FLAG-TRANSITIVE COLLINEATION GROUPS OF FINITE PROJECTIVE SPACES

BY

D. G. HIGMAN[1]

1. Introduction

A *flag* in a projective space $\mathcal{P}$ of dimension $d \geqq 2$ is a sequence

$$S_0 \subset S_1 \subset \cdots \subset S_{d-1}$$

of linear subvarieties of $\mathcal{P}$ such that S_i has dimension i $(i = 0, 1, \cdots, d-1)$. Thus, for example, a flag in a projective plane is an incident point-line pair. A collineation group G of $\mathcal{P}$ will be called *flag-transitive* if any one flag can be carried onto any other by some collineation in G. The *little projective group* of a Desarguesian $\mathcal{P}$ (i.e., the group generated by all elations of $\mathcal{P}$, isomorphic with $PSL_{d+1}(F)$, where F is the coordinatizing field) is flag-transitive. Thus the following theorem can be considered as giving a geometric characterization of $PSL_{d+1}(F)$ for finite F. If the number of points on each line of $\mathcal{P}$ is $n+1$ we will refer to n as the *order* of $\mathcal{P}$. (For $d > 2$ this differs from the order of the symmetric design formed by the points and hyperplanes of $\mathcal{P}$.)

THEOREM. *A flag-transitive collineation group G of a Desarguesian projective space $\mathcal{P}$ of dimension $d \geqq 2$ and finite order n must contain the little projective group of $\mathcal{P}$ unless*

(a) $d = 2$, $n = 2$, *and* $|G| = 3 \cdot 7$, *or*

(b) $d = 2$, $n = 8$, *and* $|G| = 9 \cdot 73$, *or*

(c) $d = 3$, $n = 2$, *and G is isomorphic with the alternating group A_7 of degree* 7.

For $d = 2$ this theorem coincides with Theorem 1 of [6]. The extension to dimensions $\geqq 3$ (where, of course, the Desarguesian property necessarily holds) is obtained in this paper as an application of extensions of results of André [1], Gleason [5], and Wagner [9] concerning perspectivities, together with a special result about embeddings of $PSL_k(F)$ in $PGL_{k+l}(F)$.

The exceptions stated in the theorem are real. In case (c), $G \approx A_7$ is doubly transitive on the points of $\mathcal{P}$. Concerning the question whether the Desarguesian condition can be moved from the hypotheses to the conclusion of the theorem, i.e., whether the existence of a flag-transitive collineation group on a finite projective plane implies Desargues' Theorem, see [6].

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2. Perspectivities

In this section we obtain the needed extensions to higher dimensions of results of André, Gleason, and Wagner concerning perspectivities of projective planes.

If a perspectivity (elation, homology) has center C and fixes a hyperplane H pointwise, we shall refer to it as a (C, H)*-perspectivity* (*-elation*, *-homology*).

We refer to the group of collineations generated by all elations of a Desarguesian projective space $\mathcal{P}$ as the *little projective group* of $\mathcal{P}$; this group is isomorphic with $PSL_{d+1}(F)$ if $\mathcal{P}$ has dimension d and is coordinatized by the field F.

If G is a collineation group of a projective space $\mathcal{P}$, and S is a linear subvariety of $\mathcal{P}$, we denote by G_S the subgroup of G consisting of all elements $g \in G$ such that $g(S) = S$.

The case $d = 2$ of the following proposition is due to André [1]. His proof carries over with only verbal changes; we include the details for the convenience of the reader.[2]

PROPOSITION 1. *Let $\mathcal{P}$ be a finite projective space of dimension $d \geqq 2$, H a hyperplane of $\mathcal{P}$, G a group of perspectivities with hyperplane H, and T the normal subgroup of G consisting of all elations in G. Then*

(a) *G is transitive on the points outside H if and only if T is, i.e., if and only if G contains all possible elations with hyperplane H, and*

(b) *T is transitive on the set of all centers of homologies in G.*

Proof. First we prove (a). Let A be the totality of points not in H, and let A_ρ $(\rho = 1, 2, \cdots, r)$ be the orbits of A under G. Since T is a subgroup of G, each A is a union of orbits under T, say u_ρ in number. Since T is regular on A, every orbit under T contains exactly t points, where $t = |T|$. Hence $|A_\rho| = u_\rho t$ and $|A| = \sum_{\rho=1}^{r} |A_\rho| = \sum_{\rho=1}^{r} u_\rho t$.

For P in A_ρ we have $|A_\rho| = G:G_P = g/s_\rho$, where $s_\rho = |G_P|$, $g = |G|$. Hence

$$g = s_\rho u_\rho t. \tag{1}$$

But G_P is the totality of homologies in G with center P; hence

$$g = |T| + \textstyle\sum_{P \in A}(|G_P| - 1),$$

i.e.,

$$g = t + \textstyle\sum_{\rho=1}^{r} u_\rho t(s_\rho - 1). \tag{2}$$

[2] The referee informs the author that in a paper to appear in Mathematische Zeitschrift, H. Lüneburg has proved Propositions 1 and 2 as well as Lemma 1 under the more general hypothesis that $\mathcal{P}$ is a symmetric design. That $\mathcal{P}$ is actually a projective space can be proved. Lüneburg has also given a partial answer to the question raised in the remark following Proposition 3.

By writing $R = \sum_{\rho=1}^{r} u_\rho$, the number of orbits under T, (2) gives $g = t + rg - Rt$, or

$$(3) \qquad g(1 - r) = t(1 - R).$$

Hence $r = 1$ if and only if $R = 1$, proving (a).

To prove (b), note first that if $R = 1$, T is transitive on A, and there is nothing to prove. Assume $R > 1$. If $s_1 = s_2 = \cdots = s_r = 1$, then $G_P = 1$ for all $P \in A$, and G contains no homologies $\neq 1$. Assume therefore that $s_1, \cdots, s_{r'} > 1$, $s_{r'+1} = s_{r'+2} = \cdots = s_r = 1$ $(r' \geqq 1)$. We want to prove that $r' = 1$. By (1) and (3),

$$u_\rho s_\rho t = g = t(1 - R)/(1 - r),$$

and therefore

$$1/s_\rho = u_\rho(1 - r)/(1 - R).$$

Hence

$$\sum_{\rho=1}^{r} 1/s_\rho = R(1 - r)/(1 - R) > r - 1 \qquad \text{since } R > 1.$$

But

$$\sum_{\rho=1}^{r} 1/s_\rho = \sum_{\rho=1}^{r'} 1/s_\rho + r - r' \leqq r'/2 + r - r' = r - r'/2.$$

Hence

$$r - r'/2 > r - 1, \qquad r' < 2, \qquad r' = 1.$$

We now have $s_1 > 1$, $s_2 = s_3 = \cdots = s_r = 1$, and A_1 contains all centers of homologies in G. By (1) and (2), $u_1 t s_1 = g = t + u_1 t(s_1 - 1)$, whence $u_1 = 1$. Thus A_1 is an orbit under T, proving (b).

As an immediate corollary of Proposition 1 we have the following proposition, due to André [1] in case $d = 2$.

PROPOSITION 2. *A group G of collineations of a finite projective space $\mathcal{P}$ of dimension $\geqq 2$ which contains a (P, H)-homology for every point P not on some given hyperplane H must contain every possible elation with hyperplane H.*

The case $d = 2$ of the following lemma is due to Gleason [5].

LEMMA 1. *Let G be a collineation group of a projective space $\mathcal{P}$ of dimension $d \geqq 2$ and finite order n, and let H be a hyperplane of $\mathcal{P}$. If for each point X on H, G contains the same number $h > 0$ of (X, H)-elations $\neq 1$, then G contains all possible elations with hyperplane H.*

Proof. The subgroup T of all elations in G with hyperplane H is the disjoint union of the identity and the sets of (X, H)-elations $\neq 1$ with X on H. Hence the order t of T is given by

$$t = h \cdot N_{d-1} + 1,$$

where $N_{d-1} = n^{d-1} + n^{d-2} + \cdots + 1$, the number of points on H, and $t > n^{d-1}$ since $h > 0$. Since T acts regularly on the points not on H, t divides the number of these points, which is n^d,

$$n^d = tm,$$

and $m < n$ since $t > n^{d-1}$. Thus $n^d = (h \cdot N_{d-1} + 1)\, m \equiv m \pmod{N_{d-1}}$. But $n^d \equiv 1 \pmod{N_{d-1}}$; hence $N_{d-1} \mid m - 1$, giving $m = 1$ since $m < n < N_{d-1}$. Hence $n^d = t$, and T is transitive on the points not on H.

PROPOSITION 3. *If a collineation group G of a finite projective space $\mathcal{P}$ of dimension $d \geqq 2$ contains a (P, H)-elation $\neq 1$ for every incident point-hyperplane pair (P, H), then $\mathcal{P}$ is Desarguesian, and G contains the little projective group of $\mathcal{P}$.*

Proof. The result is obtained easily by induction on d, from the important case $d = 2$ due to Gleason [5].

Assume $d \geqq 3$; then of course we automatically have that $\mathcal{P}$ is Desarguesian. Let H be a hyperplane of $\mathcal{P}$, L a hyperplane of H, and P a point of L. Let K be a hyperplane $\neq H$ of $\mathcal{P}$ such that $K \cap H = L$, and let σ be a (P, K)-elation in G. Then $\sigma \mid H$ is a (P, L)-elation of H. Hence the induction hypothesis implies in particular that G is transitive on the points of H. Hence the number h of (X, H)-elations of $\mathcal{P}$ is the same for each X on H, and therefore Lemma 1 implies that G contains all possible elations with hyperplane H. Since this is true for every hyperplane, we conclude that G contains the little projective group of $\mathcal{P}$.

Remark.[2] For $d = 2$ the conclusion of Proposition 3 holds under the weaker hypothesis that every point is the center of some elation, and every line is the axis of some elation (Wagner [9]). This cannot be extended to $d > 2$, at least not for odd d, because the symplectic group has the corresponding property.

PROPOSITION 4. *Let G be a collineation group of a finite projective space $\mathcal{P}$ of dimension $d \geqq 2$, transitive on incident point-hyperplane pairs. If G contains a perspectivity $\neq 1$, then $\mathcal{P}$ is Desarguesian, and G contains the little projective group of $\mathcal{P}$.*

Proof. We deduce the result from Proposition 3 by adapting an argument of Wagner [9]. Because of the assumed transitivity property of G, the result follows at once by Proposition 3 if G contains an elation $\neq 1$.

Suppose we know only that G contains homologies $\neq 1$. If we assume that every homology with a given center P has the same hyperplane $\vartheta(P)$, and every homology with a given hyperplane H has the same center $\vartheta(H)$, then we can easily check that ϑ is a polarity. By its construction, ϑ has no absolute points, contrary to the fact that a polarity of a finite projective space of dimension $\geqq 2$ always has absolute points (that this is true for arbitrary finite projective planes is due to Baer [2]). If some hyperplane is the hyperplane of homologies with different centers, Proposition 1(b) implies that G

contains an elation $\neq 1$. Otherwise some point is the center of homologies with different hyperplanes, and the dual of Proposition 1(b) gives the same result.

For completeness we include the following result, immediately derivable from Propositions 2 and 3.

PROPOSITION 5. *If a finite projective space $\mathcal{P}$ of dimension $d \geqq 2$ admits a group G of collineations such that for every nonincident point-hyperplane pair (P, H), G contains a (P, H)-homology, then $\mathcal{P}$ is Desarguesian, and G contains the little projective group of $\mathcal{P}$.*

Remark. For $d = 2$ it is known (Wagner [9]) that if every point is a center, and every line an axis of some homology $\neq 1$ in G, then (a) if G fixes a line L of $\mathcal{P}$, then G contains every possible elation with axis L, (b) dual of (a), and (c) if G fixes no point or line of $\mathcal{P}$, then $\mathcal{P}$ is Desarguesian, and G contains the little projective group of $\mathcal{P}$. Hence, for $d = 2$, the conclusion of Proposition 4 holds under the assumption of transitivity on points instead of on incident point-line pairs [9]. Again there appear to be some difficulties in extending these results to higher dimensions.

3. Embedding $PSL_k(F)$ in $PGL_{k+l}(F)$

The following result is given in a more general form than actually needed for the proof of our theorem, since the cost is slight and the result may find additional applications. Here we will apply only the case in which S is a hyperplane.

PROPOSITION 6. *Let $\mathcal{P}$ be a projective space of dimension $d \geqq 3$ and finite order n, and let S be a linear subvariety of $\mathcal{P}$ of dimension $\geqq d/2$. If G is a group of collineations of $\mathcal{P}$ such that G_S is faithful on S, and $G_S \mid S$ contains the little projective group of S, then if we exclude the case $d = 3$, $n = 2$, an element $\gamma \in G_S$ is an elation of $\mathcal{P}$ whenever $\gamma \mid S$ is an elation of S. In any case, if Γ denotes the intersection of G_S with the little projective group of $\mathcal{P}$, $\Gamma \mid S$ contains the little projective group of S.*

Proof. The first step is to reduce the problem to one about linear groups. We may realize $\mathcal{P}$ as the geometry PV of subspaces of a vector space V of dimension $d + 1$ over the field F_n of n elements, and then $S = PW$ where W is a subspace of V of dimension $1 + \dim S \geqq 1 + d/2 > (d + 1)/2$, i.e., $\dim W \geqq \frac{1}{2} (\dim V)$.

The hypotheses imply the existence of an isomorphism σ of $PSL(W)$ into the collineation group of PV such that $\gamma^\sigma \mid PW = \gamma$ for every $\gamma \in PSL(W)$. Clearly $PSL(W)^\sigma \subseteq PGL(V)$, since a collineation which is a projectivity on a subvariety of dimension $\geqq 1$ is necessarily a projectivity. Suppose $f \in SL(W)$ induces $\gamma \in PSL(W)$, and $g \in GL(V)$ induces $\gamma^\sigma \in PGL(V)$ (f and g exist since $\dim PV \geqq \dim PW \geqq 2$). Then $g \mid W$ induces $\gamma^\sigma \mid PW = \gamma$, and hence $g \mid W = \lambda f$, $\lambda \in F_n^*$. Let $f^\sigma = \lambda^{-1} g \in GL(V)$; f^σ induces γ^σ and $f^\sigma \mid W = \lambda^{-1}(g \mid W) = f$. If $h \in GL(V)$ induces γ^σ and $h \mid W = f$, we must have $h = \mu f^\sigma$, $\mu \in F_n^*$, and $f = h \mid W = \mu f^\sigma \mid W = \mu f$, whence $\mu = 1$ and $h = f^\sigma$. Hence

(1) *for each $f \in SL(W)$ there is a unique $f^\sigma \in GL(V)$ such that*
  (i) *if f induces $\gamma \in PSL(W)$, then f^σ induces γ^σ, and*
  (ii) $f^\sigma \mid W = f$.

We observe that

(2) *the mapping $\sigma: SL(V) \rightarrow GL(V)$ defined in* (1) *is an isomorphism.*

In fact, if $f^\sigma = g^\sigma$, $f, g \in SL(W)$, then $f = f^\sigma \mid W = g^\sigma \mid W = g$; hence σ is one-to-one. If f and g in $SL(W)$ induce γ and δ in $PSL(W)$ respectively, then fg induces $\gamma\delta$, and $f^\sigma g^\sigma$ induces $\gamma^\sigma \delta^\sigma = (\gamma\delta)^\sigma$. Moreover, $f^\sigma g^\sigma \mid W = fg$, and hence $f^\sigma g^\sigma = (fg)^\sigma$, proving that σ is an isomorphism.

In view of (1) and (2), our proposition will be an immediate consequence of

LEMMA 2. *Let W be a subspace of dimension $k \geqq 3$ of a vector space V of dimension $k + l$, $l < k$, over the field F_n of n elements. Let σ be an isomorphism of $SL(W)$ into $GL(V)$ such that $f^\sigma \mid W = f$ for all $f \in SL(W)$. Then, $SL(W)^\sigma \subseteq SL(V)$, and, if we exclude the case $k = 3$, $n = 2$, f^σ is a transvection of V for every transvection f of W.*

We postpone the proof of this lemma, which is computational, and proceed to derive the following needed consequence of Propositions 4 and 6. In the next section we give the proof of our theorem, and finally in §5 we complete the discussion by proving Lemma 2.

LEMMA 3. *Let G be a collineation group of a Desarguesian projective space $\mathcal{P}$ of dimension $d \geqq 3$ and finite order n, such that for each hyperplane H of $\mathcal{P}$, G_H induces the little projective group on H. Then G contains the little projective group of $\mathcal{P}$ unless $d = 3$, $n = 2$, and $G \approx A_7$, the alternating group of degree 7.*

Proof. Clearly G is transitive on points, and hence on hyperplanes ([4], [7], [8]). Hence G is transitive on incident point-hyperplane pairs, since G_H is transitive on the points of H for every hyperplane H. If G contains perspectivities $\neq 1$, Proposition 4 implies that G contains the little projective group of $\mathcal{P}$. Otherwise, for each hyperplane H, G_H is faithful on the points of H. By hypothesis, $G_H \mid H$ contains the little projective group of H, and hence, by Proposition 6 we must have $d = 3$ and $n = 2$. Moreover, the order of G will in this case be given by

$$\begin{aligned} |G| &= (\text{number of planes in } \mathcal{P})(|G_H|) \\ &= 15 \cdot 168 = 7!/2. \end{aligned}$$

But G is a subgroup of $PSL_4(2)$, which is isomorphic with A_8, and the only subgroups of A_8 of index 8 are those of the single conjugate class of subgroups isomorphic with A_7. Hence $G \approx A_7$ in this case, proving Lemma 3.

4. Proof of the theorem

We now complete the proof of the theorem stated in the introduction. Thus we assume that

$\mathcal{P}$ is a Desarguesian projective space of dimension $d \geqq 2$ and finite order n, and G is a flag-transitive collineation group of $\mathcal{P}$.

We must prove that if $(d, n) \neq (2, 2)$, $(2, 8)$, and $(3, 2)$, then G contains the little projective group of $\mathcal{P}$.

If G contains a perspectivity $\neq 1$, it follows from Proposition 4 that G contains the little projective group of $\mathcal{P}$. We assume therefore that *G_H is faithful on H for each hyperplane H.* The assumption that G is flag-transitive certainly implies that G_H is flag-transitive on H for each hyperplane H.

The case $d = 2$ is Theorem 1 of [6]. We establish successfully the cases $d = 3$ and $d = 4$, and complete the proof by induction on d.

Case $d = 3$. For $d = 3$, Theorem 1 of [6] implies that, if G_H does not contain the little projective group of H, then $n = 2$ or 8, and G_H is regular on the flags of H. Thus $|G_H| = 3\cdot 7$ or $9\cdot 73$ and $|G| = 15\cdot 3\cdot 7$ or $585\cdot 9\cdot 73$. G is primitive by Proposition 3 of [4]. But it is easily seen that there is no primitive group of order $15\cdot 3\cdot 7$ and degree 15, nor is there one of order $585\cdot 9\cdot 73$ and degree 585. For groups of these orders are solvable, and a solvable primitive group must have prime power degree ([3; §154, Theorem XIII, Corollary]). Hence G_H contains the little projective group of H. Since G is assumed to contain no perspectivities $\neq 1$, Lemma 3 implies that $n = 2$ and $G \approx A_7$. The theorem is therefore established for $d = 3$.

Case $d = 4$. If $d = 4$, the result already established implies that either G_H contains the little projective group of H, or $n = 2$ and $G_H \approx A_7$ is doubly transitive on the points of H. Since G is assumed to contain no perspectivities $\neq 1$, Lemma 3 rules out the first possibility. Thus $G_H \approx A_7$, and G is doubly transitive on the points of $\mathcal{P}$. Therefore G is doubly transitive on the hyperplanes of $\mathcal{P}$, i.e., there are exactly two G_H-orbits of points. One of these must consist of the 15 points on H; hence the other consists of the 16 other points. It follows that $16 \mid |G_H|$, i.e., $16 \mid (7!/2)$, a contradiction, which completes the proof for $d = 4$.

Case $d > 4$. If $d > 4$, the results already established, together with the induction hypothesis, imply that for each hyperplane H, $G_H \mid H$ contains the little projective group of H. By Lemma 3, this contradicts the assumption that G contains no perspectivities $\neq 1$.

Remark. It would be interesting to know if transitivity on incident point-line pairs is sufficient for the conclusion of the theorem for dimension $d > 3$. Wagner [10] has announced that for $d = 3, 4$, double transitivity of G implies that G contains the little projective group unless $d = 3$, $n = 2$, $G \approx A_7$.

5. Embedding $SL_k(F)$ in $SL_{k+l}(F)$

We still have to give a proof of Lemma 2. We will derive it as a corollary of

LEMMA 4. *Let W be a subspace of dimension $k \geqq 2$ in a vector space V of dimension $k + l$, $l > 0$, over a (not necessarily commutative) field F, and let $\sigma : SL(W) \rightarrow GL(V)$ be an isomorphism such that for each $f \in SL(W)$, $f^\sigma | W = f$ and f^σ induces the identity transformation on V/W. Then, if we exclude the cases*

(a) $k = 2$, *characteristic* $F = 2$, *and*

(b) $k = 3$, $F = F_2$,

there exists a subspace U of V such that $V = W \oplus U$ and $f^\sigma \mid U$ is the identity transformation for every $f \in SL(W)$, and f^σ is a transvection of V for every transvection $f \in SL(W)$. If we exclude the case (a), $SL(W)^\sigma \subseteq SL(V)$.

Proof. We shall assume that $l = 1$ since the result for general l follows at once from this case. Representing linear transformations of W and V by their matrices with respect to bases

$$\{a_1, \cdots, a_k\} \quad \text{and} \quad \{a_1, \cdots, a_k, a_{k+1}\}$$

of W and V respectively, we can interpret the isomorphism σ as an isomorphism

$$\sigma : SL_k(F) \rightarrow GL_{k+1}(F)$$

such that for $A \in SL_k(F)$,

$$A^\sigma = \begin{pmatrix} A & 0 \\ K(A) & 1 \end{pmatrix},$$

$K(A)$ being a $(k \times 1)$-matrix in F.

The group $SL_k(F)$ is generated by the matrices $B_{ij}(\lambda) = I + \lambda E_{ij}$ $(i \neq j = 1, 2, \cdots, n; \lambda \in F)$, E_{ij} being the $(k \times k)$-matrix with 1 in the intersection of the ith row and jth column and all other entries 0. Write

$$B_{ij}(\lambda)^\sigma = \begin{pmatrix} B_{ij}(\lambda) & 0 \\ K_{ij}(\lambda) & 1 \end{pmatrix}$$

and

$$K_{ij}(\lambda) = (K_{ij}^1(\lambda), K_{ij}^2(\lambda), \cdots, K_{ij}^k(\lambda)).$$

The cases $k \geqq 4$, $k = 3$, and $k = 2$ require separate treatment, and we will consider them in order. Suppose for the moment that $k \geqq 3$. Then for $j = 3, 4, \cdots, k$, $B_{k1}(\lambda)$ and $B_{j2}(\mu)$ commute. Hence the same is true of $B_{k1}(\lambda)^\sigma$ and $B_{j2}(\mu)^\sigma$, giving

$$K_{k1}(\lambda)B_{j2}(\mu) + K_{j2}(\mu) = K_{j2}(\mu)B_{k1}(\lambda) + K_{k1}(\lambda),$$

whence

$$K_{k1}^2(\lambda) + \mu K_{k1}^j(\lambda) + K_{j2}^2(\mu) = K_{j2}^2(\mu) + K_{k1}^2(\lambda).$$

Hence $\mu K^j_{k1}(\lambda) = 0$ for all μ, i.e.,

(1) *If* $k \geqq 3$, $K^j_{k1}(\lambda) = 0 \quad (j = 3, \cdots, k)$.

Now we shall dispose of

The case $k \geqq 4$. Since $k \neq 2, 3$, $B_{k1}(\lambda)$ and $B_{23}(\mu)$ commute. From the same relation for their images under σ we obtain

$$K_{k1}(\lambda)B_{23}(\mu) + K_{23}(\mu) = K_{23}(\mu)B_{k1}(\lambda) + K_{k1}(\lambda),$$

whence $\mu K^2_{k1}(\lambda) + K^3_{k1}(\lambda) + K^3_{23}(\mu) = K^3_{23}(\mu) + K^3_{k1}(\lambda)$, or

$$K^2_{k1}(\mu) = 0. \tag{2}$$

By (1) and (2) we have $K^j_{k1}(\lambda) = 0$ for all $j \neq 1$, and hence by symmetry

$$K^t_{rs}(\lambda) = 0 \quad \text{for all} \quad t \neq s. \tag{3}$$

Hence

$$B_{ij}(\lambda)^\sigma = I + \lambda E_{ij} + K^j_{ij}(\lambda)E_{k+1,j}\,. \tag{4}$$

(Here, of course, I and E_{rs} now represent $(k + 1) \times (k + 1)$-matrices.)

Since $B_{ij}(\lambda)^{-1} = B_{ij}(-\lambda)$, $B_{ij}(\lambda)^{-\sigma} = B_{ij}(-\lambda)^\sigma$, and hence by (4), $K^j_{ij}(-\lambda) = -K^j_{ij}(\lambda)$, and

$$B_{ij}(\lambda)^{-\sigma} = I - \lambda E_{ij} - K^j_{ij}(\lambda)E_{k+1,j}\,. \tag{5}$$

For $r \neq s$ we have $(B_{ri}(\lambda)^\sigma, B_{is}(\mu)^\sigma) = B_{rs}(\lambda\mu)^\sigma$ since this relation holds before σ is applied. Using (4) and (5) we obtain

$$(B_{ri}(\lambda)^\sigma, B_{is}(\mu)^\sigma) = I + \lambda\mu E_{rs} + K^i_{ri}(\lambda)\cdot\mu E_{k+1,s}\,,$$

and hence $K^s_{rs}(\lambda\mu) = K^i_{ri}(\lambda)\cdot\mu$. Setting

$$\alpha_r = K^s_{rs}(1) = K^i_{ri}(1)$$

we have therefore

$$K^s_{rs}(\mu) = \alpha_r\,\mu \quad \text{for all} \quad s \neq r. \tag{6}$$

Now set

$$\overset{1}{a}_{k+1} = a_{k+1} - \{\alpha_1\, a_1 + \cdots + \alpha_k\, a_k\}.$$

If f is the transformation of V determined by the matrix $B_{rs}(\lambda)^\sigma$ with respect to the original basis, then

$$\begin{aligned} f(\overset{1}{a}_{k+1}) &= [K^s_{rs}(\lambda)\cdot a_s + a_{k+1}] - [\alpha_1\, a_1 + \cdots + \alpha_r(\lambda a_s + a_r) + \cdots + \alpha_k\, a_k] \\ &= (\alpha_r\,\lambda)\cdot a_s + a_{k+1} - [\alpha_1\, a_1 + \cdots + \alpha_k\, a_k] - (\alpha_r\,\lambda)a_s \\ &= \overset{1}{a}_{k+1}. \end{aligned}$$

The proof for the case $k \geqq 4$ is now complete on taking $U = \langle \overset{1}{a}_{k+1} \rangle$.

Next we consider

The case $k = 3$. By (1) we have $K^3_{31}(\lambda) = 0$, and hence by symmetry,

$$K^i_{ij}(\lambda) = 0 \quad \text{for all } i,\ j. \tag{7}$$

From the fact that $B_{31}(\lambda)$ and $B_{21}(\mu)$ commute we obtain

$$K_{31}(\lambda)B_{21}(\mu) + K_{21}(\mu) = K_{21}(\mu)B_{31}(\lambda) + K_{31}(\lambda);$$

hence

$$K^1_{31}(\lambda) + \mu K^2_{31}(\lambda) + K^1_{21}(\mu) = K^1_{21}(\mu) + \lambda K^3_{21}(\mu) + K^1_{31}(\lambda),$$

and therefore

$$\mu K^2_{31}(\lambda) = \lambda K^3_{21}(\mu).$$

Setting $x = K^2_{31}(1) = K^3_{21}(1)$ we have therefore

$$K^2_{31}(\lambda) = \lambda x_1 = K^3_{21}(\lambda). \tag{8}$$

Similarly,

$$K^3_{12}(\lambda) = \lambda x_2 = K^1_{32}(\lambda), \tag{9}$$

$$K^2_{13}(\lambda) = \lambda x_3 = K^1_{23}(\lambda). \tag{10}$$

Comparing the last rows of the relation $B_{32}(\lambda\mu)^\sigma = (B_{31}(\lambda)^\sigma, B_{12}(\mu)^\sigma)$ and writing $u_\lambda = K^1_{31}(\lambda)$, $v_\lambda = K^2_{12}(\lambda)$, and $w_\lambda = K^2_{32}(\lambda)$, we have

$$\begin{aligned}
&(\lambda\mu x_2 \quad w_{\lambda\mu} \quad 0 \quad 1)\\
&\quad = (K_{32}(\lambda\mu), 1)\\
&\quad = (K_{31}(\lambda), 1)B_{12}(\mu)^\sigma B_{31}(-\lambda)^\sigma B_{12}(-\mu)^\sigma\\
&\quad = (u_\lambda \quad \lambda x_1 \quad 0 \quad 1)\begin{pmatrix} 1 & \mu & & \\ & 1 & & \\ & & 1 & \\ & v_\mu & x_2 & 1 \end{pmatrix} B\cdots^\sigma B\cdots^\sigma\\
&\quad = (u_\lambda \quad \mu u_\lambda + \lambda x_1 + v_\mu \quad \mu x_2 \quad 1)\begin{pmatrix} 1 & & & \\ & 1 & & \\ -\lambda & & 1 & \\ -u_\lambda & -\lambda x_1 & 0 & 1 \end{pmatrix} B\cdots^\sigma\\
&\quad = (-\lambda\mu x_2 \quad \mu u_\lambda + v_\mu \quad \mu x_2 \quad 1)\begin{pmatrix} 1 & -\mu & & \\ & 1 & & \\ & & 1 & \\ & -v_\mu & -\mu x_2 & 1 \end{pmatrix}\\
&\quad = (-\lambda\mu x_2 \quad \lambda\mu^2 x_2 + \mu u_\lambda \quad 0 \quad 1).
\end{aligned}$$

Hence $w_{\lambda\mu} = \lambda\mu^2 x_2 + \mu u_\lambda$, or $K^2_{32}(\lambda\mu) = \lambda\mu^2 x_2 + \mu K^1_{31}(\lambda)$. By symmetry we have therefore

$$K^j_{ij}(\lambda\mu) = \lambda\mu^2 x_j + \mu K_{is}(\lambda), \text{ for every permutation } (i, j, s) \text{ of } (1, 2, 3). \tag{11}$$

Moreover, $\lambda\mu x_2 = -\lambda\mu x_2$, $x_2 = -x_2$, and therefore $x_2 = 0$ if F has characteristic $\neq 2$; similarly $x_1 = x_3 = 0$ in that case.

Assume now for the moment that F has characteristic 2. By (11),

$$K_{12}^{2}(1) = x_2 + K_{13}^{3}(1) \quad \text{and} \quad K_{13}^{3}(1) = x_3 + K_{12}^{2}(1),$$

giving $x_2 = x_3$. Similarly, $x_2 = x_1$. Writing $x = x_1 = x_2 = x_3$, and reverting to the notation used above, (11) gives

$$w_{\lambda\mu} = \lambda\mu^2 x + \mu u_\lambda \quad \text{and} \quad u_{\lambda\mu} = \lambda\mu^2 x + \mu w_\lambda .$$

Putting $\lambda = t^{-1}$, $\mu = t$ in the first of these relations, and $\lambda = 1$, $\mu = t^{-1}$ in the second, we obtain

$$w_1 = tx + tu_{t^{-1}} \quad \text{and} \quad u_{t^{-1}} = t^{-2}x + t^{-1}w_1 .$$

Hence $x = t^{-2}x$, for all $t \in F$, implying that $x = 0$ unless $F = F_2$.

We now have for all $F \neq F_2$ that $x_1 = x_2 = x_3 = 0$, and hence by (11), $K_{ij}(\lambda\mu) = \mu K_{is}^{s}(\lambda)$. Putting $\alpha_i = K_{ij}^{i}(1) = K_{is}^{s}(1)$ we have $K_{ij}^{j}(\mu) = \mu\alpha_i$, $i \neq j$. The case $k = 3$ can now be completed in the same way as the case $k \geqq 4$, taking $U = \langle a_4^1 \rangle$, $a_4^1 = a_4 - \{\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3\}$.

The cases so far considered are the ones needed for our proof of the main theorem. For completeness we include

The case $k = 2$, characteristic $F \neq 2$. $SL_2(F)$ is generated by the matrices $B_{12}(\lambda)$ and $B_{21}(\lambda)$, $\lambda \in F$. Write

$$B_{12}(\lambda)^\sigma = \begin{pmatrix} 1 & \lambda & \\ & 1 & \\ x_\lambda & y_\lambda & 1 \end{pmatrix} \quad \text{and} \quad B_{21}(\lambda)^\sigma = \begin{pmatrix} 1 & & \\ \lambda & 1 & \\ u_\lambda & v_\lambda & 1 \end{pmatrix}.$$

From $B_{12}(\lambda)^\sigma B_{12}(\mu)^\sigma = B_{12}(\lambda + \mu)^\sigma = B_{12}(\mu)^\sigma B_{12}(\lambda)^\sigma$ we obtain

$$\mu x_\lambda + y_\lambda + y_\mu = y_{\lambda+\mu} = \lambda x_\mu + y_\mu + y_\lambda ,$$

whence

$$x_\lambda = \lambda x_1 \quad \text{and} \quad x_1 = y_1 + y_{-1} . \tag{12}$$

Similarly,

$$v_\lambda = \lambda v_1 \quad \text{and} \quad v_1 = u_1 + u_{-1} . \tag{13}$$

Let

$$T = \begin{pmatrix} 1 & & \\ -1 & 1 & \\ u_{-1} & v_{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & \\ & 1 & \\ x_1 & y_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ -1 & 1 & \\ u_{-1} & v_{-1} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ r & s & 1 \end{pmatrix},$$

where $r = x_1 - y_1 + v_1 - u_1$, $s = y_1 - 2v_1 - u_1$. Then

$$T^2 = \begin{pmatrix} -1 & & \\ & -1 & \\ a & b & 1 \end{pmatrix},$$

where $a = r - s$, $b = r + s$, and T^2 must belong to the center of $SL(F)^\sigma$. From $B_{12}(\lambda)^\sigma T^2 = T^2 B_{12}(\lambda)^\sigma$ we get

$$-x_\lambda + a = a + x_\lambda, \quad \text{or} \quad x_\lambda = -x_\lambda$$

and

$$-y_\lambda + b = \lambda a + b + y_\lambda, \quad \text{or} \quad 2y_\lambda = -\lambda a.$$

Hence, since F is assumed to have characteristic $\neq 2$, $x_\lambda = 0$ for all $\lambda \in F$ and

$$2y_\lambda = -\lambda[r - s] = 2\lambda y_1,$$

giving

$$y_\lambda = \lambda y_1. \tag{14}$$

Similarly,

$$u_\lambda = \lambda u_1. \tag{15}$$

The case $k = 2$, characteristic $F \neq 2$ can now be completed in the same way as the other cases.

If F has characteristic 2, (12) and (14) give immediately that $x_\lambda = v_\lambda = 0$ for all $\lambda \in F$ and hence in this case $B_{ij}(\lambda)^\sigma$ and $B_{21}(\mu)^\sigma$ represent transvections. If $F = F_2$, the relations (14) and (15) are of course immediate.

COROLLARY. *Let W be a subspace of dimension $k \geqq 2$ of a vector space V of dimension $k + l$, $l < k$, over the field F_n of n elements. Let*

$$\sigma : SL(V) \to GL(V)$$

be an isomorphism such that $f^\sigma \mid W = f$ for all $f \in SL(W)$. Then all the conclusions of Lemma 4 hold.

Proof. If $l = 1$, the homomorphism $\sigma^* : SL(W) \to GL(V/W)$ induced by σ is trivial since $SL(W)$ is generated by elements of order dividing n. If $l > 1$ we must have $k > 2$. Hence, if σ^* is nontrivial in this case, its kernel must be contained in the center of $SL(W)$. Therefore,

$$| SL(W)^{\sigma^*} | \geqq | PSL(W) |,$$

contrary to the fact that, since dim $W >$ dim V/W,

$$| PSL(W) | > | GL(V/W) |.$$

Hence σ^* is trivial in any case, and the hypotheses of Lemma 4 hold, proving the corollary.

The proof of our theorem is now complete since the corollary certainly includes Lemma 2.

Added in proof. Since this paper was submitted, a paper by A. WAGNER, *On collineation groups of projective spaces. I*, Math. Zeitschrift, vol. 76 (1961), pp. 411–426, has appeared, containing the results of the above §2 with essentially the same proofs, and an entirely different proof of Proposition 6. The main results are different.

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THE UNIVERSITY OF MICHIGAN
ANN ARBOR, MICHIGAN