IMBEDDING DELETED 3-MANIFOLD NEIGHBORHOODS IN E^3

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The following question was asked by D. R. McMillan: Does every compact set in an orientable 3-manifold have a deleted neighborhood imbeddable in E^3? We will exploit a result of Haken [1] (using a technique developed in [3] and used subsequently in [2]) to answer in the affirmative.

DEFINITIONS. We will be working with a compact $X \subseteq M^3$, a compact orientable 3-manifold, and X may be written as $X = \bigcap_{i=0}^{\infty} N_i$ where each N_i is a polyhedral neighborhood of N_{i+1} and $N_0 = M^3$.

A *surface* is a closed connected 2-manifold. A surface S is *incompressible* in a compact 3-manifold M^3 if (1) S is not a 2-sphere and when $D \subseteq M^3$ is a disk with $D \cap S = \operatorname{Bd} D$ then $\operatorname{Bd} D$ bounds a disk in S or (2) S is a 2-sphere that bounds no 3-cell in M^3. Two surfaces S_1 and S_2 in a 3-manifold M^3 bound a *parallelity component* of the disjoint collection $\{S_1, \cdots, S_m\}$ if $S_1 \cup S_2$ bounds a component U of $M - \bigcup_{i=1}^{m} S_i$ and $\bar{U}$ is homeomorphic to the product of S_1 and an interval. And $\{S_1, \cdots, S_r\}$ forms a *parallel system* in $\{S_1, \cdots, S_m\}$ if $S_i \cup S_{i+1}$ bounds a parallelity component of $\{S_1, \cdots, S_m\}$ in M^3 for $1 \leq i \leq r - 1$.

Suppose there is a disk $D \subseteq M^3$ so that, $D \cap \operatorname{Bd} N_j = \operatorname{Bd} D$ for some j and $\operatorname{Bd} D$ bounds no disk in $\operatorname{Bd} N_j$, and either (1) $D \subseteq (N_j - N_{j+1})^-$ or (2) $D \subseteq (N_{j-1} - N_j)^-$. Then the collection $\{N_1, N_2, \cdots\}$ is changed by a *simple move* if N_j is respectively replaced by (1) $N_j - N(D)$ (where $N(D)$ is a regular neighborhood of D in $(N_j - N_{j+1})^-$) or by (2) $N_j \cup N(D)$ (where $N(D)$ is a regular neighborhood of D in $(N_{j-1} - N_j)^-$.

Consider $(N_1 - N_n)^-$ for $n > 0$. It is possible to change $\{N_1, N_2, \cdots, N_n\}$ by a finite number of simple moves to make each component of $\bigcup_{i=1}^{n} \operatorname{Bd} N_i$ either incompressible in M^3 or a 2-sphere bounding a 3-cell in M^3, because a simple move along a simple closed curve bounding no disk in $\bigcup_{i=1}^{n} \operatorname{Bd} N_i$ reduces the sum $\sum_{S \in C} (\chi(S) - 2)^2$ where C is the collection of surfaces in $\bigcup_{i=1}^{n} \operatorname{Bd} N_i$ and $\chi(S)$ is the Euler characteristic of S.

THEOREM. *Suppose X is a compact subset of M^3, a closed orientable 3-manifold. Then there is a neighborhood N of X in M^3 such that $N - X$ can be imbedded in E^3.*

Proof. For each positive integer n we define a process for changing $(N_1 - N_n)^-$ by a finite sequence of simple moves to make each surface of $\{\operatorname{Bd} N_1, \cdots, \operatorname{Bd} N_n\}$ either incompressible in M^3 or a 2-sphere bounding a 3-cell in M^3. We will denote the set $Y \subseteq N_1 - N_n$ after j steps of Process n by (Y, n, j).

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Process n. Step 1. Make each surface of Bd N_n either incompressible in N_{n-1} or a 2-sphere.

Step 2. Make each surface of $(\mathrm{Bd}\, N_n, n, 1) \cup \mathrm{Bd}\, N_{n-1}$ either incompressible in N_{n-2} or a 2-sphere.

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Step j. Make each surface of $\bigcup_{i=n-j+1}^{n} (\mathrm{Bd}\, N_i, n, j-1)$ either incompressible in N_{n-j} or a 2-sphere.

Step n. Make each surface of $\bigcup_{i=1}^{n} (\mathrm{Bd}\, N_i, n, n-1)$ either incompressible in N_0 or a 2-sphere.

It is further required that the processes be compatible according to the following condition.

Condition. $((N_1 - N_n)^-, n, j)$ is homeomorphic to $((N_1 - N_n)^-, n+1, j+1)$.

The following lemma shows that the construction can be performed so that it satisfies the condition.

LEMMA 1. *Suppose $N_3 \subseteq \mathrm{Int}\, N_2 \subseteq \mathrm{Int}\, N_1$ where N_1, N_2, N_3 are compact orientable polyhedral 3-manifolds. Suppose there is a finite sequence of simple moves in N_1 changing N_2 to N_2' so each surface of* Bd N_2' *is either incompressible in N_1 or a 2-sphere. Then (N_2, N_3) can be changed by a finite sequence of simple moves in N_1 to (N_2'', N_3') so* (1) *each surface of* Bd $N_2'' \cup$ Bd N_3' *is incompressible in N_1 or a 2-sphere and* (2) $(N_1 - N_2'')^-$ *is homeomorphic to* $(N_1 - N_2')^-$.

Proof. First make a slight change in the given sequence. The first simple move removes an annulus from Bd N_2 and adds two disks D_1 and D_2. Suppose the second move results from a disk D attached to Bd N_2 along Bd D. Then before performing this second move, move Bd D off $D_1 \cup D_2$. Continue in this way; for example, before the mth move, move the curve in question off the $2m - 2$ disjoint disks of previous moves.

The new sequence of moves can now be described. First make each surface of Bd N^3 incompressible in N_2 or a 2-sphere. The first move of the given sequence results from a disk D attached to Bd N_2 along Bd D.

Case 1. If $D \subseteq (N_1 - N_2)^-$ then perform the move in $(N_1 - N_2)^-$ as in the given sequence.

Case 2. If $D \subseteq N_2$ put D in general position with Bd N_3. Suppose K is an innermost (in D) curve of $D \cap \mathrm{Bd}\, N_3$. Then K bounds a disk E in Bd N_3. Let L be an innermost in E curve of $E \cap D$. Then L bounds subdisks E' of E and D' of D. Change D by replacing D' with E' and moving E' off Bd N_3. Continuing in this way D is made to miss Bd N^3. Now perform the simple move by removing a regular neighborhood of D from N_2. In either case Bd N_2 has an annulus removed and two disks D_1 and D_2 added.

Second, make each surface of Bd N_3 incompressible in N_2 or a 2-sphere. Again think of the second move of the given sequence as resulting from a disk D attached to Bd N_2.

Case 1. If $D \subseteq (N_1 - N_2)^-$ in the original sequence, then make the simple move along a corresponding disk in the $(N_1 - N_2)^-$ constructed in the previous paragraph.

Case 2. If $D \subseteq N_2$ in the original sequence, proceed as in Case 2 above to make D miss the disjoint sets Bd N_2, D_1, and D_2. Then remove a regular neighborhood of D from N_2.

Continue in this way until all the given moves have been changed and finally make each surface of Bd N_3 incompressible in N_2 (which is now N_2'') or a 2-sphere.

Now $(N_1 - N_2'')^-$ is homeomorphic to $(N_1 - N_2')^-$ since things were kept homeomorphic move by move. Take a disk D with $D \cap \text{Bd}\, N_2'' = \text{Bd}\, D$ and move Bd D off the disks from the construction of Bd N_2''. Then D can be made to miss the disks in the construction of Bd N_2' as above (Int D may now hit Bd N_2''). So Bd D bounds a disk in Bd N_2', and therefore Bd D bounds a disk in Bd N_2''. Then each surface of Bd N_2'' is incompressible in N_1 or a 2-sphere. The lemma is proved.

Next notice that for a given n, $((N_1 - N_n)^-, n, 0)$ can be reconstructed from $((N_1 - N_n)^-, n, n)$ in M^3 by reversing Process n. That is, perform the simple moves of Process n in reverse order and backwards. If a simple move consists of removing the regular neighborhood of a disk from N_j, then the move is performed backwards by adding a 1-handle to N_j. If the move consists of adding a regular neighborhood of a disk to N_j, then the move is performed backwards by digging a tunnel in N_j (removing a regular neighborhood of a properly imbedded arc).

Suppose there is a pair of integers (L, M) such that in the reconstruction of $((N_L - N_M)^-, M, 0)$ a 1-handle is attached to different boundary components of the same component of $(N_L - N_M)^-$. Then the original neighborhood is reduced from N_1 to N_M. This is done again if the problem occurs for some (J, K) with $J, K > M$, and the problem occurs only finitely many times by [2, Lemma 3]. The neighborhoods are renumbered so that the reduced neighborhood is now N_1. So the following lemma holds for the new neighborhood.

LEMMA 2. *If $(N_L - N_M, M, M)$ is imbedded in E^3, then $(N_L - N_M, M, 0)$ can be constructed in E^3 by adding* 1*-handles and digging tunnels.*

Now fix j. All the incompressible in N_j surfaces of

$$(\text{Bd}\, N_n \cup \cdots \cup \text{Bd}\, N_{j+1}, n, n - j)$$

lie in at most $\alpha(N_j)$ parallel systems in N_j where $\alpha(N_j)$ is an integer depending only on N_j, by [1, p. 91]. Then all but at most $\alpha(N_j) + \beta_0(N_j) + 1$ (where $\beta_0(N_j)$ is the number of components of N_j) of the sets $((N_{i-1} - N_i)^-, n, n - j)$ where $j + 1 \leqq i \leqq n$ have each component made up of a punctured 3-cell or a punctured product (a 3-cell with 3-cells removed from the interior or the product of a surface and an interval with 3-cells removed from the interior). Allow n to vary. Using the condition, there is an integer 1_j so that if $n > 1_j$ then each component of $((N_{n-1} - N_n), n, n - j)$ is a punctured 3-cell or a punctured product. Also notice that

$$((N_1 - N_j)^-, n, n - j)$$

has only finitely many components. Obviously each component of

$$((N_1 - N_n)^-, n, n - j)$$

is made up of (1) components of $((N_1 - N_{n-1})^-, n, n - j)$ and (2) components of $((N_{n-1} - N_n)^-, n, n - j)$. Choose $n_j > 1j$ so that if $n > n_j$ then for each component of $((N_1 - N_n)^-, n, n - j)$, all but at most one of its type(1) components miss $((N_1 - N_j)^-, n, n - j)$ and are punctured 3-cells and punctured products. This proves Lemma 3.

LEMMA 3. *For each positive integer j there is a positive integer n_j so that if $n > n_j$ and if $((N_1 - N_n)^-, n, n - j)$ is imbedded in E^3, then*

$$((N_1 - N_{n+1})^-, n + 1, n + 1 - j)$$

can be imbedded in E^3 by adding collars to some of the boundary components of $((N_1 - N_n)^-, n, n - j)$, removing 3-cells from the collars, and by adding some new components, all of which are punctured 3-cells or punctured products. This is done without changing the imbedding of $(N_1 - N_j)^-$.

Now choose N_{n_1} to be the N of the conclusion of the theorem. Use Lemma 3 to imbed $((N_{n_1} - N_{n_2})^-, n_2, n_2 - 1)$ in E^3. Use Lemma 2 to imbed $((N_{n_1} - N_{n_2})^-, n_2, n_2 - 2)$ in E^3. Use Lemma 3 to imbed

$$((N_{n_1} - N_{n_3})^-, n_3, n_3 - 2)$$

in E^3. Again use Lemma 2 to imbed $((N_{n_1} - N_{n_3})^-, n_3, n_3 - 3)$ in E^3. Continuing in this way, for a given positive integer K there will eventually be an imbedding of

$$((N_{n_1} - N_{n_K})^-, n_K, n_K - K)$$

in E^3. Use Lemma 3 to imbed $((N_{n_1} - N_{n_{K+1}})^-, n_{K+1}, n_{K+1} - K)$ in E^3 without changing the imbedding of $(N_{n_1} - N_K)^-$. Now use Lemma 2 to imbed

$$((N_{n_1} - N_{n_{K+1}})^-, n_{K+1}, n_{K+1} - K - 1).$$

Notice the imbedding of $(N_{n_1} - N_K)^-$ is not changed. This construction imbeds $N_{n_1} - X$ in E^3.

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