}^r$. For any such f we write

$$f^\# = h^\#; \qquad Hf = Hh^\#.$$

This double use of the operator H will cause no confusion in practice.

For f in $\mathbf{F}^r$ the processes $Mf(t)$, $M_c f(t)$, $\langle Mf\rangle(t)$ and $\langle M_c f\rangle(t)$ are defined in the obvious way on Ω and then on Ω_∞ with the convention

$$\langle Mf\rangle(\zeta^*) = I(X_{\zeta^*} \ne \partial)f^2(X_{\zeta^*}).$$

9.2. DEFINITION. The *reflected Dirichlet form* E^r is defined on $\mathbf{F}^r$ by

$$E^r(f, f) = \tfrac{1}{2}\mathcal{E}\langle Mf\rangle(\zeta)$$

THEOREM 9.1. *If f belongs to $\mathbf{F}^r$ then*

$$E^r(f, f) = E(f - Hf^{\#}, f - Hf^{\#}) + N(f^{\#}, f^{\#}).$$

In particular the extended Dirichlet space $\mathbf{F}_{(e)}$ and the image $H\mathbf{N}$ of the universal Dirichlet space are mutually orthogonal relative to E^r.

Proof. It suffices to consider $f = Hh^{\#} + G\varphi$ with φ in $\mathcal{S}$. Then

$$\mathcal{E}\{\langle Mf\rangle(\zeta) - \langle Mf\rangle(\sigma(D_k))\} = \mathcal{E}\left\{h^{\#} + \int_{\sigma(D_k)}^{\zeta} dt\varphi(X_t) - f(X_{\sigma(D_k)})\right\}^2$$

and after passing to the limit $k \uparrow \infty$ as in the proof of Theorem 8.2

$$\begin{aligned}
&\mathcal{E}\{\langle Mf\rangle(\zeta) - \langle Mf\rangle(\zeta^*)\}\\
&= \mathcal{E}\left\{h^{\#} + \int_{\zeta^*}^{\zeta} dt\varphi(X_t) - h^{\#}\circ\rho - I(X_{\zeta^*} \neq \partial)f(X_{\zeta^*})\right\}^2\\
&= \mathcal{E}\{h^{\#} - h^{\#}\circ\rho\}^2 + E\left\{\int_{\zeta^*}^{\zeta} dt\varphi(X_t)\right\}^2 + \mathcal{E}I(X_{\zeta^*} \neq \partial)f^2(X_{\zeta^*})\\
&\qquad - 2\mathcal{E}\{h^{\#} - h^{\#}\circ\rho\}\int_{\zeta^*}^{\zeta} dt\varphi(X_t)\\
&\qquad - 2\mathcal{E}\left\{h^{\#} + \int_{\zeta^*}^{\zeta} dt\varphi(X_t)\right\} I(X_{\zeta^*} \neq \partial)f(X_{\zeta^*}).
\end{aligned} \tag{9.2}$$

The theorem follows since

$$\tfrac{1}{2}\mathcal{E}\{h^{\#} - h^{\#}\circ\rho\}^2 = N(h^{\#}, h^{\#}),$$

$$\frac{1}{2}\mathcal{E}\left\{\int_{\zeta^*}^{\zeta} dt\varphi(X_t)\right\}^2 = E(G\varphi, G\varphi),$$

$$\mathcal{E}\left\{h^{\#} + \int_{\zeta^*}^{\zeta} dt\varphi(X_t)\right\} I(X_{\zeta^*} \neq \partial)f(X_{\zeta^*}) = \mathcal{E}I(X_{\zeta^*} \neq \partial)f^2(X_{\zeta^*}),$$

and since

$$\mathcal{E}\{h^{\#} - h^{\#}\circ\rho\}\int_{\zeta^*}^{\zeta} dt\varphi(X_t)$$

changes sign under time reversal and therefore is zero.

In preparation for our classification results we turn our attention now to the appropriate "active" Dirichlet spaces.

First let $\mathfrak{I}$ be the σ-algebra of subsets Γ of $\Omega \cap [\zeta < +\infty]$ whose indicators 1_Γ are terminal variables. It is easy to check that there is a unique measure ν on $\mathfrak{I}$ determined by

$$\nu(\Gamma) = \int m(dx)\mathcal{E}_x e^{-\zeta}1_\Gamma. \tag{9.3}$$

We refer to $L^2(\nu)$, the real Hilbert space of $\mathfrak{J}$ measurable square integrable functions, as the *terminal Hilbert space.* For $u \geq 0$ and Φ in $L^2(\nu)$ let

$$H_u \Phi(x) = \mathcal{E}_x e^{-u\zeta}\Phi. \tag{9.4}$$

Clearly H_1 is bounded from $L^2(\nu)$ to $L^2(m)$ and from the easily verified identity

$$H_u - H_v = (v - u)G_v H_u, \quad u, v > 0, \tag{9.5}$$

it follows that also for $u > 0$ the operator H_u is bounded from $L^2(\nu)$ to $L^2(m)$. Let π_u be the corresponding adjoint operators from $L^2(m)$ to $L^2(\nu)$ defined by

$$\int m(dx)f(x)H_u\varphi(x) = \int \nu(d\omega)\pi_u f(\omega)\varphi(\omega). \tag{9.6}$$

Note that (9.5) is also valid for $u = 0$ but is valid for $v = 0$ only for functions concentrated on $\Omega \cap [\zeta < +\infty]$.

9.3. Definition. The *active universal Dirichlet space* $\mathbf{N}_a$ is the subcollection of $h^\#$ in $\mathbf{N}$ such that:

9.3.1. For quasi-every x the function $h^\# = 0$ on $[\zeta = +\infty]$ except for a $\mathcal{P}_x$ null set.

9.3.2. $h^\#$ is in $L^2(\nu)$.

9.4. Definition. The *active reflected Dirichlet space* $\mathbf{F}_a^r$ is the subcollection of f in $\mathbf{F}^r$ such that $f^\#$ belongs to the active universal Dirichlet space $\mathbf{N}_a$.

Lemma 9.2. *If Φ belongs to $L^2(\nu)$ then*
(i) *$H_u\Phi$ is in $L^2(m)$ for $u > 0$,*
(ii) *$\int m(dx)H\Phi(x)H_u\Phi(x) < +\infty$ for $u > 0$,*
(iii) *$H_u\Phi - H_v\Phi$ belongs* to $\mathbf{F}$ for $u, v > 0$ and to $\mathbf{F}_{(e)}$ for $u, v \geq 0$.

Proof. (i) follows from the trivial estimate

$$\int m(dx)\{H_1\Phi(x)\}^2 \leq \int \nu(d\omega)\{\Phi(\omega)\}^2$$

and the identity (9.5). Conclusion (ii) follows with the help of (9.5) since for $\Phi \geq 0$, clearly

$$\int m(dx)H\Phi(x)H_1\cdot\Phi(x) \leq \int m(dx)H_1\{\Phi\}^2(x)$$

$$= \int \nu(d\omega)\{\Phi(\omega)\}^2$$

and then (iii) follows since $H_u\Phi - H_v\Phi = (v - u)G_u H_v\Phi$.

Lemma 9.3. *The pair (N_a, N) is a Dirichlet space on $L^2(\nu)$.*

Proof. It is clear that if Φ belongs to $\mathbf{N}_a$ and if $\Psi = T\Phi$ with T a normalized contraction, then also Ψ belongs to $\mathbf{N}_a$ and $N(\Psi, \Psi) \leq N(\Phi, \Phi)$. Therefore it suffices to show that if $\{\Phi_n\}$ is a sequence in $\mathbf{N}_a$ which is Cauchy relative to both the Dirichlet form N and the standard inner product on $L^2(\nu)$, then there exists Φ in $\mathbf{N}_a$ such that

$$\int \nu(d\omega)\,\{\Phi(\omega) - \Phi_n(\omega)\}^2 \to 0 \tag{9.7}$$

$$N(\Phi - \Phi_n, \Phi - \Phi_n) \to 0. \tag{9.7'}$$

To show this first choose Φ in $L^2(\nu)$ such that (9.7) is true. After selecting a subsequence we can assume that $H_1\Phi_n \to H_1\Phi$ except for an m null set and by the estimate

$$\int \nu^{\sim k}(dx)\{H_1\Phi(x) - H_1\Phi_n(x)\}^2 \leq \int \nu^{\sim k}(dx)H_1\{\Phi - \Phi_n\}^2(x)$$

$$= \int \nu(d\omega)\{\Phi(\omega) - \Phi_n(\psi)\}^2$$

we can assume that for every k also $H_1\varphi_n \to H_1\varphi$ except for a $\nu^{\sim k}$ null set. By Lemma 9.2(iii), $(H - H_1)\Phi_n \to (H - H_1)\Phi$ in $\mathbf{F}_{(e)}$ and therefore we can assume that $(H - H_1)\Phi_n \to (H - H_1)\Phi$ quasi-everywhere and finally that $H\Phi_n \to H\Phi$ except for a set which is m null and also $\nu^{\sim k}$ null for all k. Now (9.7′) follows from Fatou's lemma because of Corollary 8.4.

Imitating the local theory in Section 5 we define for $0 \leq u < v$ the symmetric measure

$$U_{u,v}(d\omega, d\omega') = (v - u)\int m(dx)H_u(x, d\omega)H_v(x, d\omega')$$

on $\{\Omega \cap [\zeta < +\infty]\} \times \{\Omega \cap [\zeta < +\infty]\}$ and the bilinear forms

$$U_{u,v}(\Phi, \Phi) = \iint U_{u,v}(d\omega, d\omega')\Phi(\omega)\Phi'(\omega)$$

$$U_{u,v}\langle\Phi, \Phi\rangle = \iint U_{u,v}(d\omega, d\omega')\{\Phi(\omega) - \Phi(\omega')\}^2$$

for terminal variables Φ concentrated on $\Omega \cap [\zeta < +\infty]$ and note the relations

$$U_{u,v}(1, \Phi^2) = \tfrac{1}{2}U_{u,v}\langle\Phi, \Phi\rangle + U_{u,v}(\Phi, \Phi)$$

$$U_{u,w}(d\omega, d\omega') = U_{u,v}(d\omega, d\omega') + U_{v,w}(d\omega, d\omega'), \quad 0 \leq u < v < w.$$

It follows from the last relation that the above makes sense for $v = +\infty$. We emphasize that also for $u = 0$ the measure $U_{u,v}(d\omega, d\omega')$ is concentrated on

$$\{\Omega \cap [\zeta < +\infty]\} \times \{\Omega \cap [\zeta < +\infty]\}.$$

Next we imitate some of the local theory in Section 3. For $u > 0$ and for quasi-every x

$$p(x) = uG_u p(x), \qquad r(x) = uG_u r(x) + \mathcal{E}_x[X_{\zeta-0} \neq \partial; e^{-u\zeta}]$$

and with the help of

$$(9.5') \qquad \pi - \pi_v = (v-u)\pi_u G_v$$

which is dual to (9.5) follows

$$(9.8) \qquad v\pi_v p = u\pi_u p, \qquad 0 < u < v,$$

$$(9.9) \qquad v\pi_v r \geq u\pi_u r, \qquad 0 < u < v.$$

As in Section 3 the latter can be refined to

$$(9.9') \qquad v\pi_v r + \pi_v \kappa = u\pi_u r + \pi_u \kappa, \qquad 0 < u < v$$

with $\pi_u \kappa$ defined as a measure on the terminal σ-algebra $\mathfrak{J}$ by

$$\pi_u \kappa(\Gamma) = \int \kappa(dx)\mathcal{E}_x e^{-u\zeta} 1_\Gamma .$$

THEOREM 9.4. *If Φ belongs to the active universal Dirichlet space $\mathbf{N}_a$ then*

$$(9.10) \qquad \begin{aligned} N(\Phi, \Phi) = \frac{1}{2} U_{0,\infty}\langle\Phi, \Phi\rangle &+ \int \nu(d\omega)\{\pi_1 p(\omega) + \pi_1 r(\omega)\}\Phi^2(\omega) \\ &+ \int \pi_1 \kappa(d\omega)\Phi^2(\omega). \end{aligned}$$

Conversely if Φ is a terminal variable supported by $\Omega \cap [\zeta < +\infty]$ such that the right side of (9.10) *converges, then Φ belongs to $\mathbf{N}_a$.*

This theorem is a global version of Theorem 7.2 and is important for the classification theory of Section 10. To prove it we consider a terminal variable Φ concentrated on $\Omega \cap [\zeta < +\infty]$ and argue as in Section 7. For $u > 0$,

$$\begin{aligned} U_{0,u}\langle\Phi, \Phi\rangle &= \mathcal{E} \int_{\zeta^*}^{\zeta} dt \int uH_u(X_t, d\omega) \\ &\qquad \cdot \int H(X_t, d\omega')I(\zeta(\omega') < +\infty)\{\Phi(\omega) - \Phi(\omega')\}^2 \\ &= \mathcal{E} \int_{\zeta^*}^{\zeta} dt \int uH_u(X_t, d\omega)I(\zeta < +\infty)\{\Phi(\omega) - \Phi\}^2 \end{aligned}$$

which after application of time reversal

$$\begin{aligned} &= \mathcal{E} \int_{\zeta^*}^{\zeta} dt \int uH_u(X_t, d\omega)I(\zeta^* > -\infty)\{\Phi(\omega) - \Phi \circ \rho\}^2 \\ &= \mathcal{E}I(\zeta^* > -\infty)\{\Phi - \Phi \circ \rho\}^2 \int_{\zeta^*}^{\zeta} dt\, ue^{-u[\zeta-t]} \\ &= \mathcal{E}I(-\infty < \zeta^* < \zeta < +\infty)\{\Phi - \Phi \circ \rho\}^2\{1 - e^{-u(\zeta-\zeta^*)}\} \end{aligned}$$

and after passing to the limit $u \uparrow \infty$,

$$(9.11) \qquad U_{0,\infty}\langle\Phi, \Phi\rangle = \mathcal{E}I(-\infty < \zeta < \zeta^* < +\infty)\{\Phi - \Phi \circ \rho\}^2.$$

Similarly

$$(9.12) \qquad \begin{aligned} \int \nu(d\omega)\{u\pi_u(p + r)(\omega)\}\Phi^2(\omega) &= \mathcal{E}\int_{\zeta^*}^{\zeta} dt\{1 - h_0(X_t)\}uH_u\,\Phi^2(X_t) \\ &= \mathcal{E}\int_{\zeta^*}^{\zeta} dt\,\{1 - h_0(X_t)\}ue^{-u(\zeta-t)}\Phi^2 \end{aligned}$$

which after application of time reversal

$$\begin{aligned} &= \mathcal{E}\int_{\zeta^*}^{\zeta} dt\,\{1 - h_0(X_t)\}\,ue^{-u(t-\zeta^*)}\,\{\Phi \circ \rho\}^2 \\ &= \mathcal{E}I(\zeta^* > -\infty)I(\zeta = +\infty \quad \text{or} \quad X_{\zeta-0} \neq \partial)\,\{\Phi \circ \rho\}^2\{1 - e^{-u[\zeta-\zeta^*]}\} \end{aligned}$$

and after passage to the limit $u \uparrow \infty$,

$$(9.13) \qquad \begin{aligned} &\int \nu(d\omega)\{\pi_1(p + r)(\omega)\}\Phi^2(\omega) + \int \pi_1\,\kappa(d\omega)\Phi^2(\omega) \\ &\qquad = \mathcal{E}I(\zeta^* > -\infty)I(\zeta = +\infty \quad \text{or} \quad X_{\zeta-0} \neq \partial)\{\Phi \circ \rho\}^2. \end{aligned}$$

The theorem follows with the help of time reversal after combining $\frac{1}{2}$ of (9.11) with (9.13).

10. Classification

We begin by considering a pair $(\mathbf{H}^*, Q^*)$ where

10.1.1. $(\mathbf{H}^*, Q^*)$ is a Dirichlet space on $L^2(\nu)$,
10.1.2. $\mathbf{H}^*$ is a subset of the active universal Dirichlet space $\mathbf{N}_a$,
10.1.3. $Q^* - N$ is contractive on $\mathbf{H}^*$.

One example is $(\mathbf{H}^*, Q^*) = (\mathbf{N}_a, N)$. For $u > 0$ and for Φ in $\mathbf{H}^*$ define

$$Q^*_{(u)}(\Phi, \Phi) = Q^*(\Phi, \Phi) + U_{0,u}(\Phi, \Phi)$$

and note that

$$\begin{aligned} Q^*_{(u)}(\Phi, \Phi) &= Q^*(\Phi, \Phi) - N(\Phi, \Phi) + \tfrac{1}{2}U_{0,u}\langle\Phi, \Phi\rangle + U_{0,u}(\Phi, \Phi) \\ &\quad + \int \nu(d\omega)\,\{u\pi_u\,p(\omega) + u\pi_u\,r(\omega)\}\,\Phi^2(\omega) \\ &\quad + \int \pi_u\,\kappa(d\omega)\Phi^2(\omega) + \tfrac{1}{2}U_{u,\infty}\langle\Phi, \Phi\rangle \end{aligned}$$

$$(10.1) \qquad \begin{aligned} &= Q^*(\Phi, \Phi) - N(\Phi, \Phi) + \tfrac{1}{2}U_{u,\infty}\langle\Phi, \Phi\rangle + U_{0,u}(1, \Phi^2) \\ &\quad + \int \nu(d\omega)u\pi_u(1 - h_0)(\omega)\Phi^2(\omega) \\ &\quad + \int \pi_u\,\kappa(d\omega)\Phi^2(\omega). \end{aligned}$$

Since

$$U_{0,1}(1, \Phi^2) + \int \nu(d\omega)\pi_1(1 - h_0)(\omega)\Phi^2(\omega) = \int \nu(d\omega)\Phi^2(\omega)$$

and since

$$U_{0,1} \leq \max(1, (1/u)U_{0,u}$$

the form Q^*_u dominates the standard inner product on $L^2(\nu)$ and so there is a unique bounded operator $R_{(u)}$ mapping $L^2(\nu)$ into $\mathbf{H}^*$ and satisfying

$$Q^*_{(u)}(R_{(u)}\Psi, \Phi) = \int \nu(d\omega)\Psi(\omega)\Phi(\omega) \tag{10.2}$$

for Ψ in $L^2(\nu)$ and for Φ in $\mathbf{H}^*$. We define G^*_u on $L^2(m)$ by

$$G^*_u = G_u + H_u R_{(u)} \pi_u. \tag{10.3}$$

and note that for f in $L^2(m)$ the image $G_u f$ is in $\mathbf{F}^r_a$ and

$$G^*_u f = G_u f + H_u \{G^*_u f\}^{\#}.$$

The arguments of Section 5 in [15] suffice to establish the identity

$$R_{(u)} = R_{(v)} + (v - u)R_{(u)} \pi_u H_v R_{(v)} \tag{10.4}$$

which leads in turn to the resolvent identity

$$G^*_u = G^*_v + (v - u)G^*_u G^*_v. \tag{10.5}$$

The proof that the G^*_u are submarkovian is identical with the proof of the corresponding result in Section 6 of [15]. Thus the G^*_u, $u > 0$, form a symmetric submarkovian resolvent on $L^2(m)$. We denote the associated Dirichlet space by $(\mathbf{F}^*, E^*)$. The argument at the end of Section 6 in [15] shows that for f in $L^2(m)$ and for $u > 0$ the functions $G_u f$ and $H_u \{G_u f\}^{\#}$ both belong to $\mathbf{F}^*$ and

$$E^*_u(G^*_u f, G^*_u f) = E_u(G_u f, G_u f) + Q^*_{(u)}(\{G^*_u f\}^{\#}, \{G^*_u f\}^{\#})$$

and it follows that $\mathbf{F}^*$ contains $\mathbf{F}$, that E^* restricted to $\mathbf{F}$ is E, that for $u > 0$ the operator H_u projects $\mathbf{F}^*$ onto the E^*_u orthogonal complement of F which is precisely $H_u \mathbf{H}^*$ and that

$$E^*(H_u g, H_u g) = Q^*_{(u)}(g^{\#}, g^{\#}).$$

for g in F^*. This proves the direct part of the following theorem.

THEOREM 10.1. *Let* $(\mathbf{H}^*, Q^*)$ *be a pair satisfying* 10.1.1 *through* 10.1.3. *Then there is a unique symmetric submarkovian resolvent* G^*_u, $u > 0$ *on* $L^2(m)$ *determined by* (10.2) *and* (10.3). *The associated Dirichlet space* $(\mathbf{F}^*, E^*)$ *satisfies the following conditions.*

(i) $\mathbf{F}^*$ *contains* $\mathbf{F}$ *and the restriction of* E^* *to* $\mathbf{F}$ *is* E.
(ii) *For* $u > 0$ *the operator* H_u *projects* $\mathbf{F}^*$ *onto the* E^*_u *orthogonal comple-*

ment of F. The range of $H_u\mathbf{F}^*$ is precisely $H_u\mathbf{H}^*$ and

$$E^*(H_u\Phi, H_u\Phi) = Q^*_{(u)}(\Phi, \Phi)$$

for Φ in $\mathbf{H}^*$.

Conversely if G^*_u, $u > 0$, is a symmertic submarkovian resolvent on $L^2(m)$ such that $G^*f - G_uf$ is nonnegative and u-harmonic for $u > 0$ and for $f \geq 0$ in $L^2(m)$, then there is a unique pair $(\mathbf{H}^*, Q^*)$ such that (10.2) and (10.3) are satisfied.

Before proving the converse, we apply the direct part of Theorem 10.1 to the special case $(\mathbf{H}^*, Q^*) = (\mathbf{N}_\alpha, N)$. For $h^\#$ in $\mathbf{N}_\alpha$ and for $0 < u < v$

$$\begin{aligned}
E^r_v(H_v h^\#, H_v h^\#) &= v\int m(dx)H_v h^\#(x)H_u h^\#(x)\\
&\quad + v\int m(dx)H_v h^\#(x)\{H_v h^\#(x) - H_u h^\#(x)\}\\
&\quad + E^r(Hh^\#, Hh^\#) + E^r(H_v h^\# - Hh^\#, H_v h^\# - Hh^\#)\\
&= v\int m(dx)H_v h^\#(x)H_u h^\#(x)\\
&\quad - v\int m(dx)H_v h^\#(x)\{H_u h^\#(x) - H_v h^\#(x)\}\\
&\quad + N(h^\#, h^\#)\\
&\quad + v\int m(dx)H_v h^\#(x)\{Hh^\#(x) - H_v h^\#(x)\}
\end{aligned}$$

and after passage to the limit $u \downarrow 0$,

$$E^r_v(H_v h^\#, H_v h^\#) = N_{(v)}(h^\#, h^\#). \tag{10.6}$$

Also for φ in $L^2(m)$

$$\begin{aligned}
E^r_v(H_v h^\#, G_v\varphi) &= E^r(H_v h^\#, G_v\varphi) + v\int m(dx)H_v h^\#(x)G_v\varphi(x)\\
&= E^r(H_v h^\# - Hh^\#), G_v\varphi) + v\int m(dx)H_v h^\#(x)G_v\varphi(x)\\
&= -E^r(G\{Hh^\# - H_v h^\#\}, G_v\varphi) + v\int m(dx)H_v h^\#(x)G_v\varphi(x)
\end{aligned}$$

and therefore

$$E^r_v(H_v h^\#, G_v\varphi) = 0. \tag{10.7}$$

The identities (10.6) and (10.7) together identify the pair

$$(\mathbf{F}^r_a \cap L^2(m), E^r)$$

as the appropriate pair $(\mathbf{F}^*, E^*)$ in Theorem 10.1 and we conclude in particular that $(\mathbf{F}_a^r \cap L^2(m), E^r)$ is a Dirichlet space relative to $L^2(m)$. From this it follows with the help of an elementary random time change that if f belongs to the reflected Dirichlet space $\mathbf{F}^r$ and if T is a normalized contraction, then also Tf belongs to $\mathbf{F}^r$ and $E^r(Tf, Tf) \leq E^r(f, f)$. It seems to us that a direct proof of this using only the techniques of Sections 8 and 9 would be of considerable interest.

Added in proof. This and more is done in the monograph now in preparation which is mentioned at the end of the introduction.

We turn now to the converse of Theorem 10.1. Let G_u^*, $u > 0$ be a submarkovian symmetric resolvent on $L^2(m)$ such that $G_u^* f - G_u f$ is nonnegative and u-harmonic for $u > 0$ and for $f \geq 0$ in $L^2(m)$. Let $(\mathbf{F}^*, E^*)$ be the associated Dirichlet space. It follows either from Proposition 1.1 in [15] or from Theorem 4.1 in [19] that $\mathbf{F}^*$ contains $\mathbf{F}$ and that the restruction of E^* to $\mathbf{F}$ is dominated by E. Fix D open with compact closure and note that

$$G_u^* = G_u^D + H_u^M G_u^*. \tag{10.8}$$

We show that for $g \geq 0$ in $L^2(m)$,

$$vG_{u+v}^* H_u^M G_u^* g \uparrow H_u^M G_u^* g \tag{10.9}$$

quasi-everywhere as $v \uparrow \infty$. First

$$\begin{aligned} vG_{u+v}^* H_u^M G_u^* g &= vG_{u+v}^D H_u^M G_u^* g + vH_{u+v}^M G_{u+v}^* H_u^M G_u^* g \\ &= H_u^M G_u^* g - H_{u+v}^M G_u^* g + vH_{u+v}^M G_{u+v}^* H_u^M G_u^* g \\ &\leq H_u^M G_u^* g - H_{u+v}^M \{G_u^* - vG_{u+v}^* G_u^*\} g \\ &\leq H_u^M G_u^* g \end{aligned}$$

and then the relation

$$vG_{u+v}^* H_u^M G_u^* g = wG_{u+w}^* \{(v/w) + [1 - (v/w)]vG_{u+v}^*\} H_u^M G_u^* g$$

shows that

$$vG_{u+v}^* H_u^M G_u^* g \leq wG_{u+w}^* H_u^M G_u^* g \leq H_u^M G_u^* g \tag{10.9'}$$

for $0 < v < w$. Convergence quasi-everywhere in (10.9) follows on D since it is true with G_{u+v}^* replaced by G_{u+v}^D and follows on $\mathbf{X}$ upon considering open D' containing D and noting that

$$H_u^M G_u^* g - H_u^{M'} G_u^* g = G_u^{D'} g - G^D g$$

belongs to $\mathbf{F}$. For f, $g \geq 0$ in $L^2(m)$.

$$\begin{aligned} v\int m(dx) \{H_u^M G_u^* gx) - vG_{u+v}^* H_u^M G_u^* g(x)\} G_u^D f(x) \\ = v\int m(dx) \{H_u^M G_u^* g(x) - vG_{u+v}^* H_u^M G_u^* g(x)\} G_u^D f(x) \\ - v^2\int m(dx) H_{u+v}^M G_{u+v}^* H_u^M G_u^* g(x) G_u^D f(x). \end{aligned}$$

The first term on the right

$$= v\int m(dx)G_u^D H_{u+v}^M G_u^* g(x)f(x)$$

$$= v\int m(dx)G_{u+v}^D H_u^M G_u^* g(x)f(x)$$

$$\to \int_D m(dx)H_u^M G_u^* g(x)f(x)$$

as $v \uparrow \infty$. The second term on the right equals

$$-\int m(dx)vG_{u+v}^D H_u^M vG_{u+v}^* H_u^M G_u^* g(x)f(x)$$

which with the help of (10.9) approaches

$$-\int_D m(dx)H_u^M G_u^* g(x)f(x)$$

as $v \uparrow \infty$ and we conclude that

$$E_u^*(H_u^M G_u^* g, G_u^D f) = 0. \tag{10.10}$$

It follows that E^* agrees with E on $\mathbf{F}^D$ and that H_u^M implements E_u^* orthogonal projection onto the complement of $\mathbf{F}^D$ at least when applied to the image $G_u^* L^2(m)$. From the estimate

$$\int \nu^M(dy)f^2(y) \leq \int_M m(dx)f^2(x) + \| f \|_\infty \int_D m(dx)H_1^M |f|(x)$$

it follows that every bounded f in $\mathbf{F}^*$ has a refinement in $L^2(\nu^M)$ defined by considering the limits $uG_u^* f$ and then general f in $\mathbf{F}^*$ has a refinement specified up to ν^M equivalence defined by truncation and passage to the limit. Let $\mathbf{H}^{*M}$ be the set of φ in $L^2(\nu^M)$ such that $H_u^M\varphi$ belongs to $\mathbf{F}^*$ for one and therefore all $u > 0$ and for φ in $\mathbf{H}^{*M}$ let

$$Q_{(u)}^{*M}(\varphi, \varphi) = E_u^*(H_u^M\varphi, H_u^M\varphi).$$

From the existence of the above refinements it follows easily that the pairs $(\mathbf{H}^{*M}, Q_{(u)}^{*M})$ are Dirichlet spaces relative to $L^2(\nu^M)$. For each u denote the corresponding resolvent by $\{R_{(u)a}^{*M}, a > 0\}$. For $g \geq 0$ in $L^2(m)$ and for φ in $\mathbf{H}^{*M}$,

$$\begin{aligned} \int \nu^M(dy)\pi_u^M g(y)\varphi(y) &= \int m(dx)g(x)H_u^M \varphi(y) \\ &= Q_{(u)}^* (G_u^* g, H_u^M \varphi). \end{aligned} \tag{10.11}$$

which is enough to guarantee that

$$R_{(u)}^M \pi_u^M G = G_u^* g. \tag{10.12}$$

To justify (10.12) we argue indirectly. First

$$Q^*_{(u)1}(G^*_u\, g,\, H^M_u\varphi) = \int \nu^M(dy)\,\{\pi^M_u g(y) + G^*_u\, g(y)\}\,\varphi(y)$$

and therefore

$$G^*_u\, g = R^{*M}_{(u)1}\,\{\pi^M_u g + G^*_u\, g\}.$$

But $R^{*M}_{(u)1}\,\pi^M_u g$ is the minimal nonnegative solution of

$$\psi = R^{*M}_{(u)1}\,\{\pi^M_u g + \psi\}$$

and we conclude that

$$R^{*M}_{(u)}\,\pi^M_u\, g \le G^*_u\, g.$$

After considering the special case $0 \le g \le 1$ we conclude that $R^{*M}_{(u)}$ is bounded on $L^\infty(\nu^M)$ and therefore by symmetry on $L^2(\nu^M)$ and (10.12) follows easily. Also $Q^{*M}_{(u)}$ dominates a multiple of the standard inner product on $L^2(\nu^M)$ and therefore every function in $\mathbf{F}^*$ has a refinement in $L^2(\nu^M)$. Indeed the relevant constant is 1 for $u = 1$ and therefore

$$\int \nu^M(dy)f^2(y) \le E^*_1\,(f, f) \tag{10.13}$$

for any f in $\mathbf{F}^*$ which is 1-harmonic. The proof of Theorem 5.4 can be applied with notational changes only to show that $\mathbf{H}^{*M}$ is contained in $\mathbf{N}^M$ and that $\mathbf{N}^M - Q^{*M}$ is contractive on $\mathbf{H}^{*M}$. From this and (10.13) the converse to Theorem 10.1 follows after passage to the limit in M.

We remark that a local version of Theorem 10.1 can be proved by modifying the arguments of this section in an obvious way. The active universal Dirichlet space $(\mathbf{N}_a, N)$ must be replaced by the local universal Dirichlet space $(\mathbf{N}^M, N^M)$ for $M = \mathrm{X} - D$ with D a general open subset of X. Indeed the converse is actually proved above in the case when D has finite measure.

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University of Southern California
Los Angeles, California