ILLINOIS JOURNAL OF MATHEMATICS
Volume 25, Number 3, Fall 1981

ON THE BOUNDARY BEHAVIOR OF FUNCTIONS IN THE SPHERICAL DIRICHLET CLASS

BY

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A classical result of Fatou has the almost immediate consequence that a holomorphic injection of the unit disc, Δ, in the Riemann sphere, $\hat{\mathbf{C}}$, had radial limits almost everywhere on $\partial\Delta$. This theorem is quite striking since the cluster sets of such "schlicht" functions may be quite bizarre. Later Beurling [1] showed that radial limits exist except on a set of logarithmic (inner) capacity zero for the wider class of meromorphic functions in the spherical Dirichlet class. Tsuji [2] extended Beurling's argument to show that we have limits in any Stolz region at points in $\partial\Delta$, and the images of radial segments to the boundary are of finite spherical length. Tsuji's exceptional set is also of capacity zero.

Here we examine the question of the boundary behavior of normally convergent (i.e., uniformly convergent in the spherical metric on compact subsets of Δ) sequences of meromorphic functions in the spherical Dirichlet class. First we set the notation. Let $\| \quad \|$ be the L^2 norm with respect to Lebesgue measure $dx\,dy$ in Δ. If f is almost everywhere differentiable on Δ let

$$Tf = |f'|/(1 + |f|^2).$$

The spherical Dirichlet class D^* is the set of functions f which are meromorphic in Δ and satisfy $A[f] = \|Tf\|^2 < \infty$. $A[f]$ is the spherical area of the Riemann surface of f^{-1} spread over $\hat{\mathbf{C}}$. For $f \in D^*$, set

$$f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$$

whenever the limit exists. For $B \subset \partial\Delta$, cap B is the logarithmic inner capacity.

We will prove the following two theorems.

THEOREM 1. *Let $f_n \in D^*$ and suppose $f_n \to f$ normally and*

$$(1) \qquad \sum \|Tf_n - Tf\|^2 < \infty.$$

Then there exists a set $E \subset \partial\Delta$ with cap $E = 0$ *such that $f_n(e^{i\theta}) \to f(e^{i\theta})$ for all $e^{i\theta} \in \partial\Delta \backslash E$.*

THEOREM 2. *Let $f_n, f \in D^*$ and assume $f_n \to f$ normally and $A[f_n] \to A[f]$. Then there is a subsequence f_{n_k} and a set $E \subset \partial\Delta$ of capacity zero, so that*

Received August 29, 1979.

[1] Research partially supported by the National Science Foundation and the Alfred P. Sloan Foundation.

$f_{n_k}(e^{i\theta}) \to f(e^{i\theta})$ for all $e^{i\theta} \in \partial\Delta \backslash E$. In particular, this result is true for schlicht functions f_n, f provided $f_n \to f$ normally and $A[f_n] \to A[f]$.

These theorems permit us to interchange radial limits and limits of a sequence subject to the stated restrictions. An application of Theorem 2 to Kleinian groups will be presented elsewhere.

I would like to express my gratitude to Bob Kaufman and Joe Doob for sharing their insights with me and to Albert Baernstein whose comments and suggestions resulted in a considerable strengthening of the original result.

1. An estimate

Let $g: \Delta \to \mathbf{R}^+ \cup \{0\}$ be measurable and let $\|g\|$ denote the L^2 norm of g. We define

$$\mathscr{L}g(\theta) = \int_{-\pi/4}^{\pi/4} d\psi \int_{l_\psi} g(z)\, |dz| \quad \text{for } -\pi < \theta \leq \pi$$

where l_ψ is the linear segment from $e^{i\theta}$ to $\{|z| = .9\}$ having an angle ψ with the inner normal to $\partial\Delta$ at $e^{i\theta}$. Suppose μ is a probability measure on $\partial\Delta$. There is an associated conductor potential

$$u(z) = \int_{\partial\Delta} \log \frac{1}{|z-t|}\, d\mu(t).$$

The function u is harmonic in Δ. Let

$$V(\mu) = \sup_{|z|<1} u(z).$$

The argument given below is basically due to Tsuji [2, p. 344 ff]. We refer the interested reader to the arguments and diagrams contained therein.

LEMMA 1. *In the above notation,*

$$\int_{\partial\Delta} \mathscr{L}g(\theta)\, d\mu(\theta) \leq 4\pi^{1/2}\|g\|[\max\,(V(\mu), 2)]^{1/2}.$$

Proof. If

$$J = \int_\Delta g\, \frac{\partial u}{\partial r}\, r\, dr\, d\theta$$

then Tsuji's argument shows that

$$J^2 \leq \frac{\pi}{2}\, \|g\|^2 V(\mu) \tag{2}$$

and also

$$J = \int_{\partial\Delta} J(\phi)\, d\mu(\phi) \quad \text{where}$$

$$J(\phi) = \int_0^1 dr \int_0^{2\pi} g(re^{i\theta})(-d_\theta \arg (re^{i\theta} - e^{i\phi}).$$

He also shows

$$J(0) \geq \int_{|z-1/2| \leq 1/2} g(z)\, dr\, d\psi = \pi^{1/2}\|g\|. \tag{3}$$

Now

$$\mathscr{L}g(0) = \int_{-\pi/4}^{\pi/4} d\psi \int_{l_\psi} g(z)\, |dz| \leq \sqrt{2} \int_{-\pi/4}^{\pi/4} \cos\psi\, d\psi \int_{l_\psi} g(z)\, |dz|$$

which Tsuji shows is less than or equal to $2\sqrt{2} \int_{|z-1/2|<1/2} g(z)\, dr\, d\psi$.

Combining this with (3) we obtain

$$\mathscr{L}g(0) \leq 2\sqrt{2}[J(0) + \pi^{1/2}\|g\|]. \tag{4}$$

Equation (4) may be rotated by angle θ to show that

$$\mathscr{L}g(\theta) \leq 2\sqrt{2}[J(\theta) + \pi^{1/2}\|g\|].$$

Integrating with respect to $d\mu$, we obtain

$$\int_{\partial\Delta} \mathscr{L}g(\theta)\, d\mu(\theta) \leq 2\sqrt{2}[J + \pi^{1/2}\|g\|].$$

Inequality (2) then yields

$$\int_{\partial\Delta} \mathscr{L}g(\theta)\, d\mu(\theta) \leq 2\pi^{1/2}\|g\|[V(\mu)^{1/2} + \sqrt{2}]$$

and the lemma follows immediately.

2. Proof of Theorem 1

We denote by χ the chordal metric on $\hat{\mathbf{C}}$. Fix $\varepsilon > 0$ and let $e = e_\varepsilon$ be the set of all $e^{i\theta} \in \partial\Delta$ so that $f(e^{i\theta})$ and $f_n(e^{i\theta})$ exist, but

$$\overline{\lim_{n\to\infty}}\, \chi[f_n(e^{i\theta}), f(e^{i\theta})] > \varepsilon.$$

It suffices to show that cap $e = 0$, for then the set E may be taken to be a countable union of sets e_ε together with the set where $f(e^{i\theta})$ or some $f_n(e^{i\theta})$ fails to exist.

For $0 < R < 1$ and $f \in D^*$, let

$$T_R\, f(z) = \begin{cases} Tf(z) & \text{for } |z| \in [R, 1) \\ 0 & \text{for } |z| < R. \end{cases}$$

LEMMA 2. *For any $\delta > 0$ and $f \in D^*$ there exist $R \in (.9, 1)$ and $e' \subset \partial\Delta$ so that* cap $e' < \delta$ *and $\mathscr{L}(T_R\, f)(\theta) < \varepsilon/8$ for $e^{i\theta} \in \partial\Delta \backslash e'$.*

Proof. Since $f \in D^*$, $\lim_{R \to 1} \|T_R\, f\| = 0$. Let

$$E(R) = \{e^{i\theta} \mid \mathscr{L}(T_R\, f)(\theta) \geq \varepsilon/8\}.$$

Let μ_R be the equilibrium distribution of a closed subset of $E(R)$. Then, by Lemma 1,

$$\frac{\varepsilon}{8} \leq \int_{\partial\Delta} \mathscr{L}(T_R\, f)(\theta)\, d\mu_R(\theta) \leq 4\pi^{1/2}\{\max\,[V(\mu_R), 2]\}^{1/2}\|T_R\, f\|.$$

It follows that $\lim_{R \to 1} V(\mu_R) = \infty$ and consequently cap $E(R) \to 0$. Setting $e' = E(R)$ for some well-chosen R completes the proof of the lemma.

Let $e'' = e \backslash e'$ and $E_n = \{e^{i\theta} \in e'' \mid \chi[f_n(e^{i\theta}), f(e^{i\theta})] > \varepsilon\}$.

LEMMA 3. *For $f \in D^*$ and n sufficiently large, $\mathscr{L}(\,|T_R\, f_n - T_R\, f\,|)(\theta) > \varepsilon/8$ for all $e^{i\theta} \in E_n$.*

Proof. Since $\mathscr{L}$ is linear and $\mathscr{L}(T_R\, f)(\theta) < \varepsilon/8$ on E_n, the lemma follows from showing that $\mathscr{L}(T_R\, f_n) \geq \varepsilon/4$ for n sufficiently large.

Let $l'_\psi = l_\psi \cap \{R \leq |z| \leq 1\}$ and for a smooth curve $\alpha \subset \hat{\mathbf{C}}$, let $L(\alpha)$ denote its chordal length. Then

$$L[f_n(l'_\psi)] = \int_{l_\psi} T_R\, f_n\, |dz|. \tag{5}$$

A similar statement holds for f. A curve is no shorter than the distance between its endpoints, so

$$L[f_n(l'_\psi)] \geq \chi[f_n(e^{i\theta}), f_n(a_\psi)] \quad \text{where } a_\psi = l_\psi \cap \{|z| = R\}.$$

Since $f_n \to f$ uniformly on $|z| \leq R$, for n sufficiently large we may assume

$$\chi[f_n(a_\psi), f(a_\psi)] < \varepsilon/4\pi. \tag{6}$$

The triangle inequality shows that

$$L[f_n(l'_\psi)] \geq \chi[f_n(e^{i\theta}), f(e^{i\theta})] - \chi[f(e^{i\theta}), f(a_\psi)] - \chi[f(a_\psi), f_n(a_\psi)].$$

Now

$$\frac{\varepsilon}{8} > \mathscr{L}(T_R\, f)(\theta) = \int_{-\pi/4}^{\pi/4} L[f(l'_\psi)]\, d\psi \geq \int_{-\pi/4}^{\pi/4} \chi[f(e^{i\theta}), f(a_\psi)]\, d\psi$$

and using (5) and (6), we obtain, for $e^{i\theta} \in E_n$,

$$\mathscr{L}(T_R\, f_n)(\theta) = \int_{-\pi/4}^{\pi/4} L[f_n(l'_\psi)]\, d\psi > \frac{\pi}{2}\varepsilon - \frac{\varepsilon}{8} - \frac{\pi}{2}\frac{\varepsilon}{4\pi} > \frac{\varepsilon}{4}$$

which is the desired result.

Conclusion of the proof of Theorem 1. For all θ, $\mathscr{L}(Tf_n - Tf) \geq \mathscr{L}(T_R f_n - T_R f)$. If μ_n is the equilibrium distribution of a closed subset F_n of E_n, Lemma 1 shows that

$$\frac{\varepsilon}{8} \leq 4\pi^{1/2}\{\max [V(\mu_n), 2]\}^{1/2}\| Tf_n - Tf \|. \tag{7}$$

Equation (1) implies $\lim_{n\to\infty} \| Tf_n - Tf \| = 0$, consequently $\lim_{n\to\infty} V(\mu_n) = \infty$. By Frostman's Theorem (see Tsuji [2, p. 60]), $V(\mu_n) = v(F_n)$ where $v(F_n) = -\log(\text{cap } F_n)$. Since E_n is inner capacitable, we may choose F_n to have capacity arbitrarily close to cap E_n. From (7), we obtain

$$v(E_n)^{-1} < \left(\frac{100}{\varepsilon}\right)^2 \| Tf_n - Tf \|^2 \quad \text{for } n > n_0 = n(\varepsilon, \delta).$$

It follows that $\sum_{n=n_0}^{\infty} v(E_n)^{-1} < \infty$. By Tsuji [2, p. 56 and 63],

$$\left[v\left(\bigcup_{n=1}^{\infty} E_n \right) + \log 2 \right]^{-1} \leq \sum_{n=1}^{\infty} [v(E_n) + \log 2]^{-1}.$$

We immediately obtain $\lim_{n\to\infty} v(\bigcup_{k=n}^{\infty} E_k) = \infty$. Since for every n, $e'' \subset \bigcup_{k=n}^{\infty} E_k$,

$$\text{cap } e'' \leq \text{cap} \left(\bigcup_{k=n}^{\infty} E_k \right),$$

which converges to zero as $n \to \infty$. Theorem 1 is proved.

Proof of Theorem 2. Fix $\varepsilon > 0$. Since $f_n \to f$ normally, there is a compact subset $F \subset \Delta$ so that, for n large,

$$\| Tf \|_{L^2(F)}^2 > A[f] - \varepsilon^2 \quad \text{and} \quad \| Tf_n - Tf \|_{L^2(F)}^2 < \varepsilon^2.$$

It follows from the Minkowski inequality that

$$\begin{aligned} \| Tf_n - Tf \|^2 &< \varepsilon^2 + \| Tf_n - Tf \|_{L^2(\Delta\setminus F)}^2 \\ &< \varepsilon^2 + (\| Tf_n \|_{L^2(\Delta\setminus F)} + \| Tf \|_{L^2(\Delta\setminus F)})^2 \\ &< 5\varepsilon^2 \end{aligned}$$

for n sufficiently large. By passing to a subsequence n_k, $\sum_{k-1}^{\infty} \| Tf_{n_k} - Tf \|^2 < \infty$ and the result follows from Theorem 1.

References

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