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ON THE EXPONENTIAL STABILITY OF SET-VALUED DIFFERENTIAL EQUATIONS

BY

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Introduction

In this paper, we consider necessary and sufficient conditions for the (local) exponential stability of set-valued differential equations defined in $\mathbf{R}^k$.

To this aim, we introduce the concepts of local spectrum $\Sigma_p(F)$ and numerical range $N_p(F)$ for a set-valued Lipschitz-like map F defined in a neighborhood of a point $p \in \mathbf{R}^k$. We show that $\Sigma_p(F) \subset N_p(F)$ and that the condition $N_p(F) \subset \mathbf{R}^- = \{x \in \mathbf{R},\ x < 0\}$ ensures the exponential stability of the set-valued differential equation $\dot{x} \in F(x)$. On the other hand, we show that exponential stability implies $\Sigma_p(F) \subset \mathbf{R}^-$. Section 3 contains some applications of our results to the stability of nonlinear control systems.

We note that for linear maps the concept of local spectrum and numerical range are well known [1]. For nonlinear maps such concepts were introduced by Furi and Vignoli [6]. For set-valued maps, the definition of asymptotic spectrum was proposed in [3]. An approach to stability problems for set-valued functions can be found in [2] and in [4].

1. Definition and preliminary results

DEFINITION 1. We consider $\mathbf{R}^k$ with the standard norm and set-valued maps

$$F\colon U \to 2^{\mathbf{R}^k}\backslash\{\phi\} = S(\mathbf{R}^k),$$

where U is open in $\mathbf{R}^k$. Such a map is continuous if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|x - y\| < \delta$ implies $h(F(x), F(y)) < \varepsilon$, where h denotes the Hausdorff distance on $S(\mathbf{R}^k)$.

Henceforth we shall only consider continuous set-valued maps F on $\mathbf{R}^k$ where $F(x)$ is a compact set for every x.

For $p \in \mathbf{R}^k$, let $Q_p(\mathbf{R}^k)$ be the set of all such continuous set-valued functions defined in some neighborhood of p for which

(I) $F(p) = \{0\}$,

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(II)

$$|F| = \limsup_{x\to p} \frac{\|F(x)\|}{\|x-p\|} < +\infty \quad \text{where } \|F(x)\| = \sup\{\|y\|, y \in F(x)\}.$$

We introduce a semi-metric in $Q_p(\mathbf{R}^k)$ by setting

$$\Delta(F, G) = \limsup_{x\to p} \frac{h(F(x), G(x))}{\|x-p\|} \text{ for } F, G \in Q_p(\mathbf{R}^k).$$

It is easy to see that Δ defines an equivalence relation on $Q_p(\mathbf{R}^k)$ which induces a metric on the corresponding quotient space.

DEFINITION 2. For $F \in Q_p(\mathbf{R}^k)$, the *local spectrum* of F at the point p is

$$\Sigma_p(F) = \left\{\lambda \in \mathbf{R}: \liminf_{x\to p} \frac{h(F(x), \lambda(x-p))}{\|x-p\|} = 0\right\}.$$

$\Sigma_p(F)$ may be empty.

Set

$$\mathscr{D}(\lambda, F) = \liminf_{x\to p} \frac{h(F(x), \lambda(x-p))}{\|x-p\|}$$

and

$$r(F) = \sup\{|\lambda|, \lambda \in \Sigma_p(F)\},$$

the local spectrum has the following properties.

PROPOSITION 1. *Let $F, G \in Q_p(\mathbf{R}^k)$.*
(i) *If $\Delta(F, G) = 0$, then $\Sigma_p(F) = \Sigma_p(G)$.*
(ii) *$r(F) < |F|$.*
(iii) *$\Sigma_p(F)$ is a compact set.*
(iv) *$\Sigma_p(\alpha F) = \alpha\Sigma_p(F)$ for all $\alpha \in \mathbf{R}$.*
(v) *$\Sigma_p(\alpha(x-p) + F) = \alpha + \Sigma_p(F)$.*
(vi) *$\lambda \in \Sigma_p(F)$ implies $\mathscr{D}(0, F) < |\lambda|$.*

Proof. (i) By symmetry, it suffices to show that if $\lambda \in \Sigma_p(F)$ then $\lambda \in \Sigma_p(G)$. Indeed,

$$0 \le \liminf_{x\to p} \frac{h(G(x), \lambda(x-p))}{\|x-p\|} \le \limsup_{x\to p} \frac{h(F(x), F(y))}{\|x-p\|} + \liminf_{x\to p} \frac{h(F(x), \lambda(x-p))}{\|x-p\|} = 0.$$

(ii) For $\lambda \in \Sigma_p(F)$, we have

$$0 \le \frac{h(\lambda(x-p), 0)}{\|x-p\|} \le \frac{h(\lambda(x-p), F(x))}{\|x-p\|} + \frac{h(F(x), 0)}{\|x-p\|}.$$

Since

$$\frac{h(\lambda(x-p), 0)}{\|x-p\|} = |\lambda| \frac{h(x-p, 0)}{\|x-p\|} = |\lambda|,$$

we have

$$|\lambda| \le \liminf_{x \to p} \frac{h(\lambda(x-p), F(x))}{\|x-p\|} + \limsup_{x \to p} \frac{h(F(x), 0)}{\|x-p\|} = |F|.$$

(iii) We prove first the continuity in each variable of the map

$$\tilde{\mathscr{D}}: Q_p(\mathbf{R}^k) \times Q_p(\mathbf{R}^k) \to \mathbf{R}$$

defined by

$$\tilde{\mathscr{D}}(F, G) = \liminf_{x \to p} \frac{h(F(x), G(x))}{\|x-p\|}.$$

For F, F', G, $G' \in Q_p(\mathbf{R}^k)$, we have

$$h(F, G) \le h(F', G') + h(G, G') + h(F, F'),$$

and hence

$$\tilde{\mathscr{D}}(F, G) \le \tilde{\mathscr{D}}(F', G') + \Delta(G, G') + \Delta(F, F').$$

Interchanging the pairs (F, G) and (F', G'), we obtain

$$|\tilde{\mathscr{D}}(F, G) - \tilde{\mathscr{D}}(F', G')| \le \Delta(G, G') + \Delta(F, F').$$

It is now clear that the function $\lambda \mapsto \mathscr{D}(\lambda, F)$ is continuous for every F, and hence $\Sigma_p(F)$ is closed. By (ii), $\Sigma_p(F)$ is bounded and therefore compact.

Properties (iv), (v), (vi) follow easily from Definition 2. ■

DEFINITION 3. For $F \in Q_p(\mathbf{R}^k)$, we define the *numerical range* $N_p(F)$ of F at the point p by

$$N_p(F) = \bigcap_{r>0} \operatorname{cl} \phi(B_r \backslash \{p\}), \quad B_r = \{x \in \mathbf{R}^k : \|x-p\| \le r\},$$

where

$$\phi(x) = \frac{\langle F(x), x-p \rangle}{\|x-p\|^2}, \quad \langle F(x), x-p \rangle = \{\langle y, x-p \rangle, y \in F(x)\},$$

and cl A denotes the closure of the set $A \subset \mathbf{R}^k$.

For $F, G \in Q_p(\mathbf{R}^k)$, we define the *generalized numerical range* of F and G at p by

$$N_p(F, G) = \bigcap_{r>0} \operatorname{cl} \psi(B_r \backslash \{p\}),$$

where

$$\psi(x) = \frac{\langle F(x), G(x)\rangle}{\|x-p\|^2}, \quad \langle F(x), G(x)\rangle = \{\langle y, z\rangle, y \in F(x), z \in G(x)\}.$$

PROPOSITION 2. *Let F, $G \in Q_p(\mathbf{R}^k)$ have connected values. If $k > 1$ then $N_p(F, G)$ is a non-empty, compact and connected set.*

Proof. For every $x \in B_r\backslash\{p\}$,

$$\|\psi(x)\| \le \frac{\|\langle F(x), G(x)\rangle\|}{\|x-p\|^2} \le \frac{\|F(x)\|\,\|G(x)\|}{\|x-p\|^2} \le \frac{\beta_1\beta_2\|x-p\|^2}{\|x-p\|^2} = \beta_1\beta_2$$

for some constants β_1, β_2. From the continuity of F and G and the connectedness of $B_r\backslash\{p\}$ it is easy to see that $\operatorname{cl}\psi(B_r\backslash\{p\})$ is a compact connected set and $\bigcap_{r>0}\operatorname{cl}\psi(B_r\backslash\{p\})$ is compact and connected as the intersection of a nested family of compact intervals. ■

COROLLARY 1. *If $k > 1$ then $N_p(F) \subset \mathbf{R}$ is a non-empty, compact, connected set.*

Proof. It suffices to observe that $x \mapsto (x-p)$ lies in $Q_p(\mathbf{R}^k)$. ■

PROPOSITION 3. *If F, $G \in Q_p(\mathbf{R}^k)$ then $\alpha \in N_p(F, G)$ if and only if there exist sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ such that $x_n \mapsto p$, $x_n \ne p$, $y_n \in F(x_n)$, $z_n \in G(x_n)$ and*

$$\frac{\langle y_n, z_n\rangle}{\|x_n - p\|^2} \to \alpha.$$

Proof. If $\alpha \in N_p(F, G)$ then

$$\alpha \in \bigcap_{r>0} \operatorname{cl}\psi(B_r\backslash\{p\}).$$

Let $\{r_n\} \subset \mathbf{R}^+ = \{x \in \mathbf{R}, x > 0\}$ with $r_n \to 0$. Since $\alpha \in \operatorname{cl}\psi(B_{r_n}\backslash\{p\})$ for every n, every interval centered at α intersects $\psi(x)$ for some $x \in B_{r_n}\backslash\{p\}$. Therefore we can find a sequence $\varepsilon_n \to 0$, $\varepsilon_n > 0$, and points $x_n \in B_{r_n}\backslash\{p\}$, $w_n \in \psi(x_n)$, such that $|w_n - \alpha| < \varepsilon_n$. Since

$$w_n \in \frac{\langle F(x_n), G(x_n)\rangle}{\|x_n - p\|^2},$$

it follows that there exist $y_n \in F(x_n)$ and $z_n \in G(x_n)$ such that

$$w_n = \frac{\langle y_n, z_n\rangle}{\|x_n - p\|^2}.$$

The converse is obvious. ■

PROPOSITION 4. *Let F, $G \in Q_p(\mathbf{R}^k)$. For every $\lambda \in \Sigma_p(F)$ there exist $\alpha \in N_p(F, G)$ and $\beta \in N_p(G)$ such that $\beta\lambda = \alpha$.*

Proof. By definition of $\mathscr{D}(\lambda, F)$ there exists a sequence $\{x_n\}_{n \in \mathbf{N}}$, $x_n \neq p$, $x_n \to p$, such that

$$\frac{h(F(x_n), \lambda(x_n - p))}{\|x_n - p\|} \to 0.$$

Choose $y_n \in F(x_n)$ and $z_n \in G(x_n)$ arbitrarily. Then it follows from the definition of $Q_p(\mathbf{R}^k)$ and from Proposition 3, that, by possibly selecting appropriate subsequences,

$$\frac{\langle y_n, z_n \rangle}{\|x_n - p\|^2} \to \alpha \in N_p(F, G), \qquad \frac{\langle z_n, x_n - p \rangle}{\|x_n - p\|^2} \to \beta \in N_p(G).$$

Thus,

$$\text{(a)} \quad \left| \frac{\langle \lambda(x_n - p) - y_n, z_n \rangle}{\|x_n - p\|^2} \right| = \left| \frac{\langle \lambda(x_n - p), z_n \rangle}{\|x_n - p\|^2} - \frac{\langle y_n, z_n \rangle}{\|x_n - p\|^2} \right| \to |\lambda\beta - \alpha|,$$

$$\begin{aligned} \text{(b)} \quad \left| \frac{\langle \lambda(x_n - p) - y_n, z_n \rangle}{\|x_n - p\|^2} \right| &\leq \frac{\|\lambda(x_n - p) - y_n\|}{\|x_n - p\|} \frac{\|G(x_n)\|}{\|x_n - p\|} \\ &\leq \frac{h(F(x_n), \lambda(x_n - p))}{\|x_n - p\|} \\ &= 0, \end{aligned}$$

and hence $\lambda\beta - \alpha = 0$. ∎

COROLLARY 2. *If $F \in Q_p(\mathbf{R}^k)$ then $\sum_p (F) \subset N_p(F)$.*

Proof. It follows from Proposition 4 if we take $G(x) = x - p$, since $N_p(x - p) = 1$. ∎

2. Main results

DEFINITION 5. Let $F \in Q_p(\mathbf{R}^k)$. We say that the constant solution $\gamma(t, p) = p$ of the set-valued differential equation $\dot{\gamma} \in F$ is *exponentially stable* if there exist numbers $\delta > 0$ and $\alpha > 0$ such that any solution of $\dot{\gamma} \in F$, $\gamma(0) = x$, with $0 \leq \|x - p\| \leq \delta$ satisfies the condition

$$\|\gamma(t, x) - p\| \leq \|x - p\| e^{-\alpha t} \quad \text{for all } t \geq 0.$$

DEFINITION 6. We say that $F\colon U \to S(\mathbf{R}^k)$ has the Lipschitz selection property if, for every $x_0 \in U$ and for every $y_0 \in F(x_0)$, there exist a neighborhood W of x_0 and a locally Lipschitz map $f\colon W \to \mathbf{R}^k$ such that $f(x_0) = y_0$ and $f(x) \in F(x)$ for all $x \in W$.

We recall that the set-valued function induced by a Lipschitz control system has the Lipschitz selection property, [5].

THEOREM 1. *Let $F \in Q_p(\mathbf{R}^k)$. Assume that F admits the Lipschitz selection property and that the constant solution of $\dot{\gamma} \in F$ is exponentially stable. Then $\lambda < -\alpha$ for all $\lambda \in \Sigma_p(F)$, if $\Sigma_p(F) \neq \phi$.*

Proof. For $r < \delta$, consider the problem $\dot{\gamma} \in F$, $\gamma(0) = w \in B_r$. Let $\gamma(t, w)$ be a solution of this problem, (such a solution exists for the properties of F), and let $m = \sup \lambda$, $\lambda \in \Sigma_p(F)$. By Proposition 3 and Corollary 2, for every $r' < r$ there exist a sequence $\{x_n\} \subset B_{r'}\backslash\{P\}$, $x_n \to p$, and a sequence $\{y_n\}$, $y_n \in F(x_n)$, such that

$$m = \lim_n \frac{\langle y_n, x_n - p\rangle}{\|x_n - p\|^2}.$$

Using the semigroup property, the property of Lipschitz selection for F, and the hypothesis of exponential stability, for every $x_n \in B_{r'}\backslash\{p\}$ we can choose $t_n \to +\infty$ and $w_n \in B_r$ such that $x_n = \gamma(t_n, w_n)$, $\dot{\gamma}(t_n, w_n) = y_n$ and

$$\begin{aligned} m &= \frac{1}{2} \lim \frac{2\langle y_n, x_n - p\rangle}{\|x_n - p\|^2} \\ &= \frac{1}{2} \lim \frac{2\langle \dot{\gamma}(t_n, w_n), \gamma(t_n, w_n) - p\rangle}{\|x_n - p\|^2} \\ &= \frac{1}{2} \lim_n \frac{d}{dt} \log \|\gamma(t_n, w_n) - p\|^2; \end{aligned}$$

hence

$$2m = \lim_n \frac{d}{dt} \log \|\gamma(t_n, w_n) - p\|^2.$$

Now, for any $\varepsilon > 0$, there exists n_0 such that if $n > n_0$ we have

$$2(m - \varepsilon) < \frac{d}{dt} \log \|\gamma(t_n, w_n) - p\|^2 < 2(m + \varepsilon),$$

and, by continuity, there exists a σ_n such that

$$2(m - \varepsilon) < \frac{d}{dt} \log \|\gamma(t, w_n) - p\|^2 \leq 2(m + \varepsilon) \quad \text{for } t \in [t_n - \sigma_n, t_n + \sigma_n]$$

Integrating on this interval we get

$$e^{2(m-\varepsilon)2\sigma n} < \frac{\|\gamma(t_n + \sigma_n, w_n) - P\|^2}{\|\gamma(t_n - \sigma_n, w_n) - p\|^2} < e^{2(m+\varepsilon)2\sigma_n}.$$

On the other hand let $f\colon V_n \to \mathbf{R}^k$ be the Lipschitz selection corresponding to w_n,

t_n, y_n, where V_n is a neighborhood of w_n. Let $\gamma(t, x)$ be the unique solution of the differential equation

$$\dot{x} = f(x), \ \gamma(t_n, x) = x, \quad \text{where} \quad -\varepsilon + t_n < t < t_n + \varepsilon.$$

Choose $\sigma_n > 0$, $2|\sigma_n| < \varepsilon$. Then $\bar{\gamma}(t, x) = \gamma(t + t_n, x)$ is the solution of $\dot{x} = f(x)$ with initial time 0.

Now

$$\begin{aligned}\|\gamma(t_n + \sigma_n, w_n) - p\| &= \|\bar{\gamma}(\sigma_n, w_n) - p\| \\ &= \|\bar{\gamma}(2\sigma_n, \bar{\gamma}(-\sigma_n, w_n)) - p\| \\ &\leq \|\bar{\gamma}(-\sigma_n, w_n)\|e^{-2\sigma_n\alpha} \\ &= \|\gamma(t_n - \sigma_n, w_n) - p\|e^{-2\sigma_n\alpha}.\end{aligned}$$

It follows that $m - \varepsilon < -\alpha$ for every ε, and the theorem is proved. ∎

THEOREM 2. *Let $F \in Q_p(\mathbf{R}^k)$ with connected values, and let $N_p(F) \subset \mathbf{R}^-$. Then the constant solution of $\dot{\gamma} \in F$ is exponentially stable.*

Proof. Let $\alpha > 0$ and $\delta > 0$ be such that

$$\sup \frac{\langle w - p, F(w)\rangle}{\|w - p\|^2} \leq -\alpha \quad \text{for all } w \in B_\delta \backslash \{p\}.$$

If $\gamma(t, w)$ is a solution of $\dot{\gamma} \in F$, $\gamma(0) = w \in B_\delta \backslash \{p\}$ then for every t for which $\gamma(t, w) \in B_\delta$ we get

$$\frac{1}{2}\frac{d}{dt}\log \|\gamma(t, w) - p\|^2 < -\alpha$$

and so

$$\|\gamma(t, w) - p\|^2 \leq \|w - p\|^2 e^{-2\alpha t}.$$

Now let $t_1 = \sup\{t \geq 0: \gamma(t, w) \ \varepsilon \ B_\delta\}$ and assume that $t_1 < +\infty$. Then

$$\|\gamma(t, w) - p\| < \delta \text{ for } t \in [0, t_1) \quad \text{and} \quad \|\gamma(t_1, w) - p\| = \delta.$$

The continuity of $\gamma(t, w)$ yields the contradiction

$$\delta = \|\gamma(t_1, w) - p\| < \|w - p\|e^{-2\alpha t_1} < \delta.$$

THEOREM 3. *Let $F \in Q_p(\mathbf{R}^k)$ with connected values and $N_p(F) \subset (0, +\infty)$. Then there exists a $\delta > 0$ such that any solution $\gamma(t, w)$ of $\dot{\gamma} \in F$, $\gamma(0) = w$, with $\|w - p\| > \delta$, satisfies the condition*

$$\|\gamma(t, w) - p\| > \delta > \|w - p\| \quad \textit{for every } t > 0.$$

This theorem can be proved by techniques similar to those in the proof of Theorem 2.

3. Applications

We now consider an application of the preceding results to certain control problems. To this aim we state the following propositions.

PROPOSITION 5 [7]. *Consider the control system* $\dot{x} = A(x)u$ *where* $A(x)$ *is a linear map from* $\mathbf{R}^n$ *to* $\mathbf{R}^k$ *for every* x *in some open set* U *contained in* $\mathbf{R}^k$, *and the map* $x \mapsto A(x)$ *is locally Lipschitz. For every compact set* $K \subset \mathbf{R}^n$, *the map* $F\colon U \to S(\mathbf{R}^k)$, *defined by* $F(x) = A(x)K$, *is locally Lipschitz.*

If, in addition, $A(p)K = 0$ *for some* $p \in U$ *then* F *satisfies both conditions* (I) *and* (II) *in Section* 2.

Example. Consider the control problem

(1) $\quad \dot{x} = \|x\|u,\ x(0) = 0, \quad$ where $x \in \mathbf{R}^k$, $u \in B_1 = \{v\colon \|v\| \le 1,\ v \in \mathbf{R}^k\}$.

The hypotheses of Proposition 5 are satisfied and hence we can compute the spectrum and the numerical range of the multivalued function $F(x) = \|x\|B_1$ at the point 0: $\Sigma_0 F = 1$ and $N_0(F) = [-1, 1]$. We now perturb equation (1) by means of a Lipschitz map $f\colon U \to \mathbf{R}^k$ defined in an open neighborhood of $0 \in \mathbf{R}^k$ and such that $f(0) = 0$:

(2) $\quad \dot{x} = \|x\|u + \beta f(x),\ x(0) = 0,\ \beta \in \mathbf{R}, \quad$ where $x \in \mathbf{R}^k$, $u \in B_1$.

The multivalued function associated with (2) is now $G(x) = F(x) + \beta f(x)$. It is easy to see from Proposition 1 that $\Sigma_0(G) = 1 + \beta k$, and that $N_0(G)$ is the interval $[-1 + \beta k,\ 1 + \beta k]$.

Thus, if $\beta k < -1$, then $N_0(G) \subset \mathbf{R}^-$ and the solution $x(t) = 0$ of (2) is exponentially stable.

PROPOSITION 6 [7]. *Consider the control system* $\dot{x} = f(x, u)$ *with* $f\colon U \times K \to \mathbf{R}^k$, U *open in* $\mathbf{R}^k$, K *compact set in* $\mathbf{R}^n$.

Assume that f *is continuous in* u *for each* $x \in U$, *and uniformly Lipschitz on* U. *Assume further that there exists a* $p \in U$ *such that* $f(p, u) = 0$ *for every* $u \in K$.

Then $F(x) = \{f(x, u)\colon u \in K\}$ *satisfies conditions* (I) *and* (II) *of Section* 2.

Finally, notice that the passage from a control system to a set-valued function leads to essentially the same statement of the problem if we consider a variable control region. It suffices to assume that $U(x)$ is a Lipschitz map of x (in the Hausdorff metric) and takes compact values for every x.

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