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FLAT FAMILIES OF AFFINE LINES ARE AFFINE-LINE BUNDLES

BY

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By a *fibration* over a scheme S we mean a faithfully flat morphism $X \to S$ of finite type, and one such morphism will be called an A^n*-bundle* over S if the S-scheme X is locally isomorphic to the affine n-space A^n_S and S relative to the Zariski topology. The purpose of this note is to establish the following theorem.

THEOREM. *If $X \to S$ is a fibration such that for each point $s \in S$ the fiber of s in X is isomorphic to the affine line $A^1_{k(s)}$ and if the base S is a Noetherian normal scheme, then X is an A^1-bundle over S.*

Earlier, Kambayashi and Miyanishi proved this theorem under the additional hypotheses that the morphism $X \to S$ be affine and the base S be locally factorial [7; Theorem 1]. On the other hand, they assumed only the generic fiber to be an A^1 and all other fibers to be geometrically integral. The proof in the present paper reduces to the case where S is the spectrum of a local ring, then proceeds by induction on the dimension of S, which was the approach of [7]. The proof for $\dim S = 1$ when $X \to S$ is assumed to be affine was quite elementary (see [7; Lemma 1.3]); without this assumption it is much less so. The crux of our argument for the $\dim S = 1$ case (see §1) employs a lemma of Swan, which appears with proof as Lemma 1.1 below. The main task when $\dim S > 1$ is to weaken the assumption of "locally factorial" down to "normal" for the base scheme S. This is done in §2 following an idea of V. Danilov, contained in a letter to one of the authors. His idea involves a reduction to the case of a Henselian base scheme S, and a clever "two section" argument for that case. We are much indebted to him and want him to receive the appropriate credit for his vital contribution to this proof. We also thank I. Dolgachev for clarifying some of the points in Danilov's letter.

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1. The case $\dim S = 1$

We begin by establishing a result due to Swan.

1.1 LEMMA (Swan [11]). *Let A be a Noetherian domain containing another domain R over which it is of finite transcendence degree $d \geqq 1$. Let $t \in R$ be a prime element, and assume there exists a prime ideal P in A of height $d + 1$ such that $P \cap R = tR$. Then any sequence $f_0, f_1, f_2, \ldots$ of non-zero elements of A, starting with f_0 transcendental over R, and satisfying*

$$(1) \qquad f_i - c_i = tf_{i+1} \quad \text{for some } c_i \in R, \qquad i = 0, 1, 2, \ldots,$$

is a finite sequence.

Proof. Suppose that an infinite such sequence exists. Let $\mathbf{q} = tR$. By localizing at $\mathbf{q}$ we may assume $\mathbf{q}$ is a maximal ideal of R. Let

$$B = R[f_0, f_1, \ldots] \subseteq A.$$

It follows from the equations (1) that B is generated birationally over R by f_0; hence the transcendence degree of B over R is 1. Let $Q = tB$, we have that the composite of homomorphisms $R \to B \to B/Q$ is surjective, since by the equations (1), $f_i \equiv c_i \pmod{Q}$ for all i. Clearly $Q \neq B$, since t is not a unit (even in A). Thus the kernel $Q \cap R$ of this homomorphism must equal $\mathbf{q}$, since $\mathbf{q}$ is maximal. Hence $R/\mathbf{q} \cong B/Q$, so Q is a maximal ideal and $Q = P \cap B$.

We claim that the localization B_Q is a principal valuation ring. Clearly its maximal ideal is tB_Q; our claim will be established once we show that $\cap_{n=1}^{\infty} t^n B_Q = \{0\}$, for it follows from this that any element of B_Q has a unique factorization ut^r where u is a unit in B_Q and $r \geqq 0$. To see this, note that

$$\bigcap_{n=1}^{\infty} t^n B_Q = \left(\bigcap_{n=1}^{\infty} t^n B \right)_Q$$

(this follows easily from the fact that t is prime in B) and

$$\bigcap_{n=1}^{\infty} t^n B \subset \bigcap_{n=1}^{\infty} t^n A = \{0\}$$

since A is a Noetherian domain and t is not a unit in A. Therefore

$$\bigcap_{n=1}^{\infty} t^n B_Q = \{0\},$$

and the claim is verified. (Here let us note that in the lemma the hypothesis "A is Noetherian" can be replaced by "$\bigcap_{n=1}^{\infty} t^n A = \{0\}$".)

The extension $B_Q \subseteq A_P$ has transcendence degree $d - 1$. Upon choosing a transcendence basis $x_1, \ldots, x_{d-1} \in A_P$ for this extension, we have

$$B_Q[x_1, \ldots, x_{d-1}] \subseteq A_P, \quad \dim B_Q[x_1, \ldots, x_{d-1}] = d,$$

and

$$\dim A_P = d + 1.$$

This contradicts the following lemma, so such an infinite sequence could not exist.

1.2 LEMMA (Swan [11] or [12; Theorem 5.4]). *Let* $\Lambda \subseteq \Gamma$ *be domains with* Λ *Noetherian,* $\dim \Lambda < \infty$, *and* Γ *algebraic over* Λ. *Then* $\dim \Gamma \leqq \dim \Lambda$.

Proof. Given a chain of prime ideals $P_0 \subset P_1 \subset \cdots \subset P_n$ in Γ, choose $y_i \in P_i - P_{i-1}$ for $i = 1, \ldots, n$. Then, clearly, $\dim \Lambda[y_1, \ldots, y_n] \geqq n$. We may therefore assume Γ to be finitely generated over Λ; by induction, we may even take it to be simply generated. Thus $\Gamma = \Lambda[y] \cong \Lambda[Y]/I$ for an ideal I in the polynomial ring $\Lambda[Y]$. Since Γ is algebraic over Λ, $I \neq \{0\}$. Therefore $\dim \Gamma < \dim \Lambda[Y] = 1 + \dim \Lambda$ (since Λ is Noetherian). This finishes the proof.

Let f: $X \to S$ be a fibration and F a scheme. If for each $s \in S$ the (scheme-theoretic) fiber $f^{-1}(s)$ is isomorphic to $F_{k(s)} = F \times k(s)$ over the residue field $k(s)$ of s, we call f a *fibration of fiber type F*. Thus the morphism of our theorem is a fibration of fiber type $A^1 = \operatorname{Spec} Z[T]$. We will call such a morphism, more simply, an *A^1-fibration*.

1.3 *Remark.* If f: $X \to S$ is any faithfully flat morphism of Noetherian schemes such that S is normal and each fiber of f is normal, then X is also normal. This results from a standard fact of commutative algebra, which can be found, for example, in [9; Corollary 21.E]. We will be using the consequent fact that if f is an A^1-fibration with S normal and Noetherian, then X is normal. It is then clear that X is connected if S is. Another fact which will be used is that an A^1-fibration is a smooth morphism. This results from [4; IV, Theorem 17.5.1].

1.4 PROPOSITION. *Let* $C = \operatorname{Spec} R$ *where* R *is a principal valuation ring. Then every* A^1*-fibration over* C *is an* A^1*-bundle over* C.

Proof. Let K and k be, respectively, the field of fractions and the residue field of R. Let $Y \to C$ be an A^1-fibration, with the generic fiber and the closed fiber denoted by Y_K and Y_k, respectively. Our assumptions imply Y is normal and integral (see Remark 1.3), and that $Y_K = \operatorname{Spec} K[f]$ for some f in the function field of Y, transcendental over K. Since Y is normal, f possesses well-defined zeros and poles. Since f has no poles in Y_K, it can have a pole only at the prime divisor Y_k. Let $t \in R$ be a uniformizing parameter. Identifying R as a subring of $\Gamma(Y, \mathcal{O}_Y)$, Y_k is defined everywhere by t, and we may replace f by $t^m f$ where m is greater than or equal to the order of the pole f has at Y_k. Since f now has no poles on Y, $f \in \Gamma(Y, \mathcal{O}_Y)$ (and $Y_K = \operatorname{Spec} K[f]$ still holds).

Consider the homomorphism φ: $\Gamma(Y, \mathcal{O}_Y) \to \Gamma(Y_k, \mathcal{O}_{Y_k})$ arising from the closed embedding $Y_k \to Y$. By hypothesis, $\Gamma(Y_k, \mathcal{O}_{Y_k}) = k[u]$ where u is transcendental over k. If $\varphi(f) = \alpha_0 \in k$, then lifting α_0 to some $c_0 \in R$, we have $\varphi(f - c_0) = 0$. Since t defines Y_k, t divides $f - c_0$ in each local ring of Y. Hence t divides $f - c_0$ in $\Gamma(Y, \mathcal{O}_Y)$. Thus we have $f - c_0 = tf_1$, where $f_1 \in \Gamma(Y, \mathcal{O}_Y)$. If $\varphi(f_1) \in k$, we repeat this process to get $f_1 - c_1 = tf_2$, where $c_1 \in R$, $f_2 \in \Gamma(Y, \mathcal{O}_Y)$. We can continue choosing f_{i+1} accordingly, as long as $\varphi(f_i) \in k$; the following argument will show that the resulting sequence $f_0(=f), f_1 f_2, \ldots$ must be finite. Let U be any affine open set in Y containing a closed point y in Y_k, let $A = \Gamma(U, \mathcal{O}_Y)$, and let $P \subset A$ be the (maximal) ideal in A corresponding to y. Then A has transcendence degree 2 over R, P restricts to tR (since y maps to the closed point of C), and P has height 2 (the containments $\{y\} \subset Y_k \subset Y$ show that $\operatorname{ht}(P) \geqq 2$. That Y is two-dimensional follows from that fact that both fibers are one-dimensional.). Replacing f_i and c_i by their images in A, for $i = 1, 2, \ldots,$ we see that we are in the situation of Lemma 1.1, with $d = 1$, and we thereby conclude that the sequence is finite, i.e., $\varphi(f_n) \notin k$ for some $n \geqq 0$. Replacing f by f_n, we may assume $\varphi(f) \notin k$.

The containment $R[f] \subseteq \Gamma(Y, \mathcal{O}_Y)$ of R-algebras induces a birational morphism h: $Y \to Z$ of C-schemes, where $Z = \operatorname{Spec} R[f]$ $(\cong A^1_C)$. Our choice of f clearly implies that h restricts to an isomorphism of the generic fibers in Y and Z. Moreover, the morphism induced by h over the closed fibers corresponds to the k-algebra homomorphism $k[\bar{f}] \to k[u]$ which sends $\bar{f}$ to $\varphi(f)$. (Here $k[\bar{f}] = R[f] \otimes_R k$, and $\bar{f} = f \otimes 1$.) Since $\varphi(f) \notin k$, this is a finite morphism. In particular, we see that the morphism h is surjective and has finite fibers. By Zariski's Main Theorem [4, III, Corollary 4.4.9], h is an isomorphism.

1.5 *Remark.* The reader will easily verify that a morphism $X \to S$ of finite type, with S Noetherian, will be an A^1-bundle if (and only if) for every point $s \in S$, the induced morphism $X \times \operatorname{Spec} \mathcal{O}_{S,s} \to \operatorname{Spec} \mathcal{O}_{S,s}$ is an A^1-bundle. Thus Proposition 1.4 holds where C is any one-dimensional, normal, Noetherian scheme. Moreover, this observation will enable us to proceed with the proof of our theorem by induction on the dimension of S, even though, before localizing, S could be infinite dimensional.

2. The case $\dim S > 1$

2.1. We now proceed to prove the main theorem inductively for the general case. In lieu of Remark 1.5 and Proposition 1.4, we may assume $S = \operatorname{Spec} R$ where R is a normal, Noetherian local domain of dimension $n \geqq 2$, and we may assume the theorem holds for base schemes of lower dimension. In particular, the A^1-fibration $f\colon X \to S$ restricts to an A^1-bundle over $S' = S - \{p\}$, where p is the closed point of S, since $\dim S' = n - 1$. Letting $X' = Xx_S S'$ $(= f^{-1}(S'))$ and $f'\colon X' \to S'$ the induced morphism (which makes X' an A^1-bundle over S'), we will first show that X is indeed an A^1-bundle over S provided X' is a *trivial* A^1-bundle over S'.

In this case $X' \cong S' \times A^1$, f' corresponding to the projection $S' \times A^1 \to S'$. Let $Z = S \times A^1$ and $Z' = S' \times A^1$, so that Z' is an open set of Z. The isomorphism $X' \cong Z'$ gives a birational correspondence $g\colon X \to Z$ over S. Since X and Z are both normal, and since the fiber of p in both X and Z has codimension $\geqq 2$, we have $\Gamma(Z, \mathcal{O}_Z) = \Gamma(Z', \mathcal{O}_Z) \cong \Gamma(X', \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)$ [4; IV, Theorem 5.10.5], the middle isomorphism being induced by g. Since Z is affine, the existence of the composite homomorphism $\Gamma(Z, \mathcal{O}_Z) \to \Gamma(X, \mathcal{O}_X)$ implies that g is a morphism.

So we now have a birational morphism $g\colon X \to Z$ of S-schemes which restricts to the isomorphism $X' \cong Z'$, and we claim that g is an isomorphism. It clearly suffices to show that g is an open immersion (since, in that case, g would restrict to an open immersion $g_k\colon X_k \to Z_k$ of the closed fibers, both of which are isomorphic to A^1_k; g_k would necessarily be an isomorphism, making g an isomorphism). Since Z is normal, it suffices, by Zariski's Main Theorem, to show that g has finite fibers [4; III, 4.4.9]. The only way this can fail is if g carries X_k to a closed point z (necessarily k-rational) in Z_k, so we assume this is the case. Since $Z = S \times A^1$, we have $Z = \operatorname{Spec} R[t]$, and we may choose t so that the point z is defined by $t = 0$ on Z_k.

Let U be an affine open neighborhood of some point of X_k. Then U (as well as Z) is smooth over S, hence $\Omega^1_{U/S}$ and $\Omega^1_{Z/S}$ are locally free sheaves of rank one over U and Z, respectively [1; Ch. VII, Theorem 5.3]. Since $g|_U\colon U \to Z$ is an S-morphism which is an open immersion outside the closed fiber U_k in U, the map

$$(1) \qquad (g|_U)^*\Omega^1_{Z/S} \to \Omega^1_{U/S}$$

is an isomorphism away from U_k. Since U is normal and U_k has codimension $\geqq 2$, (1) is an isomorphism [4; IV Theorem 5.10.5]. Letting $A = R[t]$ and $B = \Gamma(U, \mathcal{O}_X)$, we identify A as a subring of B. The isomorphism (1) says that $\Omega^1_{B/R} = B \otimes_A \Omega^1_{A/R}$, from whence it follows that $\Omega^1_{B/R} = B\,dt$.

On the other hand, the fact that $g|_U$ carries U_k to z says that

$$(\mathbf{m}B) \cap A = \mathbf{m}A + tA,$$

where $\mathbf{m}$ is the maximal ideal of R; therefore $t \in \mathbf{m}B$. Writing

$$t = m_1 b_1 + \cdots + m_r b_r,$$

where each $m_i \in m$, $b_i \in B$, we have $dt = m_1 db_1 + \cdots + m_r db_r$, which shows that $dt \in (\mathbf{m}B)\Omega_{B/R}$. This is impossible, since $\Omega_{B/R}$ is free on the generator dt and $\mathbf{m}B \subsetneqq B$. This concludes the argument at hand.

2.2. The proof is now reduced to proving that $X' = X \times_S S'$ is a trivial A^1-bundle over $S' = S - \{p\}$. We first show this for the case where R is Henselian. Then, since $f\colon X \to S$ is a smooth morphism (Remark 1.3), any section for the restriction $f^{-1}(p) \to \{p\}$ of f to the closed fiber (where p is again the closed point of S) extends to a section for f [4; IV, Theorem 18.5.17]. A section on the closed fiber corresponds to a k-rational point on $f^{-1}(p)$ ($\cong A^1_k$), where $k = k(p)$. We choose two distinct such points, and extend to get two sections $\sigma_0\colon S \to X$ and $\sigma_1\colon S \to X$ whose images are non-intersecting in X. These restrict to sections $\sigma_0'\colon S' \to X'$ and $\sigma_1'\colon S' \to X'$ for the A^1-bundle $X' \to S'$. Regarding σ_0' as the "zero-section", X' can be given the structure of rank one vector bundle over S'.[2] As such, the section σ_1' becomes a "non-vanishing" section on X'. Inasmuch as a rank one vector bundle with a non-vanishing global section is trivial, X' is a trivial A^1-bundle. This settles the argument at hand when R is Henselian.

2.3. For the general case, let $\tilde{R}$ be the Henselization of R, $\tilde{S} = \operatorname{Spec}\tilde{R}$, $\tilde{X} = X \times_S \tilde{S}$, and $\tilde{f}\colon \tilde{X} \to \tilde{S}$ the resulting A^1-fibration of $\tilde{S}$. We know that $\tilde{R}$ is a normal, Noetherian, local domain, faithfully flat over R (see [4; IV, 18.6.6, 18.6.9, 18.6.12]). Let $\tilde{p}$ be the closed point if $\tilde{S}$, and put $\tilde{S}' = \tilde{S} - \{\tilde{p}\}$, $\tilde{X}' = \tilde{X} \times_{\tilde{S}} \tilde{S}'$. By the result of 2.2, $\tilde{X}'$ is a trivial A^1-bundle over $\tilde{S}'$ (and in fact $\tilde{X}$ is an A^1-bundle over $\tilde{S}$).

Now, for any ring A, let $G(A)$ be the matrix group

$$\left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in A^*,\ b \in A\right\}$$

(where A^* denotes the units of A). Let $T = \operatorname{Spec} A$. If A is a domain, $G(A)$ is identified with the group of T-automorphisms of A^1_T. We have a split exact sequence

$$1 \to A \xrightarrow{u} G(A) \xrightarrow{v} A^* \to 1 \tag{2}$$

of groups, where

$$u(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad v\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = a.$$

[2] It is an easy exercise to prove that an A^1-bundle which admits a section is a vector bundle. This fact is peculiar to rank one, however.

Now let W be any scheme and let $\mathcal{G}_W$ be the sheaf of groups on W arising from the functor G. From (2) we see that there is a split exact sequence

$$1 \to \mathcal{O}_W \to \mathcal{G}_W \to \mathcal{O}_W^* \to 1 \tag{3}$$

which together with the map $\tilde{S}' \to S'$, gives rise to the commuting diagram of cohomology sets

$$\begin{array}{ccccc} H^1(S', \mathcal{O}_{S'}) & \xrightarrow{w} & H^1(S', \mathcal{G}_{S'}) & \longrightarrow & H^1(S', \mathcal{O}_{S'}^*) \\ {\scriptstyle r}\downarrow & & {\scriptstyle s}\downarrow & & {\scriptstyle t}\downarrow \\ H^1(\tilde{S}', \mathcal{O}_{\tilde{S}'}) & \xrightarrow{\tilde{w}} & H^1(\tilde{S}', \mathcal{G}_{\tilde{S}'}) & \longrightarrow & H^1(\tilde{S}', \mathcal{O}_{\tilde{S}'}^*) \end{array} \tag{4}$$

the rows being exact sequences of sets with base point. The symbol $*$ will be used indiscriminately to denote the base point of any one of the cohomology sets occurring in (4). We observe three facts:

(a) *The kernel of* $\tilde{w}$ *is* $\{*\}$. This is due to the fact that, in the exact sequence

$$H^0(\tilde{S}', \mathcal{G}_{\tilde{S}'}) \to H^0(\tilde{S}', \mathcal{O}_{\tilde{S}'}^*) \xrightarrow{\tilde{\delta}} H^1(\tilde{S}', \mathcal{O}_{\tilde{S}'})$$

the first map is surjective, since (3) is a split exact sequence. Consequently, the image of $\tilde{\delta}$ is $\{*\}$, which is the kernel of $\tilde{w}$.

(b) *The map* t *is injective*. Since S is normal and its closed point has codimension $\geqq 2$, and the same holds for $\tilde{S}$, we have

$$H^1(S', \mathcal{O}_{S'}^*) = \mathrm{Pic}(S') \subseteq \mathrm{Cl}(S') = \mathrm{Cl}(S) = \mathrm{Cl}(R)$$

and likewise replacing S, S', and R by $\tilde{S}$, $\tilde{S}'$, and $\tilde{R}$. (Cl denotes the ideal class group.) But, because R is a local, Noetherian, normal (hence Krull) domain, and $\tilde{R}$ is faithfully flat over R, $\mathrm{Cl}(R) \to \mathrm{Cl}(\tilde{R})$ is injective [3; Corollary 6.11].

(c) *The map* r *is injective*. This follows from the isomorphism

$$H^1(S', \mathcal{O}_{S'}) \cong H^2_{\{p\}}(S, \mathcal{O}_S) \simeq \varinjlim_{q} \mathrm{Ext}_R^2(R/\mathbf{m}^q, R)$$

where $\mathbf{m}$ is the maximal ideal of R, and the same isomorphisms replacing S, S', R, and $\mathbf{m}$ by $\tilde{S}$, $\tilde{S}'$, $\tilde{R}$, and $\tilde{\mathbf{m}}$, where $\tilde{\mathbf{m}}$ is the maximal ideal of $\tilde{R}$ (cf. [6; Proposition 2.2 and Theorem 2.8]). We have

$$\begin{aligned} \varinjlim_{q} \mathrm{Ext}_{\tilde{R}}^2(\tilde{R}/\tilde{\mathbf{m}}^q, \tilde{R}) &\cong \lim\left[\mathrm{Ext}_R^2(R/\mathbf{m}^q, R) \otimes_R \tilde{R}\right] \\ &\cong \left[\varinjlim_{q} \mathrm{Ext}_R^2(R/\mathbf{m}^q, R)\right] \otimes_R \tilde{R}. \end{aligned}$$

(The first isomorphism uses the flatness of $\tilde{R}$ over R; the second uses the fact that tensor product commutes with direct limits.) Because of the faithful flatness of $\tilde{R}$ over R, the map $E \to E \otimes_R \tilde{R}$ is injective, for any R-module E. Setting

$$E = \varinjlim_{q} \operatorname{Ext}^2_R(R/\mathbf{m}^q, R)$$

gives us the desired result.

Returning to the main line of the proof, the A^1-bundle $X' \to S'$ corresponds to an element $\alpha \in H^1(S', \mathcal{G}_{S'})$. The fact that $\tilde{X}' \to \tilde{S}'$ is a trivial A^1-bundle says that $s(\alpha) = *$ in (4), and we have

$$\begin{tikzcd} \alpha \arrow[r, mapsto] \arrow[d, mapsto] & \beta \in H^1(S', \mathcal{O}^*_{S'}) \arrow[d, mapsto] \\ r(\alpha) = * \arrow[r, mapsto] & * \in H^1(\tilde{S}', \mathcal{O}^*_{\tilde{S}'}) \end{tikzcd}$$

so that, by (b) above, $\beta = *$. Hence $\alpha = w(\gamma)$ for some $\gamma \in H^1(S', \mathcal{O}_{S'})$, so that we have

$$\begin{tikzcd} \gamma \arrow[r, mapsto] \arrow[d, mapsto] & \alpha \arrow[d, mapsto] \\ r(\gamma) \arrow[r, mapsto] & * = \tilde{w}(r(\gamma)) \end{tikzcd}$$

which shows, in view of (a), that $r(\gamma) = *$. But then, by (c), we find $\gamma = *$, and hence $\alpha = *$. This proves $X' \to S'$ is a trivial A^1-bundle and thus establishes our main theorem.

3. Comments and discussions

3.1. The main theorem of this paper is likely to be true even if the fibers are affine n-spaces A^n and not just affine line A^1 (cf. Veisfeiler-Dolgachev [13]). But the only published attempt to prove this is [8], which assumes $\dim S = 1$ and $n = 2$; the proof there contains a gap (p. 279 following (3)). Meanwhile Avinash Sathaye has proved this special case (which also assumes the map $X \to S$ is affine). This theorem will presumably be published soon.

3.2. As we stated in our opening remarks, the theorem is true when only the generic fiber is assumed to be A^1 and all other fibers are geometrically integral, provided the morphism $X \to S$ is affine and S is locally factorial [7]. One should take note, however, that when the morphism $X \to S$ is not assumed to be affine, this weaker hypothesis on the fibers is not enough to preserve the theorem's validity. This is seen by the following example, pointed out to us by A. Bialynicki-Birula.

Example. Let k be a field and $X = A_k^2 - \{x\}$, where x is any closed point. Let $f\colon X \to S = A_k^1$ be the projection onto any axis in A_k^2. Clearly the generic fiber of f is of type A^1 and all other fibers are geometrically integral, but X is not an A^1-bundle over S.

3.3. In Vladimir Danilov's letter, mentioned earlier, he claims to have a proof of the main theorem. For the case $\dim S = 1$ he gives no indication of his proof, and we have no idea whether it substantially agrees with ours in §1.

3.4. If the assumption that S is normal is dropped, the theorem is no longer true. To see this one needs only to consider the well-known examples (first discovered by E. Hamann) of a (non-normal) Noetherian local ring R which has non-trivial stably polynomial algebras. Specifically, there exists, for such R, an algebra A such that $A[T] \cong R[V, W]$ as R-algebras. (See [2] for a thorough exposition on this phenomenon.) Setting $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} R$, the map $X \to S$ satisfies the hypothesis of the theorem except for the normality of S. However, X is not an A^1-bundle, since A is not a polynomial ring over R.

A more geometric example of an A^1-fibration which is not an A^1-bundle has been provided by Madhav Nori, as follows. (We leave some details for the reader to verify.) We take k to be an algebraically closed field. Let C be the curve in $A_k^2 = \operatorname{Spec} k[V, W]$ defined by $V^2 = W^3$, a curve which has a singularity at one point p (the origin), and consider the map $A_k^1 \to C \times \mathbf{P}_k^1$ sending t to $((t^3, t^2), (t:1))$. This morphism is a closed immersion. Denote its image by Z, and let $X = (C \times \mathbf{P}_k^1) - Z$. The map $X \to C$ which is the restriction of the projection $\mathbf{P}_k^1 \times C \to C$ to X is clearly an A^1-fibration, but we claim it is not an A^1-bundle in any neighborhood of the singular point p. For if it were we could shrink C about p and have an isomorphism f: $C \times A^1 \to X$ over C. Consider the composite

$$C \times A^1 \xrightarrow{f} X \subset C \times \mathbf{P}^1$$

restricted to the fiber over any point x in C. We get an open immersion $A_k^1 \to \mathbf{P}_k^1$, which extends uniquely to an isomorphism $\mathbf{P}_k^1 \to \mathbf{P}_k^1$. This isomorphism will be denoted σ_x. The map $C \times \mathbf{P}^1 \to C \times \mathbf{P}^1$ sending (x, y) to $(x, \sigma_x(y))$ is a morphism which extends f, and therefore carries the singular curve $C \times \{\infty\}$ birationally onto the non-singular curve Z, which is impossible.

3.5. Let $\mathbf{P}^n = \operatorname{Proj} Z[T_0, \ldots, T_n]$ be the projective n-space. The fibrations $X \to S$ of fiber type $\mathbf{P}^n$ are much easier to handle because of the availability of powerful tools such as Brauer groups, and also the projective and general linear groups over S (cf. Grothendieck [5]). Specifically, if $f\colon X \to S$ is a

$\mathbf{P}^n$-fibration and if S is a normal, Noetherian scheme, one can deduce, by way of Nagata's Embedding Theorem in the relative case [10], that f is a proper morphism. Then, a theorem of Grothendieck [5; I, Theorem 8.2, p. 64] applies, and tells us that X is a $\mathbf{P}^n$-bundle over S, relative to the étale-finite topology. In case S is one-dimensional, we know furthermore that X is a $\mathbf{P}^n$-bundle (relative to the Zariski topology). For, in that case, $X \to S$ has a rational section, because the generic fiber is $\mathbf{P}^n_K$ over the field K of rational functions of S; but such a section is defined everywhere on S, since S is a normal curve and $X \to S$ is proper. (This follows from the valuative criterion for properness.) We then again draw upon an argument of Grothendieck in [5; II, pp. 68–69].

Unfortunately, Grothendieck's Theorem 8.2 of [5], mentioned above, is not accompanied by a proof, even though such a proof appears to be well known in certain circles. Meanwhile, we have obtained an elementary proof of a special case by a method quite similar to the one employed in Proposition 1.4 above, the result being that a $\mathbf{P}^1$-fibration over a Noetherian, normal, integral scheme S of dimension one is a $\mathbf{P}^1$-bundle. The proof may be published elsewhere.

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