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ARONSSON'S EQUATIONS ON CARNOT–CARATHÉODORY SPACES

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ABSTRACT. Let $(\mathbf{R}^n, d_X)$ be a Carnot–Carathéodory metric space generated by a family of smooth vector fields $\{X_i\}_{i=1}^m$ satisfying Hörmander's finite rank condition, and $\mathcal{H}_X = \{(x, \sum_{i=1}^m a_i X_i(x)) | x \in \mathbf{R}^n, (a_i)_{i=1}^m \in \mathbf{R}^m\}$ be the horizontal tangent bundle generated by $\{X_i\}_{i=1}^m$. Assume that $H = H(x,p) \in C^1(\mathcal{H}_X)$ is quasiconvex in p-variable. We prove that any absolute minimizer $u \in W_X^{1,\infty}(\Omega)$ to $F_\infty(v,\Omega) = \operatorname{ess\,sup}_{x\in\Omega} H(x, Xv(x))$ is a viscosity solution of the Aronsson equation

$$\mathcal{A}^X[u] := X(H(x, Xu(x))) \cdot H_p(x, Xu(x)) = 0 \quad \text{in } \Omega.$$

1. Introduction

For $1 \le m, n$, let $\{X_i\}_{i=1}^m \subset C^\infty(\mathbf{R}^n, \mathbf{R}^n)$ be a family of smooth vector fields satisfying Hörmander's finite rank condition, i.e., there is an integer $r \ge 1$ such that $\{X_i\}_{i=1}^m$ and their commutators up to order r span $\mathbf{R}^n$ everywhere. For $x \in \mathbf{R}^n$, let

$$\mathcal{H}(x) = \operatorname{span}\{X_1(x), \dots, X_m(x)\}$$

be the horizontal tangent space at x. Let

$$\mathcal{H}_X = \{(x, \mathcal{H}(x)) | x \in \mathbf{R}^n\}$$

be the subbundle of the tangent bundle $T\mathbf{R}^n$ generated by $\{X_i\}_{i=1}^m$, called a horizontal tangent bundle. Endow an inner product on $\mathbf{R}^n$ such that $\{X_i\}_{i=1}^m$ be an orthonormal set. Recall that an absolutely continuous curve $\xi : [0,T] \to \mathbf{R}^n$ is a *horizontal* curve, if there are measurable functions

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$a_i(t) : [0,T] \to \mathbf{R}$, $1 \le i \le m$, such that

$$\sum_{i=1}^{m} a_i^2(t) = 1, \qquad \xi'(t) = \sum_{i=1}^{m} a_i(t) X_i(\xi(t)) \quad \text{for a.e., } x \in [0,T]. \tag{1.1}$$

It is readily seen from (1.1) that $t \in [0,T]$ is the arclength parameter of ξ, whose length is T. Since $\{X_i\}_{i=1}^m$ satisfies Hörmander's condition, it is well known (cf. Nagel–Stein–Wainger [NSW]) that there exists at least one *horizontal* curve joining any pair of points in $\mathbf{R}^n$. Hence, we can introduce the Carnot–Carathéodory distance (cf. [NSW]):

$$\begin{aligned} d_X(x,y) = \inf\{T \ge 0 | \exists \text{ a horizontal curve } \xi : [0,T] \to \mathbf{R}^n \\ \text{with } \xi(0) = x, \xi(T) = y\} \end{aligned} \tag{1.2}$$

for any $x, y \in \mathbf{R}^n$. Moreover, for any compact set $K \subset \mathbf{R}^n$, there exists $C_K > 0$ such that

$$C_K^{-1} \|x-y\| \le d_X(x,y) \le C_K \|x-y\|^{\frac{1}{r}} \quad \forall x, y \in K, \tag{1.3}$$

where $\|\cdot\|$ is the Euclidean distance on $\mathbf{R}^n$.

Typical examples of Carnot–Carathéodory metric spaces include (i) the Euclidean space $(\mathbf{R}^n, \|\cdot\|)$ generated by $\{\frac{\partial}{\partial x_i}\}_{i=1}^n$, and (ii) the Heisenberg group $\mathbf{H}^n \equiv \mathbf{C}^n \times \mathbf{R}$, the simplest Carnot group of step two, endowed with the group law:

$$\begin{aligned} (z,t) \cdot (z',t') = \left(z_1 + z_1', \ldots, z_n + z_n', t + t' + 2\,\mathrm{Im}\left(\sum_{i=1}^n z_i \bar{z}_i' \right) \right) \\ \forall (z,t), (z',t') \in \mathbf{C}^n \times \mathbf{R}, \end{aligned}$$

whose Lie algebra $h = V_1 + V_2$ with $V_1 = \mathrm{span}\{X_i, Y_i\}_{1 \le i \le n}$ and $V_2 = \mathrm{span}\{T\}$, where

$$X_i = \frac{\partial}{\partial x_i} = 2y_i \frac{\partial}{\partial t}, \qquad Y_i = \frac{\partial}{\partial y_i} + 2x_i \frac{\partial}{\partial t}, \quad 1 \le i \le n, \qquad T = 4\frac{\partial}{\partial t}.$$

For any bounded domain $\Omega \subset \mathbf{R}^n$ and $u : \Omega \to \mathbf{R}$, denote by $Xu := (X_1 u, \ldots, X_m u)$ the horizontal gradient of u. The horizontal Sobolev space, $W_X^{1,\infty}(\Omega)$, is defined by

$$W_X^{1,\infty}(\Omega) := \left\{ u : \Omega \to \mathbf{R} \middle| \|u\|_{W_X^{1,\infty}(\Omega)} \equiv \|u\|_{L^\infty(\Omega)} + \|Xu\|_{L^\infty(\Omega)} < +\infty \right\},$$

and the horizontal Lipschitz space is defined by

$$\mathrm{Lip}_X(\Omega) := \left\{ u : \Omega \to \mathbf{R} \middle| \|u\|_{\mathrm{Lip}_X(\Omega)} \equiv \sup_{x,y \in \Omega, x \ne y} \frac{|u(x) - u(y)|}{d_X(x,y)} < +\infty \right\}.$$

It is known (cf. Garofalo–Nieu [GN], Franchi–Serapioni–Serra [FSS]) that $u \in W_X^{1,\infty}(\Omega)$ iff $u \in \mathrm{Lip}_X(\Omega)$.

Definition 1.1. For a continuous function $H \in C(\mathcal{H}_X)$, define the L^∞-functional

$$F_\infty(v,\Omega) = \operatorname{ess\,sup}_{x\in\Omega} H(x, Xv(x)) \quad \forall v \in W^{1,\infty}_X(\Omega).$$

A function $u : \Omega \to \mathbf{R}$ is an *absolute minimizer* of H if, for any $U \subset\subset \Omega$, $u \in W^{1,\infty}_X(U)$ and

$$F_\infty(u,U) \le F_\infty(v,U) \quad \forall v \in W^{1,\infty}_X(U), \qquad v = u \quad \text{on } \partial U. \tag{1.4}$$

Formal calculations yield that an absolute minimizer $u : \Omega \to \mathbf{R}$ of H satisfies the (subelliptic) Aronsson equation:

$$\mathcal{A}^X[u] := \sum_{i=1}^m X_i(H(x, Xu(x))) \cdot H_{p_i}(x, Xu(x)) = 0 \quad \text{in } \Omega. \tag{1.5}$$

Let $\mathcal{S}^m$ be the set of symmetric $m \times m$ matrices, equiped with the usual order. Note that the Aronsson operator $\mathcal{A}^X : \Omega \times \mathbf{R}^m \times \mathcal{S}^m \to \mathbf{R}$ given by

$$\mathcal{A}^X(x,p,M) = \sum_{i,j=1}^m H_{p_i}(x,p)H_{p_j}(x,p)M_{ij} + \sum_{i=1}^m X_iH(x,p)H_{p_i}(x,p)$$

is *degenerately elliptic*, i.e., for any $(x,p) \in \Omega \times \mathbf{R}^m$,

$$\mathcal{A}^X(x,p,M) \le \mathcal{A}^X(x,p,N) \quad \forall M,N \in \mathcal{S}^m, \text{ with } M \le N. \tag{1.6}$$

Therefore, we can adapt the notion of viscosity solutions by Crandall–Lions [CL] (cf. also [CIL]) to define the following definition.

Definition 1.2. A function $u \in C(\Omega)$ is a viscosity subsolution (or supersolution, resp.) of (1.5), if for any $(x_0, \phi) \in \Omega \times C^2(\Omega)$ such that

$$0 = (\phi - u)(x_0) \le (\text{or } \ge) (\phi - u)(x) \quad \forall x \in \Omega,$$

then $\mathcal{A}^X[\phi](x_0) \ge (\text{or } \le) 0$. A function $u \in C(\Omega)$ is a viscosity solution of (1.5) if it is both a viscosity subsolution and a viscosity supersolution of (1.5).

Definition 1.3. A function $f : \mathbf{R}^m \to R$ is *quasiconvex* if

$$\{p \in \mathbf{R}^m | f(p) \le \lambda\} \text{ is convex} \quad \text{for any } \lambda \in \mathbf{R}, \tag{1.7}$$

or equivalently,

$$f\big(tp + (1-t)q\big) \le \max\{f(p), f(q)\} \quad \text{for any } p,q \in \mathbf{R}^m \text{ and } t \in [0,1]. \tag{1.8}$$

A typical quasiconvex function f, which may not be convex, can be constructed by letting $f(p) = g \circ h(p)$, where $g : \mathbf{R} \to \mathbf{R}$ is a monotone function and $h : \mathbf{R}^m \to \mathbf{R}$ is a convex function.

The second author has proved in Wang [W] that *any absolute minimizer* $u : \Omega \to \mathbf{R}$ *of* H *is a viscosity solution to the Aronsson equation* (1.5), *provided that* (i) $H = H(x,p) \in C^2(\mathcal{H}_X)$ *is quasiconvex in* p*-variable, and* (ii) $H_p(0, 0) = 0$ *and* $H(x,\cdot)$ *is homogeneous of degree* $\alpha \ge 1$. See Bieske [B1], [B2] and Bieske–Capogna [BC] for earlier works on *absolutely minimal horizontally Lipschitz extensions* on Carnot groups.

Since equation (1.5) is defined for $H \in C^1(\mathcal{H}_X)$, it is a very natural question to ask *whether the above result by* [W] *remains true if we weaken* $H \in C^1(\mathcal{H}_X)$.

In this paper we answer this question affirmatively by proving the following theorem.

THEOREM 1.4. *For any family of vector fields* $\{X_i\}_{i=1}^m$ *satisfying Hörmander's finite rank condition, if* $H = H(x,p) \in C^1(\mathcal{H}_X)$ *is quasiconvex in* p*-variable for any* $x \in \Omega$*, then any absolute minimizer* $u : \Omega \to \mathbf{R}$ *is a viscosity solution of the Aronsson equation* (1.5).

The study of absolute minimizers was initiated by Aronsson [A1], [A2], [A3] in dimension one. Jensen established in his seminal paper [J] the equivalence between infinity harmonic functions and absolute minimizing Lipschitz extensions, and their uniqueness as well. Later, Juutinen [Jp] extended the main theorem of [J] to Riemannian manifold settings. In the Euclidean setting, Barron–Jensen–Wang [BJW] provided a general study on absolute minimizers and established that any absolute minimizer for suitable $H(p,z,x) \in C^2(\mathbf{R}^n \times \mathbf{R} \times \Omega)$ is a viscosity solution of the Aronsson equation:

$$H_p(\nabla u, u, x) \cdot (H(\nabla u, u, x))_x = 0. \tag{1.9}$$

Subsequently, Crandall [C] gave a simpler proof of this result of [BJW] under weaker hypotheses. The techniques employed by [BJW] and [C] rely crucially on $H \in C^2(\mathbf{R}^n \times \mathbf{R} \times \Omega)$, because of the construction of local, C^2 solutions to the Hamilton–Jacobi equation $H(\nabla\psi, \psi, x) = k$. Very recently, Crandall–Wang–Yu [CWY] found a new proof of this theorem even for $H \in C^1(\mathbf{R}^n \times \mathbf{R} \times \Omega)$. The new observation made by [CWY] is to use global, viscosity solutions to the Hamilton–Jacobi equation associated with $H \in C^1(\mathbf{R}^n \times \mathbf{R} \times \Omega)$ as comparison functions to absolute minimizers.

Bieske–Capogna [BC] extended the idea of [C] to derive the subelliptic infinity Laplace equation for an absolute minimizing horizontal Lipschitz extension on Carnot groups. Wang [W] made a new observation based on [C] to derive the Aronsson equation for any absolute minimizer of $H \in C^2(\mathcal{H}_X)$ associated with any family of Hörmander's vector fields. Here, we aim to modify and extend the observation made in [CWY] to the Carnot–Carathéodory space $(\mathbf{R}^n, d_X)$. Roughly speaking, if $\phi \in C^2(\Omega)$ is a upper test function for an absolute minimizer $u \in W_X^{1,\infty}(\Omega)$, at x_0, then we show in Section 3 below that there exists $x_r \neq x_0$, such that

$$\begin{aligned} \phi(x_r) - \phi(x_0) \\ \geq \max_{\{p \in \mathcal{H}(x_0), H(x_0,p) \leq H(x_0, X\phi(x_0))\}} \langle p, P_{\mathcal{H}(x_0)}(x_r - x_0)\rangle_{\mathcal{H}(x_0)}. \end{aligned} \tag{1.10}$$

Here, $P_{\mathcal{H}(x_0)} : \mathbf{R}^n \to \mathcal{H}(x_0)$ is the orthogonal projection map. Roughly speaking, $x_r \in \partial B_r(x_0)$ is a maximal point of u restricted on $\partial B_r(x_0)$. It can be seen from the sections below that the right-hand side of (1.10) is the Finsler metric function $L(x_0, x_r - x_0, H(x_0, X\phi(x_0)))$. It turns out that (1.10) is a crucial ingredient to show that u is a viscosity subsolution of the Aronsson equation (1.5).

We would like to point out that Crandall–Evans–Gariepy [CEG] has shown that an absolute minimizing Lipschitz extension can also be characterized by the comparison principle with cones, which has been subsequently extended by Gariepy–Wang–Yu [GWY] to absolute minimizers to quasiconvex Hamiltonians. This characterization for absolute minimizers in term of comparison principle with cone type functions has also been obtained for some noneuclidean spaces including Grushin spaces by [B2], Finsler metric spaces by Champion–De Pascale [CD], and metric-measure spaces by Juutinen–Shanmugalingam [JS].

The paper is organized as follows. In Section 2, we establish some preliminary properties of absolute minimizers. In Section 3, we give a proof of Theorem 1.4.

2. Some preliminary results follow

This section is devoted to some basic facts on absolute minimizer and the construction of viscosity solutions to Hamilton–Jacobi equation $H(x, Xv) = k$.

Let d_X be the Carnot–Carathéodory distance given by Section 1, and define subelliptic balls

$$B_r(x_0) = \{x \in \mathbf{R}^n | d_X(x, x_0) < r\}, \qquad \overline{B}_r(x_0) = \{y \in \mathbf{R}^n | d_X(x, x_0) \le r\}.$$

First, we have the following proposition.

PROPOSITION 2.1. *Let $H = H(x,p) \in C(\mathcal{H}_X)$ be quasiconvex in p-variable. Let $U \subset\subset \Omega$ be a bounded open set.*

(a) *Suppose $(x_0, \phi) \in U \times C^1(U)$, and $v \in \mathrm{Lip}_X(U)$. If ϕ touches v at x_0 from above, i.e.,*

$$0 = (\phi - v)(x_0) \le (\phi - v)(x) \quad \forall x \in U \tag{2.1}$$

then

$$H(x_0, X\phi(x_0)) \le \lim_{r\downarrow 0} \operatorname{ess\,sup}_{B_r(x_0)} H(x, Xv(x)). \tag{2.2}$$

(b) *Let u be an absolute minimizer for H in Ω. Assume that $x_0 \in U$ and $w \in \mathrm{Lip}_X(U)$ satisfy*

$$(w - u)(x_0) \le 0 \le (w - u)(x) \quad \forall x \in \partial U, \tag{2.3}$$

then

$$\lim_{r\downarrow 0} \operatorname{ess\,sup}_{B_r(x_0)} H(x, Xu(x)) \le \operatorname{ess\,sup}_U H(x, Xw(x)). \tag{2.4}$$

Proof. First, observe that by continuity of H, we have

$$\lim_{r\downarrow 0} \operatorname{ess\,sup}_{B_r(x_0)} H(x, Xv(x)) = \lim_{r\downarrow 0} \operatorname{ess\,sup}_{B_r(x_0)} H(x_0, Xv(x)). \tag{2.5}$$

By replacing ϕ by $\phi(x) + \|x - x_0\|^2$, we may assume that for $r > 0$ small,

$$0 = (\phi - v)(x_0) < (\phi - v)(x) \quad \forall x \in \overline{B}_r(x_0) \setminus \{x_0\}. \tag{2.6}$$

For $0 < \varepsilon \le \frac{r}{2}$, let $v_\varepsilon(x) = \int_{\mathbf{R}^n} \eta_\varepsilon(x-y)v(y)\,dy \in C^\infty(B_{\frac{r}{2}}(x_0))$ be a standard mollification of v and $x_\varepsilon \in \overline{B}_{\frac{r}{2}}(x_0)$ satisfy

$$(\phi - v_\varepsilon)(x_\varepsilon) = \min_{x \in \overline{B}_{\frac{r}{2}}(x_0)} (\phi - v_\varepsilon)(x).$$

It follows from (2.6) that $\lim_{\varepsilon\downarrow 0} x_\varepsilon = x_0$. Hence, for small ε, we have $x_\varepsilon \in B_{\frac{r}{4}}(x_0)$, so that $X\phi(x_\varepsilon) = Xv_\varepsilon(x_\varepsilon)$ and

$$H(x_\varepsilon, X\phi(x_\varepsilon)) = H(x_\varepsilon, X(v_\varepsilon)(x_\varepsilon)). \tag{2.7}$$

We claim

$$|X(v_\varepsilon)(x_\varepsilon) - (Xv)_\varepsilon(x_\varepsilon)| \le C\|X\|_{C^1(B_r(x_0))}\|u\|_{W_X^{1,\infty}(B_r(x_0))}\omega(r), \tag{2.8}$$

where $\omega(r)$ denotes the modular of continuity of d_X with respect to $\|\cdot\|$.

The proof of (2.8) was originally due to Friederichs [F] (see also [FSS] and [GN]). Here, for the convenience of readers, we outline it as follows. Let $X_i(x) = \sum_{j=1}^n a_{ij}(x)\frac{\partial}{\partial x_j}$ for $x \in \mathbf{R}^n$ and $1 \le i \le m$, with $(a_{ij}) \in C^\infty(\mathbf{R}^n, \mathbf{R}^{nm})$. Then for $1 \le i \le m$ and $x \in B_{\frac{r}{2}}(x_0)$, we have

$$\begin{aligned}
&(X_i v)_\varepsilon(x) - X_i(v_\varepsilon)(x)\\
&\quad = \int_{\mathbf{R}^n} \eta_\varepsilon(x-y)\left(\sum_{j=1}^n a_{ij}(y)\frac{\partial}{\partial y_j}\right)\{v(y) - v(x)\}\,dy\\
&\qquad - \int_{\mathbf{R}^n} \sum_{j=1}^n a_{ij}(x)\frac{\partial \eta_\varepsilon(x-y)}{\partial x_j}\{v(y) - v(x)\}\,dy\\
&\quad = \sum_{j=1}^n \int_{\mathbf{R}^n}\left[-\frac{\partial}{\partial y_j}\big(a_{ij}(y)\eta_\varepsilon(x-y)\big) - a_{ij}(x)\frac{\partial \eta_\varepsilon(x-y)}{\partial x_j}\right]\big(v(y) - v(x)\big)\,dy\\
&\quad = \sum_{j=1}^n \int_{\mathbf{R}^n}\big(a_{ij}(y) - a_{ij}(x)\big)\frac{\partial \eta_\varepsilon(x-y)}{\partial x_j}\big(v(y) - v(x)\big)\,dy\\
&\qquad + \sum_{j=1}^n \int_{\mathbf{R}^n}\frac{\partial a_{ij}(y)}{\partial y_j}\eta_\varepsilon(x-y)\big(v(y) - v(x)\big)\,dy.
\end{aligned}$$

This implies

$$\begin{aligned}
&|(X_i v)_\varepsilon(x) - X_i(v_\varepsilon)(x)|\\
&\quad \le C\max_{1\le j\le n}\|\nabla a_{ij}\|_{L^\infty(B_r(x_0))}
\end{aligned}$$

$$\begin{aligned}
&\times \int_{\mathbf{R}^n} \{\eta_\varepsilon(x-y)|v(y)-v(x)| + \|y-x\||\nabla\eta_\varepsilon(x-y)||v(y)-v(x)|\}\,dy \\
&\le C\|X_i\|_{C^1(B_r(x_0))}\|v\|_{\mathrm{Lip}_X(B_r(x_0))} \max_{\|y-x\|\le r} d_X(y,x) \\
&\le C\|X_i\|_{C^1(B_r(x_0))}\|v\|_{\mathrm{Lip}_X(B_r(x_0))}\omega(r),
\end{aligned}$$

and hence (2.8) follows. Since $\|(Xv)_\varepsilon\|_{L^\infty(B_{\frac{r}{2}}(x_0))} \le \|Xv\|_{L^\infty(B_r(x_0))}$, it follows from (2.8) that for small $r>0$, $|(Xv_\varepsilon)(x_\varepsilon)| \le \|Xv\|_{L^\infty(B_r(x_0))}+1$. Hence,

$$\begin{aligned}
(2.9)\quad & H(x_\varepsilon, X(v_\varepsilon)(x_\varepsilon)) \\
&\le H(x_\varepsilon,(Xv)_\varepsilon(x_\varepsilon)) \\
&\quad + \max_{x\in B_r(x_0)} \max_{\{|p|\le\|Xv\|_{L^\infty(B_r(x_0))}+1\}} \{|H_p(x,p)| \\
&\quad \times |X(v_\varepsilon)(x_\varepsilon)-(Xv)_\varepsilon(x_\varepsilon)|\} \\
&\le H(x_\varepsilon,(Xv)_\varepsilon(x_\varepsilon)) + C\omega(r) \\
&\le H(x_0,(Xv)_\varepsilon(x_\varepsilon)) + \Big\{\max_{x\in B_r(x_0)} \max_{\{|p|\le\|Xv\|_{L^\infty(B_r(x_0))}+1\}} |\nabla_x H(x,p)|\Big\} r \\
&\quad + C\omega(r) \\
&\le \operatorname{ess\,sup}_{x\in B_r(x_0)} H(x_0, Xv(x)) + C\big(r+\omega(r)\big)
\end{aligned}$$

where we have used the quasiconvexity of $H(x_0,p)$ in p-variable:

$$\begin{aligned}
H(x_0,(Xv)_\varepsilon(x_\varepsilon)) &\le \operatorname{ess\,sup}_{B_{\frac{r}{2}}(x_0)} H(x_0, Xv_\varepsilon(x)) \\
&\le \operatorname{ess\,sup}_{B_r(x_0)} H(x_0, Xv(x)).
\end{aligned}$$

Taking r into zero and noting $\lim_{r\downarrow 0}\omega(r)=0$, (2.9) and (2.7) imply (2.2).

To prove (b), set for small $\varepsilon>0$, $\delta>0$, and

$$w_{\varepsilon,\delta}(x) = w(x) + \varepsilon\|x-x_0\|^2 - \delta, \quad x\in U.$$

Then $u(x_0) - w_{\varepsilon,\delta}(x_0) \ge \delta > 0$, and for $x\in\partial U$,

$$\begin{aligned}
u(x) - w_{\varepsilon,\delta}(x) &\le u(x) - w(x) - \varepsilon \min_{\partial U} \|x-x_0\|^2 + \delta \\
&\le \delta - \varepsilon \min_{x\in\partial U} \|x-x_0\|^2 < 0
\end{aligned}$$

provided that we choose ε and δ, such that

$$\delta - \varepsilon \min_{x\in\partial U}\|x-x_0\|^2 < 0. \tag{2.10}$$

Hence, there exists another open connected component V of $\{x\in U | u(x) - w_{\varepsilon,\delta}(x) > 0\}$, such that $x_0\in V$ and $V\subset\subset U$. Since $u = w_{\varepsilon,\delta}$ on ∂V, the absolute minimality of u implies that

$$\begin{aligned}
\operatorname{ess\,sup}_{B_r(x_0)} H(x, Xu(x)) &\le \operatorname{ess\,sup}_{B_r(x_0)} H(x, Xw_{\varepsilon,\delta}(x)) \\
&\le \operatorname{ess\,sup}_V H(x, Xw_{\varepsilon,\delta}(x)) \\
&\le \operatorname{ess\,sup}_U H(x, Xw_{\varepsilon,\delta}(x)).
\end{aligned}$$

By sending $r \downarrow 0$ and then $\varepsilon, \delta \downarrow 0$, (2.4) then follows. □

Similar to [CWY], the second observation is that we may assume

$$\lim_{\{p \in \mathcal{H}(x): \|p\| \to +\infty\}} H(x,p) = +\infty \quad \text{uniformly for } x \in \overline{\Omega}. \tag{2.11}$$

In fact, as in [CWY], Section 2, let $u \in W_X^{1,\infty}(\Omega)$ be the absolute minimizer of H under consideration and

$$\begin{aligned} R &= \|Xu\|_{L^\infty(\Omega)} + 1, \\ M &= \min\{H(x,p) | x \in \overline{\Omega}, p \in \mathcal{H}(x), \text{with } \|p\| \le R\}, \end{aligned} \tag{2.12}$$

and define

$$\hat{H}(x,p) = \max\{H(x,p), \|p - P_R(p)\| + M\} \quad \forall (x,p) \in \mathcal{H}_X, \tag{2.13}$$

where $P_R : \mathbf{R}^m \to \mathbf{R}^m$ is given by

$$P_R(p) = p \quad \text{for } |p| \le R; \qquad R\frac{p}{\|p\|} \quad \text{for } |p| \ge R.$$

It is easy to see that $\hat{H}$ is quasiconvex in p-variable and satisfies (2.11), $H \le \hat{H}$, and

$$H(x, Xu(x)) = \hat{H}(x, Xu(x)) \quad \text{for a.e., } x \in \overline{\Omega}.$$

Thus, u is also an absolute minimizer for $\hat{H}$. Finally, if $\phi \in C^1(\Omega)$ touches u from above at x_0, then Proposition 2.1(a) implies that $|X\phi|(x_0) < R$, and hence $\hat{H}_p(x_0, X\phi(x_0))$ $(= H_p(x_0, X\phi(x_0))$ exists.

Now, we indicate how to construct viscosity solutions of the Hamiltonian–Jacobi equation $H(x, X\Phi(x)) = k$. Let $P_{\mathcal{H}(x)} : \mathbf{R}^n \to \mathcal{H}(x)$, $x \in \mathbf{R}^n$, be the orthogonal projection map. For $k \in \mathbf{R}$, $x \in B_r(x_0)$ and $p \in \mathbf{R}^n$, define

$$L(x,p,k) = \max_{\{q \in \mathcal{H}(x) | H(x,q) \le k\}} \langle q, P_{\mathcal{H}(x)}(p) \rangle_{\mathcal{H}(x)}. \tag{2.14}$$

Notice that the standard method to construct viscosity solutions to the Hamilton–Jacobi equation $H(x, X\psi(x)) = k$ is through minimization of the action functional among all admissable paths, which can be closedly related to the existence of minimal geodesic in the subriemannian setting. From this view of point, $L(x,p,k)$ comes naturally since it is the Finsler metric on the Carnot–Carathéodory space $(\mathbf{R}^n, d_X)$.

Set

$$k_0(r) = \max_{x \in \overline{B}_r(x_0)} \min_{q \in \mathcal{H}(x)} H(x,q). \tag{2.15}$$

Notice that by (2.11), $k_0(r) < +\infty$.

For $L(x,p,k)$, we have the following proposition.

PROPOSITION 2.2. *If $H = H(x,p) \in C(\mathcal{H}_X)$ is quasiconvex in p-variable and satisfies the coercivity condition (2.11). Then for any $x \in \overline{B}_r(x_0)$, $p \in \mathbf{R}^n$ and $k \ge k_0(r)$, we have:*

(1) *$x \to L(x,p,k)$ is upper-semicontinuous,*

(2) *$p \to L(x,p,k)$ is Lipschitz continuous with respect to the Euclidean distance $\|\cdot\|$, and its Lipschitz constant depends only on k,*

(3) *$p \to L(x,p,k)$ is convex, positively 1-homogeneous, and $L(x,p,k) = 0$ for any $p \perp \mathcal{H}(x)$,*

(4) *If $M > 0$, then there is $k_M > 0$ such that for any $k \geq k_M$, $L(x,p,k) \geq M|P_{\mathcal{H}(x)}(p)|$ for any $(x,p) \in \mathbf{R}^n \times \mathbf{R}^n$,*

(5) *$k \to L(x,p,k)$ is nondecreasing and continuous from the right.*

Proof. In view of (1.7) and (2.11), the proof is straightforward. We leave the detail to readers. □

Definition 2.3. For $r > 0$ and $x \in \overline{B}_r(x_0)$, a *horizontal path* from x_0 to x in $\overline{B}_r(x_0)$ is a *horizontal curve* $\xi : [0,T] \to \overline{B}_r(x_0)$ such that $\xi(0) = x_0$ and $\xi(T) = x$. The set of such horizontal paths is denoted by

$$hp(x,r) := \{\text{horizontal paths } \xi \text{ from } x_0 \text{ to } x \text{ in } \overline{B}_r(x_0)\}.$$

Now, we define for $k \geq k_0(r)$ and $x \in B_r(x_0)$,

$$C_{k,r}(x,x_0) = \inf\left\{ \int_0^T L(\xi(t),\xi'(t),k)\,dt \,\middle|\, \xi \in hp(x,r) \right\}. \tag{2.16}$$

Notice that $C_{k,r}(x,x_0)$ is well-defined and finite, since $(\mathbf{R}^n, d_X)$ is a length space, i.e., the distance between any two points can be realized by the length of a horizontal curve joining the two points. In particular, for any $x \in \overline{B}_r(x_0)$, there exists a horizontal curve $\gamma : [0,T] \to \mathbf{R}^n$ joining x_0 to x such that $T = d_X(x,x_0) \leq r$. By Proposition 2.2(5), we have $k \to C_{k,r}$ is nondecreasing. We set

$$C_{k-,r}(x,x_0) = \lim_{l \uparrow k} C_{l,r}(x,x_0), \qquad C_{k+,r}(x,x_0) = \lim_{l \downarrow k} C_{l,r}(x,x_0). \tag{2.17}$$

Proposition 2.4. *Under the assumptions as in Proposition 2.2, for any $k \geq k_0(r)$, we have* (i) $C_{k,r}(x_0,x_0) \leq 0$, (ii) $C_{k,2r}(x_2,x_0) \leq C_{k,r}(x_1,x_0) + C_{k,r}(x_2,x_1)$ *for any $x_1, x_2 \in B_r(x_0)$, and* (iii) $C_{k,r}(x,x_0) \in W_X^{1,\infty}(B_r(x_0))$.

Proof. Since $L(x,0,k) = 0$, $C_{k,r}(x_0,x_0) \leq 0$. To see (ii), for $\varepsilon > 0$ be arbitrarily small, let $\xi_1 : [0,T_1] \to B_r(x_0)$ be a horizontal curve connecting x_0 to x_1 and $\xi_2 : [0,T_2] \to B_r(x_1)$ be another horizontal curve connecting x_1 to x_2, such that

$$\int_0^{T_1} L(\xi_1,\xi_1',k)\,dt \leq C_{k,r}(x_1,x_0) + \varepsilon,$$

$$\int_0^{T_2} L(\xi_2,\xi_2',k)\,dt \leq C_{k,r}(x_2,x_1) + \varepsilon.$$

If we define $\xi_3 : [0,T_1+T_2] \to B_{2r}(x_0)$ by letting $\xi_3(t) = \xi_1(t)$ for $0 \leq t \leq T_1$ and $\xi_3(t) = \xi_2(t - T_1)$ for $T_1 \leq t \leq T_1 + T_2$, then ξ_3 is a horizontal curve connecting x_0 to x_2, and

$$\begin{aligned} C_{k,2r}(x_2,x_0) &\le \int_0^{T_1+T_2} L(\xi_3,\xi_3',k)\,dt \\ &= \int_0^{T_1} L(\xi_1,\xi_1',k)\,dt + \int_0^{T_2} L(\xi_2,\xi_2',k)\,dt \\ &\le C_{k,r}(x_1,x_0) + C_{k,r}(x_2,x_1) + 2\varepsilon. \end{aligned}$$

This implies (ii). To see (iii), for $y,z \in B_r(x_0)$, let $\eta : [0,S] \to B_r(x_0)$ another horizontal curve connecting y to z such that $d_X(z,y) = S$. Define

$$K = \max_{x\in\overline{B}_r(x_0)} \max_{q\in\mathcal{H}(x):H(x,q)\le k} |q|.$$

Then similar to (ii), we have

$$\begin{aligned} C_{k,r}(z,x_0) &\le C_{k,r}(y,x_0) + \int_0^S L(\eta,\eta',k)\,dt \\ &\le C_{k,r}(y,x_0) + K\int_0^S |\eta'(t)|\,dt \\ &= C_{k,r}(y,x_0) + KS \\ &= C_{k,r}(y,x_0) + K d_X(y,z). \end{aligned}$$

This implies that $C_{k,r}(y,x_0)$ is Lipschitz continuous in $B_r(x_0)$ with respect to d_X. □

It follows from Proposition 2.4 and Rademacher's theorem on $(\mathbf{R}^n, d_X)$, which was first proved by Pansu [P] and later by Garofalo–Nieu [GN], that $XC_{k,r}(x,x_0)$ exists for a.e., $x \in B_r(x_0)$.

The main result of this section is the following proposition.

Proposition 2.5. *Under the same assumptions as in Proposition 2.2, for any $k \ge k_0(r)$, $C_{k,r}$ is a viscosity solution of*

$$H(x, XC_{k,r}(x,x_0)) = k \quad \text{in } B_r(x_0)\setminus\{x_0\}. \tag{2.18}$$

In particular, $H(x, XC_{k,r}(x,x_0)) = k$ for a.e., $x \in B_r(x_0)$.

Proof. For any $x_1 \in B_r(x_0)\setminus\{x_0\}$, let $\phi \in C^1(B_r(x_0)$ touch $C_{k,r}(x,x_0)$ at x_1 from above. Let $\xi \in C^1([0,T],\mathbf{R}^n) \cap hp(x_1,r)$. For $0 < t_0 < T$, we have

$$\begin{aligned} &\int_{t_0}^T \langle X\phi(\xi(t)), \xi'(t)\rangle_{\mathcal{H}(\xi(t))}\,dt \\ &\quad= \phi(x_1) - \phi(\xi(t_0)) \le C_{k,r}(x_1,x_0) - C_{k,r}(\xi(t_0),x_0) \\ &\quad\le C_{k,2r}(x_1,\xi(t_0)) \le \int_{t_0}^T L(\xi(t),\xi'(t),k)\,dt. \end{aligned} \tag{2.19}$$

Dividing (2.19) by $T-t_0$, taking $t_0\uparrow T$, and applying Proposition 2.2(4), we obtain

$$\begin{aligned}(2.20)\qquad \langle X\phi(x_1),\xi'(T)\rangle_{\mathcal{H}(x_1)} &\le L(x_1,\xi'(T),k)\\ &= \max_{\{q\in\mathcal{H}(x_1),H(x_1,q)\le k\}}\langle q,\xi'(T)\rangle_{\mathcal{H}(x_1)}.\end{aligned}$$

This and the quasiconvexity of $H(x_1,\cdot)$ imply $H(x_1,X\phi(x_1))\le k$, i.e., $C_{k,r}$ is a viscosity subsolution of (2.18).

To prove that $C_{k,r}$ is a viscosity supersolution of (2.18), let $\psi\in C^1(B_r(x_0))$ touch $C_{k,r}$ from below at $x_1\in B_r(x_0)\setminus\{x_0\}$. Let $\xi\in C([0,T],\mathbf{R}^n)\cap hp(x_1,r)$ be such that

$$(2.21)\qquad C_{k,r}(x_1,x_0)=\int_0^T L(\xi(t),\xi'(t),k)\,dt.$$

Then for any $t_0\in(0,T)$ we have

$$\begin{aligned}&\int_{t_0}^T\langle X\psi(\xi(t)),\xi'(t)\rangle_{\mathcal{H}(\xi(t))}\,dt\\ &\quad=\psi(x_1)-\psi(\xi(t_0))\\ &\quad\ge C_{k,r}(x_1,x_0)-C_{k,r}(\xi(t_0),x_0)\\ &\quad\ge\int_0^T L(\xi(t),\xi'(t),k)\,dt-\int_0^{t_0}L(\xi(t),\xi'(t),k)\,dt\\ &\quad=\int_{t_0}^T L(\xi(t),\xi'(t),k)\,dt\\ &\quad=\int_{t_0}^T\max_{\{p\in\mathcal{H}(\xi(t)),H(\xi(t),p)\le k\}}\langle p,\xi'(t)\rangle_{\mathcal{H}(\xi(t))}\,dt.\end{aligned}$$

This implies that there exist $t_r\uparrow T$ such that $\xi'(t_r)$ exist, and

$$(2.22)\qquad \langle X\psi(\xi(t_r)),\xi'(t_r)\rangle_{\mathcal{H}(\xi(t_r))}\ge\max_{\{p\in\mathcal{H}(\xi(t_r)),H(\xi(t_r),p)\le k\}}\langle p,\xi'(t_r)\rangle_{\mathcal{H}(\xi(t_r))}.$$

Since $\langle\xi'(t_r),\xi'(t_r)\rangle_{\mathcal{H}(\xi(t_r))}=1$, we assume that there is $q\in\mathcal{H}(x_1)$ with $\langle q,q\rangle_{\mathcal{H}(x_1)}=1$ such that $\lim_{t_r\uparrow T}\xi'(t_r)=q$. Taking $t_r\uparrow T$, (2.22) implies

$$(2.23)\qquad \langle X\phi(x_1),q\rangle_{\mathcal{H}(x_1)}\ge\max_{\{p\in\mathcal{H}(x_1):H(x_1,p)\le k\}}\langle p,q\rangle_{\mathcal{H}(x_1)}.$$

Hence, we conclude $H(x_1,X\psi(x_1))\ge k$. The proof is complete. □

3. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. We begin with some lemmas. For $x_0\in\Omega$, let $r>0$ be such that $B_r(x_0)\subset\Omega$ and let $\phi\in C^2(B_r(x_0))$ be such that

$$(3.1)\qquad 0=(\phi-u)(x_0)<(\phi-u)(x)\quad\text{for } x\in B_r(x_0)\setminus\{x_0\}.$$

For $k_0(r)$, given by (2.15), define

$$k_r = \inf\{k | k \geq k_0(r), u(x) \leq u(x_0) + C_{k,r}(x, x_0) \text{ for } x \in \partial B_r(x_0)\}. \tag{3.2}$$

Notice, that it follows from Proposition 2.2(iv) that for any $M > 0$, we have

$$C_{k,r}(x, x_0) \geq Mr \quad \text{for } x \in \partial B_r(x_0)$$

provided that $k > 0$ is sufficiently large. This implies that the quantity k_r is well defined.

LEMMA 3.1. *Let $H = H(x, p) \in C(\mathcal{H}_X)$ be quasiconvex in p-variable and satisfy (2.11). If $u \in W^{1,\infty}_X(\Omega)$ be an absolute minimizer of H, then $H(x_0, X\phi(x_0)) \leq k_r$.*

Proof. For any $k > k_r$, let $w(x) \equiv u(x_0) + C_{k,r}(x, x_0)$. Then it is easy to see that $u(x_0) \geq w(x_0)$ and

$$u(x) \leq w(x) \quad \text{for } x \in \partial B_r(x_0), \tag{3.3}$$

Hence, by Proposition 2.1(b), we have

$$\begin{aligned} H(x_0, X\phi(x_0)) &\leq \lim_{s \downarrow 0} \operatorname{ess\,sup}_{B_s(x_0)} H(x, Xu(x)) \\ &\leq \operatorname{ess\,sup}_{B_r(x_0)} H(x, XC_{k,r}(x, x_0)) = k. \end{aligned} \tag{3.4}$$

Taking $k \downarrow k_r$, this yields the result. □

Notice that if $H_p(x_0, X\phi(x_0)) = 0$, then $\mathcal{A}^X(\phi)(x_0) = 0$ and Theorem 1.4 is proved. Hence, we assume $H_p(x_0, X\phi(x_0)) \neq 0$.

LEMMA 3.2. *Let $H = H(x, p) \in C^1(\mathcal{H}_X)$ be quasiconvex in p-variable and satisfy (2.11). Assume $H_p(x_0, X\phi(x_0)) \neq 0$, if $u \in W^{1,\infty}_X(\Omega)$ is an absolute minimizer of H, then for any sufficiently small $r > 0$,*

$$H(x_0, X\phi(x_0)) > k_0(r). \tag{3.5}$$

Proof. It follows from $H_p(x_0, X\phi(x_0)) \neq 0$ that there is $p_0 \in \mathcal{H}(x_0)$ such that $H(x_0, p_0) < H(x_0, X\phi(x_0))$. By continuity of H, this implies that for a sufficiently small $r > 0$ and any $x \in B_r(x_0)$, there exists $p_x \in \mathcal{H}(x)$ such that $H(x, p_x) < H(x_0, X\phi(x_0))$. Hence, $H(x_0, X\phi(x_0)) > k_0(r)$. □

Proof of Theorem 1.4. Denote $h_0 = H(x_0, X\phi(x_0))$. For any $k < h_0 \leq k_r$ and m sufficiently large, there exist $x^k_m \in \partial B_{\frac{1}{m}}(x_0)$, such that

$$C_{k, \frac{1}{m}}(x^k_m, x_0) \leq u(x^k_m) - u(x_0). \tag{3.6}$$

For $k \uparrow h_0$, assume $x^k_m \to x_m \in \partial B_r(x_0)$. Then (3.6) yields

$$C_{h_0^-, \frac{1}{m}}(x_m, x_0) \leq u(x_m) - u(x_0). \tag{3.7}$$

Let $\varepsilon_m > 0$ be sufficiently small such that $u(x_m) - u(x_0) + \varepsilon_m < \phi(x_m) - \phi(x_0)$. By definition of $C_{h_0^-, \frac{1}{m}}$, there is $\xi_m \in C([0, T_m], \mathbf{R}^n) \cap hp(x_m, \frac{1}{m})$ such that

$$\begin{aligned}(3.8) \qquad & \int_0^{T_m} L(\xi_m(t), \xi_m'(t), h_0^-)\, dt \\ & \leq C_{h_0^-, \frac{1}{m}}(x_m, x_0) + \varepsilon_m \leq u(x_m) - u(x_0) + \varepsilon_m \\ & < \phi(x_m) - \phi(x_0) \\ & = \int_0^{T_m} \langle X\phi(\xi_m(t)), \xi_m'(t) \rangle_{\mathcal{H}(\xi_m(t))}\, dt.\end{aligned}$$

Thus, there are $t_m \in (0, T_m]$ such that $\xi_m'(t_m)$ exists, and

$$(3.9) \qquad L(\xi_m(t_m), \xi_m'(t_m), h_0^-) < \langle X\phi(\xi_m(t_m)), \xi_m'(t_m) \rangle_{\mathcal{H}(\xi_m(t_m))}.$$

This implies that $h_0 \leq H(\xi_m(t_m), X\phi(\xi_m(t_m)))$. Assume that t_m be the largest value of $t \in (0, T_m]$ such that $h_0 \leq H(\xi_m(t), X\phi(\xi_m(t)))$. Then we have $H(\xi_m(t), X\phi(\xi_m(t))) < h_0$ for a.e., $t \in (t_m, T_m]$, and hence

$$\begin{aligned}(3.10) \qquad \phi(x_m) - \phi(\xi_m(t_m)) &= \phi(\xi_m(T_m)) - \phi(\xi_m(t_m)) \\ &= \int_{t_m}^{T_m} \langle X\phi(\xi_m(t)), \xi_m'(t) \rangle_{\mathcal{H}(\xi_m(t))}\, dt \\ &\leq \int_{t_m}^{T_m} L(\xi_m(t), \xi_m'(t), h_0^-)\, dt.\end{aligned}$$

Therefore, we have

$$(3.11) \qquad \begin{aligned} &\int_0^{t_m} L(\xi_m(t), \xi_m'(t), h_0^-)\, dt < \phi(\xi_m(t_m)) - \phi(x_0), \\ &\qquad H(x_0, X\phi(x_0)) \leq H(\xi_m(t_m), X\phi(\xi_m(t_m))).\end{aligned}$$

Set $y_m = \xi_m(t_m)$. It is easy to see $y_m \neq x_0$. By Proposition 2.2(4), we can find $c(h_0) > 0$ such that

$$(3.12) \qquad L(\xi_m(t), \xi_m'(t), h_0^-) \geq c(h_0) \quad \text{for all } t \in [0, t_m].$$

Therefore, (3.11) implies

$$(3.13) \qquad c(h_0) < \frac{\phi(y_m) - \phi(x_0)}{t_m} \left(= \frac{1}{t_m} \int_0^{t_m} \langle X\phi(\xi_m(t)), \xi_m'(t) \rangle_{\mathcal{H}(\xi_m(t))}\, dt \right).$$

Set $q_m = \frac{y_m - x_0}{t_m}$. Since $\|q_m\| \leq 1$, we may assume that there exist $q \in \mathbf{R}^n$, with $\|q\| \leq 1$, such that $\lim_{m\to\infty} q_m = q$. Taking m to infinity, (3.13) implies

$$(3.14) \qquad c(h_0) \leq \left\langle X\phi(x_0), P_{\mathcal{H}(x_0)}(q) \right\rangle_{\mathcal{H}(x_0)}.$$

This implies

$$(3.15) \qquad X\phi(x_0) \neq 0, \qquad P_{\mathcal{H}(x_0)}(q) \neq 0.$$

For any $\delta > 0$, it also follows from (3.11) that

$$\begin{aligned}(3.16)\quad & \max_{\{p\in\mathcal{H}(x_0):H(x_0,p)\le h_0-\delta\}} \big\langle p, P_{\mathcal{H}(x_0)}(y_m - x_0)\big\rangle_{\mathcal{H}(x_0)} \\ & \le \int_0^{t_m} \max_{\{p\in\mathcal{H}(x_0):H(x_0,p)\le h_0-\delta\}} \langle p, \xi_m'(t)\rangle_{\mathcal{H}(x_0)}\, dt \\ & \le \int_0^{t_m} L(\xi_m(t), \xi_m'(t), h_0^-)\, dt \\ & < \phi(y_m) - \phi(x_0).\end{aligned}$$

Dividing (3.16) by t_m and sending $m \to \infty$, we have

$$\begin{aligned}(3.17)\quad & \max_{\{p\in\mathcal{H}(x_0):H(x_0,p)\le h_0-\delta\}} \big\langle p, P_{\mathcal{H}(x_0)}(q)\big\rangle_{\mathcal{H}(x_0)} \\ & \le \big\langle X\phi(x_0), P_{\mathcal{H}(x_0)}(q)\big\rangle_{\mathcal{H}(x_0)}.\end{aligned}$$

Thus,

$$(3.18)\quad \big\langle p, P_{\mathcal{H}(x_0)}(q)\big\rangle_{\mathcal{H}(x_0)} \le \big\langle X\phi(x_0), P_{\mathcal{H}(x_0)}(q)\big\rangle_{\mathcal{H}(x_0)}$$

holds for any $p \in \mathcal{H}(x_0)$ with $H(x_0,p) < h_0$. Notice that (3.18) remains true for any $p \in C$, where C is the convex set

$$C \equiv \overline{\{p \in \mathcal{H}(x_0) : H(x_0,p) < H(x_0, X\phi(x_0))\}}.$$

Since $H_p(x_0, X\phi(x_0)) \neq 0$, we have $X\phi(x_0) \in C$. Hence, (3.18) implies

$$\big\langle X\phi(x_0), P_{\mathcal{H}(x_0)}(q)\big\rangle_{\mathcal{H}(x_0)} = \max_{p\in C} \big\langle p, P_{\mathcal{H}(x_0)}(q)\big\rangle_{\mathcal{H}(x_0)}.$$

Therefore, by the Lagrange multiplier theorem, we have

$$(3.19)\quad P_{\mathcal{H}(x_0)}(q) = \lambda H_p(x_0, X\phi(x_0))$$

for some $\lambda > 0$.

Since $H(x, X\phi(x)) \in C^1(B_r(x_0))$, we have

$$\begin{aligned}(3.20)\quad 0 & \le \frac{H(y_m, X\phi(y_m)) - H(x_0, X\phi(x_0))}{t_m} \\ & = \big\langle X(H(x, X\phi(x)))|_{x=x_0}, P_{\mathcal{H}(x_0)}(q_m)\big\rangle_{\mathcal{H}(x_0)} + o(1).\end{aligned}$$

Sending $m \to \infty$ and using (3.19) lead to

$$\lambda \mathcal{A}^X[\phi](x_0) = \lambda \langle X(H(x, X\phi(x)))|_{x=x_0}, H_p(x_0, X\phi(x_0))\rangle_{\mathcal{H}(x_0)} \ge 0.$$

Since $\lambda > 0$, we have $\mathcal{A}^X[\phi](x_0) \ge 0$ and u is a viscosity subsolution of (1.5). Similarly, one can prove that u is also a viscosity supersolution. This completes the proof of Theorem 1.4. □

References

[A1] G. Aronsson, *Minimization problem for the functional* $\sup_x F(x, f(x), f'(x))$, Ark. Mat. **6** (1965), 33–53. MR 0196551

[A2] G. Aronsson, *Minimization problem for the functional* $\sup_x F(x, f(x), f'(x))$. *II*, Ark. Mat. **6** (1966), 409–431. MR 0203541

[A3] G. Aronsson, *Minimization problem for the functional* $\sup_x F(x, f(x), f'(x))$. *III*, Ark. Mat. **7** (1969), 509–512. MR 0240690

[BJW] N. Barron, R. Jensen and C. Y. Wang, *The Euler equation and absolute minimizers of L^∞-functionals*, Arch. Ration. Mech. Anal. **157** (2001), 255–283. MR 1831173

[B1] T. Bieske, *On ∞-harmonic functions on the Heisenberg group*, Comm. Partial Differential Equations **27** (2002), 727–761. MR 1900561

[B2] T. Bieske, *Lipschitz extensions on generalized Grushin spaces*, Michigan Math. J. **53** (2005), 3–31. MR 2125531

[BC] T. Bieske and L. Capogna, *The Aronsson–Euler equation for absolutely minimizing Lipschitz extensions with respect to Carnot–Carathéodory metrics*, Trans. Amer. Math. Soc. **357** (2005), 795–823 (electronic). MR 2095631

[CD] T. Champion and L. De Pascale, *A principle of comparison with distance function for absolute minimizers*, J. Convex. Anal. **14** (2007), 515–541.

[C] M. Crandall, *An efficient derivation of the Aronsson equation*, Arch. Ration. Mech. Anal. **167** (2003), 271–279. MR 1981858

[CEG] M. Crandall, L. Evans and R. Gariepy, *Optimal Lipschitz extensions and the infinity laplacian*, Cal. Var. PDE **13** (2001), 123–139. MR 1861094

[CIL] M. Crandall, H. Ishii and P. L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. **27** (1992), 1–67. MR 1118699

[CL] M. Crandall and P. L. Lions, *Viscosity solutions of Hamilton–Jacobi equations*, Trans. Amer. Math. Soc. **277** (1983), 1–42. MR 0690039

[CWY] M. Crandall, C. Y. Wang and Y. Yu, *Derivation of the Aronsson equation for C^1-Hamiltonians*, Trans. Amer. Math. Soc. **36** (2009), 103–124.

[FSS] B. Franchi, R. Serapioni and F. Serra Cassano, *Meyers–Serrin type theorems and relaxation of variational integrals depending on vector fields*, Houston J. Math. **22** (1996), 859–890. MR 1437714

[F] K. Friedrichs, *The identity of weak and strong extensions of differential operators*, Trans. Amer. Math. Soc. **55** (1944), 132–151. MR 0009701

[GWY] R. Gariepy, C. Y. Wang and Y. Yu, *Generalized cone comparison, Aronsson equation, and absolute minimizers*, Comm. Partial Differential Equations **31** (2006), 1027–1046. MR 2254602

[GN] N. Garofalo and D. Nhieu, *Lipschitz continuity, global smooth approximations and extension theorems for Sobolev functions in Carnot–Carathéodory spaces*, J. Anal. Math. **74** (1998), 67–97. MR 1631642

[J] R. Jensen, *Uniqueness of Lipschitz extensions: Minimizing the sup norm of the gradient*, Arch. Ration. Mech. Anal. **123** (1993), 51–74. MR 1218686

[Jp] P. Juutinen, *Minimization problems for Lipschitz functions via viscosity solutions*, Dissertation, University of Jyväskulä, Jyväskulä, 1998; Ann. Acad. Sci. Fenn. Math. Diss. **115** (1998), 53 pp. MR 1632063

[JS] P. Juutinen and N. Shanmugalingam, *Equivalence of AMLE, strong AMLE, and comparison with cones in metric measure spaces*, Math. Nachr. **279** (2006), 1083–1098. MR 2242966

[NSW] A. Nagel, E. Stein and S. Wainger, *Balls and metrics defined by vector fields. I. Basic properties*, Acta Math. **155** (1985), 103–147. MR 0793239

[P] P. Pansu, *Carnot–Caratheodory metrics and quasi-isometries of rank-one symmetric spaces*, Ann. of Math. (2) **129** (1989), 1–60. MR 0979599

[W] C. Y. Wang, *The Aronsson equation for absolute minimizers of L^∞-functionals associated with vector fields satisfying Hörmander's condition*, Trans. Amer. Math. Soc. **359** (2007), 91–113 (electronic). MR 2247884

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