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EXTREME OPERATORS ON FUNCTION SPACES

BY

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1. Introduction

Let A, B be Banach algebras and let i be the identity of A. The closed unit ball of A is denoted by $S(A)$ and the set of extreme points of $S(A)$ is denoted by ext $S(A)$. The closed unit ball of $\mathscr{L}[A, B]$, the Banach space of bounded linear operators from A to B, is denoted by $S[A, B]$. If X is a locally compact Hausdorff space let $C(X, A)$ stand for the space of continuous functions from X to A and let $C_0(X, A)$ stand for the subspace of continuous functions vanishing at infinity. Then $C_0(X, A)$ is a Banach algebra under the supremum norm. If $A = \mathbf{C}$, the set of all complex numbers, we simply write $C(X)$ and $C_0(X)$. The σ-algebra of Borel sets of X is denoted by $\mathscr{B}(X)$ and the set of bounded regular Borel measures is denoted by $M(X)$.

For bounded linear operator T: $C_0(X, A) \to B$ let m: $\mathscr{B}(X) \to \mathscr{L}[A, B^{**}]$ be its representing measure and let $|m|$, $\tilde{m}$ be its total variation and semivariation respectively, i.e.,

$$|m|(X) = \sup\left\{ \sum \|m(e_i)\|: \{e_i\} \in \pi(X)\right\},$$

$$\tilde{m}(X) = \sup\left\{\left\|\sum m(e_i)x_i\right\|: \{e_i\} \in \pi(X), x_i \in S(A)\right\};$$

where $\pi(X)$ denotes the collection of all the (disjoint) finite Borel-partitions of X.

It is known that $\|T\| = \tilde{m}(X)$ and that if T is weakly compact then m: $\mathscr{B}(X) \to \mathscr{L}[A, B]$ (see [1], for example).

An extreme point of $S[A, B]$ is called an *extreme operator* from A to B. An operator T in $S[A, B]$ is a *nice operator* if $T^*[\text{ext } S(B^*)] \subset \text{ext } S(A^*)$, where T^* is the adjoint operator of T. It is known that every nice operator is an extreme operator and that the converse assertion is not true in general. Several authors studied the relationship between these two operators on various settings (see [2] and [9], for example). Extreme operators from function spaces

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to $\mathbf{C}$ are characterized in, Dinculeanu [6] and Brooks and Lewis [1], for example. In Section 2 we identify all the extreme operators from $C_0(X, A)$ to B for the case where $|m|(X) = 1$. As an application we then characterize the set $PW(X, B)$ of B-valued probability measures introduced by Husain and the author [4] in terms of extreme operators in Remarks 3.5. The intrinsic differences between extreme operators and nice operators are then given for this case in Section 3.

Throughout this paper multiplication in the second dual of a Banach algebra is defined by the Arens product. Readers are referred to [7] for notations not explained here.

2. Extreme operators

Let m be the representing measure of the operator T: $C_0(X, A) \to B$. Recall first that if m has finite total variation then $|m| \in M(X)$ and m maps $\mathscr{B}(X)$ into $\mathscr{L}[A, B]$ [1, p. 148 and p. 155]. For $x \in A, e \in \mathscr{B}(X), 1_e \otimes x$ can be viewed as an element of $C_0^{**}(X, A)$:

$$(1_e \otimes x)(f') = \mu(x, f')(e) \quad \text{for } f' \in C_0^*(X, A),$$

where $\mu(x, f') \in M(X)$ is defined by

$$\int f \, d\mu(x, f') = f'(f, x) \quad (f \in C_0(X)).$$

DEFINITION 2.1. For T: $C_0(X, A) \to B, e \in \mathscr{B}(X), x \in A$ define

$$\hat{T}_{\langle e, x\rangle}: C_0^{**}(X, A) \to B^{**}$$

by

$$\hat{T}_{\langle e, x\rangle}(F) = T^{**}(F(1_e \otimes x)) \quad (F \in C_0^{**}(X, A)),$$

where the multiplication is defined by the Arens product. The restriction of $\hat{T}_{\langle e, x\rangle}$ to $C_0(X, A)$ will be denoted by $T_{\langle e, x\rangle}$.

LEMMA 2.2. *For $e \in \mathscr{B}(X), x \in A$ the mapping $T_{\langle e, x\rangle}$ is a bounded linear operator $C_0(X, A) \to B$ if m has finite total variation. In fact*

$$\|T_{\langle e, x\rangle}\| \leq \|T\| \, \|x\|.$$

Proof. For $f \in C_0(X, A), T_{\langle e, x\rangle}(f) = T^{**}(f(1_e \otimes x))$. Let

$$\left\{ \sum_{i=1}^{N_j} \left(1_{e_{i,j}} \otimes x_{i,j}\right) \right\}_j$$

be a sequence converging uniformly to f, where $e_{i,j} \cap e_{k,j} = \phi$ $(i \neq k)$, $x_{i,j} \in A$ [1, Lemma 2.1]. It is proved in [3, Lemma 2.3] that

$$\left(1_{e_{i,j}} \otimes x_{i,j}\right)\left(1_e \otimes x\right) = 1_{e_{i,j} \cap e} \otimes x_{i,j}x.$$

Since m has finite total variation, m maps $\mathscr{B}(X)$ into $\mathscr{L}[A, B]$. Thus

$$T^{**}\left(\sum_{i=1}^{N_j}\left(1_{e_{i,j}} \otimes x_{i,j}\right)\left(1_e \otimes x\right)\right) = \sum_{i=1}^{N_j} m\left(e_{i,j} \cap e\right)x_{i,j}x$$

is in B. Since B is closed in B^{**}, by a limit process, we see that $T_{\langle e, x\rangle}(f) \in B$.

The linearity of $T_{\langle e, x\rangle}$ follows from that of T^{**}. For the boundedness of $T_{\langle e, x\rangle}$ we note that

$$\begin{aligned}
\|T_{\langle e, x\rangle}\| &= \sup\left\{\|T_{\langle e, x\rangle}(f)\| : f \in S\left(C_0(X, A)\right)\right\} \\
&= \sup\left\{\|T^{**}\left(f(1_e \otimes x)\right) : f \in S\left(C_0(X, A)\right)\right\} \\
&\leq \|T\| \sup\left\{\|f(1_e \otimes x)\| : f \in S\left(C_0(X, A)\right)\right\} \\
&\leq \|T\|\ \|x\|
\end{aligned}$$

since $\|1_e \otimes x\| \leq \|x\|$ [1, Lemma 2.1].

LEMMA 2.3. *If the representing measure m of T has finite total variation, then $m_{\langle e, x\rangle}$: $\mathscr{B}(X) \to \mathscr{L}[A, B]$ defined by*

$$m_{\langle e, x\rangle}(e_1)y = m(e \cap e_1)yx \quad \left(e_1 \in \mathscr{B}(X),\ y \in A\right)$$

is the representing measure of $T_{\langle e, x\rangle}$.

Proof. Let $\{\sum_{i=1}^{N_j}(1_{e_{i,j}} \otimes x_{i,j})\}_j$ be a sequence converging uniformly to f. Now

$$\begin{aligned}
\int \chi_{e_{i,j}} x_{i,j}\, dm_{\langle e, x\rangle} &= m_{\langle e, x\rangle}\left(e_{i,j}\right)x_{i,j} \\
&= m\left(e \cap e_{i,j}\right)x_{i,j}x \\
&= T^{**}\left(1_{e \cap e_{i,j}} \otimes x_{i,j}x\right) \\
&= T^{**}\left(\left(1_{e_{i,j}} \otimes x_{i,j}\right)\left(1_e \otimes x\right)\right) \\
&= \hat{T}_{\langle e, x\rangle}\left(1_{e_{i,j}} \otimes x_{i,j}\right)
\end{aligned}$$

by [3, Lemma 2.3]. By a limit process we see that

$$\int f\,dm_{\langle e,x\rangle} = \hat{T}_{\langle e,x\rangle}(f) = T_{\langle e,x\rangle}(f) \quad (f \in C_0(X, A)).$$

Thus $m_{\langle e,x\rangle}$ is the representing measure of $T_{\langle e,x\rangle}$.

LEMMA 2.4. *For disjoint* $e_1, e_2 \in \mathscr{B}(X)$, $x \in A$, $F \in C_0^{**}(X, A)$, *we have*
(a) $(1_{e_1} \otimes x) + (1_{e_2} \otimes x) = 1_{e_1 \cup e_2} \otimes x$,
(b) $F(1_X \otimes i) = F$.

Proof. (a) For $f' \in C_0^*(X, A)$,

$$\begin{aligned}\left[(1_{e_1} \otimes x) + (1_{e_2} \otimes x)\right](f') &= \mu(x, f')(e_1) + \mu(x, f')(e_2)\\ &= \mu(x, f')(e_1 \cup e_2)\\ &= (1_{e_1 \cup e_2} \otimes x)(f').\end{aligned}$$

(b) For $f' \in C_0^*(X, A)$,

$$F(1_X \otimes i)(f') = F((1_X \otimes i)f').$$

For $f \in C_0(X, A)$,

$$(1_X \otimes i)f'(f) = \hat{f}((1_X \otimes i)f') = \hat{f}(1_X \otimes i)(f'),$$

where $\hat{f}$ is the cannonical image of f in $C_0^{**}(X, A)$. Since there is a net of A-valued simple functions over $\mathscr{B}(X)$ approximating f uniformly in $C_0^{**}(X, A)$, by a limit process and from [3, Lemma 2.3], we see that $\hat{f}(1_X \otimes i) = \hat{f}$. Thus

$$(1_X \otimes i)f'(f) = \hat{f}(f') = f'(f)$$

and so $F(1_X \otimes i)(f') = F(f')$.

LEMMA 2.5. *Let T be an extreme operator from $C_0(X, A)$ to B. If $|m|(X) = 1$, then, for each $e \in \mathscr{B}(X)$, either $\|T_{\langle e,i\rangle}\| = 0$ or $\|T_{\langle e',i\rangle}\| = 0$; where e' is the complement of e.*

Proof. We note first from Lemma 2.4 that

$$\begin{aligned}T^{**}(f) &= T^{**}(f(1_e \otimes i)) + T^{**}(f(1_{e'} \otimes i))\\ &= T_{\langle e,i\rangle}(f) + T_{\langle e',i\rangle}(f)\end{aligned}$$

for $f \in C_0(X, A)$. Now suppose the contrary that

$$\|T_{\langle e, i\rangle}\| \neq 0 \quad \text{and} \quad \|T_{\langle e', i\rangle}\| \neq 0.$$

Then $|m_{\langle e, i\rangle}|(X)$ and $|m_{\langle e', i\rangle}|(X)$ are finite and positive. Let

$$T_1 = T_{\langle e, i\rangle}/|m_{\langle e, i\rangle}|(X) \quad \text{and} \quad T_2 = T_{\langle e', i\rangle}/|m_{\langle e', i\rangle}|(X).$$

Then $T_1, T_2 \in S[C_0(X, A), B]$ and

$$T = T_{\langle e, i\rangle} + T_{\langle e', i\rangle} = |m_{\langle e, i\rangle}|(X)T_1 + |m_{\langle e', i\rangle}|(X)T_2.$$

For $e_j \in \mathscr{B}(X)$.

$$\begin{aligned} \|m_{\langle e, i\rangle}(e_j)\| &= \sup\{\|m_{\langle e, i\rangle}(e_j)x\| : x \in S(A)\} \\ &= \sup\{\|m(e \cap e_j)x\| : x \in S(A)\} = \|m(e \cap e_j)\|. \end{aligned}$$

Thus

$$\begin{aligned} |m_{\langle e, i\rangle}|(X) &= \sup\left\{\sum_{j=1}^{N} \|m_{\langle e, i\rangle}(e_j)\| : \{e_j\} \in \pi(X)\right\} \\ &= \sup\left\{\sum_{j=1}^{N} \|m(e \cap e_j)\| : \{e_j\} \in \pi(X)\right\}, \\ &= |m|(e). \end{aligned}$$

Similarly $|m_{\langle e', i\rangle}|(X) = |m|(e')$. Since $|m|(X) = 1$, $|m| \in M(X)$ and so

$$|m_{\langle e, i\rangle}|(X) + |m_{\langle e', i\rangle}|(X) = 1.$$

It is easy to see from [3, Lemma 2.3] and [1, Theorem 2.1] that $T_1 \neq T$ and $T_2 \neq T$ if $e \neq X$. Thus T is not an extreme operator from $C_0(X, A)$ to B. This contradiction shows that either $\|T_{\langle e, i\rangle}\| = 0$ or $\|T_{\langle e', i\rangle}\| = 0$.

Definition 2.6. For $s \in X$ and $l \in S[A, B]$ let $L(s, l) \in S[C_0(X, A), B]$ be defined by

$$L(s, l)(f) = l(f(s)) \quad (f \in C_0(X, A)).$$

A routine argument shows that if $l \neq 0$, then the support, $\operatorname{supp} L(s, l)$, of $L(s, l)$ is $\{s\}$. Conversely let

$$T \in S[C_0(X, A), B]$$

be such that $\operatorname{supp} T = \{s\}$ and let $m\colon \mathscr{B}(X) \to \mathscr{L}[A, B^{**}]$ be the representing measure of T. Then $\operatorname{supp} m = \{s\}$ [1, Theorem 2.6]. Let

$$m(\{s\}) = l \in S[A, B^{**}].$$

Since $T(f) = \int f\,dm$ for $f \in C_0(X, A)$, we see that $l \in S[A, B]$. Thus $T = L(s, l)$ and T is weakly compact iff $l \in S[A, B]$ is weakly compact. Furthermore we see $\|T\| = \|l\|$.

THEOREM 2.7. *Let T: $C_0(X, A) \to B$. Then T is an extreme operator and $|m|(X) = 1$ iff $T = L(s, l)$ for some $s \in X$ and some extreme operator l from A to B.*

Proof. Suppose T is an extreme operator and $|m|(X) = 1$. We claim first that $\operatorname{supp} T$ is a singleton. For if $x, y \in \operatorname{supp} T$ let U_x, U_y be disjoint open sets containing x and y respectively. Then, since T is an extreme operator,

$$\|T_{\langle U_x, i\rangle}\| = 0 \quad \text{or} \quad \|T_{\langle U_x', i\rangle}\| = 0$$

by Lemma 2.5. If $\|T_{\langle U_x, i\rangle}\| = 0$ then

$$\tilde{m}_{\langle U_x, i\rangle}(X) = \|T_{\langle U_x, i\rangle}\| = 0.$$

Thus, from Lemma 2.3, $\tilde{m}(U_x) = 0$ and so $x \notin \operatorname{supp} m = \operatorname{supp} T$ [1, Lemma 2.2]. Similarly if $\|T_{\langle U_x', i\rangle}\| = 0$ then $y \notin \operatorname{supp} T$.

Now, since $\operatorname{supp} T$ is a singleton, from the remarks after Definition 2.6 we see $T = L(s, l)$ for some $s \in X$ and some $l \in S[A, B]$. Let

$$l = \lambda l_1 + (1 - \lambda) l_2$$

for some $l_1, l_2 \in S[A, B]$ and $0 \le \lambda \le 1$. Define m_1, m_2: $\mathscr{B}(X) \to \mathscr{L}[A, B]$ by

$$\operatorname{supp} m_1 = \operatorname{supp} m_2 = \{s\} \quad \text{and} \quad m_1(\{s\}) = l_1,\ m_2(\{s\}) = l_2.$$

Then $\tilde{m}_1(X) \le 1$, $\tilde{m}_2(X) \le 1$ and $m = \lambda m_1 + (1 - \lambda) m_2$. Since m is an extreme point, $m = m_1 = m_2$ and so $l = l_1 = l_2$ and l is an extreme operator from A to B.

Conversely let $T = L(s, l)$ with $s \in X$ and l an extreme operator from A to B. Let

$$m = \lambda m_1 + (1 - \lambda) m_2$$

for some representing measures

$$m_1, m_2\colon \mathscr{B}(X) \to \mathscr{L}[A, B^{**}]$$

of operators in $S[C_0(X, A), B]$. Then, since l is an extreme point, $m_1(E) = m_2(E) = l$ for any $E \in \mathscr{B}(X)$ such that $s \in E$. Since m_1, m_2 are finitely additive, from

$$m(X) = \lambda m_1(X) + (1 - \lambda) m_2(X)$$

we see $\operatorname{supp} m_1 = \operatorname{supp} m_2 = \{s\}$ and $m_1 = m_2 = m$. Thus T is an extreme operator. Clearly $|m|(X) = \|m(\{s\})\| = \|l\| = 1$, since l is an extreme point [6, p. 159].

3. Nice operators

THEOREM 3.1. *Let T: $C_0(X, A) \to B$ be such that $|m|(X) = 1$. Then T is nice iff $T = L(s, l)$, where $s \in X$ and l is an extreme operator from A to B such that $y^*l \in$ ext $S(A^*)$ for each $y^* \in$ ext $S(B^*)$.*

Proof. Suppose T is a nice operator. Then T is an extreme operator and so $T = L(s, l)$ for some $s \in X$ and some $l \in \operatorname{ext} S[A, B]$ from Theorem 2.7. Since T is nice, for each $y^* \in \operatorname{ext} S(B^*)$,

$$L^*(s, l)(y^*) \in \operatorname{ext} S(C_0^*(X, A)).$$

For $f \in C_0(X, A)$,

$$L^*(s, l)(y^*)(f) = y^*(L(s, l)(f)) = y^*(l(f(s)) = L(s, y^*l)(f)$$

and so $y^*l \in \operatorname{ext} S(A^*)$ by Theorem 2.7 again.
The rest of the proof is similar.

COROLLARY 3.2. *Let X and Y be locally compact and compact Hausdorff spaces respectively and let T: $C_0(X) \to C(Y)$. If $|m|(X) = 1$ and T is an extreme operator then T is a nice operator.*

Proof. Recall first that

$$\operatorname{ext} S(C_0^*(Y)) = \{\lambda \delta_y\colon \lambda \in \mathbf{C} \quad \text{with } |\lambda| = 1,\ y \in Y\}$$

and that

$$\operatorname{ext} S(C(Y)) = \{f \in C(Y)\colon |f(y)| = 1 \quad \text{for } y \in Y\},$$

where δ_y is the (scalar) unit point mass at y. For $f \in C(Y)$ let T_f: $\mathbf{C} \to C(Y)$ be defined by $T_f(\alpha) = \alpha f$. It is easy to see that T_f is an extreme operator from $\mathbf{C}$ to $C(Y)$ iff $f \in \operatorname{ext} S(C(Y))$. Suppose T is an extreme operator. Then $T = L(s, T_f)$ for some $f \in \operatorname{ext}(C(Y))$ and $s \in X$ by Theorem 2.7. Now for $\lambda \in \mathbf{C}$ with $|\lambda| = 1$ and $y \in Y$,

$$(\lambda\delta_y T_f)(\alpha) = \lambda\delta_y(\alpha f) = (\lambda f(y)(\alpha)) \quad \text{for } \alpha \in \mathbf{C}.$$

Since $|f(y)| = 1$, we see $\lambda\delta_y T_f \in \operatorname{ext} S(\mathbf{C})$ and T is nice by Theorem 3.1.

It should be remarked that if Y is locally compact but not compact then $\operatorname{ext} S(C_0(Y))$ is empty and so there is no such T in Corollary 3.2.

If $B = \mathbf{C}$, then $m : \mathscr{B}(X) \to A^*$ and it is shown in [6, p. 54] that $|m| = \tilde{m}$. Thus we obtain the following result of Brooks and Lewis [1, Theorem 5.4] as

COROLLARY 3.3. *If T: $C_0(X, A) \to \mathbf{C}$ is an extreme operator then* supp T *is a singleton.*

COROLLARY 3.4. *If T: $C_0(X, A) \to \mathbf{C}$, then T is an extreme operator iff T is a nice operator.*

Proof. We note first that $\lambda \in \mathbf{C}$ is an extreme point of $S(\mathbf{C})$ iff $|\lambda| = 1$. Suppose T is an extreme operator then, since $|m|(X) = \tilde{m}(X) = 1$, $T = L(s, l)$ for some $s \in X$, $l \in \operatorname{ext} S(A^*)$. Thus, for complex λ with $|\lambda| = 1$ and $f \in C_0(X, A)$,

$$T^*(\lambda)(f) = \lambda(Tf) = \lambda(l(f(s)) = L(s, \lambda l)(f).$$

Clearly $\lambda l \in \operatorname{ext} S(A^*)$ and so $T^*(\lambda) \in \operatorname{ext} S(C_0^*(X, A))$. Thus T is nice.

Remarks 3.5. (1) It is tempting to conjecture that Theorems 2.7 and 3.1 are true in general without the assumption that $|m|(X) = 1$. However, Corollary 2.7 of [9] shows that it is not the case.

(2) If A is $\mathbf{C}$ and B is a (abstract) convolution measure algebra defined in [10, Definition 2.1], the set of all the probability measures m: $\mathscr{B}(X) \to B$ defined in [4] is denoted by $PW(X, B)$. Then, since $|m|(X) = 1$ by the definition of probability measures, from similar arguments as in Theorem 2.7, we see that m is an extreme point of $PW(X, B)$ iff $\operatorname{supp} m = \{s\}$ for some $s \in X$ and $m(\{s\}) = l$ for some positive $l \in \operatorname{ext} S(B)$. We denote such measures by $m(s, l)$. Then

$$PW(X, B) = \overline{CO}\{m(s, l) \colon s \in X, l \text{ is positive and in } \operatorname{ext} S(B)\};$$

where the closure is taken in the weak operator topology.

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