CATEGORY AND GENERALIZED HOPF INVARIANTS

BY

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1. Introduction

The Lusternik-Schnirelmann category of a topological space X is usually defined as follows: cat $X \leqq n$ if X may be covered by n open sets each of which is contractible in X. The general reference for the properties of this homotopy invariant is [4]. It was observed by G. W. Whitehead in [9] that, for a certain class of spaces including polyhedra, this definition is equivalent to the one given below (Definition 2.1), which we make the starting point of our investigation.

If we attach a cone CA to X by means of a map $f:A \rightarrow X$, it is trivial to verify with the original definition of category—and easy to verify with ours—that cat $Y \leqq$ cat $X + 1$, where $Y = X \cup_f CA$. Our interest centers in the problem of establishing conditions under which, in fact, cat $Y \leqq$ cat X. We are motivated partly by the wish to compute the category of a 1-connected polyhedron as a function of the terms in a homology decomposition (see [3]) and partly by the observation that, in the important case $n = 2$, our problem dualizes, in the sense of [2], a familiar problem of homotopy theory. Namely, if X admits a multiplication and Y is the fibre space over X induced by a map $f:X \rightarrow A$, under what circumstances does Y admit a multiplication. Answers to this question, under certain restrictions, have been given by Copeland [1] and others in terms of the concept of *primitivity* of cohomology classes. As expected, our solution of the dual problem is primarily in terms of a concept of primitivity which we introduce for homotopy classes. Moreover if we specialize A to be a Moore space $K'(G, m - 1)$, we get fairly complete results in which the primitivity property of the map f turns out to be equivalent to the vanishing of a generalized Hopf invariant which we define for elements of homotopy groups (with coefficients) of spaces of specified category. Just as in the dual situation, if X is a suspension space, all suspension elements of $\pi_{m-1}(G; X)$ are primitive, but the converse is false. We are thus enabled to construct spaces of category 2 which are not equivalent to suspensions, answering a question first raised by T. Ganea.

The definition of category which we give suggests a related notion of *weak category* (Definition 2.2) which is a weaker hypothesis on a space X in that w cat $X \leqq n$ if cat $X \leqq n$, but the converse is, in general, false. Nevertheless certain well-known properties[1] of category generalize to weak category, includ-

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[1] The Whitehead theorem (see [9]) on the nilpotency class of $\pi(X, Y)$ where Y is a group-like space also generalizes to spaces X of weak category n. See a forthcoming paper by I. Berstein and T. Ganea.

ing the fact, first pointed out by Eilenberg, that n-fold cup products vanish if cat $X \leqq n$. We obtain a condition under which w cat $Y \leqq$ cat X where $Y = X \cup_f CA$, which is expressed in terms of a homomorphism which also deserves to be regarded as a generalized Hopf invariant. The last section of the paper is largely devoted to a construction which gives rise to spaces Y such that w cat $Y = 2$ but, in general, cat $Y = 3$.

Throughout the paper it is understood that maps and homotopies take place in the "category" of spaces with base point.

2. Definitions and fundamental properties.

Let $(X, *)$ be a space with base point. Let X^n be the Cartesian product of n copies of X, and let $T^n(X) \subset X^n$ be the subspace consisting of points $(x_1, \cdots, x_n)$ such that $x_i = *$ for some i, $1 \leqq i \leqq n$. Let $\Delta: X \rightarrow X^n$ be the diagonal map

$$\Delta(x) = (x, x, \cdots, x);$$

let $X^{(n)}$ be the quotient space $X^n/T^n(X)$, and let $q: X^n \rightarrow X^{(n)}$ be the identification map. Let $j: T^n(X) \rightarrow X^n$ be the inclusion map.

Definition 2.1. X has *category* $\leqq n$ (written cat $X \leqq n$) if there exists a map $\phi: X \rightarrow T^n(X)$ with $j\phi \simeq \Delta$.

Definition 2.2. X has *weak category* $\leqq n$ (written w cat $X \leqq n$) if $q\Delta \simeq 0$.

Since $qj = 0$, it follows trivially that w cat $X \leqq n$ if cat $X \leqq n$. We offer examples later to show that the converse does not hold.

To justify the wording of these definitions we need

Proposition. 2.3. (i) *If* cat $X \leqq n$, *then* cat $X \leqq n + 1$.
(ii) *If* w cat $X \leqq n$, *then* w cat $X \leqq n + 1$.

(i) Define $\psi: X \rightarrow T^{n+1}(X)$ by $\psi(x) = (\phi(x), x)$. Then if $k: T^{n+1}(X) \rightarrow X^{n+1}$ is the inclusion, it is clear that $k\psi \simeq \Delta: X \rightarrow X^{n+1}$.

(ii) Define $\rho: X^n \rightarrow X^{n+1}$ by $\rho(x_1, \cdots, x_n) = (x_1, \cdots, x_n, x_n)$. Then ρ induces a map $\sigma: X^{(n)} \rightarrow X^{(n+1)}$ such that $\sigma q_n = q_{n+1}\rho$; and $\rho\Delta_n = \Delta_{n+1}$. Thus $q_{n+1}\Delta_{n+1} \simeq 0$ if $q_n \Delta_n \simeq 0$.

We will henceforth take the view that to assign a bound n on the category of X is to structure X with a map $\phi: X \rightarrow T^n(X)$ such that $j\phi \simeq \Delta$.

To establish the homotopy invariance of these definitions we show

Proposition 2.4. *If X is dominated by Y, then* (i) cat $X \leqq$ cat Y, (ii) w cat $X \leqq$ w cat Y.

Let $f: X \rightarrow Y$, $g: Y \rightarrow X$ be maps such that $gf \simeq 1: X \rightarrow X$.

(i) Let cat $Y \leqq n$, and let Y be structured by $\phi_Y: Y \rightarrow T^n(Y)$. Define $\phi_X = T^n(g) \circ \phi_Y \circ f$. Then $j_X \phi_X = j_X \circ T^n(g) \circ \phi_Y \circ f = g^n \circ j_Y \circ \phi_Y \circ f \simeq g^n \circ \Delta_Y \circ f = \Delta_X \circ g \circ f \simeq \Delta_X$.

(ii) Now $q_X \Delta_X \simeq q_X \Delta_X gf = g^{(n)} f^{(n)} q_X \Delta_X$. On the other hand, $f^{(n)} q_X \Delta_X = q_Y \Delta_Y f \simeq 0$. Thus $q_X \Delta_X \simeq 0$.

We next prove

PROPOSITION 2.5. *Let X be a $(q-1)$-connected polyhedron of dimension $\leqq nq-1$. Then* $\operatorname{cat} X \leqq n$. *If* $\dim X \leqq nq-2$, *then the homotopy class of the structure map $\phi: X \to T^n(X)$ is uniquely determined.*

The proof is a straightforward exercise in cellular approximation and will be omitted. In particular let X be a Moore space $K'(G, m-1)$, $m \geqq 3$. Then if we exclude the case $m = 3$, $n = 2$, G not free abelian, there is a structure map $\phi: K' \to T^n(K')$ whose homotopy class is uniquely determined. We describe such a structure map ϕ as *canonical.*

Now let $f: A \to X$ be a map, and let $Y = X \cup_f CA$ be obtained from X by attaching the cone CA to X by means of the map f. We prove

THEOREM 2.6. (i) *If* $\operatorname{cat} X \leqq n$, *then* $\operatorname{cat} Y \leqq n+1$.
(ii) *If* $\operatorname{w\,cat} X \leqq n$ *and Y is locally compact, then* $\operatorname{w\,cat} Y \leqq n+1$.

Let X' be the mapping cylinder of f, and $f': A \to X'$ the embedding. Let $Y' = X' \cup_{f'} CA$. Then $Y' \simeq Y$, $X' \simeq X$, and Y' has the property that there exists a deformation $k_t: Y' \to Y'$ with $k_0 = 1$, $k_1(CA) = *$. Thus we may suppose from the outset that Y itself has this property in proving (i) and (ii).

(i) We have a homotopy $h_t: X \to X^n$ with $h_0 = \Delta$, $h_1 = j\phi$. Now the inclusion $i: X \to Y$ is a cofibre map so we may extend h_t to a homotopy $l_t: Y \to Y^n$ with $l_0 = \Delta$, $l_1 i = i^n j\phi$. Define $m_t: Y \to Y^{n+1}$ by $m_t(y) = (k_t(y), l_t(y))$. Then $m_0 = \Delta$, and

$$m_1(y) = (k_1(y), l_1(y)).$$

If $y = x \in X$, then $m_1(x) = (k_1(x), l_1(x))$, and $l_1(x) \in T^n(X) \subseteqq T^n(Y)$, so that $m_1(x) \in T^{n+1}(Y)$. If $y = z \in CA$, then $m_1(z) = (k_1(z), l_1(z))$, and $k_1(z) = *$, so that $m_1(z) \in T^{n+1}(Y)$. Thus $m_1(Y) \subseteqq T^{n+1}(Y)$, and (i) is proved.

(ii) Since Y is locally compact, the identity map $Y \times Y^n \to Y^{n+1}$ induces a continuous map $s: Y \times Y^{(n)} \to Y^{(n+1)}$. We have a homotopy $g'_t: X \to X^{(n)}$ with $g'_0 = q\Delta$, $g'_1 = *$. We may extend g'_t to $g_t: Y \to Y^{(n)}$ with $g_0 = q\Delta$, $g_1 i = *$. We consider the homotopy $u_t = s \circ (k_t \times g_t) \circ \Delta_2: Y \to Y^{(n+1)}$, where $\Delta_2: Y \to Y \times Y$ is the diagonal map. Then plainly $u_0 = q\Delta$. If $x \in X$, $u_1(x) = s(k_1(x), g_1(x)) = s(k_1(x), *) = *$; if $z \in CA$, $u_1(z) = s(k_1(z), g_1(z)) = s(*, g_1(z)) = *$. Thus $u_1 = *$, and (ii) is proved.

This theorem shows that, by attaching a cone to a space, we can increase the category by at most one. Our object in this paper is to describe circumstances under which the category fails to increase. This description will be concerned with generalizations of the standard notions of primitivity (for cohomology classes) and Hopf invariant. We defer the definition of n-primitivity (of homotopy classes) till the next section and give now the required generalization of the Hopf invariant.

Let M be any space (with base point), let G be an abelian group, and let $p \geqq 2$ be an integer; we will suppose $p \geqq 3$ if G is not free abelian. We consider the exact homotopy sequence of homotopy groups with coefficients (see [2])

$$(2.7)\qquad \cdots \to \pi_{p+1}(G; M^n, T^n(M)) \xrightarrow{\partial} \pi_p(G; T^n(M)) \xrightarrow{j_*} \pi_p(G; M^n) \to \cdots .$$

PROPOSITION 2.8. *If $n > 1$, the sequence (2.7) splits. Precisely, there is a homomorphism $\kappa: \pi_p(G;\ M^n) \to \pi_p(G;\ T^n(M))$ with $j_*\kappa = 1$. Moreover, κ is natural in the sense that, for any map $f: M_0 \to M_1$,*

$$(2.9)\qquad T^n(f)_*\kappa = \kappa f^n_* .$$

For let $i_\lambda: M \to M^n$, $p_\lambda: M^n \to M$, $\lambda = 1, \cdots, n$, be the injection of and projection onto the λ^{th} factor in M^n. Let also $i'_\lambda: M \to T^n(M)$ be the injection of the λ^{th} factor. We put

$$\kappa = \sum_{\lambda=1}^n i'_{\lambda *} p_{\lambda *} .$$

Since the groups concerned are abelian, κ is a homomorphism. Also $ji'_\lambda = i_\lambda$, so that

$$j_*\kappa = j_* \sum_{\lambda=1}^n i'_{\lambda *} p_{\lambda *} = \sum_{\lambda=1}^n i_{\lambda *} p_{\lambda *} = 1.$$

The naturality of κ is now an immediate consequence of its definition.

COROLLARY 2.10. *There are natural homomorphisms*

$$\kappa: \pi_p(G;\ M^n) \to \pi_p(G;\ T^n(M)),$$

$$\omega: \pi_p(G;\ T^n(M)) \to \pi_{p+1}(G;\ M^n,\ T^n(M))$$

such that $j_*\kappa = 1$, $\omega\partial = 1$, and

$$1 = \kappa j_* + \partial\omega: \pi_p(G;\ T^n(M)) \to \pi_p(G;\ T^n(M)).$$

DEFINITION 2.11. Let cat $X \leqq n$ with structure map $\phi: X \to T^n(X)$, and let $\alpha \in \pi_p(G;\ X)$. The *Hopf ϕ-invariant* of α is the element $\mathcal{H}(\alpha) = \omega\phi_*(\alpha) \in \pi_{p+1}(G;\ X^n,\ T^n(X))$. The *crude Hopf ϕ-invariant* of α is the element $\bar{H}(\alpha) = q_*\,\mathcal{H}(\alpha) \in \pi_{p+1}(G;\ X^{(n)})$.

The reader will remark that if we take $G = Z$, $X = S^k$, $n = 2$, and $\phi: S^k \to S^k \vee S^k$ the map which pinches an equatorial S^{k-1} to a point, then $\mathcal{H}$ is a homomorphism which subsumes all the Hopf homomorphisms H_i, $i = 0, 1, 2, \cdots$, of [6] and $\bar{H}$ is just the homomorphism H^* of [5].

3. Primitive maps and category

We begin this section with the promised definition of n-primitivity.

DEFINITION 3.1. Let cat $X \leqq n$ with structure map $\phi: X \to T^n(X)$, and let $f: A \to X$ be a map. Then (i) if cat $A \leqq n$ with structure map

$\psi : A \to T^n(A)$, we say that f is *n-primitive* if the diagram

(D)
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\psi"'] & X \arrow[d, "\phi"] \\ T^n(A) \arrow[r, "T^n(f)"] & T^n(X) \end{tikzcd}$$

homotopy-commutes; and (ii) we say that f is *n-quasiprimitive* if there is some map ψ such that (D) homotopy-commutes.

Notice that in the definition of quasiprimitivity no assumption is made on the category of A; and that an n-primitive map is n-quasiprimitive. We also remark that if A is a Moore space, and if cat $X \leqq 2$ with structure map $\phi : X \to X \vee X$, then the notion of 2-primitivity for maps $A \to X$ dualizes the notion of primitivity for cohomology classes of H-spaces. Notice finally that if A and X are suspension spaces with suspension structure maps $\psi : A \to A \vee A$, $\phi : X \to X \vee X$, then f is 2-primitive if it is in a suspension class.

PROPOSITION 3.2. *If $f : A \to X$ is n-primitive and $\alpha \in \pi_p(G; A)$, then*

$$\mathcal{H}(f_* \alpha) = f^n_{**} \mathcal{H}(\alpha),$$

where f^n_{**} is the homomorphism induced by $f^n : A^n$, $T^n(A) \to X^n$, $T^n(X)$.

For, by the naturality of ω, $\omega T^n(f)_* = f^n_{**} \omega$, for any f. Thus, since f is n-primitive, $f^n_{**} \mathcal{H}(\alpha) = f^n_{**} \omega \psi_*(\alpha) = \omega T^n(f)_* \psi_*(\alpha) = \omega \phi_* f_* \alpha = \mathcal{H}(f_* \alpha)$.

PROPOSITION 3.3. *Let $A = K'(G, m - 1)$, where $m \geqq 4$ or $m = 3$ and G is free abelian. Let* cat $X \leqq n$, *where $n > 1$, and let $f : A \to X$ be a map. Then the following three statements are equivalent:* (a) $\mathcal{H}(f) = 0$; (b) *f is n-primitive*; (c) *f is n-quasiprimitive.*

Notice that, by Proposition 2.5, A has a canonical structure map $\psi : A \to T^n(A)$, so that n-primitivity is well-defined. In fact $\{\psi\} = \kappa\{\Delta\}$ since $j_* \kappa = 1$.

(a) $\Rightarrow$ (b). We are given $\omega\phi_*\{f\} = 0$. If ι is the class of the identity map $A \to A$, we have

$$\phi_*\{f\} = \kappa j_* \phi_*\{f\} = \kappa \Delta_*\{f\} = \kappa \Delta_* f_*(\iota) = \kappa f^n_* \Delta_*(\iota) = T^n(f)_* \kappa\{\Delta\},$$

by (2.9),

$$= T^n(f)_*\{\psi\}.$$

Thus f is n-primitive. Obviously (b) $\Rightarrow$ (c), so it remains to show

(c) $\Rightarrow$ (a). We are given that $\phi_*\{f\} = T^n(f)_*(\beta)$ for some $\beta \in \pi_{m-1}(G; T^n(A))$. By the cellular approximation theorem

$$j_*: \pi_{m-1}(G;\ T^n(A)) \rightarrow \pi_{m-1}(G;\ A^n)$$

is (1, 1) so that $\kappa: \pi_{m-1}(G;\ A^n) \rightarrow \pi_{m-1}(G;\ T_n(A))$ is an isomorphism. Thus $\beta = \kappa(\eta)$ for some $\eta \in \pi_{m-1}(G;\ A^n)$, and

$$\phi_*\{f\} = T^n(f)_*\,\kappa(\eta) = \kappa f^n_*(\eta).$$

Since $j_*\,\kappa = 1$, $\phi_*\{f\} = \kappa j_*\,\phi_*\{f\}$, whence $\partial\omega\phi_*\{f\} = 0$. Since ∂ is (1, 1), $\mathcal{H}(f) = 0$, and the proposition is proved.

We now return to the situation (and notation) of Theorem 2.6 and prove

THEOREM 3.4. *If $f: A \rightarrow X$ is n-quasiprimitive, $n > 1$, and if $Y = X \cup_f CA$, then* cat $Y \leqq n$. *Moreover we may structure Y with a map $\chi: Y \rightarrow T^n(Y)$ such that the inclusion $i: X \rightarrow Y$ is n-primitive.*

Now $i: X \rightarrow Y$ is a cofibration. Thus we have (see [2]) a commutative diagram

$$\begin{array}{ccccccc}
\pi(\Sigma A, T^n(Y)) & \rightarrow & \pi(Y, T^n(Y)) & \xrightarrow{i^*} & \pi(X, T^n(Y)) & \xrightarrow{f^*} & \pi(A, T^n(Y)) \\
\downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} \\
\pi(\Sigma A,\ Y^n) & \rightarrow & \pi(Y,\ Y^n) & \xrightarrow{i^*} & \pi(X,\ Y^n) & \xrightarrow{f^*} & \pi(A,\ Y^n)
\end{array}$$

(Σ = suspension) in which the horizonal rows are exact. Notice that, except on the extreme left of the diagrams, the sets $\pi(\ ,\ )$ are given no group structure. However, $j_*: \pi(\Sigma A,\ T^n(Y)) \rightarrow \pi(\Sigma A,\ Y^n)$ is a group-homomorphism. Indeed, since $n > 1$, it is an *epimorphism*; for we may still define $\kappa: \pi(\Sigma A,\ Y^n) \rightarrow \pi(\Sigma A,\ T^n(Y))$ as in 2.8. Although κ in general fails to be a homomorphism, it retains the property $j_*\,\kappa = 1$.

As observed by Puppe [8], there is an operation of $\pi(\Sigma A,\ Z)$ on $\pi(Y,\ Z)$ with the property that, for α, $\alpha' \in \pi(Y,\ Z)$, $i^*(\alpha) = i^*(\alpha')$ if and only if $\alpha' = \alpha^\xi$ for some $\xi \in \pi(\Sigma A,\ Z)$. Moreover it is easy to see that the operation[2] is natural in the sense that, for any $g: Z \rightarrow Z'$,

$$(3.5) \qquad g_*(\alpha^\xi) = (g_*\,\alpha)^{g_*\xi}, \qquad \alpha \in \pi(Y,\ Z),\quad \xi \in \pi(\Sigma A, Z).$$

After these preliminaries, we proceed to the proof of the theorem. Con-

[2] If $h: Y \rightarrow Z$ represents α and $u: \Sigma A \rightarrow Z$ represents ξ, then α^ξ is represented by $h^u: Y \rightarrow Z$ where

$$\begin{aligned}
h^u(x) &= h(x), && x \in X, \\
h^u(a,\ t) &= h(a,\ 2t) && a \in A,\quad 0 \leqq t \leqq \tfrac{1}{2}, \\
&= u(a,\ 2t - 1) && a \in A,\quad \tfrac{1}{2} \leqq t \leqq 1.
\end{aligned}$$

sider the map $u = T^n(i) \circ \phi : X \to T^n(Y)$. Then

$$\begin{aligned} f^*\{u\} &= \{T^n(i) \circ \phi \circ f\} \\ &= \{T^n(i) \circ T^n(f) \circ \psi\}, \quad \text{since } f \text{ is } n\text{-quasiprimitive}, \\ &= \{T^n(i \circ f) \circ \psi\} \\ &= \{*\}, \qquad \text{since } i \circ f \simeq 0 : A \to Y. \end{aligned}$$

Thus $\{u\} = i^*(\alpha)$, $\alpha \in \pi(Y, T^n(Y))$. Also

$$j_*\{u\} = \{j_Y \circ T^n(i) \circ \phi\} = \{i^n \circ j_X \circ \phi\} = \{i^n \circ \Delta_X\} = \{\Delta_Y \circ i\}.$$

Thus $j_*\{u\} = i^*\{\Delta\}$. It follows that $i^*\{\Delta\} = i^*j_*(\alpha)$, so that

$$\{\Delta\} = j_*(\alpha)^\eta,$$

for some $\eta \in \pi(\Sigma A, Y^n)$. However, as we have remarked, j_* maps $\pi(\Sigma A, T^n(Y))$ onto $\pi(\Sigma A, Y^n)$, so that $\eta = j_* \xi$ for some $\xi \in \pi(\Sigma A, T^n(Y))$.

Thus, finally, $\{\Delta\} = (j_* \alpha)^{j_* \xi} = j_*(\alpha^\xi)$, so that cat $Y \leqq n$. Moreover $i_*(\alpha^\xi) = i_*(\alpha) = \{u\} = \{T^n(i) \circ \phi\}$, so that if we structure Y by any map in the class α^ξ, i is n-primitive.

We may proceed from this theorem to construct examples of spaces of category 2 which are not equivalent to suspensions. We first prove a lemma which is probably well known but which we have not found in the literature.

LEMMA 3.6. *Let $\alpha \in \pi_{m-1}(S^q)$, $m \geqq q + 1 \geqq 3$, and let $Y = S^q \cup_f e^m$ be the space obtained by attaching e^m to S^q by a map f in the class α. Then Y is equivalent to a suspension if and only if α is a suspension class.*

Clearly Y is equivalent to a suspension if α is a suspension class. Also it is evident that α is a suspension class if $m = q + 1$. Thus it remains to consider the case in which $m > q + 1$ and Y is equivalent to a suspension, and to prove that α is a suspension class.

The case $q = 2$ is treated by a special argument (see 3.22), and we will suppose $q > 2$. We complete the proof by establishing first that Y, being 2-connected and equivalent to a suspension, is actually equivalent to the suspension of a 1-connected polyhedron. For suppose $Y \simeq \Sigma Z$, where Z is a 0-connected polyhedron. Then $Z' = Z/Z^1$ is a 1-connected polyhedron with

$$\begin{aligned} H_2(Z') &= H_2(Z) \oplus F, \quad \text{where } F \text{ is free abelian}, \\ H_r(Z') &= H_r(Z), \qquad r > 2. \end{aligned}$$

Now Z' admits a homology decomposition [3], initiated by $K'(H_2(Z) \oplus F, 2)$, where we may take

$$K'(H_2(Z) \oplus F, 2) = K'(H_2(Z), 2) \vee K'(F, 2).$$

Then $Z'' = Z'/K'(F, 2)$ is a 1-connected polyhedron such that the projection $Z \to Z''$ induces homology isomorphisms. It follows that $\Sigma Z \simeq \Sigma Z''$, so that $Y \simeq \Sigma Z''$, as asserted.

In our special case $Z'' \simeq S^{q-1} \cup_g e^{m-1}$ for some $g: S^{m-2} \to S^{q-1}$, so that there is a homotopy equivalence $h: S^q \cup_f e^m \simeq S^q \cup_v e^m$ where $v = \Sigma g$. Moreover we may suppose that $h(S^q) \subseteqq S^q$. If h induces $h': S^q \to S^q$, then h' is a homotopy equivalence and, of course, in a suspension class; similarly, by passing to quotients, h induces $h'': S^m \to S^m$ which is a homotopy equivalence and in a suspension class. Suppose $h'' \simeq \Sigma k$, $k: S^{m-1} \to S^{m-1}$.

Let $\tilde{f}: E^m, S^{m-1} \to S^q \cup_f e^m, S^q$; $\tilde{v}: E^m, S^{m-1} \to S^q \cup_v e^m, S^q$ be characteristic maps, let $\tilde{k}: E^m, S^{m-1} \to E^m, S^{m-1}$ extend k, and let $\tilde{q}$ stand for each of the projections $S^q \cup_f e^m \to S^m$, $S^q \cup_v e^m \to S^m$. Then $h''\tilde{q} = \tilde{q}h$ and $h''\tilde{q}\tilde{f} \simeq \tilde{q}\tilde{v}\tilde{k}$. Thus

$$\tilde{q}h\tilde{f} \simeq \tilde{q}\tilde{v}\tilde{k}: E^m, S^{m-1} \to S^m, *.$$

But $\tilde{q}_* : \pi_m(S^q \cup_v e^m, S^q) \cong \pi_m(S^m)$, so that

$$h\tilde{f} \simeq \tilde{v}\tilde{k}: E^m, S^{m-1} \to S^q \cup_v e^m, S^q.$$

Restricting the homotopy to S^{m-1} we find

$$h'f \simeq vk: S^{m-1} \to S^q.$$

Thus $f \simeq \bar{h}'vk$ where $\bar{h}'$ is a homotopy inverse of h'. Since $\bar{h}'$, v, and k are all in suspension classes, so is f, and the lemma is proved.[3]

Let us take in particular the case $q = 3$, $m = 2p + 1$, where p is an odd prime, and let $\alpha \in \pi_{2p}(S^3)$ be an element of order p. Then α is not a suspension class since $\pi_{2p-1}(S^2)$ contains no element of order p. On the other hand it follows readily from the general left distributive law [6; (6.1)] that α is primitive (i.e., 2-primitive); for $\pi_{2p}(S^{2k+1})$ contains no element of order p if $k > 1$. Thus if $f \in \alpha$, $S^3 \cup_f e^{2p+1}$ is of category 2 (Theorem 3.4) but is not equivalent to a suspension[4] (Lemma 3.6).

We now return to the hypotheses of Proposition 3.3 with a view to establishing a partial converse of Theorem 3.4. Let $A = K'(G, m - 1)$, where $m \geqq 4$ or $m = 3$ and G is free abelian. Let cat $X \leqq n$, where $n > 1$, with structure map $\phi: X \to T^n(X)$, and let $f: A \to X$ be a map. We construct $Y = X \cup_f CA$ with inclusion map $i: X \to Y$ inducing

$$i^n: X^n, T^n(X) \to Y^n, T^n(Y),$$
$$i^{(n)}: X^{(n)} \to Y^{(n)}.$$

We also have a characteristic map $\tilde{f}: CA, A \to Y, X$. Further we have a homotopy $\Delta_X \simeq j\phi: X \to X^n$, which may be extended to a homotopy $\Delta_Y \simeq \bar{\Delta}: Y \to Y^n$, where $\bar{\Delta}i = i^n j\phi = jT^n(i)\phi$. We regard $\bar{\Delta}$ as a map

$$\bar{\Delta}: Y, X \to Y^n, T^n(Y)$$

so that $\bar{\Delta} \mid X = T^n(i)\phi$. Then $\bar{\Delta}\tilde{f}: CA, A \to Y^n, T^n(Y)$ represents a certain

[3] Except in the case $q = 2$; but see 3.22.

[4] It has also been observed by E. H. Brown and A. H. Copeland that Theorem 3.4 would reveal the existence of such examples.

element $\eta_1 \in \pi_m(G; Y^n, T^n(Y))$, and $q\bar{\Delta}\tilde{f}$ represents $\eta_2 = q_* \eta_1 \in \pi_m(G; Y^{(n)})$. Notice that $\partial\eta_1 = \{T^n(i) \circ \phi \circ f\}$ so that $\partial\eta_1$, and hence η_1 and η_2, is entirely determined by ϕ and f.

PROPOSITION 3.7. (i) $\eta_1 = i^n_{**}\,\mathcal{H}(f)$;
(ii) $\eta_2 = i^{(n)}_* \bar{H}(f)$.

It is plainly sufficient to prove (i). Further it is sufficient to prove that $\partial\eta_1 = \partial i^n_{**}\mathcal{H}(f)$ or

$$T^n(i)_* \phi_*\{f\} = T^n(i)_* \partial\mathcal{H}(f). \tag{3.8}$$

Now $\partial\mathcal{H} = \partial\omega\phi_* = (1 - \kappa j_*)\phi_*$. Thus it remains to prove

$$T^n(i)_* \kappa j_* \phi_*\{f\} = 0. \tag{3.9}$$

But $T^n(i)_* \kappa = \kappa i^n_*$ and $j_* \phi_* = \Delta_*$. Thus

$$T^n(i)_* \kappa j_* \phi_*\{f\} = \kappa i^n_* \Delta_*\{f\} = \kappa \Delta_* i_*\{f\} = 0.$$

This establishes (3.9) and hence the proposition.

PROPOSITION 3.8. (i) *If* $\eta_1 = 0$, *then* cat $Y \leqq n$, *and we may structure* Y *so that the inclusion* $i: X \to Y$ *is* n*-primitive.*
(ii) *If* $\eta_2 = 0$, *then* w cat $Y \leqq n$.

(i) We simply reproduce the proof of Theorem 3.4; the only use we made of the n-quasiprimitivity of f was to conclude that $f^*\{u\} = 0$, but $f^*\{u\} = \partial\eta_1$.
(ii) We are given $q\bar{\Delta}\tilde{f} \simeq 0: CA, A \to Y^{(n)}, *$. Since $\tilde{f}$ is a relative homeomorphism, we conclude that $q\bar{\Delta} \simeq 0: Y, X \to Y^{(n)}, *$. Thus certainly $q\Delta \simeq 0: Y \to Y^{(n)}$, so that w cat $Y \leqq n$.

COROLLARY 3.9. (i) *If* $\mathcal{H}(f) = 0$, *then* cat $Y \leqq n$, *and we may structure* Y *so that* i *is* n*-primitive.*
(ii) *If* $\bar{H}(f) = 0$, *then* w cat $Y \leqq n$.

Now suppose X to be a $(q - 1)$-connected polyhedron, $m - 1 \geqq q \geqq 2$. We then bring Proposition 3.8 and Corollary 3.9 very close by showing

PROPOSITION 3.10. *Exclude the case* $n = 2$, $q = 2$, G *not free. Then*
(i) $i^n_{**}: \pi_m(G; X^n, T^n(X)) \cong \pi_m(G; Y^n, T^n(Y))$
(ii) $i^{(n)}_*: \pi_m(G; X^{(n)}) \cong \pi_m(G: Y^{(n)})$.

(Notice that no use is made of the assumption on cat X.)
This is an immediate consequence of the universal coefficient theorem for homotopy groups [2] and

LEMMA 3.11. *If* $p \leqq m - 2 + (n - 1)q$, *then*
(i) $i^n_{**}: \pi_p(X^n, T^n(X)) \cong \pi_p(Y^n, T^n(Y))$;
(ii) $i^{(n)}_*: \pi_p(X^{(n)}) \cong \pi_p(Y^{(n)})$.

We may assume that the $(q - 1)$-section of X is reduced to $*$. Then $Y^{(n)} - X^{(n)}$ has no cells of dimension $< m + (n - 1)q$, so that (ii) is proved.

Proof of (i). Let $X_{(n)} = E(X^n; T^n(X), *)$ be the space of paths on X^n beginning in $T^n(X)$ and terminating in $*$, and let $Y_{(n)}$ be defined similarly. Let EX be the space of paths on X ending in $*$, so that $\Omega X \subset EX$. Then $\pi_r(X_{(n)}) = \pi_{r+1}(X^n, T^n(X))$, and

$$(EX, \Omega X)^n = EX^n, X_{(n)}\,. \tag{3.12}$$

A similar statement holds for Y. Now $\pi_r(X) = \pi_r(Y) = 0$, $r \leqq q - 1$, and

$$\begin{aligned} i_* &: \pi_r(X) \cong \pi_r(Y), && r \leqq m - 2, \\ i_* &\, \pi_{m-1}(X) = \pi_{m-1}(Y). \end{aligned}$$

We infer by classical arguments that $H_r(EX, \Omega X) = H_r(EY, \Omega Y) = 0$, $r \leqq q - 1$, and[5]

$$\begin{aligned} i_* &: H_r(EX, \Omega X) \cong H_r(EY, \Omega Y), && r \leqq m - 2, \\ i_* &\, H_{m-1}(EX, \Omega X) = H_{m-1}(EY, \Omega Y). \end{aligned} \tag{3.13}$$

We now apply the relative Künneth formula to (3.12); this is justified since X, Y are polyhedra. Leaving the details to the reader, we infer that

$$\begin{aligned} i_* &: H_r(EX^n, X_{(n)}) \cong H_r(EY^n, Y_{(n)}), && r \leqq m - 2 + (n-1)q, \\ i_* &\, H_r(EX^n, X_{(n)}) = H_r(EY^n, Y_{(n)}), && r = m - 1 + (n-1)q. \end{aligned} \tag{3.14}$$

Thus

$$\begin{aligned} i_* &: H_r(X_{(n)}) \cong H_r(Y_{(n)}), && r \leqq m - 3 + (n-1)q, \\ i_* &\, H_r(X_{(n)}) = H_r(Y_{(n)}), && r = m - 2 + (n-1)q. \end{aligned} \tag{3.15}$$

Now $X_{(n)}$ and $Y_{(n)}$ are 1-connected. For $\pi_1(X_{(n)}) = \pi_2(X^n, T^n(X))$, and $X^n - T^n(X)$ has no cells of dimension $< nq$; the same argument applies to $Y_{(n)}$. Thus from (3.15) we infer

$$i_* : \pi_r(X_{(n)}) \cong \pi_r(Y_{(n)}), \qquad r \leqq m - 3 + (n-1)q, \tag{3.16}$$

and this establishes (i).

We may now prove a converse of Corollary 3.9(i). First we observe

PROPOSITION 3.17. *Suppose* $Y = X \cup_f CA$ *where* $A = K'(G, m-1)$ *and* cat $X \leqq n$, cat $Y \leqq n$. *Then if* $i: X \to Y$ *is n-primitive,* $i_{**}^n \mathcal{H}(f) = 0$.

We are given that $T^n(i) \circ \phi \simeq \chi \circ i$ for some $\chi: Y \to T^n(Y)$. Now $i_{**}^n \mathcal{H}(f) = \eta_1$ (3.7(i)), so that $i_{**}^n \mathcal{H}(f) = 0$ if and only if $\partial \eta_1 = 0$. But $\partial \eta_1 = \{T^n(i) \circ \phi \circ f\} = \{\chi \circ i \circ f\} = 0$, so the proposition is proved.

COROLLARY 3.18. *If in addition* X *is a* $(q-1)$*-connected polyhedron,* $m - 1 \geqq q \geqq 2$, *and if we exclude the case* $n = 2$, $q = 2$, G *not free, then if* $i: X \to Y$ *is n-primitive,* $\mathcal{H}(f) = 0$, *and* f *is n-primitive.*

[5] For notational simplicity in this argument we use i_* for any homomorphism effectively induced by i.

This is an immediate consequence of 3.3, 3.10(i), and 3.17. Again using 3.10 we now prove a converse of 3.9 under somewhat different hypotheses. We recall from Proposition 2.5 that if X is a $(q-1)$-connected polyhedron of dimension $\leqq nq-2$, then it possesses a canonical structure map $\phi: X \to T^n(X)$ which is unique up to homotopy. We prove further

THEOREM 3.19. *If X is a $(q-1)$-connected polyhedron, $m-1 \geqq q \geqq 2$, and if $\dim X \leqq nq-2$, then, provided we exclude the case $n=2$, $q=2$, G not free,*

(i) *if $\operatorname{cat} Y \leqq n$, $\mathcal{H}(f)=0$ and f is n-primitive;*

(ii) *if $\operatorname{w\,cat} Y \leqq n$, $\bar{H}(f)=0$.*

(Recall that $Y = X \cup_f CA$, $A = K'(G, m-1)$.)

(i) We know that Δ, and hence $\bar{\Delta}: Y \to Y^n$ is deformable into $T^n(Y)$. Now $\bar{\Delta}$ maps X into $T^n(Y)$, and $Y^n - T^n(Y)$ has no cells of dimension $< nq$. Thus we may choose the deformation to keep X in $T^n(Y)$ so that

$$\bar{\Delta} \simeq \bar{\bar{\Delta}}: Y, X \to Y^n, T^n(Y)$$

with $\bar{\bar{\Delta}}(Y) \subseteqq T^n(Y)$. Thus

$$\bar{\Delta}\tilde{f} \simeq \bar{\bar{\Delta}}\tilde{f}: CA, A \to Y^n, T^n(Y)$$

with $\bar{\bar{\Delta}}\tilde{f}(CA) \subseteqq T^n(Y)$. This implies that $\bar{\Delta}\tilde{f}$ represents the zero element of $\pi_m(G; Y^n, T^n(Y))$ or $\eta_1 = 0$. Apply 3.10(i).

(ii) We know that $q\Delta$, and hence $q\bar{\Delta}: Y \to Y^{(n)}$, is nullhomotopic and $q\bar{\Delta}(X) = *$. But $Y^{(n)}$ has no cells of positive dimension $< nq$, so we may choose the nullhomotopy to keep X at $*$. Thus $q\bar{\Delta} \simeq 0: Y, X \to Y^{(n)}, *$, whence $q\bar{\Delta}\tilde{f} \simeq 0: CA, A \to Y^{(n)}, *$, and $\eta_2 = 0$. Apply 3.10(ii).

It may be helpful to the reader at this stage for us to resume the conditions for the validity of the conclusions of 3.18 and 3.19. X is a $(q-1)$-connected polyhedron, $A = K'(G, m-1)$, $Y = X \cup_f CA$, $\operatorname{cat} X \leqq n$ where $n > 1$; further $m-1 \geqq q \geqq 2$. Then if G is free, the conclusion of 3.18 holds, and that of 3.19 under the additional hypothesis $\dim X \leqq nq-2$. If G is not free, we also need, for both 3.18 and 3.19, that $m \geqq 4$ and $nq > 4$.

The following special case should be mentioned explicitly.

THEOREM 3.20. *Let $S^q \cup_f e^m$ be obtained by attaching e^m to S^q by $f: S^{m-1} \to S^q$, $m-1 \geqq q \geqq 2$. Then*

(i) *the following four statements are equivalent:*

  (i.1) $\operatorname{cat} S^q \cup_f e^m \leqq 2$,

  (i.2) $\operatorname{cat} S^q \cup_f e^m \leqq 2$, *and we may structure $S^q \cup_f e^m$ so that the inclusion of S^q is primitive,*

  (i.3) $\mathcal{H}(f) = 0$,

  (i.4) *f is primitive;*

(ii) *the following two statements are equivalent:*

  (ii.1) $\operatorname{w\,cat} S^q \cup_f e^m \leqq 2$,

  (ii.2) $\bar{H}(f) = 0$.

M. G. Barratt has recently elucidated the relationship between the generalized Hopf invariants in the sense of Hilton and those defined by James. It follows from his results that $H(f) = 0$ if $\mathcal{H}(f) = 0$, where $H: \pi_{m-1}(S^q) \to \pi_{m-1}(S^{2q-1})$ is the homomorphism defined by James [7]. It thus follows from James's exact sequence that if q is even or if $\{f\}$ belongs to the 2-component of $\pi_{m-1}(S^q)$, then $\{f\} \in \Sigma\pi_{m-2}(S^{q-1})$ if $\mathcal{H}(f) = 0$. We conclude

PROPOSITION 3.21. *Let $f: S^{m-1} \to S^q$ where q is even or $\{f\}$ belongs to the 2-component of $\pi_{m-1}(S^q)$. Then f is primitive if and only if it is in a suspension class.*

COROLLARY 3.22.[6] *If $f: S^{m-1} \to S^2$, $m \geqq 4$, then* cat $S^2 \cup_f e^m \leqq 2$ *if and only if $f \simeq 0$.*

By this corollary we complete the proof of Lemma 3.6.

4. Weak and strong category

In this section we provide a recipe for constructing spaces Y such that w cat $Y \leqq 2$ but, in general, cat $Y = 3$.

Let X', X'' be two spaces, and $X' \vee X''$ their wedge. We may regard elements of $\pi_*(X')$ or $\pi_*(X'')$ as elements of $\pi_*(X' \vee X'')$ by means of the natural inclusions; we then consider elements $\xi \in \pi_*(X' \vee X'')$ which are representable as Whitehead products of elements drawn from $\pi_*(X')$ or $\pi_*(X'')$. The *length* of the representation is the number of elements in the product expression for ξ; and ξ is *pure* if it admits a representation as a product of elements all drawn from $\pi_*(X')$ or from $\pi_*(X'')$, and ξ is *mixed* otherwise. By abuse we will also talk of the length of ξ. Let $X' \# X''$ be the quotient space $X' \times X''/X' \vee X''$, and let $q: X' \times X'', X' \vee X'' \to X' \# X''$, $*$ be the identification map.

PROPOSITION 4.1. *If ξ is a mixed product, then $\xi \in \partial\pi_*(X' \times X'', X' \vee X'')$. If, moreover, ξ is of length $\geqq 3$, then $q_* \partial^{-1}\xi = 0$.*

If ξ is a mixed product, it is clearly annihilated by the projections $X' \vee X'' \to X'$, $X' \vee X'' \to X''$. It is thus annihilated by the inclusion $X' \vee X'' \to X' \times X''$ and so belongs to $\partial\pi_*(X' \times X'', X' \vee X'')$.

Now suppose ξ is mixed and of length $\leqq 3$. Then $\xi = [\alpha, \beta]$, and

(i) α or β is mixed, or

(ii) α and β are pure.

In case (i) we may suppose α mixed so that $\alpha = \partial\gamma$, $\gamma \in \pi_*(X' \times X'', X' \vee X'')$. Then $[\alpha, \beta] = \partial[\gamma, \beta]$, where $[\gamma, \beta]$ is the relative Whitehead product and $q_*[\gamma, \beta] = 0$ since $q_* \beta = 0$. In case (ii) we may suppose α of length $\geqq 2$. Then $\alpha = [\rho, \sigma]$, so that[7]

[6] We may also prove this without invoking Barratt's results; the latter may be found in the notes of the Chicago Summer Conference in Algebraic Topology, 1957.

[7] At this point we suppose dim $\rho, \sigma, \beta \geqq 2$; a mild modification of the argument sustains the conclusion in any case.

$$[\alpha, \beta] = [[\rho, \sigma], \beta] = \pm[[\rho, \beta], \sigma] \pm [[\sigma, \beta], \rho],$$

by the Jacobi identity. Since α, β are pure and $[\alpha, \beta]$ is mixed, it follows that $[\rho, \beta]$, $[\sigma, \beta]$ are mixed. We are thus effectively back in case (i), and the proposition is proved.

Now let cat $X \leqq 2$ with structure map $\phi: X \to X \vee X$. We will find it convenient to write X', X'' for the two copies of X in $X \vee X$, so that ϕ is a map $\phi: X \to X' \vee X''$. We will also write α', α'' for the copy of $\alpha \in \pi_*(X)$ in $\pi_*(X')$, $\pi_*(X'')$. Now let $\xi \in \pi_*(X)$ be representable as a Whitehead product $\xi = W(\alpha_1, \alpha_2, \cdots, \alpha_k)$, $\alpha_i \in \pi_*(X)$, where the α_i are primitive elements (i.e., represented by 2-primitive maps).

PROPOSITION 4.2. $\phi_*(\xi) = W(\alpha_1' + \alpha_1'', \cdots, \alpha_k' + \alpha_k'')$.

We first take for X the universal example space X_0 for the homotopy operation W. Thus X_0 is a union of spheres $S_1 \vee \cdots \vee S_k$, and $\phi_0: X_0 \to X_0' \vee X_0''$ is in the class $(\iota_1' + \iota_1'', \cdots, \iota_k' + \iota_k'')$, where ι_i is the class of the identity map of S_i and $(\gamma_1, \cdots, \gamma_k)$ is the class which restricts to γ_i on S_i. Then if $\xi_0 = W(\iota_1, \cdots, \iota_k)$, $\phi_{0*}(\xi_0) = (\iota_1' + \iota_1'', \cdots, \iota_k' + \iota_k'') \circ W(\iota_1, \cdots, \iota_k) = W(\iota_1' + \iota_1'', \cdots, \iota_k' + \iota_k'')$.

We now consider the general case. Put $\alpha = (\alpha_1, \cdots, \alpha_k)$, and consider the diagram

$$\begin{tikzcd} X_0 \arrow[r, "\phi_0"] \arrow[d, "\alpha"] & X_0' \vee X_0'' \arrow[d, "\alpha' \vee \alpha''"] \\ X \arrow[r, "\phi"] & X' \vee X''. \end{tikzcd}$$

The primitivity of the elements α_i corresponds precisely to the commutativity relation

$$(\alpha' \vee \alpha'') \circ \{\phi_0\} = \{\phi\} \circ \alpha.$$

Thus

$$\begin{aligned} \phi_*(\xi) = \{\phi\} \circ \alpha \circ \xi_0 &= (\alpha' \vee \alpha'') \circ \{\phi_0\} \circ \xi_0 \\ &= (\alpha' \vee \alpha'') \circ W(\iota_1' + \iota_1'', \cdots, \iota_k' + \iota_k''). \end{aligned}$$

But $(\alpha' \vee \alpha'') \circ (\iota_i' + \iota_i'') = (\alpha' \vee \alpha'') \circ \iota_i' + (\alpha' \vee \alpha'') \circ \iota_i'' = \alpha_i' + \alpha_i''$. Invoking this and the naturality of the Whitehead product, we conclude that $\phi_*(\xi) = W(\alpha_1' + \alpha_1'', \cdots, \alpha_k' + \alpha_k'')$.

We remark in passing that this proposition has some inherent interest since the expression on the right appears at first sight to depend on the choice of representation of ξ. We use Propositions 4.1 and 4.2 to prove

THEOREM 4.3. *Let* cat $X = 2$, *let* $\xi = W(\alpha_1, \cdots, \alpha_k)$ *be a Whitehead* *product of primitive elements α_i of $\pi_*(X)$ of dimension $\geqq 2$, and let $k \geqq 3$. Then if $f \in \xi$ and $Y = X \cup_f e^m$,* w cat $Y \leqq 2$.

Now

$$\phi_*(\xi) - \xi' - \xi''$$
$$= W(\alpha_1' + \alpha_1'', \cdots, \alpha_k' + \alpha_k'') - W(\alpha_1', \cdots, \alpha_k') - W(\alpha_1'', \cdots, \alpha_k'').$$

By the linearity of the Whitehead product this is a sum of mixed Whitehead products of length $k \geqq 3$. Thus $q_* \partial^{-1}(\phi_*(\xi) - \xi' - \xi'') = 0$. But $\partial^{-1}(\phi_*(\xi) - \xi' - \xi'') = \mathcal{H}(\xi)$, so that $\bar{H}(\xi) = q_* \mathcal{H}(\xi) = 0$, and we apply 3.9(ii).

On the other hand, it is clearly false in general that $\mathcal{H}(\xi) = 0$. If, for example, we take $X = S^q \vee S^q$ and $\xi = [\iota_1, [\iota_1, \iota_2]]$, then $\phi_*(\xi) - \xi' - \xi''$ is a sum of triple Whitehead products in $\pi_*(X' \vee X'')$ which, by the formula for the homotopy groups of a union of spheres, is an element of infinite order. It then follows from 3.19 that cat $X \cup_f e^m \nleqq 2$, so that, by 2.6, cat $X \cup_f e^m = 3$.

We mention here another rule for producing spaces of weak category 2 whose category exceeds 2. Let X be a connected but not simply-connected polyhedron of dimension $\leqq 2n - 1$ such that $H_r(X) = 0, 0 < r < n$, where $n > 1$. (Such a polyhedron, with $n = 2$, may be obtained by cutting an open 3-cell out of a Poincaré 3-sphere.) Since $\pi_1(X \vee X)$ injects onto $\pi_1(X \times X)$, and since $X \not\ast X$ is equivalent to the space obtained from $X \times X$ by erecting a cone on $X \vee X$, it follows from van Kampen's theorem that $X \not\ast X$ is 1-connected. On the other hand since $H_r(X) = 0, 0 < r < n$, it follows that the injection $H_r(X \vee X) \rightarrow H_r(X \times X)$ is an isomorphism if $r < 2n$. Thus $H_r(X \not\ast X) = 0, 0 < r < 2n$, so that $X \not\ast X$ is $(2n - 1)$-connected. It follows therefore that any map $X \rightarrow X \not\ast X$ is nullhomotopic, so that w cat $X = 2$. On the other hand, it may be shown by a purely algebraic argument that if cat $X = 2$, then $\pi_1(X)$ is free. This is certainly false since $\pi_1(X) \neq 1$ but $H_1(X) = 0$. Thus cat $X > 2$.

We close with a remark (touched on in the Introduction) intended to elicit interest in the concept of weak category. It is a standard result that if X is a polyhedron with cat $X \leqq n$, then n-fold cup products of elements of positive dimension of the cohomology of X are zero. This result also holds under the assumption w cat $X \leqq n$. For an element of $H^*(X; R)$ which is expressible as the n-fold cup product of elements of positive dimension certainly belongs to $(q\Delta)^* H^*(X^{(n)}; R)$. Thus such an element is zero if $q\Delta \simeq 0$. On the other hand it is clear that polyhedra X may be constructed such that n-fold cup products vanish but w cat $X > n$. Indeed the polyhedron $X = S^2 \cup_f e^5$ constitutes such an example, where $\{f\}$ generates $\pi_4(S^2)$; for then $\mathcal{H}(f) \neq 0$, $\bar{H}(f) \neq 0$, so that cat $X =$ w cat $X = 3$, but 2-fold cup products vanish.

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