ON THE CLASSIFICATION OF FILTERED MODULES

BY

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1. Introduction

Throughout this paper Λ will denote an algebra with unit over a field K. Modules are all left Λ-modules; vector spaces are vector spaces over K. Graded vector spaces or modules are graded by nonnegative degrees, i.e., the homogeneous components of negative degrees are 0.

We propose to study filtered modules, for which we adopt the following definition as leading to an economy of notation. A *filtered module* is a short exact sequence

$$\boldsymbol{X} = (0 \longrightarrow X \xrightarrow{\xi} X \xrightarrow{\xi''} X'' \longrightarrow 0)$$

of graded modules, the maps ξ, ξ'' being homogeneous of degrees 1, 0. This is easily seen to coincide with the more familiar notion when the maps $\xi_q : X_q \to X_{q+1}$ are interpreted as inclusions.

If also $\boldsymbol{Y} = (0 \to Y \xrightarrow{\eta} Y \xrightarrow{\eta''} Y'' \to 0)$ is a filtered module, a map $\boldsymbol{\phi} : \boldsymbol{X} \to \boldsymbol{Y}$ is a pair of maps $\phi : X \to Y$, $\phi'' : X'' \to Y''$ such that $\phi\xi = \eta\phi$, $\phi''\xi'' = \eta''\phi$. These $\boldsymbol{\phi}$ constitute, in an obvious way, a graded category F which may be given an abelian structure [1]; we shall not however make any use of this structure here. In fact F may be interpreted as a "filtered category": this notion, which will not be investigated here, the reader may supply for himself.

The functor $S''\boldsymbol{X} = X''$, $S''\boldsymbol{\phi} = \phi''$ is of course homogeneous additive. Its value on $\boldsymbol{X}$ is the *associated graded module* of $\boldsymbol{X}$; we shall also refer to $\boldsymbol{X}$ as an *extension* of $S''X$.

Two extensions of a graded module A are *equivalent* if there is an equivalence $\boldsymbol{\phi}$ of filtered modules connecting them, such that $S''\boldsymbol{\phi} = 1 : A$. The problem to which we address ourselves here is that of classifying the equivalence classes of extensions of a fixed module A. The analogous problem for abelian groups has been treated by Shih [2], who arrives at a formulation not readily comparable with that given below. He also announces (but does not state) results for the case considered here.

In the very simple case that A has only two nonvanishing homogeneous components, say A_0 and A_1, the classification, viz. $\mathrm{Ext}^1(A_1, A_0)$, is of course well known. We shall see that a similar description is also valid in the general case. In the course of the discussion we shall also solve another problem, namely that of attaching to a filtered module a strong enough

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invariant to characterize it completely, and not merely up to nonunique isomorphism. This is of course necessary if the extension, described by its invariant, is to have identifiable elements. It should be observed that the extension class in $\mathrm{Ext}^1(A_1, A_0)$ does not have this character, even in the simplest case, there being an ambiguity corresponding to the group $\mathrm{Hom}(A_1, A_0)$ in the extension described by such a class.

2. Computational preliminaries

If A is a Λ-module, we write $PA = \Lambda \otimes_K A$; if $f{:}A \to B$ over K, then $Pf = (1{:}\Lambda) \otimes f{:}PA \to PB$ is a Λ-map. The Λ-map $\pi A{:}PA \to A$ is defined by $\pi A(\lambda \otimes a) = \lambda a$; if $\phi{:}A \to B$ is a Λ-map, then $\pi B(P\phi) = \phi(\pi A)$.

We denote the kernel of πA by $\omega A{:}\Omega A \to PA$ so that

$$\boldsymbol{PA} = (0 \longrightarrow \Omega A \xrightarrow{\omega A} PA \xrightarrow{\pi A} A \longrightarrow 0)$$

is exact. Further, K-maps $tA{:}PA \to \Omega A$ and $t^*A{:}A \to PA$ are defined by

$$(2.1) \qquad \omega A(tA)(\lambda \otimes a) = \lambda \otimes a - 1 \otimes \lambda a, \qquad (t^*A)a = 1 \otimes a.$$

It is easy to see that $(\omega A, tA, t^*A, \pi A)$ is a direct sum decomposition of PA over K. If also $\mu \in \Lambda$, then 2.1 implies

$$(2.2) \qquad \mu(tA)(\lambda \otimes a) = tA(\mu\lambda \otimes a - \mu \otimes \lambda a).$$

If $f{:}A \to B$, then

$$(2.3) \qquad \pi B(Pf)t^*A = f, \qquad tB(Pf)(t^*A) = 0.$$

We define also

$$(2.4) \qquad \begin{aligned} \Omega f &= tB(Pf)\omega A{:}\Omega A \to \Omega B, \\ \Box f &= \pi B(Pf)\omega A{:}\Omega A \to B. \end{aligned}$$

Observe that $\Box f$ is always a Λ-map, and that $\Box f = 0$ if and only if f is a Λ-map. In fact,

$$(2.5) \qquad f(tA)(\lambda \otimes a) = \lambda f a - f \lambda a.$$

If further $g{:}B \to C$, then

$$(2.6) \qquad \begin{aligned} \Box(gf) &= \pi C(Pg)[\omega B(tB) + t^*B(\pi B)](Pf)\omega A \\ &= \Box g(\Omega f) + g(\Box f), \\ \Omega(gf) &= tC(Pg)[\omega B(tB) + t^*B(\pi B)](Pf)\omega A \\ &= (\Omega g)(\Omega f), \end{aligned}$$

so that Ω is a functor. Moreover if $\phi{:}A \to B$ is a Λ-map, then $(\omega B)\Omega\phi = (P\phi)\omega A$, which implies that $\Omega\phi$ is a Λ-map.

Finally, we observe that for any Λ-module A

$$(2.7)\qquad \square(t^*A) = \omega A, \qquad \square(tA) = -\Omega\pi A,$$

the latter formula following immediately from the former and 2.6; these imply

$$(2.8)\qquad \Omega\square = -\square\Omega.$$

We shall want to apply all these operations to graded modules; it is only necessary to observe that PA, ΩA are also graded if A is, that ωA, πA, tA, t^*A are all homogeneous of degree 0, and that P, $\square$, Ω preserve the degree of homogeneous maps.

3. Cocycles of a graded module

If A, B are graded Λ-modules, we denote by $\mathrm{Hom}_{\leqq q}(A, B; K)$ the set of inhomogeneous K-maps $f:A \to B$ whose homogeneous components f_p are 0 for $p > q$. The group $\mathrm{Hom}_{<q}(A, B; K)$ is defined analogously.

If A is graded by nonnegative degrees, then $\mathfrak{A}A \subset \mathrm{Hom}_{\leqq 0}(A, A; K)$ consists of those maps f such that $f_0 = 1:A$. It is a group under composition: if $g \in \mathrm{Hom}_{<0}(A, A; K)$, then $1 + \sum_{n=1}^{\infty} g^n$ is defined, and is equal to $(1 - g)^{-1}$.

Now if $z \in \mathrm{Hom}_{<0}(\Omega A, A; K)$, then $\square z:\Omega^2 A \to A$, and $\Omega z:\Omega^2 A \to \Omega A$. If

$$(3.1)\qquad \square z + z(\Omega z) = 0,$$

we shall say that z is a *cocycle* of A; the set of cocycles will be denoted by $\mathfrak{Z}A$.

If $f \in \mathfrak{A}A$ then $ff^{-1} = 1:A$ is a Λ-map. Hence by 2.6,

$$(3.2)\qquad \square f(\Omega f^{-1}) + f(\square f^{-1}) = 0; \qquad \square f^{-1} = -f^{-1}(\square f)(\Omega f^{-1}).$$

This, together with 2.8, leads immediately to the following result.

Lemma 3.3. *If $z \in \mathfrak{Z}A$, $f \in \mathfrak{A}A$, then $f * z = (fz - \square f)\Omega f^{-1} \in \mathfrak{Z}A$. Further, $(f, z) \to f * z$ defines an operation of the group $\mathfrak{A}A$ on the set $\mathfrak{Z}A$.*

The set of orbits, $\mathfrak{Z}A/\mathfrak{A}A$, will be denoted by $\mathfrak{E}A$. The reader may easily verify that if A has only two nonzero homogeneous components, say A_0, A_1, then $\mathfrak{E}A$ may be identified with $\mathrm{Ext}^1(A_1, A_0)$.

It is now possible to state the classification theorem for filtered modules.

Classification Theorem 3.4. *If A is a Λ-module graded by nonnegative degrees, then the equivalence classes of filtered modules having A as associated graded module are in canonical one-to-one correspondence with the elements of $\mathfrak{E}A$.*

Rather than prove this theorem directly, we shall deduce it as a corollary of a theorem on the structure of the category of filtered modules.

$\mathfrak{A}$, $\mathfrak{Z}$, and $\mathfrak{E}$ are not, to be sure, functors on the category of graded Λ-modules. However, if $g \in \mathrm{Hom}_{\leqq q}(A, B; K)$ is a K-isomorphism such that g_q is a Λ-map, then $(\mathfrak{A}g)f = gfg^{-1}$ defines an isomorphism $\mathfrak{A}g:\mathfrak{A}A \to \mathfrak{A}B$. Moreover $gg_q^{-1} \in \mathfrak{A}B$ and

$$(\mathfrak{Z}g)z = (gg_q^{-1}) * g_q z\Omega g_q^{-1} \tag{3.5}$$

defines a bijection $\mathfrak{Z}g:\mathfrak{Z}A \to \mathfrak{Z}B$ such that

$$[(\mathfrak{A}g)f] * [(\mathfrak{Z}g)z] = (\mathfrak{Z}g)(f * z), \qquad f \in \mathfrak{A}A, \quad z \in \mathfrak{Z}B.$$

Thus $\mathfrak{Z}g$ preserves orbits, and consequently defines $\mathfrak{E}g:\mathfrak{E}A \to \mathfrak{E}B$, once more a bijection. It is easy to see that $\mathfrak{A}$, $\mathfrak{Z}$, $\mathfrak{E}$ are functorial on the category of such maps g.

It should be noted that, in view of 3.5, $\mathfrak{E}g = \mathfrak{E}g_q$.

4. The space of splittings

If $\boldsymbol{X}$ is a filtered module we shall want to study the set $\Psi\boldsymbol{X}$ of left K-splittings of $\boldsymbol{X}$, i.e., of K-maps $s:X \to X$ of degree -1 such that $s\xi = 1:X$; this is of course in one-to-one correspondence with the set $\Psi^*\boldsymbol{X}$ of right-splittings $s^*:X'' \to X$, which are homogeneous of degree zero and satisfy $\xi''s^* = 1:X''$. The correspondence is given by $ss^* = 0$, which implies that $\xi s + s^*\xi'' = 1:X$.

We shall give to this set the structure of an affine space of a certain groupoid; we begin accordingly by defining the groupoid. If A is a graded module and $\mathfrak{e} \in \mathfrak{E}A$, then $\mathfrak{e} \subset \mathfrak{Z}A$, and $\mathfrak{A}A$ operates transitively on $\mathfrak{e}$. Let $\mathcal{G}(A, \mathfrak{e})$ have as objects the elements $z \in \mathfrak{e}$, and as maps the triples $(z'; f; z):z \to z'$ where $f \in \mathfrak{A}A$, $f * z = z'$. Composition is given by $(z''; f'; z')(z'; f; z) = (z''; f'f; z)$.

A $\mathcal{G}(A, \mathfrak{e})$-affine space structure on a set Γ is given by a map $\delta:\Gamma \times \Gamma \to \mathcal{G}(A, \mathfrak{e})$. We shall write, for $x, y \in \Gamma$,

$$\delta(y, x) = (\delta y; \delta_0(y, x); \delta x).$$

Returning now to the case of a filtered module $\boldsymbol{X}$, suppose $s \in \Psi\boldsymbol{X}$, and set

$$\delta s = \textstyle\sum_n \xi'' s^n(\Box s^*):\Omega X'' \to X'', \tag{4.1}$$

where $\sum_n$ abbreviates $\sum_{n=0}^{\infty}$, $s^0 = 1:X$, and $s^n = ss^{n-1}$. Since $\xi''s^0(\Box s^*) = \xi''\Box s^* = \Box(\xi''s^*) = 0$, this map is in $\mathrm{Hom}_{<0}(\Omega X'', X'')$.

LEMMA 4.2. *If $s \in \Psi\boldsymbol{X}$, then $\delta s \in \mathfrak{Z}X''$.*

The computation is straightforward.

If also $r \in \Psi\boldsymbol{X}$, set

$$\delta_0(r, s) = \textstyle\sum_n \xi'' r^n s^*:X'' \to X''. \tag{4.3}$$

LEMMA 4.4. *If $r, s \in \Psi\boldsymbol{X}$, then $\Box\delta_0(r, s) = \delta_0(r, s)\delta s - (\delta r)\Omega\delta_0(r, s)$.*

This is again straightforward.

In consequence of 4.2, 4.4 we have the following result.

PROPOSITION 4.5. *If $\boldsymbol{X}$ is a filtered module and $s \in \Psi\boldsymbol{X}$, then the orbit $\mathfrak{e}\boldsymbol{X}$ of $\delta s \in \mathfrak{Z}X''$ is independent of s. Further, $\delta(r, s) = (\delta r; \delta_0(r, s); \delta s)$ defines on $\Psi\boldsymbol{X}$ the structure of a $\mathcal{G}(X'', \mathfrak{e}\boldsymbol{X})$-affine space.*

5. The main lemma

We denote by $\mathfrak{F}$ the graded category of filtered Λ-modules. Our principal objective is to construct a "classification" of $\mathfrak{F}$, i.e., a functor on $\mathfrak{F}$ which is a weakly surjective local isomorphism. We begin by constructing such a functor not on $\mathfrak{F}$ but on an "enriched" category $\mathfrak{F}^\#$ whose objects are the pairs $(\boldsymbol{X}, s)$ with $\boldsymbol{X} \in \mathfrak{F}$, $s \in \Psi\boldsymbol{X}$, and whose maps are the triples

$$(t; \boldsymbol{\phi}; s):(\boldsymbol{X}, s) \to (\boldsymbol{Y}, t)$$

where $\boldsymbol{\phi}:\boldsymbol{X} \to \boldsymbol{Y}$ in $\mathfrak{F}$. Composition is defined by

$$(r; \boldsymbol{\psi}; t)(t; \boldsymbol{\phi}; s) = (r; \boldsymbol{\psi\phi}; s);$$

the additive structure is inherited from $\mathrm{Hom}(X, Y)$.

The functor will take its values in a category $\mathfrak{W}$ constructed as follows. The objects are the pairs (A, z) with A a module graded by nonnegative degrees and $z \in \mathfrak{Z}A$. The group $\mathrm{Hom}_q((A, z), (B, w))$ consists of the

$$(w; f; z)_q, \qquad f \in \mathrm{Hom}_{\leqq q}(A, B; K)$$

satisfying the condition

$$(5.1) \qquad \Box f = fz - w(\Omega f).$$

Notice that the right side is of degree $< q$, so that f_q is a Λ-map. Composition is given by

$$(v; g; w)_p(w; f; z)_q = (v; gf; z)_{p+q};$$

note that 2.6 implies that the right-hand side satisfies 5.1. The additive structure is inherited from $\mathrm{Hom}_{\leqq q}(A, B; K)$.

LEMMA 5.2. *The equations*

$$F^\#(\boldsymbol{X}, s) = (X'', \delta s),$$
$$F^\#(t; \boldsymbol{\phi}; s) = (\delta t; \textstyle\sum_n \eta'' t^n \phi s^*; \delta s)_q$$

for $\boldsymbol{\phi} = (\phi, \phi'') \in \mathrm{Hom}_q(\boldsymbol{X}, \boldsymbol{Y})$, *define a homogeneous functor* $F^\#:\mathfrak{F}^\# \to \mathfrak{W}$. *This functor is a surjective local isomorphism.*

Observe first that $0 = \Box(1:Y) = \Box(\eta t + t^*\eta'')$, whence

$$0 = \eta\Box t + \Box t^*(\Omega\eta'')$$

by 2.6, and thus $\Box t = -t(\Box t^*)\Omega\eta''$, so that

$$\Box(\textstyle\sum_n \eta'' t^n \phi s^*) = \sum_{m,k} \eta'' t^m(\Box t)\Omega(t^k\phi s^*) + \sum_n \eta'' t^n \phi(\Box s^*)$$
$$= -\textstyle\sum_{m,k} \eta'' t^m(\Box t^*)\Omega(\eta'' t^k \phi s^*) + \sum_n \eta'' t^n\phi(\Box s^*).$$

We need only evaluate

$$\textstyle\sum_n \eta'' t^n \phi s^*(\delta s) = \sum_{n,k} \eta'' t^n \phi s^* \xi'' s^k \Box s^*$$
$$= \textstyle\sum_{n,k} \eta'' t^n \phi(1 - \xi s) s^k \Box s^*$$

$$= \sum_{n,k} [\eta'' t^n \phi s^k \Box s^* - \eta'' t^n \phi s^{k+1} \Box s^*]$$
$$= \sum_n \eta'' t^n \phi \Box s^*$$

to see that $F^\#(t; \phi; s): F^\#(\boldsymbol{X}, s) \to F^\#(\boldsymbol{Y}, t)$. The fact that $F^\#$ preserves composition is proved in equally direct fashion. To see that $F^\#$ is surjective on objects, suppose $(A, z) \in \mathfrak{W}$, and define a graded module X by letting X_q be $\{(a, q) | a \in \sum_{j \leq q} A_j\}$, with the operation given by

$$(5.3) \qquad \lambda(a, q) = (\lambda a + z(tA)(\lambda \otimes a), q), \qquad a \in \textstyle\sum_{j \leq q} A_j, \lambda \in \Lambda;$$

it follows from 2.2, 3.1 that 5.3 makes X a Λ-module. The injection $X_q \to X_{q+1}$ is a Λ-map, and gives rise to $\xi: X \to X$, of degree 1. Similarly the projection $X_q \to A_q$ is a Λ-map and defines $\xi'': X \to A$. Clearly

$$\boldsymbol{X} = (0 \to X \xrightarrow{\xi} X \xrightarrow{\xi''} A \to 0)$$

is a filtered module. But if $s: X \to X$, $s^*: A \to X$ are given by the projections $X_{q+1} \to X_q$ and injections $A_q \to X_q$, then $s \in \Psi \boldsymbol{X}$, and, if $\lambda \in \Lambda$, $a \in A$,

$$\begin{aligned} s(tA)(\lambda \otimes a) &= \textstyle\sum_n \xi'' s^n (\pi X)(Ps^*)(\lambda \otimes a - 1 \otimes \lambda a) \\ &= \textstyle\sum_n \xi'' s^n z(tA)(\lambda \otimes a) \qquad \text{by 5.3} \\ &= z(tA)(\lambda \otimes a), \end{aligned}$$

so that $\delta s = z$ and $F^\#(\boldsymbol{X}, s) = (A, z)$.

Finally, for $(\boldsymbol{X}, a)$, $(\boldsymbol{Y}, t) \in \mathfrak{F}^\#$ we exhibit an inverse

$$G: \mathrm{Hom}(F^\#(\boldsymbol{X}, s), F^\#(\boldsymbol{Y}, t)) \to \mathrm{Hom}((\boldsymbol{X}, s), (\boldsymbol{Y}, t))$$

of $F^\#$. If $(\delta t; f; \delta s)_q \in \mathrm{Hom}_q(F^\#(\boldsymbol{X}, s), F^\#(\boldsymbol{Y}, t))$, we set

$$G(\delta t; f; \delta s)_q = (t; \textstyle\sum_n \sum_{k=0}^n \eta^n t^* f_{q-n+k} \xi'' s^k; s),$$

where f_j is the homogeneous component of degree j. We omit the tedious but straightforward computations which show that G is indeed inverse to $F^\#$.

We might, at the cost of additional longwindedness, have proved Lemma 5.2 by constructing explicitly a functor $G: \mathfrak{W} \to \mathfrak{F}^\#$ such that $F^\# G = 1: \mathfrak{W}$, and $GF^\#$ is naturally equivalent to $1: \mathfrak{F}^\#$.

6. The strong classification theorem

We may now consider the diagram

$$\mathfrak{F} \xleftarrow{V} \mathfrak{F}^\# \xrightarrow{F^\#} \mathfrak{W}$$

where $V(\boldsymbol{X}, s) = \boldsymbol{X}$, $V(t; \phi; s) = \phi$. The functor V is clearly a surjective local isomorphism; we have shown in 5.2 that the same is true of $F^\#$.

We shall construct a functor $F: \mathfrak{F} \to \mathfrak{V}$ where $\mathfrak{V}$ is an "enlargement" of $\mathfrak{W}$ defined as follows.

The objects of $\mathfrak{V}$ are triples $(A, \mathfrak{e}, \Gamma)$ where A is a module graded by nonnegative degrees, $\mathfrak{e} \in \mathfrak{E}A$, and Γ is a $\mathcal{G}(A, \mathfrak{e})$-affine space.

$\mathrm{Hom}_q((A, \mathfrak{e}, \Gamma), (A', \mathfrak{e}', \Gamma'))$ consists of equivalence classes of triples

$$(x'; f; x)_q, \qquad x \in \Gamma, \quad x' \in \Gamma', \quad f \in \mathrm{Hom}_{\leqq q}(A, A'; K)$$

satisfying

$$\square f = f(\delta x) - (\delta x')\Omega f, \tag{6.1}$$

the equivalence relation being

$$(x_1' ; f; x_1)_q = (x'; \delta_0(x', x_1')f\delta_0(x_1, x); x)_q . \tag{6.2}$$

This becomes a K vector space under the operations inherited from $\mathrm{Hom}_{\leqq q}(A, A'; K)$ by keeping x, x' fixed. Composition is defined by

$$(x''; f'; x')_q(x'; f; x)_r = (x'', f'f; x)_{q+r} . \tag{6.3}$$

To see that this makes $\mathfrak{V}$ a category, compare 6.1 with 5.1.

PROPOSITION 6.4. *$(A, \mathfrak{e}, \Gamma)$ and $(A', \mathfrak{e}', \Gamma') \in \mathfrak{V}$ are equivalent if and only if there is a homogeneous Λ-isomorphism $\psi : A \approx A'$ such that $(\mathfrak{S}\psi)\mathfrak{e} = \mathfrak{e}'$.*

First, suppose that ψ is such an isomorphism, its degree being q. If $z \in \mathfrak{e}$, then (see 3.5) $z' = \psi z(\Omega\psi^{-1}) \in \mathfrak{e}'$. Suppose $x \in \Gamma$, $x' \in \Gamma'$ with $\delta x = z$, $\delta x' = z'$. Then $(x'; \psi; x)_q$ is an equivalence in $\mathfrak{V}$.

Conversely, suppose $(x'; f; x)_q$ is an equivalence in $\mathfrak{V}$. Then f_q is a Λ-map (as in 5.1) and by 3.5, 6.1, $x' = (\mathfrak{Z}f)\delta x$, so that $\mathfrak{e}' = (\mathfrak{S}f)\mathfrak{e}$. But clearly $\mathfrak{S}f = \mathfrak{S}f_q$.

The strong classification theorem is the following assertion.

THEOREM 6.5. *The equations*

$$FX = (X'', \mathfrak{e}X, \Psi X),$$
$$F\boldsymbol{\phi} = (t; \textstyle\sum_n \eta'' t^n \phi s^*; s)_q$$

for $\boldsymbol{\phi} = (\phi, \phi''): X \to Y$ of degree q, $t \in \Psi Y$, $s \in \Psi X$, define a homogeneous functor $F : \mathfrak{F} \to \mathfrak{V}$. This functor is a weakly surjective local isomorphism.

That the value given for $F\boldsymbol{\phi}$ is independent of s and t follows immediately from 6.2. The functorial character of F is seen (once representatives are chosen) to be equivalent to that of $F^{\#}$, as expressed in 5.2.

The fact that F is weakly surjective, i.e., that for any $(A, \mathfrak{e}, \Gamma) \in \mathfrak{V}$ there is an X with FX equivalent to $(A, \mathfrak{e}, \Gamma)$, follows from the surjectivity of $F^{\#}$ together with 6.4.

Finally, to see that F is a local isomorphism, fix s and t, and consider the diagram

$$\begin{array}{ccc}
\mathrm{Hom}(X, Y) & \xleftarrow{V} & \mathrm{Hom}((X, s),(Y, t)) \\
\big\downarrow F & & \big\downarrow F^{\#} \\
\mathrm{Hom}(FX, FY) & \xleftarrow{W} & \mathrm{Hom}(F^{\#}(X, s), F^{\#}(Y, t))
\end{array}$$

where $W(\delta t; f; \delta s)_q = (t; f; s)_q$. This clearly commutes, and V, $F^{\#}$, W are all isomorphisms.

We observe in conclusion that Classification Theorem 3.4 follows immediately from 6.5 and 6.4. Thus $(X'', \mathfrak{e}\boldsymbol{X})$ are a complete set of invariants, *up to automorphism*, of $\boldsymbol{X}$. Theorem 6.5 asserts further that $(X'', \mathfrak{e}\boldsymbol{X}, \Psi\boldsymbol{X})$ is a complete set of invariants, determining $\boldsymbol{X}$ up to unique equivalence.

It should be remarked that the foregoing treatment lends itself to generalization, without further remark, to the case that K is a commutative ring, with the sole restriction that all sequences split as sequences of K-modules. Under suitable additional restrictions, similar observations may also be made with respect to filtered objects in more general abelian categories.

Bibliography

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