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POLYMERSIONS WITH NONTRIVIAL TARGETS

BY
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In this note we treat the following problem. Given a finite family f of closed, oriented curves in general position on an oriented surface W, what are the proper maps F from a surface M to W so that $F \mid \partial M = f$ and F is locally $1:1$ except for polynomial-like branching over a prescribed set of points on W? Historically, this work is related to Hurwitz' combinatorial classification [4] of the meromorphic functions between two closed Riemann surfaces. In his case there are no curves f, only the set A_0 of branchpoints is given. Another early version of our problem is found on p. 313 in Picard's *Traitė d'Analyse* [5], where he asks: Given a closed plane polygonal curve with selfintersections, what are the analytic functions of the upper halfplane that map the real line to the polygon? Titus [6] gave a combinatorial solution to Picard's problem. A solution to the general problem, but with trivial W (genus zero) is given in [1].

In Section 1 we review the method of assemblages, as developed in [1], for classifying polymersions that have target $W = R^2$, the complete plane. A trivial modification allows also for a finite number of punctures in W. In Section 2 we treat the case that W is a closed, oriented surface of arbitrary genus. Our strategy is to compose F with a model polymersion G of W to the sphere S^2. G is topologically equivalent to the projection of the Riemann surface of a hyperelliptic function. It then suffices to isolate the conditions on a plane assemblage of a polymersion F' so that F' factors through G. We give an example in Section 3. In Section 4 we show how our classification of branched coverings surfaces reduces to that of Hurwitz [4, pp. 51ff].

The reader is respectfully referred to [1] for certain details and examples, as well as for a review of the contributions made by other authors to the present work. The author gratefully acknowledges helpful conversations with W. Abikoff, J. Birman, W. Magnus and M. Marx. These results were first presented to the Riemann Seminar at the University of Illinois, January, 1976.

1. Plane assemblages

Define a *polymersion* $F\colon M \to W$ to be a continuous map of surfaces, so that F is locally topological except at isolated points in W, the *branchpoints* of F. Here the map looks like $w = z^n$, $n > 1$. An unbranched polymersion is, of course, an *immersion* (local homeomorphism). When W is oriented and M compact with ρ border circles, we assume that F is an immersion on a neigh-

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borhood of ∂M, and that the family $f = F \mid \partial M$ of closed curves in W, together with the set A_0 of branchpoints, lie in general position. Thus M is orientable. Orient M so that F is sense preserving and orient f so that M lies to the left of ∂M. Let $f^{\oplus}$ denote the collection of mutually noncrossing simple closed oriented curves obtained by breaking each node of f [2]. We call the curves of $f^{\oplus}$ the *Gaussian circles* of f.

Now let W denote the plane and let α denote a finite collection of disjoint *rays* (topological halflines) in general position with respect to f. Let $X(f, \alpha)$ be the set of *crossings* of f and α. A crossing x has a *sign*, $\operatorname{sgn}(x)$, which is positive (negative) if the curve f_x, in the family f, that passes through x crosses the ray α_x (in α through x) from the left to the right side (respectively, vice versa). Let a_x denote the initial point of the ray α_x and set $A = \{a_x : x \in X\}$. The algebraic number of crossings on α_x is the *circulation* $\omega(f, a_x)$ of f about a_x. It counts the number of preimages of a_x under F, with multiplicity. Each Gaussian circle also has a sign, positive (negative) if its finite complement lies to its left (right) side. The algebraic number of Gaussian circles in $f^{\oplus}$ is the *turning number* $\tau(f)$ of f. A raying α is said to be *sufficient for the data* (f, A_0) if $A_0 \subset A$, and if every member of f and every negative member of $f^{\oplus}$ is crossed at least once.

Let $\mathfrak{S}(Y)$ denote the group of permutations on a set Y, and let Q_y denote the cycle of $Q \in \mathfrak{S}(Y)$ which contains $y \in Y$. Thus Q_y is the Q-orbit of y arranged in cyclic order. The *successor permutation* $S \in \mathfrak{S}X(f, \alpha)$ is a product of ρ cycles $S_x \in \mathfrak{S}X(f_x, \alpha)$ taking each crossing into the next in the direction given by the orientation on f. An assembling permutation $P \in \mathfrak{S}X(f, \alpha)$ is the product of cycles $P \in \mathfrak{S}X(f, \alpha)$, where one of the following holds:

(1.1) x is a fixed point of P, P_x is a *singleton*.

(1.2) All crossings permuted by P_x are positive, and the *hinge* a_x of the *fan* P_x lies in A_0.

(1.3) P_x is a transposition (xy) where x is negative, y is positive and x separates y from $a_x = a_y$ on the common ray. P_x is called a *pair*.

An assembling permutation is *transitive* if the subgroup $\langle S, P\rangle$ is transitive in $\mathfrak{S}X$; it is *effective* if each of the ν negative crossings in X are paired off as in (1.3); and it is *faithful* if the product $R = SP$ has ζ cycles, where

$$(1.4) \qquad \zeta = \nu + \tau.$$

THEOREM 1 [1]. *The polymersions $F: M \to R^2$ with given set A_0 of branchpoints and family $f = F \mid \partial M$ of border curves in general position, are classified, up to a topological automorphism on M, by the transitive, effective, faithful assembling permutations on the crossings of f and a sufficient raying α.*

Moreover, there is a cell decomposition of M associated with P so that F is $1:1$ on each (open) cell. The ζ *faces* (2-cells) are in $1:1$ correspondence with the cycles of $R = SP$. One can decipher the map F from the *assemblage*

(S, P, R) on $X(f, \alpha)$ as follows. For

$$x \in X(f, \alpha), \quad y = xS, \quad z = yP = xR, \tag{1.5}$$

let $[x, y]f$ denote the oriented *section* of f_x running from x to y. (This need not be a simple arc. For example $[x, x]f = f_x$.) Let $[y, z]\alpha$ denote the *segment* of α_y oriented from y to z. (This is a simple arc, but it may be oriented towards either a_y or ∞.) Now let f^P denote the family of closed curves obtained by concatenation (denoted by $+$) of sections and segments as follows. With x, y, z as in (1.5), the section $[x, z]f^P$ is

(1.6) $\quad [x, y]f \qquad$ if P_y is a singleton;

(1.7) $\quad [x, y]f + [y, a_y]\alpha + [a_y, z]\alpha \qquad$ if P_y is a fan;

(1.8) $\quad [x, y]f + [y, z]\alpha \qquad$ if P_y is a pair.

Thus f^P has ζ members, one for each cycle of R, each bordering a simply connected but possibly slit region in R^2. These are the F-images of the faces on M. Each fan of P that permutes n crossings corresponds to a critical point of F of the type $w = z^n$. The *multiplicity* of such a fan is defined to equal $n - 1$, and the sum of the multiplicities is the *branching number* μ of F. The *genus* γ of M then satisfies the analog of the Hurwitz-Riemann relation:

$$2 + \mu = 2\gamma + \rho + \tau. \tag{1.9}$$

Remark 1. If M is a closed orientable surface of genus γ and W is the sphere, then a polymersion of M to W is just a branched covering. Let β be its degree. We produce a bordered surface M by choosing a regular value of F as the reference point ∞ on S^2 and remove β discs about the poles $F^{-1}(\infty)$ on V. Assuming that the discs are so small as to be embedded by F, we have that $\tau = \rho = \beta$ and (1.9) becomes the classical formula [4, p. 17], written as

$$2 + \mu = 2\beta + 2\gamma. \tag{1.10}$$

Remark 2. Suppose now that $W = S^2 - \{e_0, e_1, \ldots, e_m\}$ is the finitely punctured sphere. Let e_0 serve as ∞. A raying for f is deemed sufficient here if in addition to the properties in the second paragraph in this section there is also one ray from each of the m punctures e_j. Let ε denote these *exceptional* rays. Theorem 1 still holds if we add the condition:

(1.11) P has no fixed points in $X(f, \varepsilon)$.

Since none of the e_j are branch points, we have from (1.3) that $\omega(f, e_j) = 0$. So $F(M) \subset W$, since the circulations counts the number of preimages with multiplicity. We shall not consider noncompact targets further.

2. Assemblages of higher genus

For W a closed, oriented surface of genus p we use a model polymersion G: $W \to S^2$ which is two sheeted with $m = 2p + 2$ simple branchpoints e_j. So

(W, G) corresponds, topologically speaking, to the Riemann surface of a hyperelliptic function, and the e_j to its Weierstrass points, when $p > 1$.

Let $g = \{g_1, g_2\}$ be a pair of concentric, positively oriented circles around $m = 2p + 2$ points e_j in R^2. Draw a set ε of m rays, one from each e_j, to ∞. Let x_{1+j}, y_{2+j} (indices mod m) be where ε_j crosses g_1 and g_2. The successor permutation

$$S = (x_1 x_2 \cdots x_m)(y_1 y_2 \cdots y_m)$$

on $X(g, \varepsilon)$ admits only one assembling permutation with m simple fans,

$$P = (x_1 y_2)(x_3 y_4) \cdots (x_m y_1).$$

The product $R = SP$ has the $\zeta = 2$ cycles required by Theorem 1:

$$R = (y_1 x_1 y_3 x_3 \cdots)(y_2 x_2 y_4 x_4 \cdots).$$

Let G_0 be the associated polymersion to R^2 of $W_0 = W - (D_1 + D_2)$, where the D_i are two disjoint discs. (We use set theoretic addition to denote disjoint union.) Extend G_0 over W so as to embed D_i onto the polar cap outside g_i to give the model G. Note that R decomposes W into two faces, conveniently called the *odd* and the *even sheet of* G.

Now suppose we are given the data (f, A_0) on W. Choose a pair of disjoint discs D_i in the complement of the data and assign to them two nonnegative integers β_i. (D_i will have β_i preimages under the polymersion F we are about to assemble.) We may adjust the model G so that the curve family $f^* = G(f)$, the pointset $A'_0 = G(A_0)$ and the exceptional rays ε lie in general position in R^2. Now split g_i into β_i nearby concentric circles to obtain the family g^* of $\beta = \beta_1 + \beta_2$ concentric circles enclosing the data (f^*, A_0^*) on R^2. Draw rays α so that together with ε, $\alpha' = \alpha + \varepsilon$ is sufficient for $f' = f^* + g^*$ and A'_0.

In addition to a sign, each crossing in $X(f', \alpha')$ also carries a *parity* as follows. For $x \in X(g_i^*, \varepsilon_j)$, set par $(x) = i + j \pmod 2$ in accordance with the second paragraph in this section. A crossing in $X(g^*, \alpha)$ inherits the parity of the last previous exceptional crossing. For $x \in X(f^*, \alpha)$, we have guaranteed, by general position, that only one of the two points in $G^{-1}(x)$ lies on f. Assign to x the parity of the sheet of G this point lies in. Note that the parity along f^* changes each time a curve crosses an exceptional ray, because here f changes sheets. Hence we choose to assign to $x \in X(f^*, \varepsilon)$ the parity of the succeeding section, $[x, xS']f^*$. (S' is the successor permutation in $X(f', \alpha')$ and it changes parity at the exceptional crossings.) To $a' = G(a) \in A'_0$ assign the parity of the sheet containing a on W.

We now isolate a number of necessary conditions on the assembling permutation P' which also turn out to be sufficient.

Proposition 1. *If P' is the assembling permutation associated with the data (f', A'_0) by Theorem 1, where $f = F\,|\,\partial M$, F is a polymersion $M \to W$ and*

$F' = G \circ F \,|\, M - F^{-1}(D_1 + D_2)$, *then*

(2.1) P' *is parity preserving on* $X(f', \alpha)$;
(2.2) P' *is parity reversing on* $X(f', \varepsilon)$; *and*
(2.3) *all fans on* $X(f', \varepsilon)$ *have multiplicity one.*

If $X(f^*, \alpha')$ *has* v^* *negative crossings,* f^* *has turning number* τ^*,

$$\beta = \text{card } (G \circ F)^{-1}(\infty)$$

and $R' = S'P'$ *has* ζ *cycles, then*

$$\zeta = v^* + \tau^* + \beta. \tag{2.4}$$

Proof. Since $\tau(f') = \tau(f^*) + \tau(g^*) = \tau^* + \beta$, (1.4) becomes (2.4). Set

$$P'_j = P' \,|\, X(f', \varepsilon_j) \quad \text{and} \quad \delta_j = \text{card } (G \circ F)^{-1}(e_j).$$

Since G is simply branched over e_j, $G^{-1}(e_j)$ consists in a single point. This is not a branchpoint of F, by construction of G. So $G \circ F$ has δ_j simple branchpoints at e_j, reflected by δ_j fans of multiplicity one in P'_j. Hence (2.3) follows. Since P' is effective, all negative crossings are paired off as in (1.3). Thus $\omega(f', e_j) = 2\delta_j + r_j$, where r_j counts the number of fixed points of P'_j, all of which are positive. But the circulation also counts the preimages with multiplicity. Hence $r_j = 0$. This means that each of the faces in the cell decomposition of M described by R' is embedded in either the even or odd face of W.

Each nontrivial cycle of P' corresponds to a connected set of arcs on M which projects to an arc on W under F. No arc in $G^{-1}(\alpha)$ crosses from one sheet to another. Whence (2.1) holds. Each exceptional pair corresponds to an arc in M that embeds into just one of two pieces of $G^{-1}(\varepsilon_j)$. Thus it does not pass through $G^{-1}(e_j)$. Since the crossings have opposite sign they have opposite parity as well. An exceptional fan corresponds to an arc in M that also embeds under F into $G^{-1}(\varepsilon_j)$ but passes through $G^{-1}(e_j)$. Since the crossings have like sign, they have opposite parity. Hence we have (2.2). □

We now retrace our steps. In addition to the circulation

$$\omega_j^* = \omega(f^*, e_j) = \sum \text{sign } (c)\colon c \in X(f^*, \varepsilon_j),$$

we use the index

$$\kappa_j^* = (-1)^j \sum (-1)^{\text{par}(c)}\colon c \in X(f^*, \varepsilon_j),$$

which is just the difference of the "even" and "odd" circulations.

LEMMA 1. *If* P'_j *is a permutation on* $X(f', \varepsilon_j)$ *satisfying* (1.1–3) *and* (2.2), *then*

(2.5) $\beta_1 - \beta_2 = \kappa_j^*$, *and*
(2.6) $\omega_j^* + \beta = 2\delta_j$ *for some nonnegative integer* δ_j.

Proof. No element is fixed, by (2.2). Each negative crossing is paired to a positive crossing of opposite parity, by (1.3). The fans permute the remaining, even number $2\delta_j$ of positive crossings, half of which are odd, half even, by (1.2) and (2.2). The β crossings in $X(g^*, \varepsilon_j)$ are all positive. Say there are π_j^{odd} positive odd crossings in $X(f^*, \varepsilon_j)$. Then each of the $\pi_j^{\text{odd}} + \beta_j^{\text{odd}}$ positive odd crossings in $X(f', \varepsilon_j)$ is either paired to one of the ν_j^{even} negative even crossings or is one of the δ_j positive odd crossings in the fans. Thus we have

$$\pi_j^{\text{odd}} + \beta_j^{\text{odd}} = \nu_j^{\text{even}} + \delta_j. \tag{2.7 odd}$$

Trade "odd" for "even", subtract and add the two relations to get (2.5, 6). $\square$

LEMMA 2. *If an assembling permutation P' satisfies both parity conditions* (2.1, 2) *then $R' = S'P'$ is parity preserving.*

Proof. Let $y = xS'$ and assume that y is an exceptional crossing. Then par $(y) \neq$ par (x) by definition and par $(y) \neq$ par (yP') by (2.2). So par $(x) =$ par (xR'). If y is ordinary, then from (2.1) we have

$$\text{par }(x) = \text{par }(y) = \text{par }(yP') = \text{par }(xR'). \quad \square$$

Condition (2.3) keeps F from being branched over the $G^{-1}(e_j)$ and (2.4) is (1.4), as remarked above.

PROPOSITION 2. *To each abstract transitive, effective assembling permutation P' on $X(f', \alpha')$ satisfying* (2.1–4) *there is a polymersion $F\colon M \to W$ with $F \mid \partial M = f$, branched only over A_0 F is unique, up to a homeomorphism on M. The relation between the genus γ of M, the number of "poles"*

$$\beta = \text{card }(G \circ F)^{-1}(\infty),$$

the number of curves ρ in f, and the branching number μ of F, is given by

$$2 + \mu + \tfrac{1}{2} \sum \omega_j^* = (1 - p)\beta + 2\gamma + \rho + \tau^*, \tag{2.8}$$

where τ^* is the turning number of f^* and ω_j^* is the circulation of f^* about e_j.

Proof. Under the hypotheses, P' is also faithful since $\tau(f') = \tau^* + \beta$. So the Theorem 1 gives a polymersion $f'\colon M' \to R^2$ spanning f'. By Lemma 2, F' embeds each of the ζ faces of M' into the G_0-image of the W-sheet that has the same parity as its cycle in R'. Since G_0 is $1:1$ there, we can lift F' cell by cell to W_0. That is, F' factors through a polymersion $F_0\colon M' \to W_0$. Complete M_0 to M by attaching β discs to $\partial M'$ along the borders belonging to $(F')^{-1}(g^*)$. Extend F_0 over M so that F is a $\beta_i : 1$ covering over the discs D_i, $i = 1, 2$. Since (2.6) counts the F'-preimages with multiplicity, and this equals δ_j by (2.3), F cannot be branched over the $G^{-1}(e_j)$. At the other branchpoints F', G is locally $1:1$, so F is branched exactly like F' over the points in A_0. From (1.9) and (2.6) we get (2.8). $\square$

There is an obvious cell decomposition on M induced by that on M' such that F is $1:1$ on each (open) cell of dimension 0, 1 or 2. Hence two polymersions defining identical assembling permutations on the data $(G, \varepsilon, \alpha, g^*)$ differ by a cell-preserving homeomorphism in the source. However, by virtue of certain choices made in splitting g to g^*, which in effect label the poles $(G \circ F)^{-1}(\infty)$ in a certain way, different assembling permutations may assemble equivalent polymersions. The obvious action of $\mathfrak{S}(\beta_i)$ on g_i^*, $i = 1, 2$, defines a representation of the group $\mathfrak{S}(\beta_1) \times \mathfrak{S}(\beta_2)$ among the inner automorphism of $\mathfrak{S}X(f', \alpha')$. We define an *assemblage* for the data (f, A_0) on W to consist in a class Π of transitive, effective assembling permutations satisfying (2.1–4), which are equivalent under the action of $\mathfrak{S}(\beta_1) \times \mathfrak{S}(\beta_2)$. Thus we have the generalization of Theorem 1.

THEOREM 2. *The polymersions $F\colon M \to W$ with given set A_0 of branchpoints and family $f = F \mid \partial M$ of border curves in general position on a closed (oriented, connected) surface W of genus p, are classified, up to a topological automorphism on the (compact, connected) surface M, by their assemblages, as defined above.*

3. Example

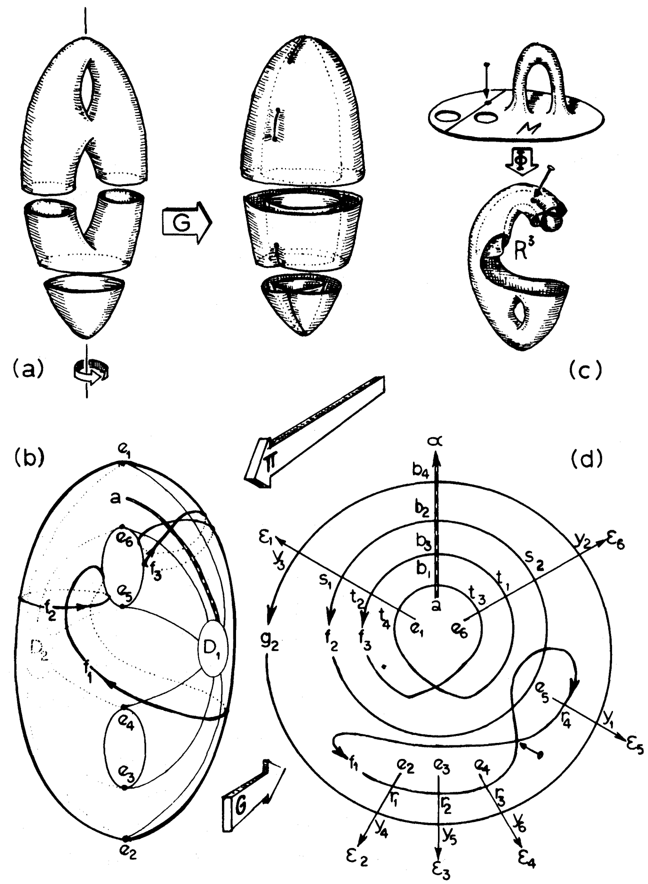
Consider the surface W of genus 2 symmetrically embedded in R^3 as shown in Figure (a). The model "hyperelliptic" map G may be visualized as the quotient projection of W modulo the involution on W obtained by a 180° rotation about the axes passing through the six "Weierstrass" points e_j. That the given family of closed curves $f = \{f_1, f_2, f_3\}$ on W, (b), bound some polymersion is demonstrated in (c). The mapping Φ of a surface M of genus 1 and 3 borders into R^3 has one Whitney umbrella (arrow) at the end of the doublepoint arc. Projecting normally to W, via Π produces F with a simple branchpoint under the umbrella point. In (d) we have drawn the G image f^* of f and the exceptional rays ε. From this alone we see that $\tau^* = 3$, $\nu^* = 1$ and $\beta_1 - \beta_2 = -1 = \kappa_j^*$, all j. Furthermore, $\omega_1^* = \omega_6^* = 3$, $\omega_2^* = \omega_3^* = \omega_4^* = 1$ and $\omega_5^* = -1$. Substituting this into (2.8) we obtain $\mu = 2(\gamma - \beta_1) - 1$. Thus there are no immersions among the polymersions extending f, nor are any of their sources planar (genus 0). Indeed $\mu = \gamma = \beta = 1$ is the simplest possible solution. However, the solution is not unique. For this we shall have to compute the assembling permutations.

For $\beta_1 = 0$ and $\beta_2 = 1$, we need only one circle in g^*, which we have drawn. For $\mu = 1$, there can be only one simple branchpoint, which we have located at the initial point a of a ray α (dashed arc), and chosen to have odd parity. The crossings have been labeled so that the subscripts have the proper parity. Note that r_4 is the only negative crossing. Curve family f has three selfintersections while f^* has four. The extra one is due to the identification under the involution. The successor permutation of $f' = \{g_2, f_1, f_2, f_3\}$ is

$$S' = (y_1 y_2 b_4 y_3 \cdots y_6)(r_1 \cdots r_4)(b_2 s_1 s_2)(t_1 b_3 t_2 t_3 b_1 t_4).$$

The parity rules on an assembling permutation require that P' have the factor

$$P_0 = (r_1 y_4)(r_2 y_5)(r_3 y_6)(r_4 y_1)(b_1 b_3).$$



However we have a choice for the partner for s_2. Say we take t_3; then $(y_2 t_1)$ is required. Indeed, some computations reveal that the remaining fans are determined, if $R' = S'P'$ is to have the five cycles stipulated by (2.4). Thus if

$$P_1 = (s_2 t_3)(y_2 t_1)(s_1 t_2)(y_3 t_4),$$

then $P' = P_0 P_1$ is a suitable assembling permutation and $R' = R_0 R_1$ where

$$R_0 = (r_2 y_6 r_4 y_4) \quad \text{and} \quad R_1 = (y_1 t_1 b_1 y_3 r_1 y_5 r_3)(y_2 b_4 t_4)(s_1 t_3 b_3)(s_2 b_2 t_2).$$

Computation again yields that the other choice, namely $(s_2 t_1)$, again determines the rest, and $P_2 = Q^{-1} P_1 Q$, where $Q = (t_1 t_3)(t_2 t_4)$.

Remark. There is no question here of "relabeling g^*", hence the two polymersions are topologically distinct. Note that if we do not specify the branchpoints, the number of inequivalent polymersions extending f becomes infinite.

To see this, attach n concentric copies of W to the surface $\Phi(M)$ by means of slits (= doublepoint arcs joining two umbrellas). This procedure extends F to a polymersion of $M \# nW$ (connected sum) with $1 + 2n$ simple branchpoints, and local degree n over D_1. We may keep D_1 out of the image and still achieve arbitrary genus by putting slits between the two sheets of $\Phi(M)$ that Π maps to the same place on W.

4. Branched coverings

If both M and W are closed then $\rho = 0$ and the degree δ is constant over each point. So $\beta = 2\delta$ and (2.8) becomes the classical Hurwitz-Riemann formula [4, p. 54], written in our symbols as

$$\gamma = \tfrac{1}{2}\mu + (p - 1)\delta + 1. \tag{4.1}$$

THEOREM 3 [4, p. 51]. *The class of branched coverings Φ: $M \to W$ of a given closed topological Riemann surface W of genus p by surfaces with δ sheets and branchpoints in a given point set $A = \{a_1, a_2, \ldots, a_w\}$ of W, is classified, up to an automorphism on M, by systems of $w + 2p$ permutations P_i, U_k, V_k in $\mathfrak{S}(\delta)$, up to conjugacy, which generate a transitive subgroup of $\mathfrak{S}(\delta)$ and satisfy the relation*

$$P_1 P_2 \cdots P_w U_1 V_1 U_1^{-1} V_1^{-1} \cdots U_p V_p U_p^{-1} V_p^{-1} = 1. \tag{4.2}$$

The classical proof is really quite simple. Let us see that our general classification also reduces to that of Hurwitz in this special case. Choose the model G and exceptional rays ε so that A lies entirely on the odd sheet of W. Draw the rays α so that g crosses the rays in the cyclic order given by

$$\alpha_1 \alpha_2 \cdots \alpha_w \varepsilon_1 \varepsilon_2 \cdots \varepsilon_m, \quad m = 2p + 2.$$

Label the crossings as follows:

$$c_i = X(\alpha_i, g_i), \quad b_i = X(\alpha_i, g_2), \quad x_{j+1} = X(\varepsilon_j, g_1) \quad \text{and} \quad y_{j+2} = X(\varepsilon_j, g_2).$$

(Subscripts are taken modulo w or m, whichever is appropriate.) Let a superscript $1 \le d \le \delta$ label the δ copies of each object obtained by splitting g to g^*. Thus the successor permutation becomes

$$S' = \prod_{d=1}^{\delta} (x_1^d c_1^d \cdots c_w^d x_2^d \cdots x_m^d)(y_1^d y_2^d b_1^d \cdots b_w^d y_3^d \cdots y_m^d).$$

Now let P' be an assembling permutation satisfying (2.1–3). For each $i = 1, \ldots, w$, the function $d \to t$, where $c_0^d P' = c_i^t$, gives a permutation $P_i \in \mathfrak{S}(\delta)$. Since the points in A have odd parity, each b_i^d is fixed by P'. For each $j = 1, \ldots, m$, the function $d \to t$, where $x_j^d P' = y_{j+1}^t$, gives a permutation $Q_{j-1} \in \mathfrak{S}(\delta)$. (To conserve printing costs we may write $\bar{Q}$ for Q^{-1} and (z, d) for z^d. Thus, by (2.3), $(y_j, d)P' = (x_{j-1}, d\bar{Q}_{j-2})$.) Evidently P' is transitive if and only if the $w + m$ permutations P_i, Q_j generate a transitive subgroup in $\mathfrak{S}(\delta)$. (Effectiveness is vacuous here.)

Next, the ζ cycles of $R' = S'P'$ have lengths which are positive multiples $l_1, \ldots, l_\zeta$ of $w + m$. So $l_1 + \cdots + l_\zeta = 2\delta$. Whence P' satisfies also condition (2.4), which here is $\zeta = 2\delta$, if and only if each $l_k = 1$. In that case, there are δ different cycles in R', each containing a different x_1^d. Let us pursue the orbit of x_1^d:

$$(x_1, d)(c_1, dP_1)(c_2, dP_1P_2) \cdots (c_w, dP_1 \cdots P_w)$$
$$(y_3', dP_1 \cdots P_wQ_1)(x_3, dP_1 \cdots P_wQ_1\bar{Q}_2) \cdots$$
$$(x_{m-1}, dP_1 \cdots P_wQ_1\bar{Q}_2 \cdots \bar{Q}_{m-2})(y_1, dP_1 \cdots P_wQ_1\bar{Q}_2 \cdots \bar{Q}_{m-2}Q_{m-1}).$$

Hence this, and a similar calculation (y_2, d), leads to identities

$$P_1 \cdots P_wQ_1\bar{Q}_2 \cdots Q_{m-1}\bar{Q}_m = 1 = \bar{Q}_1Q_2\bar{Q}_3 \cdots Q_m. \tag{4.3}$$

By coupling the δ copies of g_2 in g^* with those of g_1 we may assume that $Q_m = 1$. Eliminating Q_{m-1} from (4.3) yields

$$P_1 \cdots P_wQ_1\bar{Q}_2 \cdots \bar{Q}_{2p}\bar{Q}_1Q_2 \cdots Q_{2p} = 1. \tag{4.4}$$

A relabeling of the δ copies of g_1 in g^* merely conjugates these $w + 2p$ permutations by an element in $\mathfrak{S}(\delta)$.

Let us call a cyclically reduced word in $2p$ letters a *Dehn word* if each letter appears once to the $+1$, and once to the -1 exponent. In [3] it is shown that a Dehn word may be put into a commutator product form by a Nielsen transformation. That is, there exists an automorphism $g_i \to \phi_i(g_1, \ldots, g_{2p})$ on the free group $\langle g_1, \ldots, g_{2p} \rangle$ such that the Dehn word

$$g_1g_2^{-1}g_3g_4^{-1} \cdots g_{2p}^{-1}g_1^{-1}g_2g_3^{-1}g_4 \cdots g_{2p}$$

is the product of successive commutators in the ϕ_i. Thus we have a canonical transformation on $\mathfrak{S}(\delta)$

$$U_j = \phi_{2j-1}(Q_1, \ldots, Q_{2p}), \quad V_j = \phi_{2j}(Q_1, \ldots, Q_{2p})$$

taking (4.4) to Hurwitz' relation (4.2). $\square$

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