A CANONICAL FORM FOR ANTIDERIVATIVES

BY

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1. Introduction

In several recent studies of devices equivalent to Schwartz distributions,[2] an essential part is played by sequences of functions $f_1, f_2, \cdots$ such that on every interval J there are antiderivatives $F_1, F_2, \cdots$ of $f_1, f_2, \cdots$ of some fixed order convergent uniformly on J. The degree of arbitrariness of the k^{th} antiderivative of f_n for $k > 1$ is somewhat inconvenient in 1-space and decidedly troublesome in spaces of higher dimension, and it is desirable to have a "canonical" expression for antiderivatives that will eliminate the arbitrariness without injuring the convergence. Such an expression does in fact exist, and is exhibited in Section 5 below. However, in the process of deriving the expression some auxiliary results were obtained that led to a new proof of the "fundamental lemma of the calculus of variations"[3] with more generality and simplicity than previous proofs. This seemed worth writing up in its own right; it appears in Section 7 of this note. It also has an application, quite apart from the calculus of variations, to "weak solutions" of differential equations as devised by Bochner.[4]

2. Notation and definitions

Points in N-dimensional space R^N will be denoted by N-tuples such as $(x^1, \cdots, x^N)$, or for brevity by single letters such as x. The superfix indicates the coordinate, and is usually omitted if $N = 1$. A subset H of R^N is a closed half space if there exist an integer j $(1 \leqq j \leqq N)$ and a real number r such that $H = \{x : x^j \geqq r\}$ or $H = \{x : x^j \leqq r\}$; it is an open half space if for some j and r we have either $H = \{x : x^j > r\}$ or $H = \{x : x^j < r\}$. A set J will be called an *intersection of half spaces* if it is nonempty and is either R^N itself or else the intersection of finitely many half spaces, without restriction as to

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[2] J. Korevaar, *Distributions defined from the point of view of applied mathematics*, Nederl. Akad. Wetensch. Proc. Ser. A, vol. 58 (1955), pp. 368–378; 379–389; 483–493; 494–503; 663–674.

J. G. Mikunsiński, *Sur la méthode de généralisation de M. Laurent Schwartz et sur la convergence faible*, Fund. Math., vol. 35 (1948), pp. 235–239.

L. Schwartz, *Théorie des distributions*, Paris, 1950–51.

[3] For a detailed study of the fundamental lemma and its uses, see the dissertation of Aline Huke (*Contributions to the Calculus of Variations*, University of Chicago, 1930, pp. 45–160; University of Chicago Press).

[4] S. Bochner, *Linear partial differential equations with constant coefficients*, Ann. of Math. (2), vol. 47 (1946), pp. 202–212.

being open or closed. In particular, all intervals in R^N are intersections of half spaces.

The letters p and q will be reserved for ordered N-tuples of nonnegative integers $(p_1, \cdots, p_N)$, $(q_1, \cdots, q_N)$, and $q \leqq p$ shall mean $q_i \leqq p_i$, $i = 1, \cdots, N$. The sum $p_1 + \cdots + p_N$ is denoted by $|p|$, and the differentiation operator

$$\frac{\partial^{|p|}}{(\partial x^1)^{p_1} \cdots (\partial x^N)^{p_N}}$$

is denoted by D^p. In particular, if $p = (0, \cdots, 0)$, then $D^p f = f$ for all functions f on subsets of R^N. A (real-valued) function f defined on a subset X of R^N is of class C^p on X if $D^q f$ exists and is continuous on X whenever $q \leqq p$; it is of class C^∞ on X if it is of class C^p on X for every p.

Given N finite sets of real numbers $\{x_0^1, \cdots, x_{p_1}^1\}, \cdots, \{x_0^N, \cdots, x_{p_N}^N\}$, we use the symbol $\{x_0, \cdots, x_p\}$ to denote their cartesian product, consisting of all the points $(x_{j_1}^1, \cdots, x_{j_N}^N)$ with $0 \leqq j_i \leqq p_i$ $(i = 1, \cdots, N)$. If for each j $(j = 1, \cdots, N)$ the numbers $x_0^j, x_1^j, \cdots, x_{p_j}^j$ are all distinct, we say that $\{x_0, \cdots, x_p\}$ "satisfies the distinctness condition." Then $\{x_0, \cdots, x_p\}$ consists of $(p_1 + 1) \cdots (p_N + 1)$ points.

If f is defined on a subset X of the real numbers, we define $\Delta(x_0)f$ to be $f(x_0)$ for all x_0 in X, and by recursion, if $x_0, \cdots, x_p$ are distinct points of X, we define

$$\Delta(x_0, \cdots, x_p)f = [\Delta(x_1, \cdots, x_p)f - \Delta(x_0, \cdots, x_{p-1})f]/(x_p - x_0).$$

If $x_0, \cdots, x_p$ are distinct numbers and j is any one of the numbers $0, \cdots, p$, we define

$$\Pi_k'(x_j - x_k) = 1(x_j - x_0) \cdots (x_j - x_{j-1})(x_j - x_{j+1}) \cdots (x_j - x_p),$$

the factors following the 1 being simply omitted if $p = 0$; in this case, $\Pi_k'(x_0 - x_k) = 1$.

An easy induction on p establishes the following lemma.

Lemma 1. *If f is defined on a set X of real numbers and $x_0, \cdots, x_p$ are distinct points of X, then*

$$\Delta(x_0, x_1, \cdots, x_p)f = \textstyle\sum_{j=0}^{p} f(x_j)/\Pi_k'(x_j - x_k).$$

It follows that $\Delta(x_0, \cdots, x_p)f$ is invariant under permutation of $x_0, x_1, \cdots, x_p$.

Let $x_1, \cdots, x_p$ be distinct real numbers. To the familiar Lagrange interpolation coefficients we adjoin one nontraditional coefficient, L_0:

$$\begin{aligned} L_j(x) &= L_j(x; x_1, \cdots, x_p) \\ &= (x - x_1) \cdots (x - x_{j-1})(x - x_{j+1}) \cdots (x - x_p)/\Pi_k'(x_j - x_k), \\ &\qquad\qquad j = 1, \cdots, p; \\ L_0(x) &= L_0(x; x_1, \cdots, x_p) = -1. \end{aligned}$$

In space of N (>1) dimensions it is convenient to use a "place-marker" Greek letter to indicate the coordinate on which a difference operator acts; thus

$$\Delta_\xi(1, 2)f(\xi, y) = f(2, y) - f(1, y),$$

while

$$\Delta_\xi(1, 2)f(x, \xi) = f(x, 2) - f(x, 1).$$

If f is defined on a subset X of R^N, and the set $\{x_0, \cdots, x_p\}$ satisfies the distinctness condition and is contained in X, we define

$$\Delta(x_0, \cdots, x_p)f$$
$$= \Delta_{\xi^1}(x_0^1, \cdots, x_{p_1}^1)\Delta_{\xi^2}(x_0^2, \cdots, x_{p_2}^2) \cdots \Delta_{\xi^N}(x_0^N, \cdots, x_{p_N}^N)f(\xi^1, \cdots, \xi^N).$$

By N applications of Lemma 1 we obtain

$$\Delta(x_0, \cdots, x_p)f$$
$$= \sum_{j_1=0}^{p_1} \cdots \sum_{j_N=0}^{p_N} f(x_{j_1}^1, \cdots, x_{j_N}^N)/\Pi'_{k_1}(x_{j_1}^1 - x_{k_1}^1) \cdots \Pi'_{k_N}(x_{j_N}^N - x_{k_N}^N).$$

Hence $\Delta(x_0, \cdots, x_p)$ is invariant under change of order of application of the N difference operators $\Delta_{\xi^j}(x_0^j, \cdots, x_{p_j}^j)$.

The Lagrange interpolation coefficients have a corresponding extension to N dimensions. If the set $\{x_1, \cdots, x_p\}$ satisfies the distinctness condition, for each $q \leqq p$ we define

$$L_q(x) = L_q(x; x_1, \cdots, x_p) = -\prod_{j=1}^N [-L_{q_j}(x^j; x_1^j, \cdots, x_{p_j}^j)].$$

(Recall that $L_0 = -1$.) This is a polynomial of degree at most $p_j - 1$ in x^j $(j = 1, \cdots, N)$, and for each j, if $q_j = 0$, then $L_q(x)$ is independent of x^j. If we divide both members of the next to the last equation by the coefficient of $f(x_0^1, \cdots, x_0^N)$, we obtain, for all sets $\{x_0, \cdots, x_1\}$ satisfying the distinctness condition,

$$\begin{aligned} f(x_0) &= \sum{}' L_q(x_0; x_1, \cdots, x_p)f(x_{q_1}^1, \cdots, x_{q_N}^N) \\ &\quad + (x_0^1 - x_1^1) \cdots (x_0^1 - x_{p_1}^1)(x_0^2 - x_1^2) \cdots (x_0^N - x_1^N) \cdots \\ &\qquad (x_0^N - x_{p_N}^N)\Delta(x_0, \cdots, x_p)f, \end{aligned} \tag{1}$$

the summation $\sum'$ extending over all $q \leqq p$ except $q = (0, 0, \cdots, 0)$.

3. A mean-value theorem

THEOREM 1. *Let $J = [a, b]$ be a nondegenerate closed interval in R^N, and let $p_1, \cdots, p_N$ be positive integers. Assume that f is continuous on J, and that $D^q f$ exists on the interior of J whenever $q \leqq p$. If $\{x_0, \cdots, x_p\}$ are points of J that satisfy the distinctness condition, there exists a point $\bar{x}$ interior to J at which*

$$D^p f(\bar{x})/(p_1!) \cdots (p_N!) = \Delta(x_0, \cdots, x_p)f.$$

First we consider the case $N = 1$. By the remark after Lemma 1, there is no loss of generality in assuming $x_0^1 < x_1^1 < \cdots < x_{p_1}^1$. (For simplicity we drop the affix "1" which labels the coordinate.) Define

$$g(x) = f(x) - \textstyle\sum_{j=1}^{p} L_j(x)f(x_j) \qquad (a \leqq x \leqq b).$$

This vanishes at $x_0, x_1, \cdots, x_p$, so by Rolle's theorem there are p distinct points (one in each of the open intervals $(x_0, x_1), \cdots, (x_{p-1}, x_p)$) at which g' vanishes. By applying Rolle's theorem again, there are $p - 1$ distinct points at which g'' vanishes, and so on; finally, there is a point $\bar{x}$ at which $g^{(p)}$ vanishes. But then by Lemma 1 and the definition of L_j

$$\begin{aligned} 0 = g^{(p)}(\bar{x})/p! &= f^{(p)}(\bar{x})/p! - \textstyle\sum_{j=1}^{p} f(x_j)/\Pi_k'(x_j - x_k) \\ &= f^{(p)}(\bar{x})/p! - \Delta(x_0, \cdots, x_p)f. \end{aligned}$$

This completes the proof for $N = 1$.

For general N we proceed by induction. Assume the theorem proved for $N = M - 1$, and let $x_0^j, \cdots, x_{p_j}^j$ be distinct points of $[a^j, b^j]$ $(j = 1, \cdots, M)$. For x^1 in $[a^1, b^1]$ define

$$\psi(x^1) = \Delta_{\xi^2}(x_0^2, \cdots, x_{p_2}^2) \cdots \Delta_{\xi^M}(x_0^M, \cdots, x_{p_M}^M)f(x^1, \xi^2, \cdots, \xi^M).$$

By the proof completed, there is an $\bar{x}^1$ in the open interval (a^1, b^1) such that

$$\begin{aligned} \Delta_{\xi^1}(x_0^1, \cdots, x_{p_1}^1)\psi(\xi^1) &= \psi^{(p_1)}(\bar{x}^1)/(p_1!) \\ &= \Delta_{\xi^2}(x_0^2, \cdots, x_{p_2}^2) \cdots \Delta_{\xi^M}(x_0^M, \cdots, x_{p_M}^M) \\ &\qquad \cdot D^{(p_1,0,\cdots,0)}f(\bar{x}^1, \xi^2, \cdots, \xi^M)/(p_1!). \end{aligned}$$

The left member is $\Delta(x_0, \cdots, x_p)f$; applying the formula to the $(M - 1)$-fold difference in the numerator of the right member yields the desired conclusion.

COROLLARY 1. *Let f be continuous on a nondegenerate closed interval J in R^N and of class C^p interior to J, where $p_1, \cdots, p_N$ are positive integers. If $x_1, \cdots, x_p$ are points of J satisfying the distinctness condition and x is in J, there exists a point $\bar{x}$ interior to J such that*

$$\begin{aligned} f(x) = {\textstyle\sum_q'} L_q(x; x_1, \cdots, x_q)f(x_{q_1}^1, \cdots, x_{q_N}^N) \\ + (x^1 - x_1^1) \cdots (x^1 - x_{p_1}^1) \cdots (x^N - x_1^N) \cdots \\ (x^N - x_{p_N}^N)D^p f(\bar{x})/p_1!\, p_2! \cdots p_N!, \end{aligned}$$

the summation $\sum'$ extending over all q such that $q \leqq p$ and $q \neq (0, 0, \cdots, 0)$.

This holds at first only if $\{x, x_1, \cdots, x_p\}$ satisfies the distinctness condition. If f is of class C^p on J, it extends by a simple continuity argument to the case in which for some j $(j = 1, \cdots, N)$, x^j coincides with some $x_{q_j}^j$. In this case the coefficient of $D^p f(\bar{x})$ is 0. So even if $D^p f$ exists only interior to J, we can apply the theorem as already proved to any C^∞ function coinciding with f at the points $x, x_1, \cdots, x_p$ and obtain the desired conclusion with x an arbitrary interior point of J.

4. Pseudopolynomials

We can now characterize those functions on N-space for which the divided differences of an assigned order vanish identically.

Let J be any set in R^N, and let $p_1, \cdots, p_N$ be positive integers. A function f on J is a *pseudomonomial of degrees less than* $p_1, \cdots, p_N$ if there are an integer j $(1 \leqq j \leqq N)$ and a nonnegative integer k such that $k < p_j$ and

$$f(x) = (x^j)^k g(x) \qquad (x \text{ in } J),$$

g being independent of x^j. A function f on J is a *pseudopolynomial of degrees less than* $p_1, \cdots, p_N$ if it is the sum of finitely many pseudomonomials of degrees less than $p_1, \cdots, p_N$.

LEMMA 2. *Let $p_1, \cdots, p_N$ be positive integers, and let $\{x_1, \cdots, x_N\}$ be a set in R^N that satisfies the distinctness condition. Let f_0 be a real-valued function defined on the $p_1 + \cdots + p_N$ hyperplanes*

$$(*) \qquad x^1 = x_1^1, \cdots, x^1 = x_{p_1}^1, \quad \cdots \quad, x^N = x_1^N, \cdots, x^N = x_{p_N}^N .$$

Then there exists a pseudopolynomial f of degrees less than $p_1, \cdots, p_N$, defined on all of R^N, and coinciding with f_0 on the hyperplanes $(*)$.

Proof. We first define f_1 to be the function on R^N which for fixed $(x^2, \cdots, x^N)$ is a polynomial of degree $p_1 - 1$ in x^1 and coincides with $f_0(x^1, \cdots, x^N)$ whenever x^1 has any of the values $x_1^1, \cdots, x_{p_1}^1$. Specifically,

$$f_1(x^1, \cdots, x^N) = \textstyle\sum_{j=1}^{p_1} L_j(x^1; x_1^1, \cdots, x_{p_1}^1) f(x_j^1, x^2, \cdots, x^N).$$

Next we define f_2 to be the function on R^N which for fixed $x^1, x^3, \cdots, x^N$ is a polynomial of degree $p_2 - 1$ in x^2 and coincides with the difference $g_1 = f_0 - f_1$ whenever x^2 has any of the values $x_1^2, \cdots, x_{p_2}^2$. Specifically,

$$f_2(x^1, \cdots, x^N) = \textstyle\sum_{j=1}^{p_2} L_j(x^2; x_1^2, \cdots, x_{p_2}^2) g_1(x^1, x_j^2, x^3, \cdots, x^N).$$

When x^1 has any of the values $x_1^1, \cdots, x_{p_1}^1$ the last factor in each term of the right member has value 0, so f_2 also vanishes. Thus $f_0 - f_1 - f_2$ vanishes on the $p_1 + p_2$ hyperplanes

$$x^1 = x_1^1, \cdots, x^1 = x_{p_1}^1, \qquad x^2 = x_1^2, \cdots, x^2 = x_{p_2}^2 .$$

We repeat this process until we finally reach an f_N which is a polynomial in x^N for fixed $x^1, \cdots, x^{N-1}$, coincides with $f_0 - f_1 - \cdots - f_{N-1}$ on the hyperplanes $x^N = x_1^N, \cdots, x^N - x_{p_N}^N$, and vanishes on the other hyperplanes $(*)$. Now we define $f = f_1 + \cdots + f_N$. This is easily seen to have the desired properties.

THEOREM 2. *Let J be a nondegenerate intersection of half spaces in R^N. Let $p_1, \cdots, p_N$ be positive integers. Let f be defined on J. Then the following three statements are equivalent*:

(i) *f is a pseudopolynomial of degrees less than $p_1, \cdots, p_N$ on J.*

(ii) *If the set $\{x_0, x_1, \cdots, x_p\}$ is contained in J and satisfies the distinctness condition, then $\Delta(x_0, \cdots, x_p)f = 0$.*

(iii) *There exists a set of points $\{x_1, \cdots, x_p\}$ in J satisfying the distinctness condition and such that whenever x_0 is in J and $\{x_0, x_1, \cdots, x_p\}$ satisfies the distinctness condition, $\Delta(x_0, \cdots, x_p)f = 0$.*

Proof. (i) $\Rightarrow$ (ii). Let g be a pseudomonomial of degrees less than $p_1, \cdots, p_N$. Then for some integer j $(1 \leqq j \leqq N)$ and some integer k such that $0 \leqq k < p_j$ the equation

$$g(x) = (x^j)^k g^*(x) \qquad (x \text{ in } J)$$

holds, g^* being independent of x^j. To this we apply the differencing operator $\Delta_{\xi^j}(x_0^j, \cdots, x_{p_j}^j)$; the result is 0, by Theorem 1. Hence $\Delta(x_0, \cdots, x_p)g = 0$. Since f is a finite sum of pseudomonomials of degrees less than $p_1, \cdots, p_N$, it follows that $\Delta(x_0, \cdots, x_p)f = 0$.

(ii) $\Rightarrow$ (iii), obviously.

(iii) $\Rightarrow$ (i). Let $x_1, \cdots, x_p$ be the set of points specified in condition (iii). By Lemma 2, there is a pseudopolynomial g of degrees less than $p_1, \cdots, p_N$ which coincides with f on the hyperplanes

$$(**) \qquad x^1 = x_1^1, \cdots, x^1 = x_{p_1}^1, \quad \cdots, \quad x^N = x_1^N, \cdots, x^N = x_{p_N}^N.$$

By the part of the proof just completed, whenever $\{x_0, \cdots, x_p\}$ satisfies the distinctness condition, the difference $\Delta(x_0, \cdots, x_p)g$ vanishes, and hence so does $\Delta(x_0, \cdots, x_p)h$, where we write h for $f - g$. By equation (1), $h(x_0)$ has value 0 whenever $x_0, \cdots, x_p$ satisfies the distinctness condition. But when this set does not satisfy the distinctness condition, x_0 must belong to one of the hyperplanes in the list (**), so in this case too we have $h(x_0) = 0$. Hence $f(x) = g(x)$ for all x in R^N.

COROLLARY 2. *The pseudopolynomial f of Lemma 2 is uniquely determined.*

For if f_1, f_2 both satisfy the requirements of Lemma 2, their difference g is a pseudopolynomial vanishing on the hyperplane (*). By Theorem 2, $\Delta(x_1, x_1, \cdots, x_p)g = 0$ whenever $x_0, \cdots, x_p$ satisfy the distinctness condition, so by equation (1) $g(x_0) = 0$ for such x_0. Every other x_0 is on one of the hyperplanes (*), which implies $g(x_0) = 0$. So $f_1 - f_2$ is zero everywhere.

It follows immediately from Theorem 2 that if $f_1, f_2, \cdots$ is a sequence of pseudopolynomials on J of degrees less than $p_1, \cdots, p_N$, and for each x in J the sequence $f_n(x)$ $(n = 1, 2, \cdots)$ has a limit $f_0(x)$ as n increases, then f_0 is a pseudopolynomial of degrees less than $p_1, \cdots, p_N$. This extends easily from sequences to Moore-Smith "nets" of pseudopolynomials. A corresponding result will now be proved for sequences (or nets) converging almost everywhere; it is not quite so superficial.

THEOREM 3. *Let J be an open intersection of half spaces in R^N, and let $p_1, \cdots, p_N$ be positive integers. If $f_1, f_2, \cdots$ is a sequence of functions on J such that for each closed interval J_0 interior to J all but finitely many of the f_n are pseudopolynomials of degrees less than $p_1, \cdots, p_N$ on J_0, and f_0 is real-valued on J, and*

$$\lim_{n\to\infty} f_n(x) = f_0(x) \tag{2}$$

for almost all x in J, then f_0 is equivalent to a pseudopolynomial of degrees less than $p_1, \cdots, p_N$ on J. If (2) holds at all points of J, f_0 is itself a pseudopolynomial of degrees less than $p_1, \cdots, p_N$ on J.

Let E be a set of measure zero such that (2) holds on $J - E$. We shall reserve the letter σ for nonempty proper subsets of the set $\{1, 2, \cdots, N\}$, and shall use $\nu(\sigma)$ to mean the number of elements in σ. Also, σ' shall mean the complementary set $\{1, \cdots, N\} - \sigma$. For each such σ there is a projection P_σ of R^N into $R^{\nu(\sigma)}$ obtained by discarding the coordinates x^j with j not in σ; that is, if $\sigma = (a, b, \cdots, h)$ with $a < b < \cdots < h$, then $P_\sigma x = (x^a, x^b, \cdots, x^h)$. Each point $(x_0^s : s \in \sigma)$ in $R^{\nu(\sigma)}$ has an inverse image under P_σ which is a $\nu(\sigma')$-dimensional flat surface in R^N, defined by the equations $x^s = x_0^s$ $(s \in \sigma)$. By Fubini's theorem, for each σ there is a set E_σ in $R^{\nu(\sigma)}$ with $\nu(\sigma)$-dimensional measure zero, such that for all x_0 in $R^{\nu(\sigma)} - E_\sigma$, the flat surface $P_\sigma^{-1} x_0$ meets E in a set of $\nu(\sigma')$-dimensional measure zero. Let E_0 be the union of E and the sets $P_\sigma^{-1} E_\sigma$ for all nonempty proper subsets σ of $\{1, 2, \cdots, N\}$. This has N-dimensional measure zero.

Now, with the integers $p_1, \cdots, p_N$ of the hypothesis, we choose any set $\{z_1, \cdots, z_p\}$ of points interior to J and satisfying the distinctness condition. By rigid translation of this set by an amount $(y^1, \cdots, y^N)$ we obtain another congruent set $\{z_1 + y, \cdots, z_p + y\}$; if the $|y^i|$ are small, these points are also interior to J. For almost all choices of $(y^1, \cdots, y^N)$ in R^N, all points of the translated set $\{z_1 + y, \cdots, z_p + y\}$ will be in the complement of E_0. Therefore we can and do choose and fix a $y = (y^1, \cdots, y^N)$ such that all the points of the translated set $\{z_1 + y, \cdots, z_p + y\}$ are interior to J and in the complement of E_0. These points we rename $\{x_1, \cdots, x_p\}$. Now whenever σ is a nonempty proper subset of $\{1, 2, \cdots, N\}$ and $c = (c^s : s \in \sigma)$ is a point of $R^{\nu(\sigma)}$ such that each c^s is one of the numbers $\{x_1^s, \cdots, x_{p_s}^s\}$, the set $P_\sigma^{-1} c$ is a flat surface of dimension $N - \nu(\sigma)$. Since $c = P_\sigma x$ for some x in the set $\{x_1, \cdots, x_p\}$, and this x is not in E_0, hence not in $P_\sigma^{-1} E_\sigma$, it follows that c is not in E_σ. Therefore $P_\sigma^{-1} c$ meets E in a set of $\nu(\sigma')$-dimensional measure zero, which we call $E(c)$. Then the set $P_{\sigma'}^{-1} P_{\sigma'} E(c)$ has N-dimensional measure zero. All these sets $P_{\sigma'}^{-1} P_{\sigma'} E(c)$, for all c such as described, we adjoin to E_0, thus forming a set E_1 of N-dimensional measure zero. If (2) holds at all points of J, the sets E, E_0, E_1 can all be taken to be empty.

Let x_0 be any point of $J - E_1$. If x is in the set $\{x_0, x_1, \cdots, x_p\}$, for each j in $\{1, \cdots, N\}$ there is a number $i(j)$ in the set $\{0, 1, \cdots, p_j\}$ such that $x^j = x^j_{i(j)}$. If all the $i(j)$ are different from 0, x is in $\{x_1, \cdots, x_p\}$, hence not in E_0. If all $i(j)$ are 0, $x = x_0$; hence x is not in E_1 and not in E_0. Otherwise, let σ be the set of integers j such that $i(j) \neq 0$, and for each j in σ let c^j be $x^j_{i(j)}$. Since $P_{\sigma'} x = P_{\sigma'} x_0$, x_0 is in $P_{\sigma'}^{-1} P_{\sigma'} x$. But x_0 is not in E_1, so x is not in $E(c)$. It is in $P_\sigma^{-1} c$, so it must be in the complement of E. Thus in all cases, x is not in E, and (2) holds at x whenever x is in $\{x_0, x_1, \cdots, x_p\}$.

Let x_0 be any point in $J - E$ such that $\{x_0, x_1, \cdots, x_p\}$ satisfies the distinctness condition; and let J_0 be a closed interval contained in J and containing the set $\{x_0, \cdots, x_p\}$. Except for finitely many values of n, f_n is a pseudopolynomial of degrees less than $p_1, \cdots, p_N$ on J_0, so by Theorem 2 we have

$$\Delta(x_0, x_1, \cdots, x_p) f_n = 0.$$

If x_0 is in $J - E_1$, (2) holds at each x in $\{x_0, x_1, \cdots, x_p\}$; this implies that $\Delta(x_0, x_1, \cdots, x_p) f_0 = 0$ for almost all x_0. Now (1) expresses f_0 as the sum of a pseudopolynomial of degrees less than $p_1, \cdots, p_N$ and a function which vanishes almost everywhere, and the proof is complete.

5. A class of auxiliary functions

For each positive integer k we define the function Y_k by

$$Y_k(x) = \begin{cases} 0 & \text{if } x \leqq 0 \\ x^{k-1}/(k-1)! & \text{if } x > 0. \end{cases}$$

Then for $k = 2, 3, \cdots$ the function Y_k is of class C^{k-2}, and $Y'_{k+1} = Y_k$.

For each set of distinct real numbers $\{x_1, \cdots, x_p\}$ we define for all real x and y

$$\begin{aligned} W(x, y) &= W(x, y; x_1, \cdots, x_p) \\ &= Y_p(x - y) - \textstyle\sum_{j=1}^p L_j(x; x_1, \cdots, x_p) Y_p(x_j - y). \end{aligned}$$

This is of class C^{p-2} in both variables. By equation (1), if

$$x \neq x_j \qquad (j = 1, \cdots, p),$$

then

$$W(x, y) = (x - x_1) \cdots (x - x_p) \Delta_\xi(x, x_1, \cdots, x_p) Y_p(\xi - y).$$

On each half line $\{x : x < 0\}$ and $\{x : x > 0\}$ the functions Y_p are of class C^∞, and $D^p Y_p = 0$. So by Theorem 1, if y is above the greatest or below the least of the numbers $x, x_1, \cdots, x_p$, the difference $\Delta_\xi(x, x_1, \cdots, x_p) Y_p(\xi - y)$ vanishes, and so does $W(x, y)$. (Alternatively, for fixed y the value of $W(x, y)$ is the error at x of Lagrange interpolation for $(Y_p(x - y) : x \text{ real})$. Except for y between the least and greatest of $x, x_1, \cdots, x_p$, this function is a polynomial of degree at most $p - 1$, and the error is 0.)

To extend this to N dimensions, let $p_1, \cdots, p_N$ be positive integers and let $\{x_1, \cdots, x_p\}$ be a set of points of R^N satisfying the distinctness condition. For each x and y in R^N we define

$$W(x, y) = W(x, y; x_1, \cdots, x_p) = \prod_{j=1}^{N} W(x^j, y^j; x_1^j, \cdots, x_{p_j}^j).$$

If each p_j is greater than 1, and $p - 2$ means $(p_1 - 2, p_2 - 2, \cdots, p_N - 2)$, then W is of class C^{p-2} in both x and y. If J is any interval containing $(x, x_1, \cdots, x_p)$, $W(x, y) = 0$ for y outside of J.

Given any ordered N-tuple $p = (p_1, \cdots, p_N)$ of nonnegative integers, we define $\mathfrak{D}^p$ to be the set of all functions φ of class C^p on R^N such that $D^q\varphi$ is bounded for each $q \leqq p$, and for each φ in $\mathfrak{D}^p$ we define the norm

$$\| \varphi \| = \sup (| D^q\varphi(x) | : q \leqq p, x \text{ in } R^N).$$

$\mathfrak{D}^p$ with this norm is a familiar complete normed linear space. For every closed interval J in R^N we define $\mathfrak{D}_J^p$ to be the set of all φ in $\mathfrak{D}^p$ that vanish outside of J. This too is a complete normed linear space. If $\{x_1, \cdots, x_p\}$ is a set of points of J satisfying the distinctness condition, and each p_j is greater than 1, the mapping which to each x in J assigns the function $(W(x, y; x_1, \cdots, x_p) : y \in R^N)$ is a mapping of J into $\mathfrak{D}_J^{p-2}$. Also, if $q \leqq p - 2$, for each fixed x in J we have for all y in J

$$D^qW(x, y) = (-1)^{|q|} \prod_{j=1}^{N} [Y_{p_j-q_j}(x^j - y^j) - \sum_{h=1}^{p_j} L_h(x^j; x_1^j, \cdots, x_{p_j}^j) Y_{p_j-q_j}(x_h^j - y^j)]. \tag{3}$$

The x^j enter only in the Lipschitzian functions $Y_{p_j-q_j}(x^j - y^j)$ and $L_h(x^j, x_1^j, \cdots, x_{p_j}^j)$, which have bounded coefficients, so the mapping from J into $\mathfrak{D}_J^{p-2}$ is Lipschitzian.

We wish to investigate the growth of $W(x, y)$ as x departs from the origin. We may assume $x_1 < x_2 < \cdots < x_p$. If $N = 1$, the greatest absolute value of $W(x, y)$ is assumed for some y in the interval (x_1, x_p), for if $x_1 \leqq x \leqq x_p$, then $W(x, y)$ vanishes outside this interval, and otherwise, on the interval between x and the nearer (say x_j) of x_1 and x_p, the function $W(x, y)$ is a multiple of $(x - y)^{p-1}$ and reaches its maximum absolute value on this interval at x_j. In equation (3) the coefficients $Y_{p-q}(x_h - y)$ are bounded for $x_1 \leqq y \leqq x_p$, so the derivative is majorized by a polynomial in x of degree at most $p - 1$. If $N > 1$, we apply this to each factor in the definition of W and find that all derivatives $D_y^q W(x, y)$ $(q \leqq p)$ are majorized by polynomials of degree at most $p_j - 1$ in x^j $(j = 1, \cdots, N)$. Hence there exists a constant k such that the norm of $(W(x, y) : y \text{ in } R^N)$ in D^{p-2} does not exceed

$$k(1 + | x^1 |^{p_1-1}) \cdots (1 + | x^N |^{p_N-1}).$$

6. A "canonical" antiderivative

The next lemma is related to the theorem of mean value with integral form of the remainder, which is in fact a limiting case of it.

LEMMA 3. *Let $x_0, x_1, \cdots, x_p$ be distinct points of a closed interval $[a, b]$ in R^1, and let k be an integer such that $1 \leqq k \leqq p$. If f is of class C^k on $[a, b]$, (or, more generally, if f is of class C^{k-1} and $f^{(k-1)}$ is absolutely continuous on $[a, b]$), then*

$$\Delta(x_0, x_1, \cdots, x_p)f = \int_{-\infty}^{\infty} [\Delta_\xi(x_0, x_1, \cdots, x_p)Y_k(\xi - y)]f^{(k)}(y)\, dy.$$

As observed in the previous section, the integrand vanishes for all y not in $[a, b]$. By repeated integration by parts,

$$\begin{aligned}
\int_a^b \Delta_\xi(x_0, \cdots, x_p)Y_k(\xi - y)f^{(k)}(y)\, dy
&= \int_a^b \Delta_\xi(x_0, \cdots, x_p)Y_1(\xi - y)f'(y)\, dy \\
&= \Delta_\xi(x_0, \cdots, x_p)\int_a^b Y_1(\xi - y)f'(y)\, dy \\
&= \Delta_\xi(x_0, \cdots, x_p)\int_a^\xi f'(y)\, dy \\
&= \Delta_\xi(x_0, \cdots, x_p)[f(\xi) - f(a)] \\
&= \Delta_\xi(x_0, \cdots, x_p)f(\xi).
\end{aligned}$$

THEOREM 4. *Let f be continuous on R^N. Let $p_1, \cdots, p_N$ be positive integers, and let the set of points $\{x_1, \cdots, x_p\}$ satisfy the distinctness condition. Then there exists a unique function F on R^N which satisfies $D^pF = f$ and vanishes on the hyperplanes*

$$x^1 = x_1^1, \cdots, x^1 = x_{p_1}^1, \quad x^2 = x_1^2, \cdots, x^2 = x_{p_2}^2, \cdots, \quad x^N = x_1^N, \cdots, x^N = x_{p_N}^N.$$

This F is continuous and is determined by the formula

$$\begin{aligned}
F(x) &= \int W(x, y; x_1, \cdots, x_p)f(y)\, dy \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} W(x^1, y^1; x_1^1, \cdots, x_{p_1}^1) \cdots W(x^N, y^N; x_1^N, \cdots, x_{p_N}^N) \\
&\qquad \cdot f(y^1, \cdots, y^N)\, dx^1 \cdots dy^N.
\end{aligned} \tag{4}$$

First consider the case $N = 1$, and omit the superfix 1 on x^1, etc. Let G be any continuous function such that $D^pG(x) = f(x)$; for example, G may be obtained by p-fold integration from some point c. Let φ be a polynomial of degree less than p which coincides with G at $x_1, \cdots, x_p$, and let $F = G - \varphi$. Then F is continuous and vanishes at $x_1, \cdots, x_p$, and $D^pF = D^pG = f$. By Lemma 3 and equation (1), if $x \neq x_1, \cdots, x_p$,

$$\begin{aligned}
F(x) &= (x - x_1)\cdots(x - x_p)\Delta(x, x_1, \cdots, x_p)F \\
&= \int_{-\infty}^{\infty} W(x, y; x_1, \cdots, x_p)D^pF(y)\, dy.
\end{aligned}$$

By continuity $F(x)$ is equal to this integral for all x. For $N > 1$ we apply this to the coordinates successively; the function F defined by (4) satisfies $D^p F = f$. To prove uniqueness, note that if F_1 and F_2 both have the required properties, then for $F = F_1 - F_2$ we have $D^p F = 0$. By Corollary 1, F vanishes identically, and $F_1 = F_2$.

COROLLARY 3. *Let J be an intersection of half spaces in R^N; let f be continuous on J; let $p_1, \cdots, p_N$ be positive integers, and let $\{x_1, \cdots, x_p\}$ be a set of points in J that satisfies the distinctness condition. Then there exists a unique function F on J which satisfies $D^p F = f$ and vanishes on the intersection of J with the hyperplanes listed in Theorem 4. This F is continuous, and is determined by equation (4).*

For each x in J, let J_x be the smallest closed interval containing the set $\{x, x_1, \cdots, x_p\}$. There is a function f_1 continuous on R^N and coinciding with f on J_x. We apply Theorem 4 to f_1.

Theorem 4 has an analogue in which f is assumed only to be summable over every interval, but in order to state this theorem it is convenient to introduce a definition. A function F on R^N is *of class* $\mathrm{AC}^{(1,\cdots,1)}$ (where the superscript is an N-tuple of 1's) provided that it is continuous, and there is a function f on R^N summable over every interval in R^N such that whenever $a^j < b^j$ $(j = 1, \cdots, N)$, the equation

$$\Delta(a, b)F = \int_{a^1}^{b^1} \cdots \int_{a^N}^{b^N} f(x)\, dx \Big/ \prod_{j=1}^{N} (b^j - a^j)$$

holds. (The integral is an N-tuple integral.) In this case F is said to be an indefinite N-tuple integral of f. If $N = 1$, f is $\mathrm{AC}^{(1)}$ if and only if it is absolutely continuous on every interval. By use of the Radon-Nikodym theorem it can be shown that a continuous function F is $\mathrm{AC}^{(1,\cdots,1)}$ if and only if the associated interval function $\Delta F(J)$, defined for each interval $J = \{x : a^1 \leqq x^1 \leqq b^1, \cdots, a^N \leqq x^N \leqq b^N\}$ to be $\prod_{j=1}^{n}(b^j - a^j)\Delta(a, b)F$, has the following property: to each interval J^* and each positive ε there corresponds a positive δ such that if $J_1, \cdots, J_k$ are subintervals of J^* whose interiors are disjoint, and whose total volume is less than δ, then $\sum_{i=1}^{k} | \Delta F(J_i) | < \varepsilon$. It is also easy to see that if F is the indefinite N-tuple integral of f, then for almost all x

$$\frac{\partial}{\partial x^1} \cdots \frac{\partial}{\partial x^N} \Delta F(a, x) = f(x),$$

the equation remaining valid if the order of the differentiation in the left member is permuted in any way.

If $p_1, \cdots, p_N$ are positive integers, a function F on R^N is *of class* $\mathrm{AC}^{(p-1)}$ if it is of class $C^{(p-1)}$ and the partial derivative $D^{p-1}F$ is of class $\mathrm{AC}^{(1,\cdots,1)}$.

If $p_1, \cdots, p_N$ are positive integers and f is a function summable over every interval in R^N, a function F will be called *an* AC^{p-1} *solution of the equation*

$$D^p F(x) = f(x) \tag{5}$$

provided that F is of class AC^{p-1} and $D^{p-1}F$ is an indefinite N-tuple integral of f.

We now state our extension of Theorem 4.

THEOREM 5. *Let f be Lebesgue-measurable on R^N and summable over every interval in R^N. Let $p_1, \cdots, p_N$ be positive integers and $\{x_1, \cdots, x_p\}$ a set of points in R^n satisfying the distinctness condition. Then there is a unique AC^{p-1} solution F of equation* (5) *which vanishes on the hyperplanes*

$$x^1 = x_1^1, \cdots, x^1 = x_{p_1}^1, \quad \cdots, \quad x^N = x_1^N, \cdots, x^N = x_{p_N}^N.$$

This solution is determined by equation (4).

To prove uniqueness, let F_1 and F_2 both satisfy the requirements of the conclusion. For every interval $\{x : a^1 \leqq x^1 \leqq b^1, \cdots, a^N \leqq x^N \leqq b^N\}$ we then have

$$\Delta(a, b)D^{p-1}F_1 = \Delta(a, b)D^{p-1}F_2 = \int_{a^1}^{b^1} \cdots \int_{a^N}^{b^N} f(x)\,dx \Big/ \prod (b^j - a^j),$$

so $\Delta(a, b)D^{p-1}[F_1 - F_2] = 0$. This is still true if we relax the condition $a^j < b^j$ $(j = 1, \cdots, N)$ and ask only that $\{a, b\}$ satisfy the distinctness condition, since interchange of a^j and b^j leaves the divided difference unchanged. Suppose first that $p = (1, \cdots, 1)$. We choose $a = x_1$ and recall that $F_1 - F_2$ vanishes on the hyperplanes $x^1 = x_1^1, \cdots, x^N = x_1^N$, and find that if $\{x_1, b\}$ satisfies the distinctness condition, then $F_1(b) - F_2(b) = 0$. If $\{x_1, b\}$ does not satisfy the distinctness condition, then b is on one of the listed hyperplanes, so $F_1(b) = F_2(b) = 0$. Thus if $p = (1, \cdots, 1)$, we have $F_1 = F_2$. Second, suppose that at least two of the numbers $p_1, \cdots, p_N$ are greater than 1. Since F_1 and F_2 vanish on the hyperplanes named in the theorem, $D^{p-1}[F_1 - F_2]$ vanishes on each of the hyperplanes. As before, we find that $D^{p-1}[F_1 - F_2]$ vanishes identically. By Corollary 1, $F_1 - F_2$ vanishes identically. This leaves only the case in which exactly one of the numbers $p_1, \cdots, p_N$ exceeds 1, say $p_1 > 1$, $p_2 = \cdots = p_N = 1$. Then the operator D^{p-1} is $\partial^{p_1-1}/(\partial x^1)^{p_1-1}$, and so $D^{p-1}[F_1 - F_2]$ vanishes on the hyperplanes $x^2 = x_1^2, \cdots, x^N = x_1^N$. If $\{x_1, x\}$ satisfies the distinctness condition,

$$0 = \Delta(x_1, x)D^{p-1}[F_1 - F_2],$$

whence

$$0 = D^{p-1}[F_1 - F_2](x^1, \cdots, x^N) - D^{p-1}[F_1 - F_2](x_1^1, x^2, \cdots, x^N).$$

So for fixed $x^2, \cdots, x^N$ the function $F_1 - F_2$ is a polynomial in x^1 of degree at most $p_1 - 1$. Since it vanishes at $x_1^1, \cdots, x_{p_1}^1$, it is identically zero, and the proof of uniqueness is complete.

Since f is summable over every interval, there exists a sequence $f_1, f_2, \cdots$ of functions continuous on R^N and having

$$\lim_{n\to\infty} \int_J |f_n(x) - f(x)|\,dx = 0$$

for every interval J in R^N. (The integral is an N-tuple Lebesgue integral.) Define F by equation (4), and define F_n to be the function obtained by substituting f_n for f in the right member of (4). Suppose now that $q_1, \cdots, q_N$ are nonnegative integers such that $q_j < p_j$ ($j = 1, \cdots, N$). By standard convergence theorems we can prove that D^qF can be computed from (4) by differentiating with respect to x under the integral sign; thus

$$D^qF(x) = \int D^qW(x, y; x_1, \cdots, x_p)f(y)\, dy,$$

where the differentiation in the integrand is with respect to the variables $x^1, \cdots, x^N$. A similar equation holds with F_n, f_n in place of F, f respectively. If J^* is any interval containing the set $\{x_1, \cdots, x_p\}$, whenever x is in J^* the function $D^qW(x, y; x_1, \cdots, x_p)$ vanishes for all x outside J^*, and for y in J^* it is bounded uniformly for all x in J^*. From this and the choice of the f_n we find

$$\lim_{n\to\infty} D^qF_n(x) = D^qF(x),$$

the convergence being uniform on every interval in R^N. This implies that D^qF is continuous whenever $0 \leqq q_j < p_j$, $j = 1, \cdots, N$.

Now let J be an interval $\{x: a^1 \leqq x^1 \leqq b^1, \cdots, a^N \leqq x^N \leqq b^N\}$. Since D^pF_n is everywhere equal to the continuous function f_n, we have readily

$$\Delta(a, b)D^{p-1}F_n = \int_{a^1}^{b^1} \cdots \int_{a^N}^{b^N} f_n(x)\, dx \Big/ \prod(b^j - a^j).$$

By virtue of the limit relation for the D^qF_n this implies

$$\Delta(a, b)D^{p-1}F = \int_{a^1}^{b^1} \cdots \int_{a^N}^{b^N} f(x)\, dx \Big/ \prod(b^j - a^j),$$

so $D^{p-1}F$ is an indefinite N-tuple integral of f. That is, the function defined by (4) is an AC^{p-1} solution of equation (5).

In contrast with Theorem 4, the requirements that F be continuous, vanish on the specified hyperplanes, and satisfy $D^pF = f$ almost everywhere are insufficient to determine F uniquely, even if N and p_1 are both 1. For there are infinitely many continuous functions ($f(x)$: $-\infty < x < \infty$) such that $f(0) = 0$ and $Df(x) = 0$ almost everywhere.

As an indication of a type of use of Theorem 4, suppose that $f_1, f_2, \cdots$ is a sequence of continuous functions on an interval J such that for some $(p_1, \cdots, p_N)$, there exists a uniformly convergent sequence $G_1, G_2, \cdots$ such that $D^pG_n = f_n$. For $j = 1, \cdots, N$, choose distinct numbers $x_1^j, \cdots, x_{p_j}^j$, and define for x in J

$$F_n(x) = \int W(x, y; x_1, \cdots, x_p)f_n(y)\, dy.$$

Then $D^pF_n = f_n$, so by Theorem 1, $\Delta(x_0, \cdots, x_p)[F_n - G_n]$ vanishes identically. By equation (1), applied to $F_n - G_n$, we find that the F_n are also uniformly convergent on J. Hence if on J any uniformly convergent sequence with $D^p G_n = f_n$ exists, our sequence $F_1, F_2, \cdots$ has this property.

7. The fundamental lemma of the Calculus of Variations

In establishing our form of the fundamental lemma of the Calculus of Variations we shall need some elementary remarks on approximation of Lebesgue-summable functions. Let δ_1 be a nonnegative function of class C^∞ on R^N that vanishes outside the sphere of radius 1 about the origin and satisfies

$$\int_{R^N} \delta_1(x)\, dx = 1;$$

and let M_1 be the maximum value of δ_1. For each positive integer n we define

$$\delta_n(x) = n^N \delta_1(nx) \qquad (x \in R^N);$$

its maximum value is $n^N M_1$.

Now let f be defined on R^N and summable over every bounded measurable set. For each x in R^N we define

$$f_n(x) = \delta_n * f(x) = \int_{R^N} \delta_n(x - y) f(y)\, dy.$$

The integral obviously exists, and it is easy to show that f_n is of class C^∞ on R^N.

It is well known that for almost all x (and in particular for each point of continuity of f), if S_r is the sphere of radius r and center x, then

$$\lim_{r\to 0}\left\{\int_{S_r} |f(y) - f(x)|\, dy \Big/ \operatorname{vol} S_r\right\} = 0. \tag{6}$$

Since vol S_r is $c_N r^N$, where c_N depends only on the dimensionality N of the space, we find

$$\left|\int_{R^N} \delta_n(x - y) f(y)\, dy - f(x)\right| \leqq \int_{R^N} |f(y) - f(x)|\, \delta_n(x - y)\, dy$$
$$\leqq \int_{S_{1/n}} |f(y) - f(x)|\, n^N M_1 \cdot dy = M_1 c_N \int_{S_{1/n}} |f(y) - f(x)|\, dy \Big/ \operatorname{vol} S_{1/n},$$

so that $\lim_{n\to\infty} f_n(x) = f(x)$ except on the set of measure 0 on which (6) fails to hold.

The next theorem is a form of the fundamental lemma of the Calculus of Variations.

THEOREM 6. *Let J be a closed interval $[a, b]$ in R^N, and let f be summable over J. Assume that there exist positive integers $p_1, \cdots, p_N$ such that for every ordered N-tuple $(\varphi_1, \cdots, \varphi_N)$ of infinitely differentiable functions of one variable with the property that for each j in $\{1, \cdots, N\}$, φ_j vanishes on neighborhoods of a^j and of b^j, it is true that*

$$\int_{a^1}^{b^1} \cdots \int_{a^N}^{b^N} [D^{p_1}\varphi_1(\xi^1)] \cdots [D^{p_N}\varphi_N(\xi^N)] f(\xi^1, \cdots, \xi^N)\, d\xi^1 \cdots d\xi^N = 0.$$

Then there exists a pseudopolynomial of degrees less than $p_1, \cdots, p_N$ *that coincides with* f *at almost all points of* J; *if* f *is continuous on* J, *this pseudopolynomial coincides with* f *on all of* J.

Proof. Let ε be a positive number less than the least of the numbers $(b^j - a^j)/2$ $(j = 1, \cdots, N)$, and for each j in $\{1, \cdots, N\}$ let φ_j be a function of class C^∞ on R^1 that vanishes outside the interval $(a^j + \varepsilon, b^j - \varepsilon)$. Extend f to all R^N by assigning it the value 0 outside J. If $y^1, \cdots, y^N$ are numbers of absolute value less than ε, for each j the function

$$(\varphi_j(x - y^j): -\infty < x < \infty)$$

is of class C^∞ and vanishes near a^j and near b^j, so by hypothesis

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} [D^{p_1}\varphi_1(\xi^1 - y^1)] \cdots [D^{p_N}\varphi_N(\xi^N - y^N)] f(\xi^1, \cdots, \xi^N)\, d\xi^1 \cdots d\xi^N = 0.$$

If $n > 1/\varepsilon$, $\delta_n(-y^1, \cdots, -y^N) = 0$ if any $| y^j |$ is as great as ε, so the product of $\delta_n(-y^1, \cdots, -y^N)$ by the left member of the above equation vanishes for all y. Hence

$$\int_{R^N} \int_{R^N} \delta_n(-y^1, \cdots, -y^N)[D^{p_1}\varphi_1(\xi^1 - y^1)] \cdots [D^{p_N}\varphi_N(\xi^N - y^N)] \cdot f(\xi^1, \cdots, \xi^N)\, d\xi^1 \cdots d\xi^N\, dy^1 \cdots dy^N = 0.$$

The integrand is 0 except on a bounded subset of $R^N \times R^N$. We change variables of integration from (ξ, y) to (ξ, η), where $\eta^j = \xi^j - y^j$ $(j = 1, \cdots, N)$, and perform the integration with respect to the ξ^j; in view of the definition of f_n this gives us

$$\int_{R^N} [D^{p_1}\varphi_1(\eta^1)] \cdots [D^{p_N}\varphi_N(\eta^N)] f_n(\eta^1, \cdots, \eta^N)\, d\eta^1 \cdots d\eta^N = 0.$$

Integration by parts p_j times with respect to η^j $(j = 1, \cdots, N)$ yields

$$\int_{R^N} \varphi_1(\eta^1) \cdots \varphi_N(\eta^N) D^p f_n(\eta^1, \cdots, \eta^N)\, d\eta^1 \cdots d\eta^N = 0. \tag{7}$$

If there were a point x^* in the interval

$$J_\varepsilon = \{x : a^j + \varepsilon \leqq x^j \leqq b^j - \varepsilon, j = 1, \cdots, N\}$$

at which $D^p f_n(x^*)$ were not 0, $D^p f_n$ would remain nonzero and of one sign on some open interval $I = \{x : \alpha^j < x^j < \beta^j, j = 1, \cdots, N\}$ contained in J_ε. By choosing φ_j to be of class C^∞, zero outside the interval (α^j, β^j) and positive inside it, we would obtain a contradiction to (7). Hence $D^p f_n$ vanishes everywhere in J_ε whenever $n > 1/\varepsilon$. If J_0 is any closed interval interior to J, it is contained in J_ε for some small positive ε. Then except for finitely many n, $D^p f_n$ vanishes on J_0, and by Corollary 1, f_n is a pseudopolynomial of degrees less than $p_1, \cdots, p_N$ on J_0. By Theorem 3, f is equivalent to a pseudopolynomial on J. If f is continuous on J, so that (6) holds everywhere in J, by Theorem 3, f is a pseudopolynomial of degrees less than $p_1, \cdots, p_N$ on the interior of J. By (1) with $\Delta(x_0, \cdots, x_p)f = 0$ we express f as a pseudopolynomial of degrees less than $p_1, \cdots, p_N$ interior to J; by continuity this remains valid on the boundary of J.

8. Weak solutions

Let n_0 be a positive integer, and for each N-tuple p such that $|p| \leqq n_0$ let a_p be a real number. Then

$$\Lambda f = \sum_{|p| \leqq n_0} a_p D^p f$$

is a linear differential operator. Following Bochner,[4] if $n \geqq n_0$, a function f on an open set D in R^N is a *weak solution of class* C^n of the equation $\Lambda f = 0$ on D if it is Lebesgue-summable over every compact subset of D, and if for each x_0 in D there exist a neighborhood U of x_0 and a sequence $f_1, f_2, \cdots$ of functions, of class C^n on U and satisfying $\Lambda f_k = 0$ on U, such that, for every function ψ bounded and measurable in U,

$$\lim_{k\to\infty} \int_U f_k(x)\psi(x)\,dx = \int_U f(x)\psi(x)\,dx. \tag{8}$$

We here consider only the very special operator D^p, where p is an N-tuple of positive integers. For this we can find the form of all weak solutions of the differential equation $D^p f = 0$.

THEOREM 7. *Let D be an open set in R^N, and let p be an N-tuple of positive integers. The following statements are equivalent:*

(i) *For some q such that $q \geqq p$, f is a weak solution of class C^q of the equation $D^p f = 0$ on D.*

(ii) *f is a weak solution of class C^∞ of $D^p f = 0$ on D.*

(iii) *Each point x_0 of D has a neighborhood U in D such that f coincides almost everywhere in U with a pseudopolynomial of degrees less than $p_1, \cdots, p_N$.*

Obviously (ii) implies (i). Suppose (i) true, and let x_0 be a point of D. By hypothesis, there are a neighborhood U of x_0 and a sequence $f_1, f_2, \cdots$ of functions of class C^q on U such that $D^p f_k = 0$ on U $(k = 1, 2, \cdots)$ and (8) holds whenever ψ vanishes outside U and is of class C^∞. Let J be a closed interval contained in U and having x_0 as interior point. If ψ is of class C^∞ and vanishes with all its derivatives on the boundary of J, by integration by parts

$$\int_J f_k(x) D^p\psi(x)\,dx = (-1)^{|p|} \int_J D^p f_k(x)\psi(x)\,dx = 0 \qquad (k = 1, 2, \cdots).$$

Since (8) holds with $D^p\psi$ in place of ψ,

$$\int_J f(x)D^p\psi(x)\,dx = 0$$

whenever ψ is of class C^∞ and vanishes with all its derivatives on the boundary of J. By Theorem 6, f coincides almost everywhere on J with a pseudopolynomial of degrees less than $p_1, \cdots, p_N$. Hence (i) implies (iii).

Assume finally that (iii) holds. Let x_0 be any point of D and U a neighborhood of x_0 on which f coincides almost everywhere with a pseudopolynomial g of degrees less than $p_1, \cdots, p_N$. We may assume that U is an open interval. Let J be a closed interval contained in U, and let ε be the distance from J to the complement of U. With the function δ_k of the preceding section we define $f_k = \delta_k * f$, that is

$$f_k(x) = \int_U \delta_k(y)f(x-y)\,dy = \int_U \delta_k(y)g(x-y)\,dy,$$

for all x in J and all k greater than $1/\varepsilon$. By Lemma 1 and Theorem 2, if $\{x_0, \cdots, x_p\}$ is a set of points in J satisfying the distinctness condition then

$$\Delta(x_0, \cdots, x_p)f_k = \int_U \delta_k(y)\Delta_\xi(x_0, \cdots, x_p)g(\xi - y)\,dy = 0.$$

Now equation (1) exhibits $f_k(x_0)$ as a pseudopolynomial of degrees less than $p_1, \cdots, p_N$ and with all coefficients $f_k(x^1_{q1}, \cdots, x^N_{qN})$ of class C^∞ in $(x^1, \cdots, x^N)$. Hence the differentiation operator D^p can be applied to f_k, yielding $D^pf_k = 0$. It remains to show that (8) holds for every function ψ bounded and measurable on J. We shall prove more than this; we shall prove that f_k tends to f in the norm of L_1 over J. By definition of f_k,

$$\begin{aligned}\int_J |f_k(x) - f(x)|\,dx &= \int_J \left| \int_U \delta_k(x-y)f(y)\,dy - f(x)\right| dx \\ &= \int_J \left| \int_U \delta_k(x-y)[f(y) - f(x)]\,dy \right| dx \\ &\leqq \int_J \int_U \delta_k(x-y)\,|f(y) - f(x)|\,dy\,dx.\end{aligned}$$

To simplify notation we assume that f vanishes outside U; any of the integrals can be written as an integral over the whole space, but the integrands vanish outside some bounded set. We change the variable of integration from (x, y) to (t, y), where $t = x - y$, $y = y$; this yields

$$\int_J |f_k(x) - f(x)|\,dx \leqq \int_U \int \delta_k(t)\,|f(y) - f(y+t)|\,dy\,dt.$$

But $\int |f(y) - f(y+t)|\,dy$ is a continuous function of t which vanishes at

$t = 0$, so the right member of the last inequality approaches zero as k increases. This completes the proof.

From the proof it is apparent that when we are dealing with the equation $D^p f = 0$, it makes no difference if we change the definition of weak solution by asking only that (8) hold for functions ψ of class C^∞, or by making the apparently stronger requirement that $\int_U | f_k - f | \, dx$ shall converge to zero.

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