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ORDER OF THE CANONICAL VECTOR BUNDLE ON $C_n(k)/\Sigma_k$

BY

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Abstract

It has been proved that the order of the naturally defined vector bundle on $C_n(k)/\Sigma_k$, the configuration space of k distinct points in R^n, is helpful in finding some new elements of maps between spheres. Now, with the help of representation ring, we give some partial results concerning the largest powers of prime numbers dividing the orders.

Introduction

Let $C_n(k)$ be the configuration space of k distinct points in R^n, that is

$$\{(x_1, x_2, \ldots, x_k) \in R^{nk} \mid x_i \neq x_j, \quad \text{for all } i \neq j\}.$$

The symmetry group Σ_k acts on $C_n(k)$ by permuting the x_i's. $C_n(k)/\Sigma_k$ denotes the orbit space. Let $\zeta_{n,k}$ be the k-dimensional vector bundle

$$C_n(k) \times_{\Sigma_k} R^k \to C_n(k)/\Sigma_k$$

(Σ_k still acts on R^k by permuting the coordinates). Let $X_{n,k}(r)$ be the Thom space of $r\zeta_{n,k}$, the Whitney sum of r copies of $\zeta_{n,k}$. Snaith [14] showed that $\Omega^n S^{n+r}$ is stably homotopy equivalent to $\bigvee_{k>0} X_{n,k}(r)$, when $r > 0$. Recently the homotopy structure of the $X_{2,k}(r)$ has led to many deep results on the stable homotopy groups of spheres (Mahowald [10], R. Cohen [7], Brown and Peterson [4]). In particular, if $s\zeta_{n,k}$ is trivial, then $X_{n,k}(s+r) = \Sigma^{sk} X_{n,k}(r)$, and for each pair (n, k), only a finite number of stable homotopy types occur. In this paper we present partial results concerning the order of $\zeta_{n,k}$ and make a conjecture (see (1.4)) based on these results on the actual order, which, the author has been informed, has just been proved by F. Cohen and R. Cohen.

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1. Statement of results

Let $KO(X)$ ($K(X)$) denote the abelian group associated with the abelian semi-group of isomorphism classes of real (complex) vector bundles over the

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space X under the Whitney sum operation, and $\tilde{KO}(X)$ ($\tilde{K}(X)$) the reduced $KO(X)$ ($K(X)$).

DEFINITION (*Order of a vector bundle*). Imbed the positive integers Z^+ in $KO(X)$ ($K(X)$) by sending $m \to m\varepsilon$, where ε is the 1-dimensional trivial bundle over X. Then for any bundle η on X, the order of η is defined as the order of $\{\eta - \dim(\eta)\}$ in $\tilde{KO}(X)$ ($\tilde{K}(X)$).

Note. If $s =$ order (η) and $s \cdot \dim(\eta) > \dim(X)$, then $s\eta$, the Whitney sum of s copies of η, is a trivial bundle. The bundles $\zeta_{n,k}$ all satisfy this condition (by results of F. Adams).

Let $C_n(k)$ be the set of the ordered k-tuples of distinct points in R^n. Consider the k-plane bundle

$$\zeta_{n,k}\colon C_n(k) \times_{\Sigma_k} R^k \to C_n(k)/\Sigma_k,$$

where the symmetry group Σ_k acts on $C_n(k)$ and R^k both by permuting the k factors and on $C_n(k) \times R^k$ diagonally.

By results of Milgram [11, Theorem 1.2.2], $\zeta_{n,k}$ has finite order. We want to compute the order of $\zeta_{n,k}$. When $n \leq m$, $k \leq s$, $\zeta_{n,k}$ can be induced from $\zeta_{m,s}$, and hence order $(\zeta_{n,k})$ divides order $(\zeta_{m,s})$.

The known results of order $(\zeta_{n,k})$ are as follows:

(i) When $n = 1$ or $k = 1$, $\zeta_{n,k}$ has order 1.

(ii) By Cohen, Mahowald and Milgram [6], $\zeta_{2,k}$ has order 2.

(iii) When $k = 2$, $C_n(2)/\Sigma_2 \simeq RP^{n-1}$ and $\zeta_{n,2}$ is the Whitney sum of the trivial line bundle and the canonical line bundle. Thus, by F. Adams result in [1], $\zeta_{n,2}$ has order $2^{\phi(n-1)}$, where $\phi(n-1)$ is the number of elements in $\{1, 2, \ldots, n-1\}$ which are congruent to 0, 1, 2, 4 modulo 8.

Here we prove that when $n \geq 3$, $k \geq 3$ the odd primary part of order $(\zeta_{n,k})$ is divisible by $p^{[(n-1)/2]}$ for all odd primes $p \leq k$. Precisely we have:

THEOREM (1.1). *For an odd prime p,*

$$\text{order}(\zeta_{n,p}) = p^{[(n-1)/2]} \cdot s,$$

where s is prime to p.

Also, the prime power of order $(\zeta_{n,k})$ *is stable in certain sense.*

THEOREM (1.2). *For any prime number q, and non-negative integer k, s, satisfying $q^s \leq k < q^{s+1}$, the orders of ζ_{n,q^s} and $\zeta_{n,k}$ are divisible by the same power of q. In particular, when $k < q$ ($s = 0$), q does not divide* order $(\zeta_{n,k})$.

Combining the above two theorems, we have:

COROLLARY (1.3). (i) *For odd prime p and $p^2 > k \geq p$, $p^{[(n-1)/2]}$ is the largest power of p dividing* order $(\zeta_{n,k})$.

(ii) *For $k = 2, 3$, $2^{\phi(n-1)}$ is the largest power of 2 dividing* order $(\zeta_{n,k})$.

Thus, order $(\zeta_{n,3})$ is $2^{\phi(n-1)}3^{[(n-1)/2]}$; moreover, by computing the Atiyah-Hirzebruch Spectral Sequence of $\tilde{K}O(C_n(3)/\Sigma_3)$, we have the result, similar to that for $\tilde{K}O(C_n(2)/\Sigma_2)$, that $\{\zeta_{n,3} - 3\}$ generates $\tilde{K}O(C_n(3)/\Sigma_3)$.

For the 2-primary part of order $(\zeta_{n,k})$ we have one further result: 2-primary part of order $(\zeta_{4,4})$ is 4; thus order $(\zeta_{4,4}) = 12$. And this implies

$$\text{order } (\zeta_{3,4}) = 12,$$

$$\text{order } (\zeta_{3,5}) = \text{order } (\zeta_{4,5}) = \text{order } (\zeta_{3,6}) = \text{order } (\zeta_{4,6}) = 60,$$

$$\text{order } (\zeta_{3,7}) = \text{order } (\zeta_{4,7}) = 420.$$

The results of [6] $(n = 2)$, (1.3) and $\zeta_{4,4}$ suggests the following conjecture:

Conjecture (1.4). When $k \geq 2$,

$$\text{order } (\zeta_{n,k}) = 2^{\phi(n-1)} \prod_{\substack{p \leq k \\ p \text{ an odd prime}}} p^{[(n-1)/2]} .$$

In Section 2, we utilize the relationship between the representation ring and K-group to prove a property needed for Theorem (1.1). In Section 3, we obtain the order of p-torsion subgroup of $\tilde{K}(C_n(p)/\Sigma_p)$ and prove that $\{(C \otimes_R \zeta_{n,p}) - p\}$ generates the p-torsion subgroup of $\tilde{K}(C_n(p)/\Sigma_p)$. This leads directly to the proof of (1.1). In Section 4 we prove (1.2) and the result for $\zeta_{4,4}$.

2. A lemma for Theorem (1.1)

To prove Theorem (1.1), we consider $\zeta_{n,k}$ as the image of the permutation representation P_k of Σ_k on R^k, under the natural transformation

$$\eta\colon R_r(\Sigma_k) \to KO(C_n(k)/\Sigma_k), \qquad (P_k \to \zeta_{n,k}),$$

where $R_r(\Sigma_k)$ is the representation ring of Σ_k over the real number and the definition of η is completely similar to that of $\zeta_{n,k}$.

Let $\eta_1 : \tilde{R}_r(\Sigma_k) \to \tilde{K}O(C_n(k)/\Sigma_k)$ be the restriction of η to the virtual representation ring, (write $\eta_2 : \tilde{R}(\Sigma_k) \to \tilde{K}(C_n(k)/\Sigma_k)$ for the complex case). Then $\eta_1(P_k - k) = \zeta_{n,k} - k$. In general, for any group G and a free G-space X, we have the natural transformations

$$\eta_{X,G}\colon \tilde{R}_r(G) \to \tilde{K}O(X/G) \quad (\tilde{R}(G) \to \tilde{K}(X/G)).$$

By Milgram [11], when G is a finite group, Im $(\eta_{X,G})$ is contained in the torsion subgroup of $\tilde{K}O(X/G)$ $(\tilde{K}(X/G))$. Let β_p denote the projection of any torsion abelian group to its p-torsion component, then

$$\beta_p \circ \eta_{X,G}\colon \tilde{R}_r(G) \to {}_p\tilde{K}O(X/G) \quad (\tilde{R}(G) \to {}_p\tilde{K}(X/G))$$

is well defined (for any abelian group A, ${}_pA$ denotes its p-torsion part). Let $\eta_3 = \beta_p \circ \eta_1$, $\eta_4 = \beta_p \circ \eta_2$; also let Q_k be the permutation representation of Σ_k on C^k, that is, the complexification of P_k. Our object in the remainder of this section is to prove the following proposition.

PROPOSITION (2.1). *For any prime p, $\eta_4(Q_p - p)$ generates $\eta_4(\tilde{R}(\Sigma_p))$.*

For this, we need two more propositions, the first one proved by J. C. Becker and D. H. Gottlieb [3].

PROPOSITION (2.2) [3, Theorem 5.7]. *Suppose G is a finite group and H is a subgroup of G, such that* order (G)/order (H) *is prime to a prime q, then, for any free G-space X of finite dimension, $\pi^{\#}: {}_q\tilde{K}(X/G) \to {}_q\tilde{K}(X/H)$ is injective, where $\pi: X/H \to X/G$ is the covering projection.*

PROPOSITION (2.3). *Let π_p be the subgroup of Σ_p generated by the p-cycle $(1, 2, \ldots, p)$, and $j: \pi_p \to \Sigma_p$ be the inclusion. Then $j^*(Q_p - p)$ generates the image of $j^*: \tilde{R}(\Sigma_p) \to \tilde{R}(\pi_p)$.*

Proof of (2.3). Let α_s be the 1-dimensional representation of π_p over C,

$$\alpha_s(x_0): C \to C, \quad \alpha_s(x_0)(c) = \exp\left(\frac{2\pi i}{p} s\right)c,$$

where x_0 is a generator of π_p. Then $R(\pi_p)$ is the free abelian group generated by $\{\alpha_0, \alpha_1, \ldots, \alpha_{p-1}\}$, and $\alpha_0 = 1$, the trivial one. Also, the following equalities hold: $\alpha_s \circ \alpha_t = \alpha_{s+t}$, $(\alpha_t)^s = \alpha_{ts}$. If $\rho: \Sigma_p \to U$, the unitary group, is any representation, then $j^*(\rho) = \sum_{i=0}^{p-1} a_i\alpha_i$, for some integers a_i, $i = 0, 1, \ldots, p-1$, that is,

$$j^*(\rho)(x_0) = \bigoplus_{i=0}^{p-1} a_i\alpha_i(x_0): C^{\dim(\rho)} \to C^{\dim(\rho)},$$

where $\dim(\rho) = a_0 + a_1 + \cdots + a_{p-1}$.

LEMMA. $a_1 = a_2 = \cdots = a_{p-1}$.

Proof of lemma. For any positive integer s, $0 < s < p$, x_0^s is a p-cycle and there exists an element y in Σ_p such that $y^{-1}x_0y = x_0^s$. Let $\rho_1: \Sigma_p \to U$ be defined as $\rho_1(x) = \rho(y^{-1}xy)$, then ρ_1 is isomorphic to ρ. But

$$\begin{aligned} j^*(\rho_1)(x_0 &= \rho(y^{-1}x_0y) \\ &= \rho(x_0^s)(\text{the composite of } s \text{ copies of } \rho(x_0)) \\ &= \bigoplus_{i=0}^{p-1} a_i\alpha_{is}(x_0); \end{aligned}$$

also the map $i \to is$ is a permation of $\{1, 2, \ldots, p-1\}$. Thus, $a_1 = a_s$. This proves the lemma.

By the representation theory of finite groups (for example, [8]),

$$j^*(Q_p) = 1 + \alpha_1 + \alpha_2 + \cdots + \alpha_{p-1},$$

and hence $j^*(Q_p - p)$ generates $j^*(\tilde{R}(\Sigma_p))$.

Proof of (2.1). Consider the commutative diagram

$$\begin{tikzcd} \tilde{R}(\Sigma_p) \arrow[r, "j^*"] \arrow[d, "\eta_4"] & \tilde{R}(\pi_p) \arrow[d, "\eta_5"] \\ {}_p\tilde{K}(C_n(p)/\Sigma_p) \arrow[r, "\pi"] & {}_p\tilde{K}(C_n(p)/\pi_p), \end{tikzcd}$$

where $\pi\colon C_n(p)/\pi_p \to C_n(p)/\Sigma_p$ is the covering projection. By (2.3),

$$\pi^{\#}(\eta_4(Q_p - p)) = \eta_5(j^*(Q_p - p))$$

generates

$$\pi^{\#}(\eta_4(\tilde{R}(\Sigma_p))) = \eta_5(j^*(\tilde{R}(\Sigma_p))).$$

But, by (2.2), the property that order (Σ_p)/order (π_p) is prime to p, implies $\pi^{\#}$ is injective. Therefore, $\eta_4(Q_p - p)$ generates $\eta_4(\tilde{R}(\Sigma_p))$.

Remark on (2.1). This is only of interest when p is odd. In this case since

$$BU_{(p)} = \Omega^2(BO_{(p)}) \times BO_{(p)},$$

(2.1) implies the corresponding result of real case. The referee also points out that (2.3) is equivalent to a well-known result proved by Atiyah in 1962 that

$$\{r^*(E\Sigma_p \times_{\Sigma_p} R^p) - p\}$$

generates the image of $r^*\colon \tilde{K}(B\Sigma_p) \to \tilde{K}(B\pi_p)$, where $r\colon B\pi_p \to B\Sigma_p$ is induced from the inclusion $\pi_p \to \Sigma_p$.

3. ${}_p\tilde{K}(C_n(p)/\Sigma_p)$

From *Theory of iterated loop space*, there is a map

$$\coprod_{\substack{k>0,\\ \text{disjoint union}}} C_n(k)/\Sigma_k \to \Omega^n S^n,$$

inducing an injection in homology (in fact, $\Omega^n S^n$ is the group completion of $\coprod_{k>0} C_n(k)/\Sigma_k$) [5]. In the following, we use the image of the map to describe the homology of $C_n(p)/\Sigma_p$.

PROPOSITION (3.1) (F. Cohen [5]). *If n is odd, then*

$$H_*(C_n(p)/\Sigma_p;\, Z_p)$$

has additive basis

$$\{[p],\, \beta^{\varepsilon} Q^i[1],\, \varepsilon = 0, 1;\, 1 \le i \le (n-1)/2\},$$

dim $\beta^\varepsilon Q^i[1] = 2i(p-1) - \varepsilon$, *and* dim $[p] = 0$. *If n is even, then*

$$H_*(C_n(p)/\Sigma_p; Z_p)$$

has additive basis

$$\{[p], \beta^\varepsilon Q^i[1], \varepsilon = 0, 1; 1 \le i \le (n-1)/2; \lambda_{n-1}([1], [1]) * [p-2]\}$$

and dim $\lambda_{n-1}([1], [1]) = n - 1$. *Here $[k] \in H_0(\Omega^n S^n; Z_p)$ is the class of degree k map in $\Omega^n S^n$, $*$ denotes the loop product in $\Omega^n S^n$, β is the Bockstein operation, Q^i is the Dyer-Lashof operation and λ_{n-1} is the Browder operation.* (See [5].)

If $n < m$, Let $F_{n,m}$ be the embedding $C_n(p)/\Sigma_p \to C_m(p)/\Sigma_p$ induced from the standard embedding

$$R^n \to R^m, \quad (x_1, \ldots, x_n) \to (x_1, \ldots, x_n, 0, \ldots, 0),$$

then $(F_{n,m})_*$ sends $[p]$ to $[p]$, $\beta^\varepsilon Q^i[1]$ to $\beta^\varepsilon Q^i[1]$, and when n is even, sends $\lambda_{n-1}([1], [1]) * [p-2]$ to zero [5].

PROPOSITION (3.2). (i) *If n is odd, $\tilde{H}^*(C_n(p)/\Sigma_p; Z)$ has torsion elements only. If n is even,*

$$\tilde{H}^*(C_n(p)/\Sigma_p; Q) \simeq \tilde{H}^{n-1}(C_n(p)/\Sigma_p; Q) \simeq Q.$$

(ii) *The p-torsion part of $H^*(C_n(p)/\Sigma_p; Z)$ is*

$${}_pH^k(C_n(p)/\Sigma_p; Z) = \begin{cases} Z_p & \text{if } k = 2s(p-1), \quad 1 \le s \le [(n-1)/2] \\ 0 & \text{otherwise.} \end{cases}$$

(iii) *If $n < m$, $F^*_{n,m}$: $H^*(C_m(p)/\Sigma_p; Z) \to H^*(C_n(p)/\Sigma_p; Z)$, restricted to the p-torsion parts, is surjective.*

Proof of (3.2). By the computation of the Bockstein Spectral Sequence and the Universal Coefficient Theorem for cohomology, we have (i) and (ii). For (iii): when m, n odd, it follows from the facts that $\tilde{H}_*(C_n(p)/\Sigma_p; Z)$ has only torsion elements and $(F_{n,m})_*$ is a one-to-one map in $H_*(\ : Z_p)$; when n is even, it is proved by comparing

$$\text{order } ({}_pH^*(C_n(p)/\Sigma_p; Z))$$

with

$$\text{order } ({}_pH^*(C_{n-1}(p)/\Sigma_p; Z)).$$

PROPOSITION (3.3). ${}_p\tilde{K}(C_n(p)/\Sigma_p)$ *has order $p^{[(n-1)/2]}$, and for $n < m$, the map $(F_{n,m})^\#: {}_p\tilde{K}(C_m(p)/\Sigma_p) \to {}_p\tilde{K}(C_n(p)/\Sigma_p)$ is surjective.*

Proof of (3.3). First assume n and m are odd. Consider the Atiyah-Hirzebruch Spectral Sequence of $C_n(p)/\Sigma_p$. Because $\tilde{H}^*(C_n(p)/\Sigma_p; Z)$ has only torsion elements and the p-torsion parts exist only in dimensions multiples of 4, the p-torsion parts of E_2-term all survive to E_∞. Counting the number of Z_p's in E_∞, we have

$$\text{order } ({}_p\tilde{K}(C_n(p)/\Sigma_p)) = p^{[(n-1)/2]}.$$

Also, the surjectivity of $(F_{n,m})^*$ in ${}_p\tilde{H}^*(\ ;Z)$ implies the surjectivity of p-torsion part of E_2 and E_∞ terms and this leads to the surjectivity of $(F_{n,m})^{\#}$.

Now suppose n is even, and hence $n+1$ and $n-1$ are odd. From above,

$$(F_{n-1,n+1})^{\#} = (F_{n-1,n})^{\#} \circ (F_{n,n+1})^{\#}$$

is surjective, and hence $(F_{n-1,n})^{\#}$ is surjective. Counting also the number of Z_p's in E_2-terms of A-H S. S. of $\tilde{K}(C_n(p)/\Sigma_p)$, we have that

$$\text{order } ({}_p\tilde{K}(C_n(p)/\Sigma_p)) = p^{(n-2)/2}$$

and that $(F_{n-1,n})^{\#}$ is bijective. This implies that $(F_{n,n+1})^{\#}$ is surjective, and hence $(F_{n,m})^{\#}$ is surjective, for any $m > n$.

THEOREM (3.4) (Atiyah). *The map $\eta_4\colon \tilde{R}(\Sigma_p) \to {}_p\tilde{K}(C_n(p)/\Sigma_p)$ is surjective.*

Proof. Atiyah [2] showed that, for any positive integer m, the map

$$\eta\colon \tilde{R}(\Sigma_p) \to \tilde{K}(C_m(p)/\Sigma_p)$$

factors into

$$\bar{\eta}\colon \tilde{R}(\Sigma_p)/(\tilde{R}(\Sigma_p))^{mp} \to \tilde{K}(C_m(p)/\Sigma_p)$$

and there exist

$$s > m \quad \text{and} \quad \psi\colon \tilde{K}(C_s(p)/\Sigma_p) \to \tilde{R}(\Sigma_p)/(\tilde{R}(\Sigma_p))^{mp}$$

such that the following commutes:

$$\begin{tikzcd}
\tilde{R}(\Sigma_p)/(\tilde{R}(\Sigma_p))^{sp} \arrow[r, "\bar{\eta}"] \arrow[d] & \tilde{K}(C_s(p)/\Sigma_p) \arrow[d, "(F_{m,s})\#"] \arrow[dl, "\psi"'] \\
\tilde{R}(\Sigma_p)/(\tilde{R}(\Sigma_p))^{mp} \arrow[r, "\bar{\eta}"'] & \tilde{K}(C_m(p)/\Sigma_p).
\end{tikzcd}$$

By (3.3), Im $(F_{m,s})^{\#}$ contains ${}_p\tilde{K}(C_m(p)/\Sigma_p)$, which implies the theorem.

Proof of (1.1). By (3.4) and (2.1), $\eta_4(Q_p - p)$ has order $p^{[(n-1)/2]}$. But $\eta_4(Q_p - p)$ is the complexification of $\eta_3(P_p - p)$, and the map of complexification of vector bundles, restricted to the odd torsion part, is injective, we have (1.1).

4. Proof of (1.2) and the result of $\zeta_{4,4}$

First, we provide a general method to find the upperbounds of prime powers of the order for vector bundles obtained from representations.

In the following, q denotes a prime number.

Notation. For a vector bundle η of finite order, $L(\eta, q)$ denotes the largest power of q dividing order (η).

PROPOSITION (4.1). *Let G be a finite group, H be its subgroup. G and H act on finite dimensional spaces X and Y respectively. If* order (G)/order (H) *is prime to q and there is an H-map f: $X \to Y$, then, for any representation ρ of G,*

$$L(\eta_{X,G}(\rho), q) \leq L(\eta_{Y,H}(\rho), q).$$

Proof of (4.1). Let π: $X/H \to X/G$ be the covering projection of $\bar{f}$: $X/H \to Y/H$ be the map induced from f, then

$$\pi^{\#}(\eta_{X,G}(\rho)) = \eta_{X,H}(\rho\restriction_H) = \bar{f}^{\#}(\eta_{Y,H}(\rho\restriction_H)).$$

By (2.2), $L(\eta_{X,G}(\rho)), q) = L(\eta_{X,H}(\rho\restriction_H), q)$. But order $(\eta_{X,H}(\rho\restriction_H))$ divides order $(\eta_{Y,H}(\rho\restriction_H))$. This prove (4.1).

For Theorem (1.2), we prove the following:

PROPOSITION (4.2). *If $k_1 \geq k_2 \geq \cdots \geq k_r$, $k = \sum_{i=1}^{r} k_i$, and*

$$\frac{k!}{k_1!\,k_2!\cdots k_r!}$$

is prime to q, then

$$L(\zeta_{n,k}, q) = L(\zeta_{n,k_1}, q).$$

Proof of (4.2). For simplicity, we prove only the case $r = 2$, $s = k_1$, $t = k_2$, $k = s + t$. Let

$$A = \{1, 2, \ldots, s\} \quad \text{and} \quad H = \{\sigma \in \Sigma_k \,|\, \sigma(A) = A\}.$$

Then H is isomorphic to $\Sigma_s \oplus \Sigma_t$. Consider the H-space $Y = C_n(s) \times C_n(t)$ and the H-map f: $C_n(k) \to Y$,

$$f(x_1, x_2, \ldots, x_k) = ((x_1, x_2, \ldots, x_s), (x_{s+1}, \ldots, x_{s+t})).$$

Then, by assumption that $k!/s!\,t!$ is prime to q and (4.1),

$$L(\zeta_{n,k}, q) \leq L(\eta_{Y,H}(P_k|_H), q).$$

But $\eta_{Y,H}(P_k|_H) = \zeta_{n,s} \times \zeta_{n,t}$, and hence it has the same order as that of $\zeta_{n,s}$ (note that $s \geq t$). Thus $L(\zeta_{n,k}, q) \leq L(\zeta_{n,s}, q)$ and the equality holds.

Proof of (1.2). Consider the special case of (4.2): $r = q$, $k_1 = k_2 = \cdots = k_{q-1} = q^s$ and $k_q = q^s - 1$. Then $k = q^{s+1} - 1$. To check that

$$\frac{k!}{k_1!\,k_2!\cdots k_r!}$$

is prime to q, it is enough to note that for $x = aq^s + b$, $1 \leq x \leq q^{s+1} - 1$, $1 \leq b \leq q^s$, x has the same q-power as b. Thus, by (4.2),

$$L(\zeta_{n,q^{s+1}-1}, q) = L(\zeta_{n,q^s}, q),$$

and this proves (1.2).

In the remaining part, our object is to find the order of $\zeta_{4,4}$.

By (1.3), the odd prime part of order $(\zeta_{4,4})$ is 3. To find $L(\zeta_{4,4}, 2)$, let H_1 be the 2-Sylow subgroup of Σ_4 generated by

$$\{(1\ 2), (3\ 4), (1\ 3)(2\ 4)\};$$

then H_1 is isomorphic to $\Sigma_2 \int \Sigma_2$, the wreath product of Σ_2 with itself. If Σ_2 acts freely on X_1 and X_2, then $\Sigma_2 \int \Sigma_2$ acts freely on $X_1 \times (X_2)^2$. For any space X, let $F(X, k)$ be the subspace of X^k given by

$$\{(x_1, x_2, \ldots, x_k) \mid x_i \neq x_j, \quad \text{if } i \neq j\}$$

(when $X = R^n$, $F(R^n, k) = C_n(k)$). Then Σ_2 acts freely on $F(X, 2)$, and $\Sigma_2 \int \Sigma_2$ acts freely on $F(X, 2) \times (C_n(2))^2$; also, when $X = C_n(2)/\Sigma_2$, there is an H_1-map $g\colon C_n(4) \to F(C_n(2)/\Sigma_2, 2) \times (C_n(2))^2$ defined by

$$g(x_1, x_2, x_3, x_4) = ((\{x_1, x_2\}, \{x_3, x_4\}), (x_1, x_2), (x_3, x_4)).$$

Furthermore, $C_n(2)/\Sigma_2$ is homeomorphic to $R^{n+1} \times RP^{n-1}$. An explicit homeomorphism is induced by

$$h\colon C_n(2) \to R^n \times (R^n - \{0\}) \quad \text{where } h(x, y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right).$$

Since $R^5 \times RP^3$ embeds in R^8, we obtain a

$$\Sigma_2\text{-map}\colon F(C_4(2)/\Sigma_2, 2) \to F(R^8, 2)(= C_8(2)),$$

and hence an H_1-map

$$g_1\colon C_4(4) \to C_8(2) \times (C_4(2))^2.$$

Let $Y_1 = C_8(2) \times (C_4(2))^2$. Then $Y_1/H_1 = C_8(2) \times_{\Sigma_2} (C_4(2)/\Sigma_2)^2$ and by (4.1),

$$L(\zeta_{4,4}, 2) \leq L(\eta_{Y_1,H_1}(P_4 \restriction_{H_1}), 2).$$

To determine the order of $\eta_{Y_1,H_1}(P_4 \mid_{H_1})$, we prove the following proposition.

PROPOSITION (4.3). *Suppose ζ is an s-plane bundle over X. Let η be the sk-plane bundle*

$$C_n(k) \times_{\Sigma_k} (\zeta)^k \to C_n(k) \times_{\Sigma_k} X^k.$$

Then

$$\text{order } (\eta) = \frac{1}{s} \text{ (\textit{the least common multiple of } } s \cdot \text{order } (\zeta) \text{ \textit{and} order } (\zeta_{n,k})).$$

(Note that when $X = C_4(2)/\Sigma_2$, $\zeta = \zeta_{4,2}$, $n = 8$ and $k = 2$, then $\eta = \eta_{Y_1,H_1}(P_4 \restriction_{H_1})$.)

Proof of (4.3). The construction of η gives a natural transformation from the functor $KO(X)$ of X to the functor $KO(C_n(k) \times_{\Sigma_k} X^k)$ of X, which preserves the addition (Whitney sum) structure and, when $X = \{\text{point}\}$, the transformation

$$KO(\text{point})(\approx Z) \to KO(C_n(k)/\Sigma_k)$$

just sends r to $r\zeta_{n,k}$. Moreover, there is a commutative diagram,

$$\begin{tikzcd} KO(\text{point}) \arrow[r, "\pi^{\#}"] \arrow[d, "T_1"] & KO(X) \arrow[d, "T_2"] \\ KO(C_n(k)/\Sigma_k) \arrow[r, "\pi_1^{\#}"] & KO(C_n(k) \times_{\Sigma_k} X^k), \end{tikzcd}$$

induced from the unique map $\pi\colon X \to \text{point}$.

Let $t = \text{order } (\zeta_{n,k})$, $u = \text{order } (\zeta)$ and

$$m = \frac{1}{s}(\text{l.c.m. } \{su, t\}),$$

that is, $sm = \text{l.c.m. } \{su, t\}$. Then, $m\{\zeta\} = ms$ in $KO(X)$, and

$$m\{\eta\} = T_2(\{m\zeta\}) = T_2(\pi^{\#}(ms)) = \pi_1^{\#}(T_1(ms)) = \pi_1^{\#}(\{ms\zeta_{n,k}\}) = msk.$$

Thus, order (η) divides m. Furthermore, $s\zeta_{n,k}$ and $\zeta \oplus (k-1)s$ can be induced from η, and hence t divides $s \cdot \text{order } (\eta)$ and u divides order (η). Thus sm divides $s \cdot \text{order } (\eta)$, and hence m divides order (η) also. This prove (4.3).

Thus, $L(\eta_{Y_1,H_1}(P_4 \restriction_{H_1}), 2) = 4 \geq L(\zeta_{4,4}, 2) \geq 4$ (note that $\phi(4-1) = 2$), and we have the order of $\zeta_{4,4}$.

Remark. The proof for the 2-primary part of order $(\zeta_{4,4})$ is due to the referee of the Transactions of the American Mathematical Society.

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