VIRTUAL GROUP HOMOMORPHISMS WITH DENSE RANGE

BY

JOEL J. WESTMAN

Introduction

Let $\mathscr{F}$ be the virtual subgroup of Z (the integers) defined by an ergodic measure-preserving transformation Φ on an analytic finite (nonatomic) measure space X. A homomorphism F: $\mathscr{F} \to A$ (a locally compact second countable group) has *dense range* if the Anzai skew product transformation on $X \times A$, defined by F and Φ, is ergodic. The main results obtained in this paper are as follows. (a) For every compact second countable group A there exists a homomorphism F: $\mathscr{F} \to A$, with dense range. This responds partially to the problem raised by Mackey (1968) as to whether a virtual subgroup admits a homomorphism "onto" a particular compact group, i.e., with dense range. (b) For every countable abelian group A, there exists a homomorphism F: $\mathscr{F} \to A$, with dense range.

The basic approach to obtaining results (a) and (b) is to construct for each group, A, a virtual group and strict homomorphism of that virtual group into A with dense range. The virtual groups and homomorphisms all arise from rather conventional examples of free ergodic actions of groups. Then the results of Dye in 1959 and 1963 are applied to show that the virtual groups used for (nontrivial) A compact second countable or countable abelian are all similar to $\mathscr{F}$—in fact, isomorphic (mod i.c.'s). Since the existence of a homomorphism with dense range F: $\mathscr{F} \to A$ (fixed A) is a similarity invariant for $\mathscr{F}$ the results (a) and (b) follow immediately.

The results (a) and (b) have interesting consequences in regards to the first cohomology of $\mathscr{F}$, i.e.,

$$H^1(\mathscr{F}; B) = \{\text{homomorphisms } h\colon \mathscr{F} \to B\} \quad \text{mod similarity,}$$

where the coefficient group B is an analytic Borel group. In [W-1] we prove that if F: $\mathscr{F} \to A$ has dense range then the induced map on the cohomology

$$F^*\colon H^1(A; B) \to H^1(\mathscr{F}; B)$$

is injective. Hence, using the well-known duality theory for abelian groups (say as in [P]), for B the circle group, $H^1(\mathscr{F}; B)$ contains as subgroups every countable abelian group (by (a)) and every compact abelian second countable group (by (b)).

Received December 30, 1974.

Section 1

Suppose G and A are locally compact second countable groups, and that G acts as a Borel transformation group on an analytic Borel space X, preserving the measure class of a finite measure, m, on X. The action of G is assumed to be ergodic. We form the ergodic groupoid (or virtual group) $X \times G$ as in [M-1] or [R] (see these references for standard terminology regarding virtual groups). We use the definitions of strict homomorphism, homomorphism, strict similarity, and similarity as given in [R, 6.1–6.14]. The measure class on $X \times G$ is that of $m \times$ Haar measure for G. The composition is given by $(x, g)\cdot(x', g') = (x, gg')$ iff $x = gx'$, and $(x, g)^{-1} = (g^{-1}x, g^{-1})$. We write gx or $g(x)$ for the image of x under the action of $g \in G$.

Suppose that $F: X \times G \to A$ is a strict homomorphism, i.e., F is a Borel map and $F(x, g)F(g^{-1}x, g') = F(x, gg')$. We define an action of $G \times A$ on $X \times A$ by $(g, b)(x, a) = (gx, baF(x, g^{-1}))$, as in [M-1]. The action of $G \times A$ preserves the measure class of $m \times m'$, where m' is Haar measure on A. The action of G (via the natural identification with $G \times \{1\}$) on $X \times A$ is the skew product group action, which generalizes Anzai's construction of a skew product transformation, as described in [M-1]. The action of A (via the natural identification with $\{1\} \times A$) commutes with that of G. Both actions extend naturally to Boolean actions on the measure algebra $M(X \times A)$ of Borel sets in $X \times A$ mod $m \times m'$ null sets. Then $M(X \times A)^G$, the set of G invariant elements in $M(X \times A)$, is invariant under the action of A. By Mackey's "point realization" technique, as described in [R], the Boolean action of A on $M(X \times A)^G$ can be induced by a Borel action of A on an analytic measure space W, with an A invariant measure class. The action of A on W is ergodic, $M(W) \cong M(X \times A)^G$, and the virtual group $W \times A$ is determined by the Boolean action, up to an inessential contraction.

If F is a homomorphism, but not strict, then there is a conull Borel set $X_1 \subseteq X$ such that the restriction $F: X \times G \restriction X_1 \to A$ is a strict homomorphism. [W-1] deals with this complication in the general case, however if G is countable the simplest policy is to form $X_2 = \bigcap_{g \in G} gX_1$. Then X_2 is a Borel conull and invariant subset of X_1. Taking a further i.c. to X_2 yields $X \times G \restriction X_2 = X_2 \times G$ and the skew product action is then defined on $X_2 \times A$.

The following definitions are due to Mackey and are discussed and motivated in [M-1] and [M-2].

1.0 Definition. A homomorphism $F: X \times G \to A$ has *dense range* iff the corresponding skew product action of G is ergodic.

1.1 Definition. If a homomorphism $F: X \times G \to A$ has dense range the virtual group defined by the corresponding ergodic skew product action of G is called the *kernel* of F.

1.2 *Remarks.* In the terminology of [L], a virtual group $\mathcal{G}$, with a homomorphism $F\colon \mathcal{G} \to A$ is called a virtual group over A and is an object in the Mackey category $M(A)$. The Mackey functor M sends the pair $\mathcal{G}$, F into an ergodic action of A on an analytic Borel space, say W, with an invariant measure class. If $\mathcal{G} = X \times G$, then the action of A on the space W can be obtained by the construction described above. We observe that a homomorphism $F\colon X \times G \to A$ has dense range iff the action of A on W is isomorphic to the trivial action of A on a single point.

1.3 THEOREM. *Given a homomorphism $F_1\colon X \times G \to A$ and a closed normal subgroup C of A, let $p\colon A \to A/C$ be the natural projection. Then $F = p \cdot F_1\colon X \times G \to A/C$ has dense range iff the (restricted from A) action of C on W is ergodic.*

Proof. F has dense range iff the measure algebra $M(X \times A/C)^G$ is trivial. There is a natural isomorphism

$$\phi^*\colon M(X \times A/C)^G \to M(X \times A)^{G \times C},$$

induced by $\phi\colon X \times A \to X \times A/C$; $\phi(x, a) = (x, aC)$. Moreover, $M(X \times A)^{G \times C}$ is easily identified with $M(W)^C$. Hence F has dense range iff $M(W)^C$ is trivial iff the action of C on W is ergodic.

We find the following lemma of great value in constructing homomorphisms with dense range, in connection with the result (b) mentioned in the introduction.

1.4 LEMMA. *If H is a closed normal subgroup of G then the strict homomorphism $F\colon X \times G \to G/H$; $F(x, g) = gH$, has dense range iff the (restricted from G) action of H on X is ergodic.*

Proof. Apply Theorem 1.3 to the special case—$A = G$, $C = H$, $F_1\colon X \times G \to G$; $F_1(x, g) = g$. The action of G on W (defined by F_1) is naturally isomorphic to that of G on X. The map

$$h\colon X \times G \to X;\ h(x, g) = gx$$

induces an isomorphism $h^*\colon M(X) \to M(X \times G)^{G \times \{1\}} \cong M(W)$. Then the action of H on W is ergodic iff the action of H on X is ergodic.

Next, we sketch briefly how Dye's work on weak equivalence of groups of measure preserving transformations applies to virtual groups, using the terminology of [D-1], cf. [M-3, p. 229]. Suppose G_1 and G_2 are groups of freely acting measure-preserving automorphisms of finite separable nonatomic measure algebras M and M', respectively. We assume here that G_1 and G_2 are countable groups and that their actions are ergodic. Then the automorphisms are induced by measure-preserving transformations of (nonatomic) analytic Borel spaces X and X' with finite measures m and m', and the corresponding actions of G_1 and G_2 are ergodic and free (see [R, Section 3]). By clipping appropriate null sets we can assume all isotropy subgroups of G_1 and G_2 are trivial.

Suppose that $X = X'$ and that G_1 and G_2 are equivalent as defined in [D-1]. We can define (after clipping suitable null sets from X) an isomorphism of virtual groups $\psi\colon X \times G_1 \to X \times G_2$ by sending (x, g) into (x, g') where g' is the unique element in G_2 such that $g'(g^{-1}x) = x$. Next, suppose that G_1 and G_2 are weakly equivalent (X not necessarily $= X'$). Then there are invariant Borel conull sets X_0 and X'_0 in X and X', respectively, and a measure class preserving Borel isomorphism $\phi\colon X_0 \to X'_0$ such that $G'_1 = \phi G_1 \phi^{-1}$ is equivalent to G_2. ϕ then induces an isomorphism

$$\tilde{\phi}\colon X_0 \times G_1 \to X'_0 \times G'_1; \qquad \tilde{\phi}(x, g) = (\phi(x), \phi g \phi^{-1}),$$

of virtual groups. Accordingly, we obtain the following result.

1.5 LEMMA. *The weak equivalence of the countable freely acting ergodic automorphism* (*or transformation*) *groups G_1 and G_2 implies the isomorphism* (mod i.c.'s) *of the corresponding virtual groups $X \times G_1$ and $X' \times G_2$.*

An isomorphism (mod i.c.'s) of two virtual groups easily establishes that the virtual groups are similar.

1.6 LEMMA. *Suppose that the virtual groups $\mathscr{G}_1$ and $\mathscr{G}_2$ are similar via homomorphisms*

$$\phi\colon \mathscr{G}_1 \to \mathscr{G}_2 \quad \textit{and} \quad \psi\colon \mathscr{G}_2 \to \mathscr{G}_1$$

and that $F_2\colon \mathscr{G}_2 \to A$ is a strict homomorphism with dense range. Then $F_1 = F_2 \cdot \phi\colon \mathscr{G}_1 \to A$ is a homomorphism with dense range.

Proof. Apply Theorem 7.11 in [R, p. 299]. The Boolean A-space defined in [R] by F_i is trivial iff F_i has dense range. If F_2 were not strict, then F_2 and ϕ may not be composable. However, [R] shows that there exists ϕ' similar to ϕ such that F_2 and ϕ' are composable. Then Lemma 1.6 is valid for $F_1 = F_2 \cdot \phi'$.

Section 2

Given a compact second countable group, A, with normalized Haar measure m', we construct A^Z, the set of all sequences (a_k), $k \in Z$, $a_k \in A$, with the countable product Borel structure and measure m. A^Z is (with the countable product topology and under pointwise multiplication) a compact second countable group with normalized Haar measure m. We define $\Phi\colon A^Z \to A^Z$ by $\Phi((a_k)) = (a_{k-1})$. Then Φ is an invertible, Borel, and measure-preserving transformation. Φ defines a free ergodic action of Z on A^Z in which $n \in Z$ is sent into Φ^n. We form the principal virtual group $A^Z \times Z = \mathscr{G}(A)$.

2.0 THEOREM. *If A is a compact second countable group, then the strict homomorphism $F\colon \mathscr{G}(A) \to A$, specified by $F((a_k), 1) = a_0$, has dense range.*

Proof. Consider the map

$$V\colon A^{Z} \to A^{Z} \times A; \quad V((a_k)) = ((a_k^{-1}a_{k+1}), a_0).$$

Then it is easy to verify that V is a bijective map and a Borel isomorphism, and commutes with Φ and the skew product transformation Φ_F of $A^{Z} \times A$, i.e.,

$$V(\Phi^{-1}((a_k)) = V((a_{k+1})) = ((a_{k+1}^{-1}a_{k+2}), a_1) = \Phi_F^{-1}V((a_k)).$$

We next show that V is measure preserving. This will establish that the skew product transformation is ergodic (so F has dense range), as well as the curious result that Φ is isomorphic to Φ_F (via V). We introduce the map

$$W\colon A \times A^{Z} \to A^{Z} \times A;\ W(a, (a_k)) = ((a_k^{-1}a_{k+1}), aa_0)$$

and the map

$$c\colon A \times A^{Z} \to A^{Z};\ c(a, (a_k)) = (aa_k).$$

Then the diagram

$$\begin{tikzcd}
A \times A^{Z} \arrow[d, "c"'] \arrow[dr, "W"] & \\
A^{Z} \arrow[r, "V"'] & A^{Z} \times A
\end{tikzcd}$$

commutes, and V will be shown to be measure-preserving when we prove that $c_*(m' \times m) = m$ and $W_*(m' \times m) = m \times m'$. Take a Borel set D in A^{Z}. Then

$$c_*(m' \times m)(D) = m' \times m(c^{-1}(D)) = m' \times m(\{(a, (a_k))\colon (a_k) \in a^{-1}D\})$$

(apply the Fubini Theorem, first fixing a and then integrating with respect to m')

$$= \int m(a^{-1}D)\, dm'(a) = m(D).$$

Thus $c_*(m' \times m) = m$. Next, take a cylinder $E = \prod C_i \times C \subseteq A^{Z} \times A$, where C and the C_i's are Borel sets in A and the C_i's are equal to A except possibly for a finite number of indices (say $= A$ for $i < -n$ and $i > n$, for some positive integer n). Let

$$R_p = \{x = (a_{-n}, a_{-n+1}, \ldots, a_{n+1-p})\colon a_{i+1} \in a_iC_i\}$$

and m_p be the product measure on $A \times A \times \cdots \times A$ ($2n + 2 - p$ copies of A). Then

$$\begin{aligned} W_*(m' \times m)(E) &= m' \times m(W^{-1}(E)) \\ &= m' \times m(\{(a, (a_k))\colon a_{i+1} \in a_iC_i \text{ and } a \in Ca_0^{-1}\}) \end{aligned}$$

(apply Fubini's Theorem—first fix (a_k) and then integrate over R_0 with measure m_0)

$$= \int_{R_0} m'(Ca_0^{-1})\, dm_0(x) = m'(C)m_0(R_0)$$

(to deal with R_0 we again apply Fubini's Theorem, fixing all components of $x \in R_0$ except the last—a_{n+1} and then integrating over R_1 with measure m_1)

$$= m'(C) \int_{R_1} m'(a_n C_n)\, dm_1(x) = m'(C)m'(C_n)m_1(R_1)$$

(continuing this process eventually yields)

$$= m'(C)m'(C_n) \cdots m'(C_{-n}) = m' \times m(E).$$

2.1 COROLLARY. *If A is a compact second countable group then there is a homomorphism $F\colon \mathscr{F} \to A$, with dense range.*

Proof. If A is trivial then so is the corollary. If A is nontrivial then by [D-1] and application of Lemma 1.5 we find $\mathscr{G}(A)$ is isomorphic (mod i.c.'s) to $\mathscr{F}$. Then Corollary 2.1 follows directly from Theorem 2.0 and Lemma 1.6.

Section 3

Suppose that K is a countable group. We let S be the set of all functions $f\colon K \to Z$ with finite support. Then S is a countable abelian group under pointwise addition of functions. K acts on S, sending $f \in S$ into $L_k f$, $L_k f(h) = f(k^{-1}h)$, for each $k \in K$. We define $X(K)$ = the dual of S = the homomorphisms of S into the circle group. $X(K)$ is a compact second countable abelian group; in fact, it is just the product of n copies of the unit circle, where n is the cardinal number of K. K acts on $X(K)$, sending the character ϕ into $L_k\phi = \phi \cdot L_k^{-1}$, which is also a character since each $L_k\colon S \to S$ is an automorphism of S. The action of K on $X(K)$ preserves the Haar measure on $X(K)$ and from the proof of Lemma 2.1 in [D-1] we see that this action is free and we obtain the following statement regarding its ergodicity.

3.0 LEMMA. *A subgroup H of K acts ergodicly on $X(K)$ iff H has infinitely many elements.*

We form the corresponding virtual group $X(K) \times K = \mathscr{F}(K)$.

3.1 THEOREM. *If A is a countable group, then the strict homomorphism $F\colon \mathscr{F}(A \times Z) \to A$; $F(x, a, n) = a$, has dense range.*

Proof. We apply Lemma 1.4 using $A \times Z$ for G, and $\{1\} \times Z$ for H, so that G/H is naturally isomorphic to A. Then F has dense range if and only if the action of H on $X(A \times Z)$ is ergodic. Lemma 3.0 establishes that this action is ergodic.

3.2 COROLLARY. *If the action of $A \times Z$ on $X(A \times Z)$ is approximately finite, then there is a homomorphism $F\colon \mathscr{F} \to A$, with dense range.*

Proof. The action is type II, since $A \times Z$ is a freely acting infinite group. Theorem 5, page 154, in [D-1] together with our Lemma 1.5 yield the result that $\mathscr{F}(A \times Z)$ is isomorphic to $\mathscr{F}$ (mod i.c.'s). Then Corollary 3.2 follows from Lemma 1.6 using the homomorphism from Theorem 3.1.

3.3 COROLLARY. *If A is a countable abelian group or a finite group, then there is a homomorphism F: $\mathscr{F} \to A$, with dense range.*

Proof. The result follows from Corollary 3.2, since the action of $A \times Z$ on $X(A \times Z)$ is approximately finite if A is a finite group by Lemma 4.2, page 561, in [D-2] or if A is abelian by Corollary 4.1, page 561, in [D-2].

REFERENCES

[A] H. ANZAI, *Ergodic skew product transformations on the torus*, Osaka Math. J., vol. 3 (1951), pp. 83–89.

[D-1] H. A. DYE, *On groups of measure preserving transformations I*, Amer. J. Math., vol. 81 (1959), pp. 119–159.

[D-2] ———, *On groups of measure preserving transformations II*, Amer. J. Math., vol. 85 (1963), pp. 551–576.

[L] K. LANGE, A. RAMSAY, AND G. C. ROTA, *Frobenius reciprocity in ergodic theory*, Bull. Amer. Math. Soc., vol. 77 (1971), pp. 713–718.

[M-1] G. W. MACKEY, *Ergodic theory and virtual groups*, Math. Ann., vol. 166 (1966), pp. 187–207.

[M-2] ———, "Virtual groups," in *Topological Dynamics* (Joseph Auslander and Walter H. Gottschalk, Eds.), Benjamin, New York, 1968, pp. 331–364.

[M-3] ———, *Ergodic theory and its significance for statistical mechanics and probability theory*, Advances in Math., vol. 12 (1974), pp. 178–268.

[P] L. PONTRJAGIN, *Topological groups*, Princeton University Press, Princeton, 1958.

[R] A. RAMSAY, *Virtual groups and group actions*, Advances in Math., vol. 6 (1971), pp. 253–322.

[W-1] J. J. WESTMAN, *Skew products and similarity of virtual subgroups*, preprints.

UNIVERSITY OF CALIFORNIA
IRVINE, CALIFORNIA