# THE STRUCTURE OF UNITARY AND ORTHOGONAL QUATERNION MATRICES

BY

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## 1. Introduction

It is known that a normal quaternion matrix (and hence a unitary quaternion matrix) is unitarily similar to a diagonal matrix with complex elements [2]. It is also known that a quaternion matrix is unitarily equivalent to a diagonal matrix with nonnegative real elements [4]. Also, the transpose of a unitary quaternion matrix is not necessarily unitary; a necessary and sufficient condition that the transpose, $V^T$, of a unitary quaternion matrix $V$ be unitary is that there exist real orthogonal matrices $U$ and $W$ such that $UVW = D$ is a diagonal quaternion matrix [5].

In the present work two theorems are obtained concerning the structure of unitary and orthogonal quaternion matrices, respectively. An orthogonal quaternion matrix, $P$, is defined to be a matrix such that $PP^T = I\ (= P^TP)$, where $P^T$ denotes the transpose of $P$. In each case the essential "quaternion character" of the matrix is clearly revealed by the form obtained; and in the unitary case the form obtained gives more meaning to the above quoted theorem concerning the transpose of a unitary matrix.

## 2. The structure of a unitary matrix

The following theorem will be obtained:

THEOREM 1. *Every quaternion unitary matrix $P$ can be written in the form $P = UDW$, where $U$ and $W$ are complex unitary matrices and $D$ is a quaternion diagonal unitary matrix; conversely, every matrix of this form is a quaternion unitary matrix.*

Let $P = P_1 + jP_2$ (where $P_1$ and $P_2$ have complex elements) be a unitary quaternion matrix. Then, since $PP^{CT} = I = P^{CT}P \quad (P^{CT} = P_1^{CT} - jP_2^T$ denotes the quaternion-conjugate transpose of $P$), the following hold:

$$P_1P_1^{CT} + P_2^CP_2^T = I = P_1^{CT}P_1 + P_2^{CT}P_2\,,$$

$$P_2P_1^{CT} - P_1^CP_2^T = 0 = P_1^TP_2 - P_2^TP_1\,.$$

By a known theorem [1] for the complex matrix $P_1$ there exist two complex unitary matrices $U_1$ and $W_1$ such that $U_1P_1W_1 = D$ is a real diagonal matrix with nonnegative elements along the diagonal. There is no loss in generality in assuming that like diagonal elements are arranged together so that $D = D_1 \dotplus D_2 \dotplus \cdots \dotplus D_k$ where $D_i = c_iI_i$ where $c_i$ is nonnegative and real,

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where $c_i \neq c_j$ if $i \neq j$, and where $c_k = 0$ if zeros are present on the diagonal. From $P_2 P_1^{CT} = P_1^C P_2^T$ and $P_1^T P_2 = P_2^T P_1$ there follow:

$$U_1^C P_2 W_1 W_1^{CT} P_1^{CT} U_1^{CT} = U_1^C P_1^C W_1^C W_1^T P_2^T U_1^{CT},$$

$$W_1^T P_1^T U_1^T U_1^C P_2 W_1 = W_1^T P_2^T U_1^{CT} U_1 P_1 W_1 .$$

Set $U_1^C P_2 W_1 = M$; then the above become $MD = DM^T$ and $DM = M^T D$. Therefore $MD^2 = MDD = DM^T D = DDM = D^2 M$, so that since $D^2 = D_1^2 \dotplus D_2^2 \dotplus \cdots \dotplus D_k^2$, it follows that $M = M_1 \dotplus M_2 \dotplus \cdots \dotplus M_k$, where $M_i$ has the same order as $D_i$. Because of the nature of $D$, it follows that $MD = DM$ where $U_1(P_1 + jP_2)W_1 = U_1 P_1 W_1 + jU_1^C P_2 W_1 = D + jM$.

Since $MD = DM^T$, $M_i(c_i I) = (c_i I)M_i$ so that $M_i = M_i^T$ for $i = 1, 2, \cdots, k - 1$ where each $M_i$ is a complex matrix. Also, from the above

$$U_1(P_1 P_1^{CT} + P_2^C P_2^T)U_1^{CT} = D^2 + M^C M^T = I,$$

$$W_1^{CT}(P_1^{CT} P_1 + P_2^{CT} P_2)W_1 = D^2 + M^{CT} M = I,$$

so that $M^{CT}M = I - D^2 = M^C M^T$ is a real diagonal matrix and hence symmetric. Therefore $M^{CT}M = M^C M^T = (M^C M^T)^T = MM^{CT}$, and so $M_i^{CT} M_i = M_i M_i^{CT}$ for $i = 1, 2, \cdots, k - 1, k$, where $M_k^{CT} M_k = I$. Therefore, for $i = 1, 2, \cdots, k - 1$, $M_i$ is a complex, normal, and symmetric matrix such that $M_i M_i^{CT} = I - c_i^2 I$.

Let $A + iB$ be a complex matrix with these properties (where $A$ and $B$ are real matrices). Then $A^T + iB^T = A + iB$ implies that $A = A^T$ and $B = B^T$. Since

$$\begin{aligned}(A + iB)(A^T - iB^T) &= (A + iB)(A - iB) \\ &= A^2 + B^2 + i(BA - AB) = (1 - c^2)I\end{aligned}$$

is a real scalar matrix, $AB = BA$. Two commutative real symmetric matrices can be diagonalized by the same real orthogonal matrix $S$, and so

$$S(A + iB)S^T = D_a + iD_b$$

where $D_a$ and $D_b$ are real diagonal matrices. Therefore, for $M_i$, $i = 1, 2, \cdots, k - 1$, there exists a real orthogonal $S_i$ such that $S_i M_i S_i^T = D_i'$ is complex diagonal. Since $M_k$ is unitary and since $D_k = 0$, there exists a complex unitary matrix $S_k$ such that $S_k M_k S_k^{CT} = D_k'$ is complex diagonal. Form

$$V_1 = S_1 \dotplus S_2 \dotplus \cdots \dotplus S_{k-1} \dotplus S_k^C ,$$

$$V_2 = S_1^T \dotplus S_2^T \dotplus \cdots \dotplus S_{k-1}^T \dotplus S_k^{CT}.$$

$V_1$ and $V_2$ are complex unitary matrices tuch that $V_1 U_1(P_1 + jP_2)W_1 V_2 = V_1(D + jM)V_2 = D + V_1 jMV_2 = D + jD'$, where $D' = D_1' \dotplus D_2' \dotplus \cdots \dotplus D_k'$ is complex diagonal, and $V_1 U_1$ and $W_1 V_2$ are complex unitary matrices.

It should be noted that in the above if $P_2 = 0$, then $U_1 P_1 W_1 = D = I$ (and $P_1$ is complex unitary), so that the above is essentially concerned with the case where $P_2 \neq 0$. The converse follows immediately.

## 3. The structure of an orthogonal matrix

For this case the following holds:

THEOREM 2. *Every orthogonal quaternion matrix $P$ can be written in the form $U(I \dotplus C)W$ where $U$ and $W$ are real orthogonal matrices, where $I$ is an identity matrix, and where $C$ is a direct sum of $2 \times 2$ matrices of the form*

$$\begin{bmatrix} q & b \\ -b & q \end{bmatrix}$$

*where $b$ is real and nonzero and $q$ is a nonzero quaternion of the form*

$$a_1 i + a_2 j + a_3 ij$$

*where $a_1$, $a_2$, and $a_3$ are real, and $q^2 + b^2 = 1$; conversely, every matrix of this form is a quaternion orthogonal matrix.*

Let $P = P_1 + jP_2$ be an orthogonal quaternion matrix so that

$$(P_1 + jP_2)\ (P_1^T + jP_2^T) = I = (P_1^T + jP_2^T)\ (P_1 + jP_2),$$

where $P_1$ and $P_2$ are complex matrices. As a result,

$$P_1 P_1^T - P_2^C P_2^T = I = P_1^T P_1 - P_2^{CT} P_2\,,$$

$$P_2 P_1^T + P_1^C P_2^T = 0 = P_2^T P_1 + P_1^{CT} P_2\,.$$

Let $P_2 = T_1 + iT_2$ where the $T_i$ are real matrices.

Since $P_2^C P_2^T = P_1 P_1^T - I$, $P_2^C P_2^T$ is a complex matrix which is hermitian and symmetric so that $P_2^C P_2^T$ is a real matrix.

$$P_2^C P_2^T = (T_1 - iT_2)\ (T_1^T + iT_2^T) = T_1 T_1^T + T_2 T_2^T + i(T_1 T_2^T - T_2 T_1^T)$$

is real and so $T_1 T_2^T - T_2 T_1^T = 0$. Since $P_2^{CT} P_2$ is also real, $T_1^T T_2 = T_2^T T_1$.

Consider $P_2$ first. According to the above-mentioned result due to Eckert and Young [1] if $U$ is a unitary matrix which diagonalizes $AA^{CT}$ (where all matrices are complex), there exists a unitary matrix $V$ such that $UAV = D$ is a diagonal matrix with nonnegative real elements. If $A$ is itself real, $U$ and $V$ may be taken to be real orthogonal. ($U$ can be real since $AA^T$ is real symmetric; if $V = V_1 + iV_2$, and if $UAV = D$, then $UA = DV_1^T - iDV_2^T$ so $DV_2^T = 0$, and if the first $r$ diagonal elements of $D$ are not zero while the last $n - r$ diagonal elements are zero, only the last $n - r$ rows of $V_2^T$ may be nonzero so that the first $r$ rows of $V^{CT}$ are real. Using these rows as the first $r$ rows, a new real orthogonal matrix $W$ can be constructed so that $UA = DW$.) Let $U$ and $W$ be real orthogonal matrices such that

$$UT_2 W = D_1 = c_1 I_1 \dotplus c_2 I_2 \dotplus \cdots \dotplus c_k I_k$$

where $c_i$ are real, $c_i \neq c_j$ for $i \neq j$ and $c_k = 0$ if zero appears on the diagonal. From the relations above involving $T_1$ and $T_2$ it follows that $UT_1 WW^T T_2^T U^T = UT_2 WW^T T_1^T U^T$ and $W^T T_1^T U^T UT_2 W = W^T T_2^T U^T UT_1 W$ or, if $UT_1 W = M$, $MD_1 = D_1 M^T$ and $M^T D_1 = D_1 M$. As before, this means $MD_1^2 = D_1^2 M$ and also $MD_1 = D_1 M$ where $UP_2 W = M + iD_1$. Again, as before, if $M = M_1 \dot{+} M_2 \dot{+} \cdots \dot{+} M_k$, $M_i = M_i^T$ for $i = 1, 2, \cdots, k - 1$. Let $U_i$ be a real orthogonal matrix which diagonalizes $M_i$, $i = 1, 2, \cdots, k - 1$, and let $U_k$ and $U_k'$ be real orthogonal matrices such that $U_k M_k U_k'$ is diagonal with nonnegative real elements. Let

$$V_1 = U_1 \dot{+} U_2 \dot{+} \cdots \dot{+} U_{k-1} \dot{+} U_k,$$

$$V_2 = U_1^T \dot{+} U_2^T \dot{+} \cdots \dot{+} U_{k-1}^T + U_k'.$$

Then $V_1 UP_2 WV_2 = V_1(M + iD_1)V_2 = V_1 MV_2 + iD_1 = D_2 + iD_1$ where $V_1 U$ and $WV_2$ are real orthogonal matrices and $D = D_2 + iD_1$ is a complex diagonal matrix.

If $d_i$ and $d_j$ are two diagonal elements of $D$ such that $d_i^2 = d_j^2$, then $(d_i - d_j)(d_i + d_j) = 0$ and either $d_i = d_j$ or $d_i = -d_j$. If $d_i$ and $-d_i$ appear in $D$, any of the latter can be changed into $d_i$ by multiplying $D$ (on the right, say) by a (real orthogonal) diagonal matrix with $+1$ and $-1$ properly placed along the diagonal. Also, like diagonal elements may be grouped together so there exist real orthogonal matrices $V_3$ and $V_4$ such that $V_3 P_2 V_4 = D = d_1 I_1 \dot{+} d_2 I_2 \dot{+} \cdots \dot{+} d_t I_t$, $d_i$ complex, $d_i \neq d_j$ for $i \neq j$, $d_i^2 \neq d_j^2$ for $i \neq j$, and $d_t = 0$ if present.

Next consider $P_1$. From the relations $P_2 P_1^T = -P_1^C P_2^T$ and $P_2^T P_1 = -P_1^{CT} P_2$ there follow $V_3 P_2 V_4 V_4^T P_1^T V_3^T = -V_3 P_1^C V_4 V_4^T P_2^T V_3^T$ and

$$V_4^T P_2^T V_3^T V_3 P_1 V_4 = -V_4^T P_1^{CT} V_3^T V_3 P_2 V_4.$$

Let $V_3 P_1 V_4 = N$; then $DN^T = -N^C D$ and $DN = -N^{CT} D$. From the former $ND = -DN^{CT}$ and so $D^2 N = DDN = -DN^{CT} D = ND^2$. Therefore $N = N_1 \dot{+} N_2 \dot{+} \cdots \dot{+} N_t$ where $N_i$ has the same order as $I_i$ from the nature of $D$ and $D^2$, so that $DN = ND$. Since $P_1 P_1^T - P_2^C P_2^T = I = P_1^T P_1 - P_2^{CT} P_2$, $V_3 P_1 V_4 V_4^T P_1^T V_3^T - V_3 P_2^C V_4 V_4^T P_2^T V_3^T = I$ and

$$V_4^T P_1^T V_3^T V_3 P_1 V_4 - V_4^T P_2^{CT} V_3^T V_3 P_2 V_4 = I$$

or $NN^T - D^C D = I$ and $N^T N - D^{CT} D = I$ and so $NN^T = N^T N = I + D^C D$ is a real diagonal matrix (and $N$ and $P_1$ are therefore always nonsingular, incidentally). For $i = 1, 2, \cdots, t$, $N_i N_i^T = N_i^T N_i = I_i + d_i \bar{d}_i I_i = r_i I_i$ where $r_i$ is a positive real number. From this relation it follows that if $N_i = A_i + iB_i$, where $A_i$ and $B_i$ are real, the matrix coefficient of $i$ in the products $N_i N_i^T$ and $N_i^T N_i$ ard zero so that $A_i B_i^T = -B_i A_i^T$ and $B_i^T A_i = -A_i^T B_i$ for $i = 1, 2, \cdots, t$. Note also that $A_i A_i^T - B_i B_i^T = A_i^T A_i - B_i^T B_i =$ a diagonal matrix with positive real elements along the diagonal.

Consider the two cases:

(a) $N_i$, for $i = 1, 2, \cdots, t - 1$, is such that $d_i I_i N_i = -N_i^{CT} d_i I_i$ (since $DN = -N^{CT} D$) so that $N_i = -N_i^{CT} = A_i + iB_i = -(A_i^T - iB_i^T)$ from which $A_i = -A_i^T$ and $B_i = B_i^T$. In this case (dropping subscripts momentarily) $AB = BA$. If $R$ is a real orthogonal matrix such that $RBR^T =$

$b_1 I \dotplus b_2 I_2 \dotplus \cdots \dotplus b_s I_s$, $b_i$ real, $b_i \neq b_j$ for $i \neq j$, and $b_s = 0$ (if zeros are present along the diagonal), then $RAR^T = A_1 \dotplus A_2 \dotplus \cdots \dotplus A_s$ where $A_i$ and $I_i$ are of the same order. For each $A_i$ there exists (see [3], Theorem 9.27, for example) a real orthogonal matrix $S_i$ such that $S_i A_i S_i^T$ is a direct sum of zero elements and $2 \times 2$ matrices of the form

$$\text{(i)} \qquad \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$$

where $b$ is real. If $S = S_1 \dotplus S_2 \dotplus \cdots \dotplus S_s$, then $Q = SR$ is a real orthogonal matrix such that $Q(A + iB)Q^T = S_1 A_1 S_1^T \dotplus \cdots \dotplus S_s A_s S_s^T + iRBR^T$. For each $N_i$ there exists such a real orthogonal matrix $Q_i$, $i = 1, 2, \cdots, t - 1$. (It will be seen below that each $S_i A_i S_i^T$ here must be a direct sum of $2 \times 2$ matrices of the form (i).)

(b) $N_t$ is such that $N_t^T N_t = N_t N_t^T = I$ and, as above, $A_t B_t^T = - B_t A_t^T$ and $B_t^T A_t = - A_t^T B_t$. As before, it can be shown that there exist real orthogonal matrices $W_1$ and $W_2$ such that $W_1 B_t W_2 = h_1 I_1 \dotplus h_2 I_2 \dotplus \cdots \dotplus h_p I_p$, $h_i$ real, $h_i \neq h_j$ for $i \neq j$, and $h_p = 0$, if present, while $W_1 A_t W_2 = C_1 \dotplus C_2 \dotplus \cdots \dotplus C_p$ where $C_i = - C_i^T$ for $i = 1, 2, \cdots, p - 1$ and $C_p$ is real where $C_p^T C_p = I$. $C_p$ is then real orthogonally equivalent to an identity matrix, and each $C_i$ for $i \neq p$ can be brought under a real orthogonal similarity transformation into a direct sum of $2 \times 2$ matrices of type (i) and zero elements. There exist, then, real orthogonal matrices $Q_t$ and $Q_t'$ such that $Q_t N_t Q_t' = D_a + iD_b$ where $D_b$ is real and diagonal and $D_a$ is a direct sum of matrices of form (i), of zero elements, and of $+1$'s.

Now $V_3(P_1 + jP_2)V_4 = N + jD$. Set

$$V_5 = Q_1 \dotplus Q_2 \dotplus \cdots \dotplus Q_{t-1} \dotplus Q_t,$$

$$V_6 = Q_1^T \dotplus Q_2^T \dotplus \cdots \dotplus Q_{t-1}^T \dotplus Q_t';$$

then $V_5 V_3(P_1 + jP_2)V_4 V_6 = Z + jD$ where $V_5 V_3$ and $V_4 V_6$ are real orthogonal matrices, $D$ is complex diagonal, and $Z$ is a direct sum of $2 \times 2$ matrices of the form

$$\text{(ii)} \qquad \begin{bmatrix} ai & b \\ -b & ai \end{bmatrix}$$

(where $a$ and $b$ are real), of $+1$'s, and of elements $ci$, $c$ real. But the latter cannot appear. For $NN^T = I + D^C D$ and $V_5 NN^T V_5^T = ZZ^T = I + D^C D$ which would mean that $(ci)^2 = - c^2 = 1 + \bar{d}d$, which is not possible. The diagonal elements of $D$ which correspond to any matrix (ii) are alike and the form as described in the theorem is now obtainable. If $P$ is real, the form obtained is the identity matrix $I$. If $P$ is complex, the form is a direct sum of $+1$'s and matrices of the form (ii).

The converse follows immediately.

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