On 81 symplectic resolutions of a 4-dimensional quotient by a group of order 32

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Abstract We provide a construction of 81 symplectic resolutions of a 4-dimensional quotient singularity obtained by an action of a group of order 32. The existence of such resolutions is known by a result of Bellamy and Schedler. Our explicit construction is obtained via geometric invariant theory (GIT) quotients of the spectrum of a ring graded in the Picard group generated by the divisors associated to the conjugacy classes of symplectic reflections of the group in question. As a result we infer the geometric structure of these resolutions and their flops. Moreover, we represent the group in question as a group of automorphisms of an abelian 4-fold so that the resulting quotient has singularities with symplectic resolutions. This yields a new Kummer-type symplectic 4-fold.

1. Introduction

1.A. Background

Long before the notion of the Mori dream space was brought to life by Hu and Keel [27], put in the context of Mumford's [34] geometric invariant theory (GIT), and related to homogeneous coordinate rings, which were introduced for toric varieties by David Cox [13], a similar concept emerged in the local study of contractions in the minimal model program (MMP). In particular, in his 1992 inspiring talk, Miles Reid [38] explained how to view flips in terms of variation of GIT. Although the notion of the *total coordinate ring*, known also as the *Cox ring*, has been extensively studied for projective varieties in the last decade (see, e.g., [12], [42], [3]), apparently it has not been used for understanding local contractions nor for resolution of higher-dimensional singularities, as was proposed in [38].

In the present article we follow the ideas of [38] and construct a family of crepant resolutions of a symplectic singularity via GIT quotients. Two preceding

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papers which used similar ideas, [19] and [18], concerned the case of surface singularities for which the resolutions are classically known. The excellent book [4], which provides an exhaustive overview of the present state of knowledge about Cox rings and their applications, tackles this problem for toric and complexity-1 cases.

In the present article we focus on a 4-dimensional symplectic singularity which was shown to admit such a resolution by Bellamy and Schedler [9]. The result of Bellamy and Schedler is based on the relation of symplectic resolutions to smoothings of the singularity by a Poisson deformation (see [36], [21]). The methods which we provide in the present article are of a completely different nature. They reveal an explicit description of the resolutions, which is a significant advantage over the previous approach, which was not effective in this respect.

1.B. Resolutions via GIT quotients

The main result of the present article is the following.

THEOREM 1.1

Let V be a 4-dimensional vector space with a symplectic form ω. Assume that G is a group of order 32 defined in Section 2.C, acting on V and preserving ω. By $\mathbb{T}$ we denote a 5-dimensional algebraic torus with coordinates t_i, $i=0,\dots,4$, associated to five classes of symplectic reflections generating G. Let $\mathcal{R}$ be a $\mathbb{C}$-subalgebra generated in $\mathbb{C}[V]\otimes\mathbb{C}[\mathbb{T}]$ by t_i^{-2} for $i=0,\dots,4$ and $\phi_{ij}t_it_j$ for $0\le i<j\le 4$, where the ϕ_{ij}'s are eigenfunctions of the action of $\mathrm{Ab}(G)$ on $\mathbb{C}[V]^{[G,G]}$, as defined in Lemma 3.12. Then there are 81 GIT quotients of $\operatorname{Spec}\mathcal{R}$ which yield all crepant resolutions of V/G. A distinguished resolution of V/G has a 2-dimensional fiber which is a union of a $\mathbb{P}^2$ blown up in four points and of ten copies of $\mathbb{P}^2$.

The same number of different symplectic resolutions of V/G has been calculated by Bellamy [8] using the results of Namikawa [35]. Recently, using methods from [24] and [26], Hausen and Keicher [25] have proved that the ring $\mathcal{R}$ in our theorem is, in fact, the total coordinate ring of any resolution of V/G.

Our construction of the ring $\mathcal{R}$ is built up on the base of the total coordinate ring of the quotient V/G, which is equal to $\mathbb{C}[V]^{[G,G]}\subset\mathbb{C}[V]$ (see [5]). The generators of $\mathcal{R}$ are the classes of exceptional divisors of the resolution and strict transforms of Weil divisors associated to homogeneous generators of $\mathbb{C}[V]^{[G,G]}$. In our theorem these are the generators of types t_i^{-2} and $\phi_{ij}t_it_j$, respectively. We note that by the McKay correspondence (see [29]) the exceptional divisors are in correspondence with classes of symplectic reflections in G.

1.C. Contents of the article

In Section 2 we introduce definitions and recall results which are needed in subsequent sections. Next, in Section 3 we introduce the layout for constructing the total coordinate ring of resolutions of quotient singularities.

The idea is as follows. It is known that the Cox ring of a quotient singularity V/G is the ring of invariants of the commutator $\mathbb{C}[V]^{[G,G]}$ which decomposes into eigenspaces of the action of the abelianization $\mathrm{Ab}(G)$ which can be identified with the class group $\mathrm{Cl}(V/G)$. That is, $\mathbb{C}[V]^{[G,G]} = \bigoplus_{\mu\in G^\vee} \mathbb{C}[V]^G_\mu$ where $\mathbb{C}[V]^G_\mu$ is a rank 1 reflexive $\mathbb{C}[V]^G$-module associated to a character $\mu \in G^\vee$. Given a resolution $\varphi : X \to V/G$ the pushforward map allows one to identify spaces of sections of line bundles over X with submodules of these eigenmodules, $\Gamma(X, \mathcal{O}_X(D)) \hookrightarrow \Gamma(V/G, \mathcal{O}_{V/G}(\varphi_* D))$.

If $D = \sum_i a_i E_i$ where the E_i's are exceptional divisors of φ and the a_i's are integers, then it follows that $\Gamma(X, \mathcal{O}_X(D)) = \{f \in \Gamma(V/G, \mathcal{O}_{V/G}) : \forall_i \ \nu_{E_i}(f) \geq -a_i\}$ where the ν_{E_i}'s are divisorial valuations on the field $\mathbb{C}(X) = \mathbb{C}(V)^G$. This observation does not make much sense for a general D, but fortunately, in the case of symplectic resolutions each valuation ν_{E_i} can be related to a monomial valuation ν_{T_i} on the field of fractions of $\mathbb{C}[V]$ which comes from the action of the symplectic reflection T_i of V associated to E_i via the McKay correspondence. We use this idea to construct the ring $\mathcal{R}$ in Theorem 1.1.

The resolution $X \to V/G$ is recovered as a GIT quotient of $\mathrm{Spec}\,\mathcal{R}$ with respect to the action of the algebraic torus $\mathbb{T}_{\mathrm{Cl}(X)}$ which is associated to the grading of $\mathcal{R}$ in the class group $\mathrm{Cl}(X)$. We do it in Section 4. Although the construction of a GIT quotient and verification of its smoothness is, in general, conceptually clear, it is still computationally involved. First, we find out that a natural choice of a character κ of $\mathbb{T}_{\mathrm{Cl}(X)}$ yields a good GIT quotient (see Section 4.A). Second, we verify that the resulting GIT quotient is indeed smooth (see Section 4.C). To perform the calculations we use an embedding of $\mathrm{Spec}\,\mathcal{R}$ in an affine space in which the action of $\mathbb{T}_{\mathrm{Cl}(X)}$ is diagonal (see Section 3.D). Next, we consider a covering of $\mathrm{Spec}\,\mathcal{R}$ by sets associated to orbits of the big torus of the affine space. After all reductions and using symmetries, we are left with a manageable computational problem which is calculated and cross-calculated with standard algebraic software (see [22], [15], [41]).

A striking consequence of our calculation of the resolution $\varphi^\kappa : X^\kappa \to V/G$ is that it has the unique 2-dimensional fiber containing, as a component, the blowup of $\mathbb{P}^2$ at four general points. In terms of McKay correspondence this component is related to the only nontrivial central element of the group G. The Picard group of X^κ can be identified with the Picard group of this surface. In fact, different resolutions $\varphi : X \to V/G$ are in relation to the Zariski decomposition of divisors on that surface; we describe the geometry of all resolutions and their flops (see Section 5).

In the final section of the article we follow the program initiated in [1] and [17]. We produce a representation of the group G in the group of automorphisms of an abelian 4-fold which is the product of four elliptic curves with complex multiplication. The resulting quotient admits a symplectic resolution; hence, we obtain a new Kummer-type symplectic 4-fold (see Corollary 6.4). Thus, the second main result of our article is the following.

THEOREM 1.2

Let $\mathbb{E}$ be an elliptic curve with complex multiplication by $i = \sqrt{-1}$, and let G be the group defined in Section 2.C. Then there exists an embedding $G \to G' \subseteq \mathrm{Aut}(\mathbb{E}^4)$ such that the quotient $\mathbb{E}^4/G'$ has a resolution which is a Kummer symplectic 4-fold X with $b_2(X) = 23$ and $b_4(X) = 276$.

1.D. Notation

We use the standard notation in set and group theory. A commutator of G is denoted by $[G,G]$, and by $\mathrm{Ab}(G)$ we denote its abelianization $G/[G,G]$. A group of characters of G is $G^\vee = \mathrm{Hom}(G,\mathbb{C}^*)$. The quotient group $\mathbb{Z}/\langle r\rangle$ will be denoted by $\mathbb{Z}_r$ with $[d]_r$ denoting the class of $d \in \mathbb{Z}$. By $\lfloor d/r \rfloor$ we denote the integral part (rounded down) of the fraction d/r. If G acts on a set B, then by B^G we denote the set of the fixed points of the action. In particular, if G acts by homomorphisms on a ring B, then by B^G we denote the ring of invariants of the action.

A torus $\mathbb{T}$ means an algebraic torus with finite lattice of characters (also called monomials) $M = M_{\mathbb{T}} = \mathrm{Hom}_{\mathrm{alg}}(\mathbb{T},\mathbb{C}^*)$ and dual lattice of 1-parameter subgroups $N = M^*$. Given a finitely generated free abelian group M we define the associated torus $\mathbb{T}_M = \mathrm{Hom}(M,\mathbb{C}^*)$ with lattice of characters equal to M. We drop subscripts whenever it causes no confusion. The pairing $M \times N \to \mathbb{Z}$ is denoted by $(u,v) \mapsto \langle u,v\rangle$.

For $u \in M_{\mathbb{T}}$ by χ^u we denote the character $\chi^u : \mathbb{T} \to \mathbb{C}^*$. By $\mathbb{C}[M_{\mathbb{T}}]$ we denote the ring of Laurent polynomials graded in the lattice $M_{\mathbb{T}}$. More generally, for a cone $\sigma \subset M_{\mathbb{Q}}$ by $\mathbb{C}[M \cap \sigma]$ we understand the respective subalgebra of $\mathbb{C}[M]$. For the lattice N or M with a given basis by σ_N^+ or σ_M^+, respectively, we will denote the convex cone generated by the basis, which we refer to as a positive orthant. Thus, for a $\mathbb{C}$-linear space V we have $\mathbb{C}[V] \subset \mathbb{C}[M_{\mathbb{T}}]$, where $\mathbb{C}[V]$ is the ring of polynomials in linear coordinates of V and $\mathbb{T} = \mathbb{T}_V$ is the standard torus acting diagonally on coordinates of V.

All varieties are defined over the field of complex numbers. By $\mathbb{C}(X)$ we denote the field of rational functions on a variety X, while by $\mathbb{C}[X]$ we denote the algebra of global functions on X. By $\mathrm{Spec}\, A$ we understand the maximal spectrum of a ring A; in particular, if X is an affine variety, then $X = \mathrm{Spec}\,\mathbb{C}[X]$.

For a $\mathbb{Q}$-factorial variety X, by $\mathrm{N}^1(X)$ and $\mathrm{N}_1(X)$ we will denote the $\mathbb{R}$-linear space of divisors and of proper 1-cycles on X, respectively, modulo numerical equivalence. The class of a divisor D or a curve C will be denoted by $[D]$ and $[C]$, respectively. We have a natural intersection pairing of these two spaces.

A cone in an $\mathbb{R}$-vector space N generated by elements $v_1, v_2, \dots$ is, by definition, $\mathrm{cone}(v_1, v_2, \dots) = \sum_i \mathbb{R}_{\geq 0} v_i$. By $\mathrm{Nef}(X) \subset \mathrm{N}^1(X)$ we denote the cone of divisors which are nonnegative on effective 1-cycles. By $\mathrm{Eff}(X)$ and $\mathrm{Mov}(X)$ we denote the cones in $\mathrm{N}^1(X)$ which are $\mathbb{R}_{\geq 0}$-spanned, respectively, by the classes of effective divisors and by divisors whose linear systems have no fixed component.

A blowup of $\mathbb{P}^2$ in r general points will be denoted by $\mathbb{P}^2_r$. The blowup of $\mathbb{P}^2$ in three (different) collinear points will be denoted by $\mathbb{P}^2_{\overline{3}}$.

2. Preliminaries

2.A. The total coordinate ring

Let X be a normal $\mathbb{Q}$-factorial variety over the field of complex numbers. We assume that $\mathbb{C}[X]$ is finitely generated, its invertible elements are in $\mathbb{C}^*$, and X is projective (hence proper) over $\operatorname{Spec}\mathbb{C}[X]$. In what follows we also assume that the divisor class group $\operatorname{Cl}(X)$ is finitely generated.

To define the total coordinate ring of X, also called the Cox ring of X, we set

$$\mathcal{R}(X) = \bigoplus_{[D]\in\operatorname{Cl}(X)} \Gamma\big(X, \mathcal{O}_X(D)\big), \tag{2.1}$$

where $\Gamma(X, \mathcal{O}_X(D))$ denotes the space of global sections of the reflexive sheaf associated to the linear equivalence class of the divisor D. It is standard to use the identification $\Gamma(X, \mathcal{O}_X(D)) = \{f \in \mathbb{C}(X)^* : \operatorname{div}(f) + D \geq 0\} \cup \{0\}$ and to say that a nonzero $f \in \Gamma(X, \mathcal{O}_X(D))$ is associated with an effective divisor $D' = \operatorname{div}(f) + D$. The divisor D' is the zero locus of this section of $\mathcal{O}_X(D)$, which we will usually denote by $f_{D'}$. Note that by our assumptions the relation $D' \leftrightarrow f_{D'}$ is unique up to multiplication by a constant from $\mathbb{C}^*$.

Now, to define the multiplication in $\mathcal{R}(X)$ properly we need to set the inclusion $\Gamma(X, \mathcal{O}_X(D)) \subset \mathbb{C}(X)$ so that it does not depend on the choice of the divisor D in its linear equivalence class. If $\operatorname{Cl}(X)$ is a free finitely generated group (no torsions), then we choose divisors $D_1, \dots, D_m$ whose classes make a basis of $\operatorname{Cl}(X)$, and for D linearly equivalent to $\sum_i a_i D_i$, with $a_i \in \mathbb{Z}$, we define

$$\begin{aligned}\Gamma\big(X, \mathcal{O}_X(D)\big) &= \Big\{f \in \mathbb{C}(X)^* : \operatorname{div}(f) + \sum_i a_i D_i \geq 0\Big\} \cup \{0\} \\ &= \Big\{f \in \mathcal{O}_X\Big(X \setminus \bigcup_i D_i\Big) \subset \mathbb{C}(X) : \forall_i\ \nu_{D_i}(f) \geq -a_i\Big\},\end{aligned} \tag{2.2}$$

where the second equality makes sense if the D_i's are prime divisors and for every i we take $\nu_{D_i} : \mathbb{C}(X) \to \mathbb{Z} \cup \{\infty\}$, a valuation centered at the divisor D_i. In this way we present each graded piece of $\mathcal{R}(X)$ as a $\mathbb{C}[X]$-submodule of $\mathbb{C}(X)$, and consequently, we define the multiplication in $\mathcal{R}(X)$ as inherited from $\mathbb{C}(X)$. It can be checked that this defines a ring structure on $\mathcal{R}(X)$ which does not depend on the choice of D_i's. (Different choices give isomorphic rings.) This definition has to be adjusted if $\operatorname{Cl}(X)$ has torsions. We advise the reader to consult [4, Chapter 1] for details.

We recall that, for a Weil divisor D on a normal variety X, the sheaf $\mathcal{O}_X(D)$ is reflexive of rank 1. The association $D \to \mathcal{O}_X(D)$ determines the bijection between the class group $\operatorname{Cl}(X)$ and the isomorphism classes of reflexive rank 1 sheaves over X. Locally, in a similar manner, we can define a graded sheaf of divisorial algebras over X, which we will denote by $\mathcal{R}_X$.

Suppose that $\mathcal{R}(X)$ is a finitely generated $\mathbb{C}$-algebra. Then $\operatorname{Spec}\mathcal{R}(X)$ is an affine variety and the grading of $\mathcal{R}(X)$ in $\operatorname{Cl}(X)$ determines the action of the algebraic quasitorus $\mathbb{T}_{\operatorname{Cl}(X)} = \operatorname{Hom}(\operatorname{Cl}(X), \mathbb{C}^*)$.

As we assumed that X is projective over $\operatorname{Spec}\mathbb{C}[X]$, we can use GIT (see [34]) to recover X as a quotient of $\operatorname{Spec}\mathcal{R}(X)$. Namely, a relatively ample divisor H on X determines a linearization of this action on the trivial line bundle over $\operatorname{Spec}\mathcal{R}(X)$ and the irrelevant ideal $\mathcal{I}\mathrm{rr}_H = \sqrt{(\Gamma(X, \mathcal{O}(mH)) : m > 0)}$ which determines the set of unstable points with respect to this linearization. The quotient of its complement is X. The choice of a big but not necessarily ample divisor on X yields a GIT quotient which is birational to X.

This line of argument works more broadly as explained in [4, Chapter 1]. In the present article we will concentrate on a special situation of quotient singularity and its resolution, which will be discussed in detail in subsequent sections.

2.B. Small blowups of $\mathbb{P}^2$

The surface obtained by blowing up r general points on $\mathbb{P}^2$ will be denoted by $\mathbb{P}^2_r$. If $r \leq 3$, then $\mathbb{P}^2_r$ is a toric surface whose geometry and Cox ring are well known. Here, for the sake of completeness, we recall properties of $\mathbb{P}^2_4$.

If $\mathbb{P}^2_4$ is obtained by blowing up $\mathbb{P}^2$ at points $p_1, \ldots, p_4$, then by F_{0i} we denote the exceptional (-1)-curve over p_i and by F_{ij}, with $1 \leq i < j \leq 4$, we denote the strict transform of the line passing through $\{p_1, \ldots, p_4\} \setminus \{p_i, p_j\}$. With this notation we have the following well-known facts (see, e.g., [33]).

LEMMA 2.3

The surface $\mathbb{P}^2_4$ has the following geometry.

(1) *For two pairs of numbers $0 \leq i < j \leq 4$ and $0 \leq p < q \leq 4$ we have $F_{ij} \cdot F_{pq} = |\{i,j\} \cup \{p,q\}| - 3$.*

(2) *The surface $\mathbb{P}^2_4$ admits five distinct birational morphisms $\beta_i : \mathbb{P}^2_4 \to \mathbb{P}^2$ such that β_i contracts four (-1)-curves F_{pq} for $i \in \{p,q\}$.*

(3) *The surface $\mathbb{P}^2_4$ admits five conic fibrations $\alpha_i : \mathbb{P}^2_4 \to \mathbb{P}^1$, each of them having three reducible fibers $F_{rs} \cup F_{pq}$, where $\{i,p,q,r,s\} = \{0, \ldots, 4\}$.*

Let us consider a 5-dimensional $\mathbb{R}$-vector space W with a basis $e_0, \ldots, e_4$. We define the intersection product on W by setting $e_i^2 = -3$ and $e_i \cdot e_j = 1$ if $i \neq j$.

For $0 \leq i < j \leq 4$ we set $f_{ij} = (e_i + e_j)/2$. If moreover we set $\kappa = \sum_i e_i$ and $c_i = [\kappa - e_i]/2 = [(\sum_j e_j) - e_i]/2$, then $\kappa \cdot e_i = \kappa \cdot f_{ij} = 1$, $\kappa^2 = 5$, and $f_{ij}^2 = -1$. Also $e_i \cdot c_i = 2$ and $e_i \cdot c_j = 0$ for $i \neq j$; hence, the base $e_i/2$ for $i = 0, \ldots, 4$ is dual, in terms of the intersection product, to the base consisting of the c_i's. In particular, $\operatorname{cone}(e_0, \ldots, e_4)$ is where the intersection with the c_i's is positive.

By $\Lambda \subset W$ we denote the lattice spanned by the f_{ij}'s. We note that $\sum_i a_i(e_i/2) \in \Lambda$ if the a_i's are integral and $\sum_i a_i$ is even. The following can be easily verified.

LEMMA 2.4

The space $\mathrm{N}^1(\mathbb{P}^2_4) = \mathrm{N}_1(\mathbb{P}^2_4)$ with intersection product and the lattice of integral *divisors* $\mathrm{Pic}(\mathbb{P}^2_4)$ *can be identified with* $W \supset \Lambda$*, so that* $[F_{ij}] = f_{ij}$ *and* $[-K_{\mathbb{P}^2_4}] = \kappa$*. Under this identification the cone* $\mathrm{Eff}(\mathbb{P}^2_4)$ *is spanned by the* f_{ij}*'s. It has ten facets.*

- *Five facets of* $\mathrm{Eff}(\mathbb{P}^2_4)$ *are associated to morphisms* α_i *from Lemma 2.3, and they are contained in the facets of* σ^+*. Given* $i \in \{0, \ldots, 4\}$ *a facet of this type is spanned by six* f_{pq}*'s such that* $i \notin \{p, q\}$*, and it is perpendicular (in the sense of the intersection product) to* $c_i = (\kappa - e_i)/2$*.*
- *Five simplicial facets of* $\mathrm{Eff}(\mathbb{P}^2_4)$ *are associated to morphisms* β_i *from Lemma 2.3, and they are obtained by cutting* σ^+ *with a hyperplane perpendicular to* $(\kappa + e_i)/2$*. This facet of* $\mathrm{Eff}(\mathbb{P}^2_4)$ *is spanned by four* f_{pq}*'s such that* $i \in \{p, q\}$*.*

As a consequence, the cone $\mathrm{Nef}(\mathbb{P}^2_4)$ *is spanned by the classes of* $(\kappa \pm e_i)$*'s.*

The cone $\mathrm{Eff}(\mathbb{P}^2_4)$ is divided into Zariski chambers depending on the Zariski decomposition of the divisors whose classes are inside the interior of each chamber (see [6]). For example, the "central" chamber is the cone $\mathrm{Nef}(\mathbb{P}^2_4)$, and if $[D]$ is in the interior of $\mathrm{Nef}(\mathbb{P}^2_4)$, then the linear system $|mD|$, $m \gg 0$, determines a morphism into the projective space whose image is $\mathbb{P}^2_4$. Equivalently, $\mathrm{Proj} \bigoplus_{m \geq 0} \Gamma(\mathbb{P}^2_4, \mathcal{O}(mD)) = \mathbb{P}^2_4$. For D outside the nef cone the image of the rational map defined by $|mD|$ (or this projective spectrum) will depend on the intersection of D with (-1)-curves F_{ij}. This determines the division of $\mathrm{Eff}(\mathbb{P}^2_4)$ into the chambers in question. We summarize the information in Table 1 (see, e.g., [6]).

Table 1. Zariski chambers and birational images of $\mathbb{P}^2_4$.

$\mathrm{Proj}(\bigoplus_{m \geq 0} \Gamma(\mathbb{P}^2_4, \mathcal{O}(mD)))$	Number of chambers with D of this type
$\mathbb{P}^2_4$	1, nef cone
$\mathbb{P}^2_3$	10
$\mathbb{P}^2_2$	30
$\mathbb{P}^2_1$	20
$\mathbb{P}^2$	5, associated to simplicial facets of $\mathrm{Eff}(\mathbb{P}^2_4)$
$\mathbb{P}^1 \times \mathbb{P}^1$	10

The following result is known (see, e.g., [42]).

PROPOSITION 2.5

The total coordinate ring of $\mathbb{P}^2_4$ *coincides with the projective coordinate ring of the Grassmann variety* $\mathrm{Gr}(2, W)$ *of planes in a 5-dimensional vector space* W *which is embedded in* $\mathbb{P}(\bigwedge^2 W^*)$ *via the Plücker embedding. That is,* $\mathcal{R}(\mathbb{P}^2_4)$ *is the quotient of the polynomial ring in variables* w_{ij}*, for* $0 \leq i < j \leq 4$*, by the ideal generated by the following quadratic trinomials:*

$$w_{14}w_{23} + w_{13}w_{24} - w_{12}w_{34}, \qquad w_{04}w_{23} - w_{03}w_{24} - w_{02}w_{34},$$

$$w_{04}w_{13} + w_{03}w_{14} - w_{01}w_{34}, \qquad w_{04}w_{12} - w_{02}w_{14} - w_{01}w_{24},$$

$$w_{03}w_{12} + w_{02}w_{13} - w_{01}w_{23}.$$

The action of the Picard torus can be identified with that of $\mathbb{T}_\Lambda = \mathbb{T}_W/\langle -I_W\rangle$, *where* $\mathbb{T}_W$ *is the standard torus of the space* W *which acts on* $\bigwedge^2 W$ *with isotropy* $\langle -I_W\rangle$. *In other words, it is given by a grading in* $\Lambda \subset \mathbb{Z}^5$ *such that* $\deg w_{01} = (1,1,0,0,0), \deg w_{02} = (1,0,1,0,0), \ldots, \deg w_{34} = (0,0,0,1,1)$.

The GIT quotients of the affine variety $\operatorname{Spec}\mathcal{R}(\mathbb{P}^2_4)$ *depend on the choice of the linearization of the torus action given by a character in the lattice* Λ. *In particular, the choice of a divisor class in the interior of each of the Zariski chambers of* $\mathbb{P}^2_4$ *determines the quotient as in Table 1.*

2.C. The group of order 32

In what follows we will consider complex matrices acting on a linear space V. The case of our primary interest is when V is of even dimension and admits a nondegenerate linear 2-form, which we will call a *symplectic form*. The group of linear transformations preserving such a form we will denote by $\operatorname{Sp}(V)$ or $\operatorname{Sp}(\dim V, \mathbb{C})$.

By I_V we denote the identity matrix, and by $-I_V$ we denote its opposite. We will usually drop the subscript. Also, for every matrix A, we denote its opposite by $-A$, that is, $-I \cdot A = A \cdot (-I)$. For two matrices A and B we set $[A,B] = A \cdot B \cdot A^{-1} \cdot B^{-1}$. If the linear space fixed by a matrix is of codimension 1, then we say that it is a *quasireflection*. The groups which we will consider contain no quasireflections. If $A \in \operatorname{Sp}(V)$ and its fixed point set is of codimension 2, then we call it a *symplectic reflection*.

For V of dimension 4 with coordinates $x_1, \ldots, x_4$ we consider the following matrices in $\operatorname{GL}(V)$:

$$T_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},$$

$$T_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad T_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{2.6}$$

$$T_4 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}.$$

We note that the symplectic form $\omega = dx_1 \wedge dx_3 + dx_2 \wedge dx_4$ is preserved by each of the T_i's. Moreover, they are symplectic reflections of order 2.

We list the following facts which can be verified easily.

LEMMA 2.7

If for $i \neq j$ we set $R_{ij} = T_i \cdot T_j$, then R_{ij}'s are of order 4 and moreover the following holds:

(1) $[T_i, T_j] = R_{ij}^2 = -I$, $T_0 \cdot T_1 \cdot T_2 \cdot T_3 \cdot T_4 = I$, *and* $R_{ij} = -R_{ji} = R_{ji}^{-1}$;

(2) $[T_s, R_{ij}] = -I$ *if* $s \in \{i, j\}$ *and* $[T_s, R_{ij}] = I$ *if* $s \notin \{i, j\}$;

(3) *for two distinct pairs* i, j *and* p, q *we have* $[R_{ij}, R_{pq}] = -I$ *if* $\{i, j\} \cap \{p, q\} \neq \emptyset$ *and* $[R_{ij}, R_{pq}] = I$ *if* $\{i, j\} \cap \{p, q\} = \emptyset$.

Let G be the group generated by the reflections T_i in $\mathrm{Sp}(V)$. It is worthwhile to note that this group is conjugate in $\mathrm{Sp}(V)$ to the group generated by Dirac gamma matrices (cf. [43]).

LEMMA 2.8

The group G is of order 32 and it has the following properties.

(1) *There are 17 classes of conjugacy of elements in G: two of them consist of single elements I and $-I$, five of them contain pairs of opposite reflections $\pm T_i$, for $i = 0, \ldots, 4$, and ten conjugacy classes consist of pairs of opposite elements $\pm R_{ij}$, with $0 \leq i < j \leq 4$.*

(2) *The commutator $[G, G]$ of G coincides with its center, and it is generated by $-I$. The abelianization is* $\mathrm{Ab}(G) = G/[G, G] = \mathbb{Z}_2^4$.

(3) *If $N(T_i) < G$ is the normalizer (or centralizer) of the reflection T_i, then $N(T_i)/\langle T_i \rangle \cong Q_8$, where Q_8 is the group of quaternions, or a binary-dihedral group, of order 8.*

2.D. Quotient singularities

Let $G \subset \mathrm{GL}(V)$ be a finite group with no quasireflections acting linearly and faithfully on $V \cong \mathbb{C}^n$. By $\mathbb{C}[V] \cong \mathbb{C}[x_1, \ldots, x_n]$ we understand the coordinate ring of the linear space with the ring of invariants denoted by $\mathbb{C}[V]^G$ which, by the Hilbert–Noether theorem, is a finitely generated $\mathbb{C}$-algebra. We set $Y = \operatorname{Spec} \mathbb{C}[V]^G$.

PROPOSITION 2.9

We have the isomorphisms: $\mathrm{Pic}(Y) = 1$, $\mathrm{Cl}(Y) = G/[G, G]$, *and* $\mathcal{R}(Y) = \mathbb{C}[V]^{[G,G]}$.

Proof

The first part is classical nowadays (see, e.g., [10]); the second part is in [5]. □

Recall that the abelianization $\mathrm{Ab}(G) = G/[G, G]$ is isomorphic to the group of characters $G^\vee = \mathrm{Hom}(G, \mathbb{C}^*)$. The ring $\mathcal{R}(Y) = \mathbb{C}[V]^{[G,G]}$, as a $\mathbb{C}[Y]$-module, is a direct sum of reflexive rank 1 modules associated to characters of the group G (see [40, Theorem 1.3]). That is, the grading of $\mathcal{R}(Y)$ is into the eigenspaces associated to the characters of G:

$$\mathcal{R}(Y) = \mathbb{C}[V]^{[G,G]} = \bigoplus_{\mu \in G^\vee} \mathbb{C}[V]^G_\mu, \tag{2.10}$$

where $\mathbb{C}[V]^G_\mu$ is a rank 1 reflexive $\mathbb{C}[Y] = \mathbb{C}[V]^G$-module on which G acts with character $\mu \in G^\vee$ (for discussion, see [18, Lemma 6.2]).

The following fact is in [18, Section 2].

LEMMA 2.11

Let $\varphi : X \to Y$ be a resolution of a quotient singularity. Then $\operatorname{Pic} X = \operatorname{Cl} X$ *is a free finitely generated abelian group.*

Proof

By the Grauert–Riemenschneider theorem we know that $\mathrm{H}^1(X, \mathcal{O}) = 0$; hence, it is enough to prove that $\operatorname{Pic} X = \mathrm{H}^2(X, \mathbb{Z})$ has no finite torsion. However, by [31, Theorem 7.8] we know that the fundamental group is trivial, so by the universal coefficients theorem, $\mathrm{H}^2(X, \mathbb{Z})$ has no torsion. $\square$

We note that the exceptional set of a resolution $\varphi : X \to Y = V/G$ is a divisor with m irreducible components E_i, and thus, $\operatorname{Cl} X$ is a free abelian group of rank m. Moreover, we have the exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^{m} \mathbb{Z}[E_i] \longrightarrow \operatorname{Cl} X \cong \mathbb{Z}^m \longrightarrow \operatorname{Cl} Y = \operatorname{Ab}(G) \longrightarrow 0, \tag{2.12}$$

where the right arrow is the pushforward morphism φ_*.

Now we assume that $\dim V = 2n$, G is a finite group in $\operatorname{Sp}(V)$, and there exists a symplectic resolution $\varphi : X \to Y = V/G$. That is, X admits a closed everywhere nondegenerated 2-form which restricts to the invariant symplectic form defined on the smooth locus of V/G. Then the morphism $\varphi : X \to Y$ is semismall, which means, in particular, that every exceptional divisor of φ in X is mapped to a codimension 2 component of the singular set associated to the fixed point locus of a symplectic reflection. In fact, G has to be generated by symplectic reflections, and we have the following version of McKay correspondence by Kaledin (see [29]).

THEOREM 2.13

There exists a natural basis of Borel–Moore homology $\mathrm{H}^c_\bullet(X, \mathbb{Q})$ whose elements are in bijection with the following objects:

- *irreducible closed subvarieties $Z \subset X$ for which it holds that $2 \operatorname{codim}_X Z = \operatorname{codim}_Y \varphi(Z)$;*
- *conjugacy classes of elements g in G.*

Under this correspondence every conjugacy class $[g]$ of an element $g \in G$ is related to $Z^{[g]} \subseteq X$ such that $\varphi(Z^{[g]}) = [V^g]$, where $[V^g]$ denotes the image in V/G of the linear subspace V^g of fixed points of g.

Since the exceptional divisors E_i of φ are contracted to a codimension 2 set in Y, their configuration is modeled on 2-dimensional Du Val singularities. Namely, if $C_i \subset E_i$ is an irreducible component of a general fiber of $\varphi_{|E_i}$, then C_i is a rational curve and the intersection matrix $(E_i \cdot C_j)$ is a direct sum of known Cartan-type matrices (see [44, Theorem 1.3], [2, Theorem 4.1]). If $[L_i] \in \mathrm{N}^1(X)$, for $i = 1, \ldots, m$, is the basis dual (in terms of the intersection) to that consisting of $[C_i] \in N_1(X)$, then $\mathrm{Pic}(X)$ is a sublattice of the lattice $\langle [L_1], \ldots, [L_m] \rangle$ and we note that the left-hand side arrow in the sequence (2.12) can be written as $E_i \mapsto \sum_j (E_i \cdot C_j)[L_j]$. Thus, the sequence (2.12) can be extended to the following diagram in which the left-hand side arrow in the lower sequence is a map of lattices of the same rank given by the intersection matrix $(E_i \cdot C_j)$:

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus_i \mathbb{Z}[E_i] \arrow[r] \arrow[d, equal] & \mathrm{Cl}(X) \arrow[r] \arrow[d, hook] & \mathrm{Cl}(V/G) \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \bigoplus_i \mathbb{Z}[E_i] \arrow[r] & \bigoplus_i \mathbb{Z}[L_i] \arrow[r] & Q \arrow[r] & 0 \end{tikzcd} \tag{2.14}$$

Here Q is the quotient of lattices $\bigoplus_i \mathbb{Z}[E_i] \hookrightarrow \bigoplus_i \mathbb{Z}[L_i]$. The homomorphism $\mathrm{Cl}(V/G) \hookrightarrow Q$ associates to a class of a Weil divisor D on V/G the class of $\sum_i (\varphi_*^{-1} D \cdot C_i)[L_i]$ in the quotient group Q.

We conclude this section by observing that the existence of the symplectic form on a crepant resolution of a quotient singularity follows by a known result of Beauville [7].

LEMMA 2.15

Let $G < \mathrm{Sp}(V)$ be a finite subgroup preserving a symplectic linear 2-form on a vector space V of dimension $2n$. Suppose that $\varphi : X \to Y = V/G$ is a resolution with exceptional divisors E_i, where $i = 1, \ldots, m$. Assume that for every i the image $\varphi(E_i)$ is of codimension 2 in $Y = V/G$ and in codimension 2 the morphism φ is a minimal resolution of Du Val singularities or φ is crepant. Then $\varphi : X \to Y$ is a symplectic resolution, that is, X admits a symplectic form.

Proof

By [7, Proposition 2.4] the symplectic form ω on V descends to the smooth locus of V/G and extends to a regular closed 2-form $\widetilde{\omega}$ on every resolution $\varphi : X \to Y$. The top exterior power $\widetilde{\omega}^{\wedge n}$ does not vanish outside the E_i's and because of our assumption on the E_i's it is nonzero on the E_i's as well. Thus, $\widetilde{\omega}$ is a symplectic form on X. ∎

2.E. Cyclic group quotients and monomial valuations

In this section we discuss a fundamental example of group action and a blowup of the resulting quotient. Let $\epsilon_r = \exp(2\pi i/r) \in \mathbb{C}^*$ be the primitive rth root of unity, $r > 1$. We consider the cyclic group $\langle \epsilon_r \rangle \subset \mathbb{C}^*$. We assume that the group $\langle \epsilon_r \rangle \cong \mathbb{Z}_r$ acts diagonally on the vector space V of dimension n with nonnegative weights $(a_1, \ldots, a_n)$, that is, $\epsilon_r(x_1, \ldots, x_n) = (\epsilon_r^{a_1} x_1, \ldots, \epsilon_r^{a_n} x_n)$, with $0 \leq a_i < r$, and at least two of the a_i's positive. Moreover, we assume that the action of $\mathbb{Z}_r$ is faithful, which means that $(a_1, \ldots, a_n, r) = 1$. This action extends to the action of $\mathbb{C}^* \supset \langle \epsilon_r \rangle$ with the same weights: for $t \in \mathbb{C}^*$ we take $t(x_1, \ldots, x_n) = (t^{a_1} x_1, \ldots, t^{a_n} x_n)$.

Let us describe this situation in toric terms. By M, let us denote the lattice with the basis $(u_1, \ldots, u_n)$ consisting of characters of the standard torus $\mathbb{T}_V$ of V so that $\chi^{u_i} = x_i$. If $N = M^*$ is the lattice of 1-parameter subgroups of the standard torus, then the $\mathbb{C}^*$-action in question is defined by $v \in N$ such that for every i we have $\langle v, u_i \rangle = a_i$. Equivalently, the action of $\mathbb{C}^*$ is associated to a grading $\deg_v$ on $\mathbb{C}[V] \subset \mathbb{C}[M]$ such that $\deg_v(\chi^u) = \langle v, u \rangle$. The composition of $\deg_v$ with the residue homomorphism $\mathbb{Z} \to \mathbb{Z}_r$ determines $\mathbb{Z}_r$-grading $[\deg_v]_r$ on $\mathbb{C}[V]$ associated to the action of $\langle \epsilon_r \rangle$. By $[\deg_v]_r$ we will denote both the grading on $\mathbb{C}[V] = \mathbb{C}[M \cap \sigma^+]$ and the associated group homomorphism $M \to \mathbb{Z}_r$.

Following [39], [28], and [29, Section 2] we state this definition.

DEFINITION 2.16

The monomial valuation associated to the cyclic group action described above is $\nu_v : \mathbb{C}(V) \to \mathbb{Z} \cup \{\infty\}$ such that for $f = \sum_j c_j \chi^{m_j} \neq 0$ it holds that $\nu_v(f) = \min\{\langle v, m_j \rangle : c_j \neq 0\}$ and moreover $\nu_v(f_1/f_2) = \nu_v(f_1) - \nu_v(f_2)$.

EXAMPLE 2.17

In the situation introduced above we define a lattice $N_{v/r} = N + (v/r) \cdot \mathbb{Z}$ where the sum is taken in $N_{\mathbb{Q}}$. Its dual $M_{v/r} \subset M$ consists of characters on which v/r assumes integral values. In fact, $M_{v/r}$ is the kernel of $[\deg_v]_r$. Thus, $\mathbb{C}[M_{v/r} \cap \sigma_M^+] \subset \mathbb{C}[V]$ is the ring of invariants of the action of $\langle \epsilon_r \rangle$ on V, or $Y_{v/r} := \operatorname{Spec} \mathbb{C}[M_{v/r} \cap \sigma_M^+]$ is the quotient $V/\langle \epsilon_r \rangle$. Equivalently, $Y_{v/r}$ is an affine toric variety associated to the cone σ_N^+ and the lattice $N_{v/r}$. We define a toric variety $X_{v/r}$ whose fan is defined by taking a ray in $N_{v/r}$ that is generated by v/r and subdividing the cone σ_N^+ into a simplicial fan. The induced morphism $\varphi : X_{v/r} \to Y_{v/r}$ is proper and birational, and its exceptional set is an irreducible divisor $E_{v/r}$ which maps to $V^{\langle \epsilon_r \rangle}$ as a weighted projective space bundle, with fiber $\mathbb{P}(a_i : a_i > 0)$.

Let us consider the rank $n+1$ lattice $\widehat{M}_{v/r}$, which is a sublattice of $M \times \mathbb{Z}$ generated by (u_i, a_i), for $i = 1, \ldots, n$, and $(0, -r)$. By $\widehat{\sigma}^+$ we denote the cone in $M_{\mathbb{R}} \times \mathbb{R}$ spanned by these generators, and by $p_2 : \widehat{M}_{v/r} \to \mathbb{Z}$ we denote the projection onto the last coordinate. Let us note that the kernel of p_2 coincides with $M_{v/r}$. If fact, if $p_1 : M \times \mathbb{Z} \to M$ is the projection to the first factor, then on $\widehat{M}_{v/r}$ the composition $[\deg_v]_r \circ p_1$ coincides with p_2 composed with the residue homomorphism $\mathbb{Z} \to \mathbb{Z}_r$. By $\psi : \mathbb{C}[\widehat{M}_{v/r} \cap \widehat{\sigma}^+] \to \mathbb{C}[M \cap \sigma_M^+]$ let us denote the homomorphism of polynomial rings induced by the projection p_1, so that $\psi(\chi^{(u_i, a_i)}) = \chi^{u_i}$ and $\psi(\chi^{(0,-r)}) = 1$.

PROPOSITION 2.18

In the above situation the following statements hold.

(1) $\mathrm{Cl}(Y_{v/r}) = \mathbb{Z}_r$ *and* $\mathcal{R}(Y_{v/r}) = \mathbb{C}[M \cap \sigma_M^+]$ *with grading* $[\deg_v]_r$.

(2) $\mathrm{Cl}(X_{v/r}) = \mathbb{Z}$ *and* $\mathcal{R}(X_{v/r}) = \mathbb{C}[\widehat{M}_{v/r} \cap \widehat{\sigma}^+]$ *with grading* $\mathcal{R}(X_{v/r}) = \bigoplus_{d\in\mathbb{Z}} \mathcal{R}(X_{v/r})_d$ *associated to the projection* p_2.

(3) $\mathcal{R}(Y_{v/r})_0 = \mathcal{R}(X_{v/r})_0 = \mathbb{C}[V]^{\langle \epsilon_r \rangle}$, *and if* $\mathcal{R}(X_{v/r})^+ = \bigoplus_{d\geq 0} \mathcal{R}(X_{v/r})_d$, *then* $X_{v/r} = \operatorname{Proj} \mathcal{R}(X_{v/r})^+$ *and* $\mathcal{O}_{X_{v/r}}(-E_{v/r}) = \mathcal{O}_{\operatorname{Proj}(\mathcal{R}(X_{v/r}))^+}(r)$.

(4) *The valuation* ν_v *restricted to* $\mathbb{C}(X_{v/r}) = \mathbb{C}(Y_{v/r}) \subset \mathbb{C}(V)$ *coincides with* $r \cdot \nu_E$, *where* ν_E *is the divisorial valuation centered at* $E_{v/r}$.

Proof

The proof uses toric geometry. Let $\widehat{N}_{v/r}$ be a lattice dual to $\widehat{M}_{v/r} \subset M \times \mathbb{Z}$ with the basis $v_0, v_1, \ldots, v_n$ such that $\langle v_i, (u_j, a_j)\rangle = 1$ for $1 \leq i = j \leq n$, $\langle v_0, (0, -r)\rangle = 1$, and the other products are zero. We check that the homomorphism dual to the inclusion $M_{v/r} \hookrightarrow \widehat{M}_{v/r}$ is $\widehat{N}_{v/r} \to N_{v/r}$, where the v_i's are sent to the elements of the basis of N and $v_0 \mapsto v/r$. Indeed, if $u = b_0(0, -r) + \sum_1^n b_i(u_i, a_i)$ is in $M_{v/r} \subset \widehat{M}_{v/r}$, then $b_0 = \sum_1^n b_i(a_i/r)$; hence, the claim follows. Now we can use standard arguments in toric geometry (see, e.g., [14]). □

COROLLARY 2.19

Suppose that the situation is as introduced above. Then the projection-induced homomorphism

$$\psi : \mathbb{C}[\widehat{M}_{v/r} \cap \widehat{\sigma}^+] = \mathcal{R}(X_{v/d}) \to \mathbb{C}[M \cap \sigma_M^+] = \mathcal{R}(Y_{v/r}) \tag{2.20}$$

is a homomorphism of graded $\mathbb{C}[V]^{\langle \epsilon_r \rangle}$*-algebras. More precisely, for* $d \in \mathbb{Z}$ *we have the induced injective homomorphism of the dth graded pieces* $(\psi)_d : \mathcal{R}(X_{v/r})_d \hookrightarrow \mathcal{R}(Y_{v/r})_{[d]_r}$ *as modules over* $\mathbb{C}[V]^{\langle \epsilon_r \rangle} = \mathcal{R}(X_{v/r})_0 = \mathcal{R}(Y_{v/r})_0$, *and the following holds:*

$$\psi\big(\mathcal{R}(X_{v/r})_d\big) = \big\{ f \in \mathcal{R}(Y_{v/r})_{[d]_r} : \nu_v(f) \geq d \big\}. \tag{2.21}$$

3. The total coordinate ring of symplectic resolutions of a quotient

3.A. The pushforward map

We begin this section by discussing a somewhat more general situation than what is needed to tackle the problem of our interest. Let $\varphi : X \to Y$ be a projective birational morphism of normal $\mathbb{Q}$-factorial varieties which satisfy assumptions formulated at the beginning of Section 2.A. By the $\mathbb{Q}$-factoriality of Y we know that the exceptional set of the morphism φ is a Weil divisor with components denoted by E_i. The pushforward map of codimension 1 cycles $\varphi_* : \mathrm{Cl}(X) \to \mathrm{Cl}(Y)$ is surjective, and its kernel is generated by the classes of the E_i's (cf. (2.12)).

Moreover, φ determines the morphism of the respective total coordinate rings, which we will denote by φ_* as well. Namely, for a reflexive sheaf $\mathcal{L} = \mathcal{O}_X(D)$ its reflexive pushforward $\varphi_*\mathcal{L}^{\vee\vee}$ is isomorphic to $\mathcal{O}_Y(\varphi_* D)$. Thus, pushing down the sections determines the injective homomorphism of spaces

$$\Gamma\big(X, \mathcal{O}_X(D)\big) \to \Gamma\big(Y, \mathcal{O}_Y\big(\varphi_*(D)\big)\big) \tag{3.1}$$

associated to the inclusion

$$\big\{f \in \mathbb{C}(X)^* : \operatorname{div}(f) + D \geq 0\big\} \hookrightarrow \big\{f \in \mathbb{C}(Y)^* : \operatorname{div}(f) + \varphi_*(D) \geq 0\big\}. \tag{3.2}$$

This yields the homomorphism of graded rings $\varphi_* : \mathcal{R}(X) \to \mathcal{R}(Y)$. Note that we use the fact that $\mathbb{C}(X) = \mathbb{C}(Y)$. The well-definedness of the homomorphism φ_* follows from the construction in [4, Section 1.4] (see [26]). For example, in the case when $\mathrm{Cl}(X)$ is torsion-free, which is the case of our primary interest (cf. Lemma 2.11), we choose divisors D_i on X whose classes generate $\mathrm{Cl}(X)$ and use the construction explained in Section 2.A (see also [4, Construction 1.4.1.1]) to define $\mathcal{R}(X)$. Subsequently we use divisors $\varphi_*(D_i)$ and [4, Construction 1.4.2.1] to define $\mathcal{R}(Y)$. We summarize this short general introduction by stating a result from a paper by Hausen, Keicher, and Laface [26, Proposition 2.2] to which we refer the reader for details.

PROPOSITION 3.3

Let $\varphi : X \to Y$ be a proper birational morphism of varieties which satisfy assumptions stated at the beginning of Section 2.A. The exceptional set of φ is equal to $\bigcup_{i=1}^r E_i$, where the E_i's are prime divisors. Then there exists a canonical surjective homomorphism $\varphi_* : \mathcal{R}(X) \to \mathcal{R}(Y)$ which agrees with the homomorphism of gradings $\varphi_* : \mathrm{Cl}(X) \to \mathrm{Cl}(Y)$. The kernel of φ_* contains elements $1 - f_{E_i}$, where $f_{E_i} \in \Gamma(X, \mathcal{O}(E_i))$ is a section defining E_i. Moreover, if $\mathcal{R}(X)$ is finitely generated, then in fact $\ker \varphi_* = (1 - f_{E_i} : i = 1, \ldots, r)$.

The case of a blowup of a cyclic singularity discussed in Section 2.E is a particular example of this situation. In particular, the homomorphism ψ introduced there is the pushforward φ_* of the respective Cox rings (cf. Corollary 2.19).

EXAMPLE 3.4

Let us consider the case of a resolution of a surface A_1-singularity. That is, $Y = V/\mathbb{Z}_2$ where $V = \mathbb{C}^2$ and the nontrivial element of $\mathbb{Z}_2$ acts as the $-I$ matrix. The $\varphi : X \to Y$ is the blowdown of a (-2)-curve. Clearly the situation is toric, and the elements of both $\mathcal{R}(Y) = \mathbb{C}[V]$ and of $\mathcal{R}(X)$ are linear combinations of monomials which can be visualized as points in a lattice of rank 2 or 3, respectively.

In Figure 1 we present $\varphi_*(\mathcal{R}(X))_d$ as a submodule of $\mathcal{R}(Y)_{[d]_2} \subset \mathbb{C}[V]$. The monomials in $\mathbb{C}[V]$ are integral lattice points in the positive quadrant on the plane which is indicated by the solid line segments; the dot in the lower-left corner is $(0,0)$. These monomials which are in $\varphi_*(\mathcal{R}(X)_d)$ are denoted by $\bullet$, while those which are not in $\varphi_*(\mathcal{R}(X)_d)$ are denoted by $\circ$. The skewed dotted line indicates where the monomial valuation $\nu_{(1,1)}$ assumes value d (cf. (2.21)). For $d \leq 1$ we have $\varphi_*(\mathcal{R}(X)_d) = \mathcal{R}(Y)_{[d]_2}$.

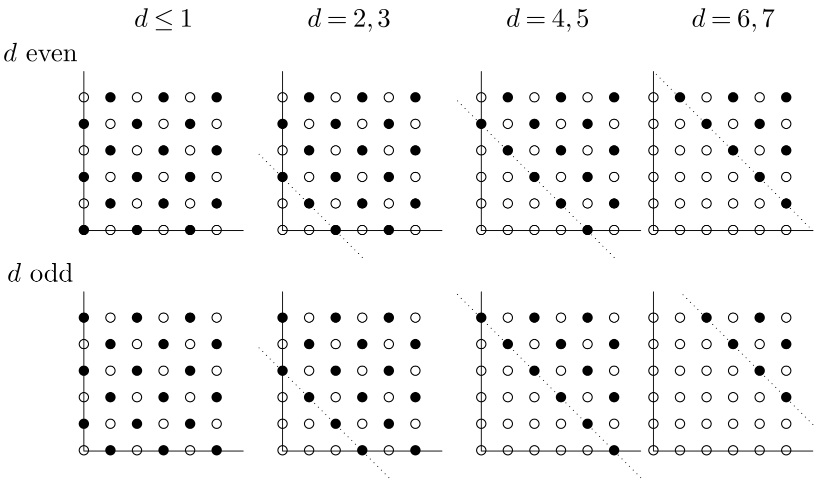

Figure 1. Resolution of A_1-singularity; the image $\varphi_*(\mathcal{R}(X)_d)$ in $\mathcal{R}(Y)_{[d]_2}$.

3.B. Cox ring of a resolution of a quotient singularity

From now on we consider the case which is of primary interest. Let $G \subset \mathrm{GL}(V)$ be a finite group without quasireflections with $Y = V/G = \operatorname{Spec} \mathbb{C}[V]^G$ the quotient. Suppose that $\varphi : X \to Y$ is a resolution of singularities. Now, because of Lemma 2.11, $\mathrm{Cl}(X) = \mathrm{Pic}(X)$ is free abelian of rank, say, m. Then the group $\mathrm{Hom}(\mathrm{Cl}(X), \mathbb{C}^*)$ is an algebraic torus $\mathbb{T} = \mathbb{T}_{\mathrm{Cl}(X)} \cong (\mathbb{C}^*)^m$ with coordinate ring $\mathbb{C}[\mathrm{Cl}(X)]$. The grading of $\mathcal{R}(X)$ in $\mathrm{Cl}(X)$ is associated to the action of $\mathbb{T}$ on $\mathcal{R}(X)$ with the ring of invariants equal to $\mathcal{R}(X)_0 = \mathbb{C}[V]^G$.

If $\mathcal{R}(X)$ is a finitely generated $\mathbb{C}$-algebra, then we can present the emerging objects in a single diagram

$$
\begin{tikzcd}
\operatorname{Spec}\mathcal{R}(X) \arrow[r, dashed] \arrow[dr, "\mathbb{T}"'] & X \arrow[d] & V \arrow[dl, "G"'] \arrow[d, "{[G,G]}"] \\
& V/G & V/[G,G] \arrow[l, "\mathrm{Ab}(G)"] \arrow[ull]
\end{tikzcd}
\tag{3.5}
$$

where $X \to V/G$ is the resolution of singularities. Moreover, $V \to V/[G,G] \to V/G$ and $\operatorname{Spec}\mathcal{R}(X) \to V/G$ are the (categorical) quotients with respect to appropriate group actions (spectra of rings of invariants), and the rational map $\operatorname{Spec}\mathcal{R}(X) \dashrightarrow X$ is a GIT quotient. The morphism of affine schemes $V/[G,G] = \operatorname{Spec}\mathcal{R}(V/G) \to \operatorname{Spec}\mathcal{R}(X)$ is the map of varieties associated to $\varphi_* : \mathcal{R}(X) \to \mathcal{R}(V/G)$ introduced above in Proposition 3.3.

The action of the torus $\mathbb{T} = \mathbb{T}_{\mathrm{Cl}(X)}$ on $\operatorname{Spec}\mathcal{R}(X)$ is associated to the multiplication homomorphism

$$
\mathcal{R}(X) \to \mathcal{R}(X) \otimes \mathbb{C}\big[\mathrm{Cl}(X)\big]
\tag{3.6}
$$

sending $f \in \Gamma(X, \mathcal{O}_X(D))$ to $f \cdot \chi^{[D]}$, with class $[D] \in \mathrm{Cl}(X)$ defining the character $\chi^{[D]}$ of the torus in question. Here we make an identification $\mathcal{R}(X) \otimes \mathbb{C}[\mathrm{Cl}(X)] = \mathcal{R}(X)[\mathrm{Cl}(X)]$ of the tensor product with Laurent polynomials with coefficients in $\mathcal{R}(X)$.

We define a map

$$\Theta : \mathcal{R}(X) \to \mathbb{C}[V]^{[G,G]} \otimes \mathbb{C}\big[\mathrm{Cl}(X)\big], \tag{3.7}$$

which to $f \in \Gamma(X, \mathcal{O}_X(D))$ associates $f \cdot \chi^{[D]} \in \Gamma(Y, \mathcal{O}_Y(\varphi_*(D))) \otimes \chi^{[D]}$, where $\chi^{[D]}$ denotes the character of $\mathbb{T}$. That is, Θ is a composition of the pushing down (3.1) and multiplication (3.6).

PROPOSITION 3.8

The map Θ defined above is injective.

Proof

If $\Theta(f_1) = \Theta(f_2)$, then they are both in the same space $\Gamma(X, \mathcal{O}_X(D))$. However, the map $\Gamma(X, \mathcal{O}_X(D)) \to \Gamma(Y, \mathcal{O}_Y(\varphi_*(D)))$ is injective—hence, the claim. $\square$

Now we know that the total coordinate ring of X can be realized as a subring of the known ring $\mathbb{C}[V]^{[G,G]} \otimes \mathbb{C}[\mathrm{Cl}(X)] = \mathcal{R}(V/G)[\mathrm{Cl}(X)]$; the problem now is to construct generators of this subring.

If $G \subset \mathrm{Sp}(V)$ is a symplectic group and $\varphi : X \to V/G$ is a symplectic resolution, then we are in the situation of (2.14) and the description of elements of $\mathcal{R}(X)$ can be made even more transparent. Recall that $[L_i]$, with $i = 1, \dots, m$, is a $\mathbb{Q}$-basis of $\mathrm{N}^1(X)$, dual to the classes of C_i's, components of fibers of $\varphi_{|E_i}$ (see (2.14)). Then we have the embedding of lattices $\mathrm{Cl}(X) \hookrightarrow \bigoplus_i \mathbb{Z}[L_i]$. This yields an embedding $\mathbb{C}[\mathrm{Cl}(X)] \hookrightarrow \mathbb{C}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ where t_i's are variables associated to $[L_i]$'s, that is, $t_i = \chi^{[L_i]}$. By

$$\overline{\Theta} : \mathcal{R}(X) \to \mathcal{R}(V/G)[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \tag{3.9}$$

we denote the composition of the homomorphism Θ with the extension of coefficients $\mathbb{C}[\mathrm{Cl}(X)] \hookrightarrow \mathbb{C}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. Let us note the following consequence of the construction of $\overline{\Theta}$.

LEMMA 3.10

The composition of $\overline{\Theta} : \mathcal{R}(X) \to \mathcal{R}(V/G)[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ with the evaluation homomorphism $\mathrm{ev}_1 : \mathcal{R}(V/G)[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \to \mathcal{R}(V/G)$ such that $\mathrm{ev}_1(t_i) = 1$, for every $i = 1, \dots, m$, is equal to the pushforward homomorphism $\varphi_* : \mathcal{R}(X) \to \mathcal{R}(V/G)$.

COROLLARY 3.11

Assume that we are in the situation discussed above. Let $f_D \in \mathcal{R}(V/G) = \mathbb{C}[V]^{[G,G]}$ be a nonzero element associated to an effective Weil divisor D on V/G. Let $\overline{D} = \varphi_*^{-1} D$ be its strict transform in X. If $f_{\overline{D}} \in \mathcal{R}(X)$ is associated to

$\overline{D}$ *in the total coordinate ring of* X*, then*

$$\overline{\Theta}(f_{\overline{D}}) = f_D \cdot \prod_i t_i^{(\overline{D}\cdot C_i)}.$$

Proof
The $\mathcal{R}(V/G)$-coefficient of $f_{\overline{D}}$ is f_D, and it remains to verify the degree of $f_{\overline{D}}$ with respect to $\mathrm{Cl}(X)$ which is provided by the homomorphism $\mathrm{Cl}(X) \hookrightarrow \bigoplus_i \mathbb{Z}[L_i]$ in (2.14). □

3.C. From the group G to a torus $\mathbb{T}$

From this point on by G we denote the group introduced in Section 2.C. The ring of invariants of $[G,G] = \langle -I_V\rangle$ is generated by quadratic forms in $\mathbb{C}[x_1,x_2,x_3,x_4]$. We note that the linear space of forms is S^2V^*. The following observation can be verified easily.

LEMMA 3.12
The action of $\mathrm{Ab}(G) = \mathbb{Z}_2^4$ *on* S^2V^* *yields a decomposition of* S^2V^* *into the sum of* 1*-dimensional eigenspaces generated by the functions* ϕ_{ij} *given in the following table. The action of the class of* T_i *in* $\mathrm{Ab}(G)$ *on the function* ϕ_{rs} *is by multiplication by* ± 1*, as indicated in the following table:*

(3.13)

Function	T_0	T_1	T_2	T_3	T_4
$\phi_{01} = -2(x_1x_4 + x_2x_3)$	$-$	$-$	$+$	$+$	$+$
$\phi_{02} = 2\sqrt{-1}(-x_1x_4 + x_2x_3)$	$-$	$+$	$-$	$+$	$+$
$\phi_{03} = 2\sqrt{-1}(x_1x_2 + x_3x_4)$	$-$	$+$	$+$	$-$	$+$
$\phi_{04} = 2(-x_1x_2 + x_3x_4)$	$-$	$+$	$+$	$+$	$-$
$\phi_{12} = 2(x_1x_3 - x_2x_4)$	$+$	$-$	$-$	$+$	$+$
$\phi_{13} = -x_1^2 - x_2^2 + x_3^2 + x_4^2$	$+$	$-$	$+$	$-$	$+$
$\phi_{14} = \sqrt{-1}(x_1^2 + x_2^2 + x_3^2 + x_4^2)$	$+$	$-$	$+$	$+$	$-$
$\phi_{23} = \sqrt{-1}(-x_1^2 + x_2^2 - x_3^2 + x_4^2)$	$+$	$+$	$-$	$-$	$+$
$\phi_{24} = x_1^2 - x_2^2 - x_3^2 + x_4^2$	$+$	$+$	$-$	$+$	$-$
$\phi_{34} = 2(x_1x_3 + x_2x_4)$	$+$	$+$	$+$	$-$	$-$

The labeling of functions ϕ_{rs} indicates an isomorphism between S^2V^* and $\bigwedge^2 W^*$ where W is a 5-dimensional space with coordinates $t_0,\dots,t_4$; under this isomorphism the function ϕ_{rs} is associated to the 2-form $t_r \wedge t_s$.

In fact, the representation of $\mathrm{Sp}(V)$ on $\bigwedge^2 V$ splits into $W \bigoplus \mathbb{C}$ where $\mathbb{C}$ stands for the trivial representation and the action of $\mathrm{Sp}(4,\mathbb{C})$ on W is associated to the double cover $\mathrm{Sp}(4,\mathbb{C}) \to \mathrm{SO}(5,\mathbb{C})$ whose kernel is $-I$. Then we have the natural isomorphism of $\mathrm{Sp}(4,\mathbb{C})$ representations $\bigwedge^2 W \cong S^2V$ (see [20, Section 16.2]). The coordinates ϕ_{rs} diagonalize the induced action of the T_i's.

Let $\mathbb{T}_W$ be the standard torus of W with $\mathrm{Hom}(\mathbb{T}_W,\mathbb{C}^*) \cong \bigoplus_{i=0}^4 \mathbb{Z}e_i$ and characters $t_0 = \chi^{e_0},\dots,t_4 = \chi^{e_4}$. Let $\Lambda \subset \mathrm{Hom}(\mathbb{T}_W,\mathbb{C}^*)$ be the index 2 sublattice of $\bigoplus_{i=0}^4 \mathbb{Z}e_i$ consisting of characters invariant by multiplication by $-I_W$; that is,

$\Lambda = \{\sum a_i e_i : a_i \in \mathbb{Z}, 2 \mid \sum a_i\}$. We have a surjective morphism $\mathbb{T}_W \to \mathbb{T}_\Lambda$ with kernel $\langle -I_W \rangle$ which is associated to the inclusion of lattices of characters. Since $-I_W$ acts trivially on $\bigwedge^2 W^*$ the action of $\mathbb{T}_W$ on $\bigwedge^2 W^*$ descends to the action of $\mathbb{T}_\Lambda$.

Let $\widetilde{T}_i : W^* \to W^*$ be a homomorphism defined as $\widetilde{T}_i(t_i) = -t_i$, $\widetilde{T}_i(t_j) = t_j$, for $j \neq i$. We have an injection $\bigoplus_{i=0}^4 \mathbb{Z}_2 \widetilde{T}_i \hookrightarrow \mathbb{T}_W$ and, thus, a morphism $\bigoplus_{i=0}^4 \mathbb{Z}_2 \widetilde{T}_i \to \mathbb{T}_\Lambda$ with kernel $\langle -I_W \rangle$. We summarize this discussion in the following.

LEMMA 3.14

The homomorphism of groups $\mathrm{Ab}(G) = G/[G,G] \to \mathbb{T}_\Lambda$ *which maps the class of* $\pm T_i$ *in* $\mathrm{Ab}(G)$ *to the class of* $\pm \widetilde{T}_i$ *in* $\mathbb{T}_\Lambda$ *makes the isomorphism* $S^2 V^* \cong \bigwedge^2 W^*$ *equivariant with respect to the action of* $\mathrm{Ab}(G)$.

We note that the above homomorphism $\mathrm{Ab}(G) \to \mathbb{T}_\Lambda$ can be described in terms of characters of these groups. Let $\bigoplus_{i=0}^4 \mathbb{Z} e_i \to \bigoplus_{i=0}^4 \mathbb{Z}_2 e_i$ be the reduction modulo 2. The latter group can be interpreted as the group of characters of $\bigoplus_{i=0}^4 \mathbb{Z}_2 \widetilde{T}_i$. The morphism $\bigoplus_{i=0}^4 \mathbb{Z}_2 \widetilde{T}_i \to \mathrm{Ab}(G)$ which maps $\widetilde{T}_i$ to $[\pm T_i]$ implies inclusion $\mathrm{Ab}(G)^\vee \hookrightarrow (\bigoplus_{i=0}^4 \mathbb{Z}_2 \widetilde{T}_i)^\vee = \bigoplus_{i=0}^4 \mathbb{Z}_2 e_i$, and because of the inclusion $\Lambda \hookrightarrow \bigoplus_{i=0}^4 \mathbb{Z} e_i$, we get a surjective homomorphism of groups of characters $\Lambda \to \mathrm{Ab}(G)^\vee$.

DEFINITION 3.15

Let $\mathcal{R} \subset \mathbb{C}[V] \otimes \mathbb{C}[\mathbb{T}_W] = \mathbb{C}[x_1, \ldots, x_4, t_0^{\pm 1}, \ldots, t_4^{\pm 1}]$ be the subring generated by the following functions:

- $\phi_{ij} \cdot t_i t_j$, where $0 \leq i < j \leq 4$;
- t_i^{-2}, where $i = 0, \ldots, 4$.

The torus $\mathbb{T}_W$ acts naturally on $\mathbb{C}[V] \otimes \mathbb{C}[\mathbb{T}_W]$ by multiplication of the right factor, and the inclusion $\mathcal{R} \subset \mathbb{C}[V] \otimes \mathbb{C}[\mathbb{T}_W]$ is $\mathbb{T}_W$-equivariant. We see that $-I_W$ acts on $\mathcal{R}$ trivially so the action of $\mathbb{T}_W$ on $\mathcal{R}$ descends to the action of $\mathbb{T}_\Lambda$. Note that we have a surjective homomorphism $\mathcal{R} \to \mathbb{C}[V]^{\langle -I_V \rangle} \subset \mathbb{C}[V]$ obtained by setting $t_i \mapsto 1$.

PROPOSITION 3.16

The induced homomorphism $\mathcal{R}^{\mathbb{T}_W} \to \mathbb{C}[V]^{[G,G]} \subset \mathbb{C}[V]$ *is an injection onto the ring of invariants* $\mathbb{C}[V]^G$. *Therefore,* $\mathcal{R}^{\mathbb{T}_W} \cong \mathbb{C}[V]^G$.

Proof

As noted in Lemma 3.14 we have an injection $\mathrm{Ab}(G) \hookrightarrow \mathbb{T}_\Lambda$, and we claim that the morphism $\mathcal{R} \to \mathbb{C}[V]^{[G,G]}$ is $\mathrm{Ab}(G)$-equivariant. Indeed, the action of $\mathrm{Ab}(G) \hookrightarrow \mathbb{T}_W$ on $\phi_{ij} t_i t_j$ agrees with that of G on ϕ_{ij}, while on t_i^2 the group $\mathrm{Ab}(G) \hookrightarrow \mathbb{T}_W$ acts trivially. Therefore, in particular, we have $\mathcal{R}^{\mathbb{T}_W} \to \mathbb{C}[V]^G$. The injectivity of this homomorphism is clear since $(t_i-1, i=0,\dots,4)\cap\mathbb{C}[\mathbb{T}_W]^{\mathbb{T}_W}=(0)$. It remains to prove surjectivity. To this end, we note that a monomial $\prod_{i,j}\phi_{ij}^{a_{ij}}\in\mathbb{C}[V]^{\langle -I\rangle}$ is G-invariant if, for every $k=0,\dots,4$, the sum $s_k=\sum_{k\in\{i,j\}}a_{ij}$ is divisible by 2. But then the monomial in question is the image of the $\mathbb{T}_W$-invariant monomial $\prod_{i,j}(\phi_{ij}\cdot t_it_j)^{a_{ij}}\prod_k(t_k^{-2})^{s_k/2}\in\mathcal{R}$. ☐

3.D. Generators of ideals

We present the ring $\mathcal{R}$ as the quotient ring of the graded polynomial ring $\mathbb{C}[w_{ij},u_k : k=0,\dots,4, 0\le i<j\le 4]$ with the grading in $\mathrm{Hom}(\mathbb{T}_W,\mathbb{C}^*)\cong\bigoplus_{m=0}^{4}\mathbb{Z}\cdot e_m$ given by the formula $\deg w_{ij}=e_i+e_j$, $\deg u_k=-2e_k$.

PROPOSITION 3.17

The homomorphism $\mathbb{C}[w_{ij},u_k : k=0,\dots,4,0\le i<j\le 4]\to\mathcal{R}$ *that sends* w_{ij} *to* $\phi_{ij}t_it_j$ *and* u_k *to* t_k^{-2} *is surjective and preserves grading. Its kernel, denoted by* $\mathcal{I}$, *is generated by the following homogeneous polynomials:*

$$w_{14}w_{23}+w_{13}w_{24}-w_{12}w_{34},\qquad w_{04}w_{23}-w_{03}w_{24}-w_{02}w_{34},$$
$$w_{04}w_{13}+w_{03}w_{14}-w_{01}w_{34},\qquad w_{04}w_{12}-w_{02}w_{14}-w_{01}w_{24},$$
$$w_{03}w_{12}+w_{02}w_{13}-w_{01}w_{23},$$
$$w_{02}w_{12}u_2-w_{03}w_{13}u_3+w_{04}w_{14}u_4,\qquad w_{01}w_{14}u_1-w_{02}w_{24}u_2+w_{03}w_{34}u_3,$$
$$w_{01}w_{13}u_1+w_{02}w_{23}u_2+w_{04}w_{34}u_4,\qquad w_{01}w_{12}u_1+w_{03}w_{23}u_3+w_{04}w_{24}u_4,$$
$$w_{03}w_{04}u_0-w_{13}w_{14}u_1+w_{23}w_{24}u_2,\qquad w_{02}w_{04}u_0+w_{12}w_{14}u_1+w_{23}w_{34}u_3,$$
$$w_{01}w_{04}u_0+w_{12}w_{24}u_2+w_{13}w_{34}u_3,\qquad w_{02}w_{03}u_0-w_{12}w_{13}u_1-w_{24}w_{34}u_4,$$
$$w_{01}w_{03}u_0+w_{12}w_{23}u_2+w_{14}w_{34}u_4,\qquad w_{01}w_{02}u_0+w_{13}w_{23}u_3-w_{14}w_{24}u_4,$$
$$w_{02}^2u_0+w_{12}^2u_1+w_{23}^2u_3+w_{24}^2u_4,\qquad w_{03}^2u_0+w_{13}^2u_1+w_{23}^2u_2+w_{34}^2u_4,$$
$$w_{01}^2u_1+w_{02}^2u_2+w_{03}^2u_3+w_{04}^2u_4,\qquad w_{04}^2u_0+w_{14}^2u_1+w_{24}^2u_2+w_{34}^2u_3,$$
$$w_{01}^2u_0+w_{12}^2u_2+w_{13}^2u_3+w_{14}^2u_4.$$

Proof

The first part is clear. The second part, that is, the generators of the kernel $\mathcal{I}$, is obtained by computer calculation. ☐

The next observation follows from Proposition 2.5.

COROLLARY 3.18

The ideal $\mathcal{I}_0=\mathcal{I}+(u_0,\dots,u_4)$ *descends to an ideal in* $\mathbb{C}[w_{ij}]=\mathbb{C}[w_{ij},u_k]/(u_0,\dots,u_4)$, *which is an ideal of the affine cone over the Grassmann variety* $\mathrm{Gr}(2,W)$ *embedded via Plücker embedding in* $\mathbb{P}(\bigwedge^2 W^*)$ *with the associated grading coming from the action of* $\mathbb{T}_W$. *In particular, we can identify* $\mathrm{Cl}(\mathbb{P}_4^2)$ *with* Λ *and we have a* $\mathbb{T}_\Lambda$*-equivariant embedding* $\mathrm{Spec}\,\mathcal{R}(\mathbb{P}_4^2)\hookrightarrow\mathrm{Spec}\,\mathcal{R}$.

3.E. Monomial valuations and the total coordinate ring

In Section 4 we will produce a GIT quotient X of $\operatorname{Spec}\mathcal{R}$ and prove that it is smooth so that the resulting morphism $\varphi: X \to V/G$ is a resolution of singularities (see Theorem 4.15). The aim of the present section is to relate $\mathcal{R}$ to $\mathcal{R}(X)$, the total coordinate ring of the resolution X.

To simplify the notation we set $\mathcal{P} := \mathbb{C}[V]^{\langle -I\rangle} = \mathcal{R}(V/G)$. The ring $\mathcal{R}$ is constructed as a subring of the Laurent polynomial ring $\mathcal{P}[t_0^{\pm 1}, \dots, t_4^{\pm 1}]$ with grading in a lattice $\Lambda \subset \mathbb{Z}^5$ inherited from the ambient polynomial ring (see Definition 3.15). In this construction each variable t_i is associated to the action of the symplectic reflection T_i (see Lemma 3.14).

On the other hand, the morphism $\overline{\Theta}: \mathcal{R}(X) \to \mathcal{P}[t_0^{\pm 1}, \dots, t_4^{\pm 1}]$, defined in (3.9), is an embedding (see Proposition 3.8). The grading of $\mathcal{R}(X)$ is in $\operatorname{Cl}(X)$, which embeds via (2.14) into $\mathbb{Z}^5$, so that $\operatorname{Cl}(X) = \Lambda$. Now, however, the t_i's are variables associated to exceptional divisors E_i of the symplectic resolution φ, which, by McKay correspondence (see Theorem 2.13), are in relation with the conjugacy classes of the T_i's. The composition of $\overline{\Theta}$ with evaluation $\mathrm{ev}_1: t_i \mapsto 1$ is $\varphi_*: \mathcal{R}(X) \to \mathcal{P}$ (see Lemma 3.10). The restriction of the evaluation ev_1 to $\mathcal{R}$ will be called $\Phi: \mathcal{R} \to \mathcal{P}$.

The definition of both $\mathcal{R}$ and $\overline{\Theta}(\mathcal{R}(X))$ depends on the meaning of the t_i's; the link is McKay correspondence. Thus, to relate these objects, following [39] and [29], we will use monomial valuations. Namely, by $\nu_i: (\mathcal{P}) = \mathbb{C}(V)^{\langle -I\rangle} \to \mathbb{Z} \cup \{\infty\}$ we denote the monomial valuation on the field of fractions of $\mathcal{P}$ associated to the action of T_i. By [29] we know that $(\nu_i)_{|\mathbb{C}(V)^G} = 2\nu_{E_i}$ (cf. Proposition 2.18).

Because of (2.10) we have the following decomposition into a sum of $\mathbb{C}[V]^G$-modules of eigenfunctions of the action of $\operatorname{Ab}(G)$:

$$\mathcal{P} = \bigoplus_{\mu \in G^\vee} \mathbb{C}[V]^G_\mu = \bigoplus_{\mu \in G^\vee} \mathcal{P}_{(\bar{\mu}(T_0), \dots, \bar{\mu}(T_4))}, \tag{3.19}$$

where $\bar{\mu}(T_i) = 0$ if $\mu(T_i) = 1$ and $\bar{\mu}(T_i) = 1$ if $\mu(T_i) = -1$. Note that this gives a $\mathbb{Z}_2^5$ grading on $\mathcal{P}$. The grading on $\mathcal{P}$ agrees with the $\mathbb{Z}^5$ grading on $\mathcal{R}$ and $\mathcal{R}(X)$ as well as with the valuations ν_i.

LEMMA 3.20

For every $f \in \mathcal{P}$ if a monomial $f t_0^{d_0} \cdots t_4^{d_4}$ is in either $\mathcal{R}$ or $\mathcal{R}(X)$, then $f \in \mathcal{P}_{([d_0]_2, \dots, [d_4]_2)}$. If $f \in \mathcal{P}_{(d_0, \dots, d_4)}$, then for every $i = 0, \dots, 4$ the valuation $\nu_i(f)$ has the same parity as d_i. In particular, $\nu_r(\phi_{ij}) = 1$ if $r \in \{i, j\}$ and $\nu_r(\phi_{ij}) = 0$ if $r \notin \{i, j\}$.

Proof

For $f t_0^{d_0} \cdots t_4^{d_4} \in \mathcal{R}$ it is enough to check the statement for generators of $\mathcal{R}$ (see Definition 3.15). If $f t_0^{d_0} \cdots t_4^{d_4} \in \mathcal{R}(X)$, then the statement follows from the definition of Θ (see (3.7)). The last part follows directly from Definition 2.16. □

EXAMPLE 3.21

Let $\overline{D}_{ij}$ be a divisor on X which is a strict transform via φ^{-1} of the Weil divisor D_{ij} on V/G associated to $\phi_{ij} \in \mathcal{R}(V/G) = \mathcal{P}$. Then the principal divisor on X of the function $\phi_{ij}^2 \in \mathbb{C}[V]^G \subset \mathbb{C}(X)$ satisfies the equality $\mathrm{div}_X(\phi_{ij}^2) = 2\overline{D}_{ij} + E_i + E_j$. Thus, if C_m is a general fiber of $\varphi_{|E_m}$, then $\overline{D}_{ij} \cdot C_m = 1$ if $m \in \{i,j\}$ and $\overline{D}_{ij} \cdot C_m = 0$ if $m \notin \{i,j\}$.

More generally we have the following.

LEMMA 3.22

Let $\overline{D}$ be a divisor in X which is a strict transform via φ^{-1} of an effective Weil divisor D on V/G. If $f_D \in \mathcal{R}(V/G) = \mathcal{P}$ is the element associated with D, then $\nu_i(f_D) = \overline{D} \cdot C_i$. Moreover, if $f_{\overline{D}} \in \mathcal{R}(X)$ is the element associated with $\overline{D}$, then

$$\overline{\Theta}(f_{\overline{D}}) = f_D t_0^{\nu_0(f_D)} \cdots t_4^{\nu_4(f_D)}.$$

Proof

Note that $f_D^2 \in \mathbb{C}[V]^G \subset \mathbb{C}(X)$ and

$$\mathrm{div}_X(f_D^2) = 2\overline{D} + \nu_{E_0}(f_D^2)E_0 + \cdots + \nu_{E_4}(f_D^2)E_4.$$

Since $\mathrm{div}_X(f_D^2) \cdot C_i = 0$ and $(\nu_i)_{|\mathbb{C}(X)} = 2\nu_{E_i}$ we get

$$2\overline{D} \cdot C_i = -\big(\nu_0(f_D)E_0 + \cdots + \nu_4(f_D)E_4\big) \cdot C_i,$$

and the first claim follows because $D_i \cdot C_i = -2$ and $D_j \cdot C_i = 0$ if $j \neq i$. The second statement follows from Corollary 3.11. $\square$

COROLLARY 3.23

In the notation introduced above the following statements hold: $\overline{\Theta}(f_{E_i}) = t_i^{-2}$ and $\overline{\Theta}(f_{\overline{D}_{ij}}) = \phi_{ij} t_i t_j$. Therefore, $\mathcal{R} \subseteq \overline{\Theta}(\mathcal{R}(X))$.

The following result has been anticipated in the cyclic quotient case (see (2.21) and Example 3.4).

PROPOSITION 3.24

The image via φ_* of the graded pieces of $\mathcal{R}(X)$ is determined by valuations ν_i in the following way:

$$\varphi_*\big(\mathcal{R}(X)_{(d_0,\ldots,d_4)}\big) = \big\{f \in \mathcal{P}_{([d_0]_2,\ldots,[d_4]_2)} : \forall_i \ \nu_i(f) \geq d_i\big\}. \tag{3.25}$$

Proof

We use the notation from Lemma 3.22. If $f = f_D$ and $d_i \leq \nu_i(f)$, then the numbers $a_i = (\nu_i(f) - d_i)/2$ are nonnegative integers because of Lemma 3.20 and

$$f_M := f_{\overline{D}} f_{E_0}^{a_0} \cdots f_{E_4}^{a_4} \in \mathcal{R}(X)_{(d_0,\ldots,d_4)}$$

is the element which is mapped to f_D via φ_*.

On the other hand, given an effective divisor M on X we can write it as $M = \overline{D} + \sum a_i E_i$, where $\overline{D}$ is the strict transform of $D := \varphi_*(M)$ and the a_i's are nonnegative integers. If $f_M = f_{\overline{D}} f_{E_0}^{a_0} \cdots f_{E_4}^{a_4}$ is in $\mathcal{R}(X)_{(d_0,\ldots,d_4)}$, then by the same arguments $\nu_i(\varphi_*(f_M)) = \nu_i(f_D) = d_i + 2a_i$. $\square$

4. GIT quotients of $\operatorname{Spec}\mathcal{R}$

We study linearizations and corresponding GIT quotients of $\operatorname{Spec}\mathcal{R}$. For a chosen one we prove its smoothness, and in this way, we obtain an explicit description of a resolution of V/G. In Section 5 we will use these results to show how to modify this resolution to obtain all other ones.

4.A. Linearization, stability, and isotropy

To construct a GIT quotient of $\operatorname{Spec}\mathcal{R}$ we need to choose a suitable linearization of the trivial line bundle. It will be represented by a character $\chi^u \colon \mathbb{T}_\Lambda \to \mathbb{C}^*$ of the 5-dimensional torus. We investigate the sets of stable and semistable points of $\operatorname{Spec}\mathcal{R}$ with respect to χ. In this section we explain how to check whether χ^u and these sets have properties needed to have a good description of the quotient, that is, satisfy Condition 4.9. Note that we do not explicitly compute the irrelevant ideal, that is, the ideal of the closed set of unstable points—we prefer to deal with the set of semistable points by using a description based on toric geometry, as explained below. As with the 2-dimensional quotients in [18, Section 4], the idea is to look at the embedding

$$\operatorname{Spec}\mathcal{R} \hookrightarrow \operatorname{Spec}\mathbb{C}[w_{ij}, u_k : k = 0,\ldots,4, 0 \le i < j \le 4] \simeq \mathbb{C}^{15}$$

such that $\mathbb{T}_\Lambda$ is a subtorus of the big torus $(\mathbb{C}^*)^{15}$ of the affine space, as described in Section 3.C. Then the set of semistable points can be presented as the intersection of $\operatorname{Spec}\mathcal{R}$ with certain orbits of the big torus (see Lemma 4.2 and Section 4.B).

We start from a few observations in a slightly more general setting. Let Z be an affine subvariety of $A \simeq \mathbb{C}^r$, invariant under a (diagonal) action of a subtorus $\mathbb{T}$ of $\mathbb{T}_A \simeq (\mathbb{C}^*)^r$. By $M_\mathbb{T}$ and $\widehat{M}$ we denote monomial lattices of $\mathbb{T}$ and $\mathbb{T}_A$, respectively, and by σ^+ and $\widehat{\sigma}^+$ we denote their positive orthants. Then both $\mathbb{C}[Z]$ and $\mathbb{C}[A]$ have a grading by $M_\mathbb{T}$ associated with the action of $\mathbb{T}$. To analyze semistability we use the notion of orbit cones (see [11, Definition 2.1]).

DEFINITION 4.1

The orbit cone $\omega_\mathbb{T}(z) \subset M_\mathbb{T} \otimes \mathbb{R}$ of $z \in Z$ is a convex (polyhedral) cone generated by

$$\bigl\{u \in M_\mathbb{T} \colon \exists f \in \mathbb{C}[Z]_u \ f(z) \neq 0\bigr\},$$

where $\mathbb{C}[Z]_u$ denotes the graded piece in degree u of $\mathbb{C}[Z]$.

That is, to prove that z is semistable with respect to χ^u it is sufficient to check that $u \in \omega_\mathbb{T}(z)$; hence, we want to describe the orbit cones for the considered action. We rely on a basic observation which follows directly from the definition of stability (see, e.g., [16, Section 8.1]).

LEMMA 4.2

Fix a character χ^u, $u \in M_{\mathbb{T}}$, which gives linearizations of actions of $\mathbb{T}$ both on Z and on A. Then the sets of stable and semistable points with respect to χ^u satisfy $Z^{\mathrm{ss}} = Z \cap A^{\mathrm{ss}}$ and $Z \cap A^s \subseteq Z^s$.

The first part can be rephrased in terms of orbit cones (cf. [11, Proposition 2.5]).

COROLLARY 4.3

The orbit cone for $z \in Z$ and the action of $\mathbb{T}$ on Z is equal to the orbit cone of z and the action on A.

Orbit cones for the affine space A are easy to describe. Let $\pi\colon \widehat{M} \to M_{\mathbb{T}}$ be the homomorphism of lattices corresponding to $\mathbb{T} \subseteq \mathbb{T}_A$; we will assume that it is given by the matrix U of weights of the action of $\mathbb{T}$ on A. By γ_z we denote the face of $\widehat{\sigma}^+$ generated by monomials that are nonvanishing on $\mathbb{T}_A \cdot z \subset A$. The proof of the following statement is straightforward.

LEMMA 4.4

Orbit cones for the action of $\mathbb{T}$ on A, hence also on Z, are images of faces of $\widehat{\sigma}^+$ under π. More precisely, $\omega_{\mathbb{T}}(z) = \pi(\gamma_z)$.

COROLLARY 4.5

A point z is semistable with respect to the $\mathbb{T}$-action on Z linearized by χ^u if and only if $u \in \pi(\gamma_z)$ (see also [11, Lemma 2.7]).

The next lemma follows from the fact that the actions of $\mathbb{T}$ and $\mathbb{T}_A$ on A commute (or for semistability from Lemma 4.4; cf. [11, Proposition 2.5]).

LEMMA 4.6

Stability, semistability, and isotropy groups of the action of $\mathbb{T}$ on A (hence also semistability and isotropy groups of the action on Z) are invariants of $\mathbb{T}_A$, that is, are properties of whole $\mathbb{T}_A$-orbits.

These observations are very useful in algorithms dealing with sets of semistable points (see [30]). Computations concerning stability are more subtle because of the orbit closedness condition: it may happen that a point is stable under the action on Z, but not on A. However, it turns out that for our purposes it is sufficient to check whether a point of Z is in A^s—by Lemma 4.2 such points are stable in Z.

LEMMA 4.7

Consider a $\mathbb{T}$-action on A linearized by χ^u, and assume that the isotropy group of a point $z \in A$ is finite. If u is in the relative interior of $\pi(\gamma_z)$, then z is stable.

Proof

A point in the boundary of $\mathbb{T} \cdot z$ is a limit of some 1-parameter subgroup of $\mathbb{T}$; hence, it belongs to an orbit corresponding to a proper face of γ_z. Thus, to prove the stability of z we want to find a $\mathbb{T}$-invariant section f of the trivial bundle on A such that $z \in A_f = \{a \in A\colon f(a) \neq 0\}$ and A_f does not contain any orbits corresponding to proper faces of γ_z. We will choose f, which is a character of $\mathbb{T}$ (regular on A). If u is in the relative interior of $\pi(\gamma_z)$, then there is some $\overline{u}$ in the relative interior of γ_z such that $\pi(\overline{u}) = u$, and we take $f = \chi^{\overline{u}}$. Because $\overline{u}$ is not contained in any face of γ_z, f vanishes on all orbits corresponding to faces of γ_z. $\square$

Next, we need to determine the orders of isotropy groups of points under the action of $\mathbb{T}$. They can be computed using the Smith normal form of a matrix (see [37, Theorem II.15]), which is implemented, for example, in Singular (see [15]). A matrix U_z is obtained from the matrix U of weights of the $\mathbb{T}$-action by choosing columns corresponding to nonzero coordinates in the orbit $\mathbb{T}_A \cdot z$.

LEMMA 4.8

Let $a_1, \ldots, a_p$ be the nonzero entries on the diagonal of the Smith normal form (over $\mathbb{Z}$) of U_z for some $z \in A$. If they fill the whole diagonal, then the order of the isotropy group of z under the $\mathbb{T}$-action is $a_1 \cdots a_p$. If there are also zeros on the diagonal, then the isotropy group of z is infinite.

Proof

Note that columns of U_z are exactly the rays of the orbit cone $\omega_{\mathbb{T}}(z)$, that is, U_z determines the homomorphism $M_z \to M_{\mathbb{T}}$ of monomial lattices of $\mathbb{T}_A \cdot z$ and $\mathbb{T}$. Then $\operatorname{Hom}(M_{\mathbb{T}}/M_z, \mathbb{C}^*)$ is isomorphic to the kernel of the corresponding morphism of the tori, which is exactly the isotropy group of z. The Smith normal form of U_z is obtained by multiplying it on both sides by some invertible integer matrices such that the result is a diagonal matrix with nonzero entries $a_1, \ldots, a_p$, where $a_i \mid a_{i+1}$ for $1 \leq i \leq p-1$. Thus, it gives the description of the quotient group $M_{\mathbb{T}}/M_z$ as a product of finite cyclic groups of orders $a_1, \ldots, a_p$ and $\mathbb{Z}^q$, where q is the number of zeros on the diagonal. $\square$

Now we can describe the algorithm which we use to determine a good linearization for constructing a GIT quotient of Z by $\mathbb{T}$ explicitly. We are looking for a linearization χ^u satisfying the following condition.

CONDITION 4.9

The semistability of a point of Z with respect to χ^u implies its stability with respect to χ^u.

This is because in such a situation we obtain a geometric quotient together with a nice description of the set of stable points. By Lemma 4.6 this set (and also the set of zeros of the irrelevant ideal) is a sum of intersections of certain $\mathbb{T}_A$-orbits in A with Z.

DEFINITION 4.10

A $\mathbb{T}_A$-orbit in A which has nonempty intersection with Z and whose points are semistable with respect to a fixed linearization χ^u will be called a $\mathbb{C}[Z]$-*relevant orbit* with respect to χ^u. (We will skip the information about the linearization whenever the choice is clear.)

Note that if a linearization χ^u satisfies Condition 4.9, then the intersections of Z with all $\mathbb{C}[Z]$-relevant orbits cover the set of stable points Z^s.

Algorithm 4.11 is implemented in the form of a small Singular package (available at www.mimuw.edu.pl/∼marysia/gitcomp.lib). The input data for the algorithm consist of

(1) the ideal $\mathcal{I}$ of Z,
(2) the matrix U defining the $\mathbb{T}$-action on A and Z,
(3) a linearization of this action, given by a character χ^u of $\mathbb{T}$, where $u \in M_{\mathbb{T}}$.

The output is whether χ^u satisfies Condition 4.9.

We start the computations by determining the set of $\mathbb{T}_A$-orbits which have nonempty intersection with Z. They are represented by convex polyhedral cones: faces of the positive orthant $\widehat{\sigma}^+$ of $\widehat{M}$. Such cones are called $\mathcal{I}$-*faces* in [30]. Note that by Lemmas 4.6 and 4.7 we need to check only properties of the whole $\mathbb{T}$-orbits; hence, the program operates on lists of $\mathcal{I}$-faces or corresponding orbit cones.

ALGORITHM 4.11

The following actions are performed.

(1) Determine the list $\mathcal{F}$ of $\mathcal{I}$-faces of $\widehat{\sigma}^+$. Here we use the Singular package `GITfan.lib` (see [30]), which for given $\mathcal{I}$ returns the desired list of cones.

(2) Determine the semistable points. For all $\mathcal{I}$-faces from $\mathcal{F}$ we compute rays of corresponding orbit cones by using the matrix U. Then, by Corollary 4.5 we check whether u is inside these cones. The result is the list $\mathcal{F}^{\mathrm{ss}}$ of orbit cones corresponding to $\mathbb{T}_A$-orbits semistable with respect to χ^u.

(3) Check the finiteness of the isotropy group. The order of the isotropy group is computed for each cone from $\mathcal{F}^{\mathrm{ss}}$, as described in Lemma 4.8. If for all cones from $\mathcal{F}^{\mathrm{ss}}$ points of corresponding orbits have finite isotropy groups, then

the next step is performed. Otherwise the negative answer is given immediately. Note that this point of the computations is independent of the linearization.

(4) Check the stability of elements of $\mathcal{F}^{\mathrm{ss}}$. By Lemma 4.7 we check whether u is in the relative interior of cones from $\mathcal{F}^{\mathrm{ss}}$. If it is true for all cones from this list, then the linearization given by χ^u satisfies Condition 4.9. In this case the program can output the list $\mathcal{F}^{\mathrm{ss}}$, which gives a useful description of the set of χ^u-stable points of Z. Also, the program returns the information on orders of isotropy groups for all stable orbits.

Finally, we reveal the main application of Algorithm 4.11. We are looking for a linearization of the $\mathbb{T} := \mathbb{T}_\Lambda$-action on $Z := \operatorname{Spec}\mathcal{R}$, embedded in $A \simeq \mathbb{C}^{15}$, which allows one to describe explicitly the geometry of the quotient. A very good candidate, because of its symmetries, is χ^κ given by the weight vector $\kappa = (2,2,2,2,2)$. The corresponding quotient makes a good starting point for performing flops leading to other resolutions (see Section 5). We prove that this is indeed the right choice.

PROPOSITION 4.12
The linearization of the $\mathbb{T}_\Lambda$-action on $\operatorname{Spec}\mathcal{R}$ given by χ^κ for $\kappa = (2,2,2,2,2)$ satisfies Condition 4.9. Therefore, the corresponding quotient is geometric. Moreover, all points of $\operatorname{Spec}\mathcal{R}$ which are semistable with respect to χ^κ have trivial isotropy group.

Proof
Computations performed using the implementation of Algorithm 4.11 give the result stated above. We use the weights of the $\mathbb{T}_W$-action instead of $\mathbb{T}_\Lambda$, which changes just the order of the isotropy group, multiplying it by 2. □

4.B. The set of stable points

Using Algorithm 4.11, for a chosen linearization satisfying Condition 4.9 one obtains an explicit description of the set $(\operatorname{Spec}\mathcal{R})^s$ of stable points. In general, it comes in the form of the list $\mathcal{F}^{\mathrm{ss}}$ of $\mathcal{I}$-faces corresponding to $\mathcal{R}$-relevant orbits (see Definition 4.10). We will describe these orbits in the case of the linearization χ^κ; by Proposition 4.12 their intersections with $\operatorname{Spec}\mathcal{R}$ cover the whole $(\operatorname{Spec}\mathcal{R})^s$.

It turns out that for χ^κ there are only 167 $\mathcal{R}$-relevant orbits in A. Because of the symmetries of generators of $\mathcal{I}$ the result may be presented as a much shorter list. The action of the permutation group on the set of indices of variables induces the action on the set of orbits. Hence, we list just combinatorial types of possible sets of vanishing variables defining orbits (see Table 2).

Note that in each description of an orbit type in Table 2 letters a, b, c, d, e stand for different elements of $\{0,1,2,3,4\}$. The division into orbit types is based on the number of u_i's equal to 0 and then on the set of w_{ij}'s equal to 0.

Table 2. $\mathcal{R}$-relevant orbits in the ambient affine space of $\operatorname{Spec}\mathcal{R}$.

Type ID	Equations	# orbits	dim	dim orbit $\cap \operatorname{Spec}\mathcal{R}$
5A	$u_0 = u_1 = u_2 = u_3 = u_4 = 0$, $w_{ab} = w_{cd} = 0$	15	8	5
5B	$u_0 = u_1 = u_2 = u_3 = u_4 = 0$, $w_{ab} = 0$	10	9	6
5C	$u_0 = u_1 = u_2 = u_3 = u_4 = 0$	1	10	7
3A	$u_a = u_b = u_c = 0$, $w_{ab} = w_{ac} = w_{bc} = 0$, $w_{de} = 0$	10	8	5
3B	$u_a = u_b = u_c = 0$, $w_{ab} = w_{de} = 0$	30	10	6
3C	$u_a = u_b = u_c = 0$, $w_{de} = 0$	10	11	7
1A	$u_a = 0$, $w_{ab} = w_{ac} = w_{bc} = 0$, $w_{de} = 0$	30	10	6
1B	$u_a = 0$, $w_{bc} = w_{de} = 0$	15	12	7
1C	$u_a = 0$, $w_{ab} = 0$	20	13	7
1D	$u_a = 0$	5	14	8
0A	$w_{ab} = w_{ac} = w_{bc} = w_{de} = 0$	10	11	7
0B	$w_{ab} = 0$	10	14	8
0C		1	15	9

4.C. Smoothness of the quotient

The last element needed in the proof that the geometric quotient $X = (\operatorname{Spec}\mathcal{R})^s/\mathbb{T}_\Lambda$ associated with the distinguished linearization χ^κ is a resolution of singularities of V/G is the smoothness of X. We follow the idea explained in [18, Proposition 4.5]: we check that X is a geometric quotient of a smooth variety by a free torus action. Since by Proposition 4.12 the action of $\mathbb{T}_\Lambda$ on $(\operatorname{Spec}\mathcal{R})^s$ is free, it is sufficient to show that X is nonsingular.

A natural approach is to compute the ideal of the set of singular points of $\operatorname{Spec}\mathcal{R}$ directly from the Jacobian criterion and show that it has empty intersection with $(\operatorname{Spec}\mathcal{R})^s$. However, the input data is too big for performing a direct computation in a reasonable amount of time. Hence, we divide the process into a few separate cases and make use of the $\mathbb{T}_\Lambda$-action and the description of $(\operatorname{Spec}\mathcal{R})^s$ in terms of the toric structure of the ambient affine space $\mathbb{C}^{15}$ in Table 2.

We rely on two basic observations. First, it is sufficient to prove the smoothness of one point in every $\mathbb{T}_\Lambda$-orbit in $(\operatorname{Spec}\mathcal{R})^s$. Thus, we may consider only points with all u_i's equal to 0 or 1; that is, using the $\mathbb{T}_\Lambda$-action we move nonzero u_i's to 1. This already simplifies the Jacobian matrix of $\operatorname{Spec}\mathcal{R}$ a lot. Then, since we do not want to treat each orbit separately, we use the symmetries of equations of Spec $\mathcal{R}$ so that we can consider certain representatives of combinatorial types of $\mathcal{R}$-relevant orbits. The following observation can be checked straightforwardly.

LEMMA 4.13

The equations of Spec $\mathcal{R}$*, listed in Proposition 3.17, are invariant under a cyclic change of indices* $i \mapsto i+1 \mod 5$.

PROPOSITION 4.14

The set $(\operatorname{Spec}\mathcal{R})^s$ *of stable points with respect to the linearization* χ^κ *is nonsingular.*

Proof

The argument is computational. We explain how to deal with the computations by using basic functions of, for example, Macaulay2 (see [22]). We assume that the equations of Spec $\mathcal{R}$ are ordered as in Proposition 3.17. To compute the Jacobian matrix we differentiate with respect to variables ordered as follows:

$$w_{01}, w_{02}, w_{03}, w_{04}, w_{12}, w_{13}, w_{14}, w_{23}, w_{24}, w_{34}, u_0, u_1, u_2, u_3, u_4.$$

Take $z \in \operatorname{Spec}\mathcal{R}$, and assume that all 6×6 minors of the Jacobian matrix vanish at z. Then we have to show that $z \notin (\operatorname{Spec}\mathcal{R})^s$, that is, z does not belong to any orbit from Table 2. Let us outline the computations. For each orbit type in Table 2 we choose representatives with respect to the cyclic group action, using Lemma 4.13, and consider only z from these chosen orbits. We simplify the Jacobian matrix by substituting 0 or 1 for some variables, looking at the $\mathbb{T}_\Lambda$-action on z and its orbit type. Then we compute some (suitably chosen) 6×6 minors of this matrix and look for monomials. Finding a monomial minor means that the product of some coordinates vanishes at z. Usually this gives a few subcases to consider. (The vanishing of each coordinate from the product has to be considered separately.) However, we obtain more precise information of the orbit type of z, which simplifies the Jacobian matrix even more. In each case, after a small number of such steps, we arrive at the conclusion that z is not contained in any $\mathcal{R}$-relevant orbit, which finishes the proof.

We are left with providing the details of the computations. To shorten the description, by $\det(i_1,\dots,i_k \mid j_1,\dots,j_k)$ we will denote the minor of the rows $i_1,\dots,i_k$ and the columns $j_1,\dots,j_k$ of the Jacobian matrix of equations of Spec $\mathcal{R}$. By $\operatorname{Mon}(x_{k_1},\dots,x_{k_n})$ we understand the set of all monomials in variables $x_{k_1},\dots,x_{k_n}$. There are four cases depending on the type of orbit from Table 2 in which z lies.

Type 5. We have $u_0 = u_1 = u_2 = u_3 = u_4 = 0$. After substituting into the Jacobian matrix, check that $\det(7,\dots,12 \mid 2,3,4,5,8,15) \in \operatorname{Mon}(w_{01}, w_{02}, w_{12})$; hence, one of these variables is 0. By permuting indices we get two cases.

(a) $w_{01} = 0$. Then we have $\det(0,1,4,11,13,14 \mid 0,1,2,9,11,13) \in \operatorname{Mon}(w_{13}, w_{14}, w_{34})$, and $\det(5,6,9,10,12,13 \mid 1,3,4,7,8,14) \in \operatorname{Mon}(w_{02}, w_{03}, w_{23})$. Hence, at least three of the w_{ij}'s are 0, which is impossible in orbits of type 5 in Table 2.

(b) $w_{02}=0$. Then we have $\det(0,1,4,10,13,14 \mid 0,1,2,6,7,9) \in \operatorname{Mon}(w_{03}, w_{04}, w_{34})$, and also $\det(2,3,9,11,12,14 \mid 0,3,4,10,11,14) \in \operatorname{Mon}(w_{12}, w_{14}, w_{24})$. Hence, again at least three w_{ij}'s vanish.

Type 3. Applying the $\mathbb{T}_\Lambda$-action we may move to the point where these u_i's that are nonzero are equal to 1. By Remark 4.13 there are two cases.

(a) $u_0=u_1=1$, $u_2=u_3=u_4=0$, $w_{01}=0$. Then $\det(0,\dots,4,14 \mid 0,1,2,9,17,18) \in \operatorname{Mon}(w_{04}, w_{34})$, but from the equations for cases 3A and 3B we see that only $w_{34}=0$ could happen. Next, $\det(0,\dots,3,6,14 \mid 0,1,3,10,18,19)$ is a monomial, so at least three w_{ij}'s are 0. This means that we are in the case 3A and $w_{23}=w_{24}=0$. However, in this case $\det(0,\dots,3,7,12 \mid 0,9,10,11,17,18)$ is a monomial and too many variables vanish.

(b) $u_0=u_2=1$, $u_1=u_3=u_4=0$, $w_{02}=0$. Then $\det(0,\dots,4,11 \mid 0,1,2,9,15,18) \in \operatorname{Mon}(w_{04}, w_{12}, w_{34})$. Again, the only possibility consistent with equations of 3A and 3B is $w_{34}=0$. Now $\det(0,\dots,3,5,11 \mid 0,9,10,11,17,18) \in \operatorname{Mon}(w_{01}, w_{04}, w_{24})$, but none of these variables can be 0 in the cases of type 3.

Type 1. By permuting indices and applying the $\mathbb{T}_\Lambda$-action we may assume that $u_0=u_1=u_2=u_3=1$ and $u_4=0$. Then $\det(5,7,\dots,11 \mid 2,3,4,5,17,19) \in \operatorname{Mon}(w_{01}, w_{03})$ and $\det(0,2,6,9,10,12 \mid 1,3,5,12,14,15) \in \operatorname{Mon}(w_{02}, w_{12})$. Hence, at least two variables vanish, and we are in the case 1A or 1B. From their description we see that the vanishing variables are w_{03} and w_{12}—two vanishing variables without index 4 must have a disjoint set of indices. Then $\det(4,7,\dots,11 \mid 2,3,4,8,17,19) = w_{01}^8$, but $w_{03}=w_{12}=w_{01}=0$ is impossible for any of these types.

Type 0. Applying the $\mathbb{T}_\Lambda$-action we may assume that $u_0=u_1=u_2=u_3=u_4=1$. Then $\det(6,\dots,11 \mid 2,3,4,5,11,17) \in \operatorname{Mon}(w_{01}, w_{04})$. Since all u_i's take the same value, the situation is symmetric with respect to the cyclic permutation of indices. Using permutations one can produce four other monomials in two variables from the given one and check that thus at least three different variables vanish. Hence, we are in the case of type 0A, and up to a cyclic permutation there are two possibilities.

(a) $w_{01}=w_{02}=w_{12}=w_{34}=0$. Then $\det(0,1,2,3,5,7 \mid 5,10,11,14,18,19) \in \operatorname{Mon}(w_{04}, w_{13})$, so five variables vanish, which is impossible in orbits of type 0A.

(b) $w_{01}=w_{13}=w_{03}=w_{24}=0$. Then $\det(0,\dots,4,9 \mid 0,1,2,9,16,18) \in \operatorname{Mon}(w_{04}, w_{34})$, a contradiction again. □

THEOREM 4.15

The GIT quotient X of $\operatorname{Spec}\mathcal{R}$ by $\mathbb{T}_\Lambda$ associated with the linearization χ^κ is a resolution of singularities of V/G.

Proof

The isomorphism $\mathcal{R}^{\mathbb{T}_\Lambda} \simeq \mathbb{C}[V]^G$ proved in Proposition 3.16 gives a proper birational morphism from X to V/G. The properness follows by [14, Proposition 14.1.12] applied to the embedding $\operatorname{Spec}\mathcal{R}$ and its quotients in the toric ambient spaces. Then by Proposition 4.14 we know that X is smooth. $\square$

5. The geometry of resolutions

5.A. The central resolution

Let us summarize the information which we obtained from previous sections. Let $G \subset \mathrm{Sp}(V)$ be the group defined in Section 2.C. The exceptional set of the resolution $\varphi : X \to V/G$ constructed as a GIT quotient in the previous section is covered by divisors $E_0, \dots, E_4$ associated to the classes of symplectic reflections in G. Each E_i is contracted by φ to a surface of A_1-singularities outside of $[0] \in V/G$. In terms of the ring $\mathcal{R}$, the divisors E_i are associated to functions t_i^{-2} (see Corollary 3.23). By Lemma 2.15 the resolution $\varphi : X \to V/G$ is symplectic. There is a unique 2-dimensional fiber of φ over $[0] \in V/G$ which has 11 components (see Theorem 2.13 and Lemma 2.8).

By C_i we denote a general fiber of $\varphi_{|E_i}$. Clearly $E_i \cdot C_j$ is -2 if $i = j$, and it is zero otherwise. Now we define $\kappa = \sum_i e_i$. In terms of the basis in $\mathrm{N}^1(X) = \mathrm{Cl}(X) \otimes \mathbb{R}$ dual to classes of C_i's, the class κ is the vector $(2,2,2,2,2)$ (see (2.14)). For $i = 0, \dots, 4$ in $\mathrm{N}^1(X)$ we consider classes $e_i = [-E_i]$. By [2, Theorem 3.5] we get the following.

LEMMA 5.1

For every resolution $X \to V/G$ the cone of movable divisors $\mathrm{Mov}(X)$ is spanned by the classes e_i.

By Theorem 4.15 the GIT quotient of $\mathcal{R}$ by $\mathbb{T}_\Lambda$ associated to the character κ is a resolution $\varphi^\kappa : X^\kappa \to V/G$.

Recall that Table 2 presents a list of big torus orbits in the affine space containing $\operatorname{Spec}\mathcal{R}$ which are relevant with respect to the finite isotropy and the semistability condition associated to κ. Note that divisors E_i are associated to relevant orbits of type 1D. The intersection $\bigcap_i E_i$ is associated to the unique relevant orbit of type 5C. In fact, from Corollary 3.18 we see that this special orbit comes from an equivariant embedding $\operatorname{Spec}\mathcal{R}(\mathbb{P}^2_4) \hookrightarrow \operatorname{Spec}\mathcal{R}$.

This gives rise to an embedding of GIT quotients $\iota : \mathbb{P}^2_4 \hookrightarrow X^\kappa$ such that $\iota^* : \operatorname{Pic} X^\kappa \to \operatorname{Pic}\mathbb{P}^2_4$ is an isomorphism. By F_0 we will denote $\iota(\mathbb{P}^2_4)$. It follows that we can identify $\mathrm{N}^1(X^\kappa) = \mathrm{N}^1(\mathbb{P}^2_4)$, and we have $\mathrm{Nef}(X^\kappa) \subseteq \mathrm{cone}(\alpha_i, \beta_i : 0 \le i \le 4)$, where $\alpha_i := (e_i + \kappa)/2$ and $\beta_i := (-e_i + \kappa)/2$ (cf. Lemma 2.4).

Dually, we have isomorphism $\iota_* : \mathrm{N}_1(X) \cong \mathrm{N}_1(\mathbb{P}^2_4) = \mathrm{N}^1(\mathbb{P}^2_4)$, and via this identification, the classes of (-1)-curves on $F_0 \cong \mathbb{P}^2_4$ are $f_{ij} = (e_i + e_j)/2$. Let $C_{ij} \subset F_0$ be one of these (-1)-curves. Then the family of deformations of C_{ij} is of dimension 2 at least (see [45, Proposition 2.3]), and it must cover a component of a 2-dimensional fiber of φ^κ. Let us call such a component F_{ij}. Since the intersection of C_{ij} with the ample class κ is 1, the family of deformations of C_{ij} in F_{ij} is unsplit and of dimension 2; hence, every curve in F_{ij} is numerically proportional to C_{ij} (see, e.g., [32, Proposition IV.3.13.3]).

Because the 2-dimensional fiber of φ^κ has 11 components we have a bijection between C_{ij}'s and components of this fiber that are different from F_0. Also, it follows that all curves in the 2-dimensional fiber of φ^κ have classes in $\mathrm{Eff}(F_0)$, and therefore, dually, $\mathrm{Nef}(X) = \mathrm{Nef}(F_0)$. Therefore, a contraction of the (-1)-curve C_{ij} in F_0 extends to a small contraction of X^κ, and by [45, Theorem 1.1], $F_{ij} \cong \mathbb{P}^2$. Again, because C_{ij} has intersection 1 with the ample class it follows that it is a line on F_{ij}. Thus, we have proved the following.

PROPOSITION 5.2

There exists a resolution $X^\kappa \to V/G$ such that κ is a class of an ample divisor on X^κ. The unique 2-dimensional fiber of the resolution $X^\kappa \to V/G$ consists of 11 components:

- *the unique component $F_0 = \bigcap_i E_i \cong \mathbb{P}^2_4$, and*
- *ten components F_{ij}, for $0 \le i < j \le 4$, which are contained in intersections of E_k's such that $k \notin \{i,j\}$, $F_{ij} \cong \mathbb{P}^2$.*

The intersection $F_0 \cap F_{ij}$ is a line on F_{ij} and, (-1)-curve on F_0. The line bundle associated to κ is $-K_{F_0}$ on F_0 and $\mathcal{O}(1)$ on every F_{ij}.

We note that curves C_{ij} can be related to orbits of type 5B, while the components F_{ij} can be related to orbits of type 3C in Table 2. In fact, the arguments above regarding F_{ij}'s can be replaced by direct calculations of quotients of respective closed subsets of $\mathrm{Spec}\,\mathcal{R}$.

5.B. Flops

We can use the identification ι^* introduced in the previous section to describe the other resolutions of V/G which will be obtained from $X^\kappa \to V/G$ by Mukai flops (see [45]).

Our description is similar to that of [2, Section 6.7]. Figure 2 illustrates the components of the 2-dimensional fiber of these resolutions and their incidence. The distinguished *central* component F_0 is always denoted by ★. Each of the diagrams in the table is described by the isomorphism class of this component. By $(\mathbb{P}^2)^\vee$ we denote the central component after the flop. Also the other components of the 2-dimensional fiber are denoted by the same name F_{ij} for every resolution. Their isomorphism types are described as follows: $\bullet = \mathbb{P}^2$, $\blacktriangle = \mathbb{P}^2_1$, ♦ $= \mathbb{P}^2_2$, ■ $=$ $\mathbb{P}^1 \times \mathbb{P}^1$, and ♠ $= \mathbb{P}^2_3$ is the blowup of $\mathbb{P}^2$ at three collinear points.

The incidence of components is denoted by line segments joining the respective symbols. The solid line denotes intersection along a rational curve, while a dotted line denotes intersection at a point. For the sake of clarity we ignore the intersection (at a point) of components which will not be flopped.

The first diagram in Figure 2 illustrates the special fiber of the unique central resolution in which the central component is $\mathbb{P}^2_4$ and the remaining components are $\mathbb{P}^2$. The other resolutions are obtained by flopping some components which are isomorphic to $\mathbb{P}^2$; the direction of the flops is indicated by arrows. That is, via

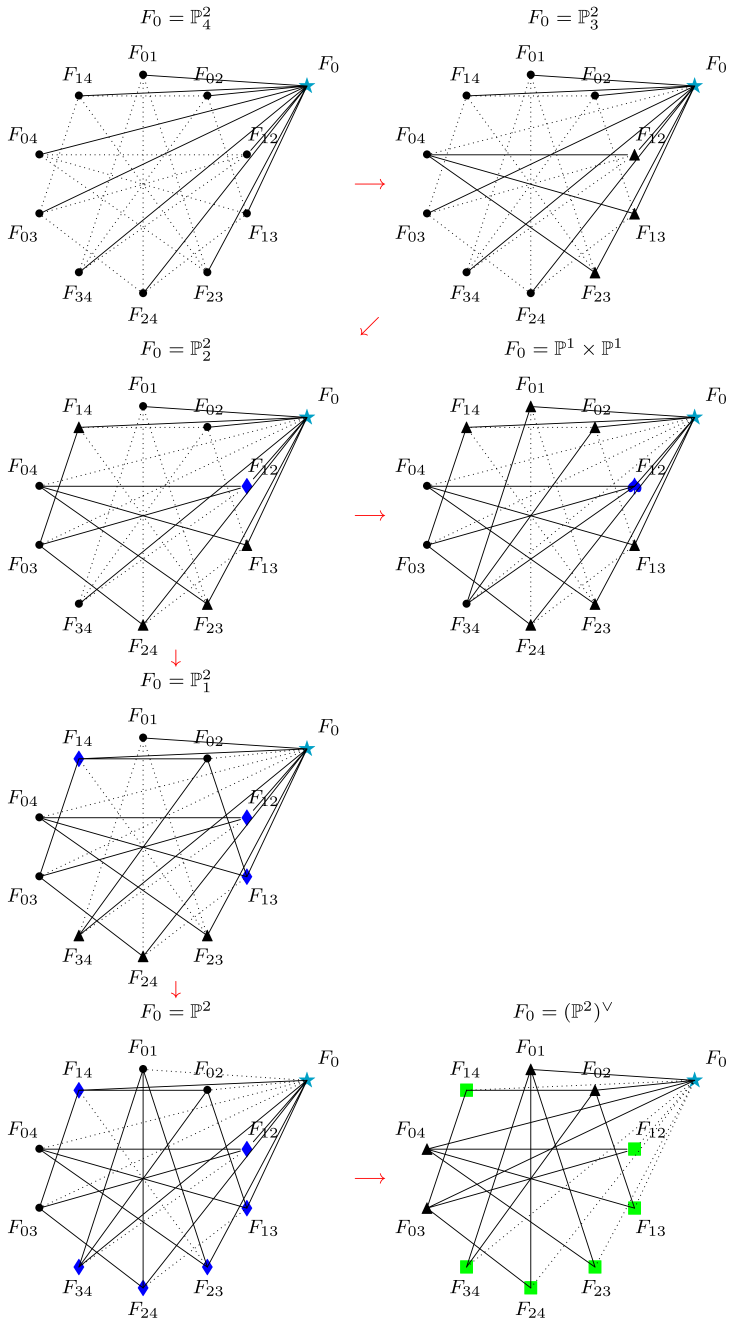

Figure 2. Flops of symplectic resolutions of V/G.

the identification of $\mathrm{N}^1(X)$ with $\mathrm{N}^1(F_0) = \mathrm{N}^1(\mathbb{P}^2_4)$, the ample cone of $\mathbb{P}^2_4$ is placed in the center of the movable cone of X and the direction of our flops points out outside the central chamber.

The first two flops are along F_{04} and F_{03}, and they lead to the central component isomorphic to, respectively, $\mathbb{P}^2_3$ (there are ten different resolutions of this type) and $\mathbb{P}^2_2$ (there are 30 different resolutions of this type). The surface $\mathbb{P}^2_2$ has three (-1)-curves which make a chain. Contracting the central one, we get $\mathbb{P}^1 \times \mathbb{P}^1$, while contracting any of the other two we get $\mathbb{P}^2_1$. Respectively, we can consider either a flop along F_{24}, which makes F_0 isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (there are ten resolutions of this type), or a flop along F_{02}, which gives $F_0 = \mathbb{P}^2_1$ (there are 20 resolutions of this type). This latter type can be further flopped along F_{01} to get $F_0 = \mathbb{P}^2$ (five resolutions of this type). Finally, F_0 can be flopped. We denote it then by $(\mathbb{P}^2)^\vee$. There are five resolutions of this type. We will denote them by $X^i \to V/G$. The cone $\mathrm{Nef}(X^i)$ is simplicial, and it is generated by e_i and by four classes $(e_i + e_j)/2$, with $j \neq i$. We will call it an *outer* chamber of $\mathrm{Mov}(X)$.

Note that, after the first flop, the surfaces F_{23}, F_{24}, and F_{34} have a common point, which comes from contracting the (-1)-curve on F_0. We ignore it in our diagram, since none of these three components will be flopped. Similarly, we will not put in the diagrams the intersection points which will be negligible from the point of view of possible flops.

Counting the number of resolutions, we obtain a result already announced by Bellamy [8].

PROPOSITION 5.3
There are 81 *symplectic resolutions of the quotient singularity* V/G.

In fact, from our construction it follows that symplectic resolutions of V/G are in bijection with chambers in the cone $\mathrm{cone}(e_0, \ldots, e_4)$ obtained by cutting this cone with hyperplanes perpendicular to classes f_{ij} and β_i, which were defined in Section 2.B.

6. A Kummer 4-fold

6.A. A Kummer surface

Let $\mathbb{E}$ be an elliptic curve with complex multiplication by $i = \sqrt{-1}$. That is, we have a linear automorphism $i \in \mathrm{Aut}(\mathbb{E})$ such that $i^2 = -\mathrm{id}_{\mathbb{E}}$. For simplicity, we can assume that $\mathbb{E} = \mathbb{C}/\mathbb{Z}[i]$ and that i acts by standard complex multiplication. The automorphism i of $\mathbb{E}$ has two fixed points $p_0 = [0]$ and $p_1 = [(1+i)/2]$, while it interchanges the other two order 2 points $i[1/2] = [i/2]$ and $i[i/2] = [1/2]$. Here the square brackets denote the classes in $\mathbb{C}/\mathbb{Z}[i]$. We see that, in fact, this multiplication yields an isomorphism of the group of order 2 points on $\mathbb{E}$ with the ring $\mathbb{Z}_2[i]$ with p_1 identified with the unique zero divisor $1+i$.

Let us recall that the standard representation of the binary dihedral group of order 8, or the quaternion group, Q_8 in $\mathrm{SL}(2, \mathbb{C})$ is given by the following 2×2

matrices over $\mathbb{C}$:

$$A_I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad A_J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad A_K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Since, in fact, this representation is in $\mathrm{SL}(2, \mathbb{Z}[i])$, we have the action of Q_8 on $\mathbb{E}^2$. It is not hard to check (see the argument below) that the group Q_8 acts on $\mathbb{E}^2$ with two fixed points, namely, (p_0, p_0) and (p_1, p_1), six points with isotropy $\mathbb{Z}_4$, and eight points with isotropy $\mathbb{Z}_2$. This makes 16 points with nontrivial isotropy. In fact, they are all order 2 points on $\mathbb{E}^2$, because $-\mathrm{id}$ is contained in every nontrivial subgroup of Q_8.

If n_8, n_4, and n_2 are the number of orbits of points with the isotropy Q_8, $\mathbb{Z}_4$, and $\mathbb{Z}_2$, respectively, then taking into consideration the ranks of the normalizers of these subgroups we get $n_8 = 2$, $n_4 = 3$, $n_2 = 2$, and $n_8 + 2n_4 + 4n_2 = 16$. Thus, on the quotient $\mathbb{E}^2/Q_8$ we have $n_8 = 2$ singularities of type D_4, $n_4 = 3$ singularities of type A_3, and $n_2 = 2$ singular points of type A_1. Resolving these singularities we obtain a K3 surface with $4 \cdot n_8 + 3 \cdot n_4 + 1 \cdot n_2 = 19$ exceptional (-2)-curves. We note that $\dim_{\mathbb{C}} H^{1,1}(\mathbb{E}^2)^{Q_8} = 1$ completes this number to the dimension of $H^{1,1}$ of a Kummer surface.

More generally, by the above argument, for any representation of Q_8 in $\mathrm{Aut}(\mathbb{E}^2)$ the numbers n_8, n_4, and n_2 defined above satisfy the two equations

$$n_8 + 2n_4 + 4n_2 = 16,$$
$$4n_8 + 3n_4 + n_2 = 19.$$

Clearly the numbers n_i are nonnegative integers and $n_8 > 0$. We find out that there are two solutions of this system:

$$(6.1) \qquad (n_8, n_4, n_2) = (2, 3, 2) \qquad \text{or} \qquad (n_8, n_4, n_2) = (4, 0, 3).$$

The latter solution is satisfied for the following representation of Q_8 in $\mathrm{Aut}(\mathbb{E}^2)$:

$$B_I = \begin{pmatrix} i & 0 \\ i+1 & -i \end{pmatrix}, \qquad B_J = \begin{pmatrix} -i & i-1 \\ 0 & i \end{pmatrix}, \qquad B_K = \begin{pmatrix} 1 & -1-i \\ 1-i & -1 \end{pmatrix}.$$

The fixed points for this representation are (p_0, p_0), (p_0, p_1), (p_1, p_0), and (p_1, p_1). Therefore, the remaining order 2 points have isotropy equal to $\mathbb{Z}_2$.

It is convenient to describe the action of the group Q_8 in terms of $\mathbb{Z}_2[i]$-modules. By M^r we will denote the free module $(\mathbb{Z}_2[i])^{\oplus r}$, and by M_0^r we denote its submodule $(1+i) \cdot M^r$, which consists of elements annihilated by $1+i$. As noted above, the points with nontrivial isotropy with respect to the action of Q_8 on $\mathbb{E}^2$ are of order 2, as $-\mathrm{id}$ is contained in every nontrivial subgroup of Q_8. Thus, in fact, to understand the isotropy of the action of Q_8 on $\mathbb{E}^2$ we can look into the action of $Q_8/\langle -\mathrm{id}\rangle$ on the set of order 2 points on $\mathbb{E}^2$, which is the free $\mathbb{Z}_2[i]$-module M^2. That is, we focus on representations of $Q_8/\langle -\mathrm{id}\rangle = \mathbb{Z}_2^{\oplus 2}$ in $\mathrm{SL}(2, \mathbb{Z}_2[i])$.

From this point of view the first representation of Q_8 is reduced to

$$A_I = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \qquad A_J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad A_K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

while the second is

$$B_I = \begin{pmatrix} i & 0 \\ i+1 & i \end{pmatrix}, \qquad B_J = \begin{pmatrix} i & i+1 \\ 0 & i \end{pmatrix}, \qquad B_K = \begin{pmatrix} 1 & i+1 \\ 1+i & 1 \end{pmatrix}.$$

Now the claims about the fixed point sets are easy to verify on M^2. Note that the second representation fixes the points whose coordinates are annihilated by $1+i$, which is the module M_0^2.

6.B. A symplectic Kummer 4-fold

After discussing the 2-dimensional case we pass to dimension 4. We use notation consistent with the preceding section. First, we prove a somewhat more general result on symplectic Kummer 4-folds.

PROPOSITION 6.2

Suppose that $G' < \mathrm{SL}(4, \mathbb{Z}[i])$ is a finite subgroup such that

- *as a subgroup of $\mathrm{SL}(4, \mathbb{C}) \supset \mathrm{SL}(4, \mathbb{Z}[i])$ G' is conjugate to the group G, and in particular, it is generated by five order 2 matrices T_i';*
- *the reduction $G'/\langle -\mathrm{id} \rangle \to \mathrm{SL}(4, \mathbb{Z}_2[i])$ acts on M^4 so that its action on M_0^4 is trivial and every element in $M^4 \setminus M_0^4$ has isotropy generated by $[\pm T_i']$ for some i.*

Then G' acts on $\mathbb{E}^4$ and the quotient $\mathbb{E}^4/G'$ admits a symplectic resolution X such that its Betti numbers are $b_2(X) = b_6(X) = 23$ and $b_4(X) = 276$.

Proof

Clearly, $G' < \mathrm{SL}(4, \mathbb{Z}[i])$ yields an action of G' on $\mathbb{E}^4$. We claim that any nontrivial isotropy group of this action which is different from a symplectic reflection T_i' is actually isomorphic and conjugate in $\mathrm{SL}(4, \mathbb{C})$ to G or to a group $\langle T_i', -T_i' \rangle$. This follows from the fact that every subgroup of G which is different from $\langle T_i \rangle$ contains $-\mathrm{id}$; hence, it is the isotropy of a point on $\mathbb{E}^4$ which is of order 2. Thus, the isotropy of the action of G' on $\mathbb{E}^4$ can be understood by looking at the action of $G'/\langle -\mathrm{id} \rangle$ on the module M^4, and the claim follows.

Locally both $\mathbb{C}^4/G$ and $\mathbb{C}^4/\langle \pm T_i \rangle$ admit symplectic resolutions. These resolutions can be glued to the global symplectic resolution X of $\mathbb{E}^4/G'$, because outside isolated points which are images of the order 2 points in $\mathbb{E}^4$, the singularities are 2-dimensional families of A_1 surface singularities whose resolution is unique.

Thus, it remains to calculate the Betti numbers of X. For this it is enough to find the number of irreducible 2-dimensional components of the singular locus of $\mathbb{E}^4/G'$. The normalization of such components is a quotient of the fixed point set of some T_i' by the group $N(T_i')/\langle T_i' \rangle = Q_8$ which, as we noted in Section 6.1, has either two or four points of isotropy equal to Q_8. On the other hand, $\mathbb{E}^4/G'$ has $16 = |M_0^4|$ singular points of type $\mathbb{C}^4/G$, and each of them belongs to five different 2-dimensional components of the singular locus of $\mathbb{E}^4/G'$. Since each such component contains four such points at most, it follows that the number of these components is $16 \cdot 5/4 = 20$ at least. Now the result follows by [23]. $\square$

From the above calculations one can conclude that any resolution $X \to \mathbb{E}^4/G'$ contracts 20 exceptional divisors, and it has 16 fibers of a type discussed in Section 5 and 30 fibers which are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Indeed, if G' satisfies the assumptions of Proposition 6.2, then 2^4 out of 2^8 2-torsion points in $\mathbb{E}^4$ have isotropy equal to G' while the remaining $2^8 - 2^4 = 240$ points have isotropy equal to the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by $\pm T_i$: these subgroups are normal and of index 8. Hence, in the quotient we get $240/8 = 30$ points with quotient singularities of type $\mathbb{C}^4/\langle \pm T_i \rangle$, the resolution of which is known to have $\mathbb{P}^1 \times \mathbb{P}^1$ as the 2-dimensional central fiber.

Although the number of such resolutions is 81^{16} (some of these resolutions lead to nonprojective varieties), there exists a unique special resolution which over each of the 16 points in $\mathbb{E}^4/G'$ with singularity of type $\mathbb{C}^4/G$ is of type φ^κ described in Section 5.A.

Let us consider the following five matrices defined over $\mathbb{Z}[i]$:

$$T_0' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1+i & 1 & 0 \\ 1-i & 0 & 0 & -1 \end{pmatrix}, \qquad T_1' = \begin{pmatrix} i & -1-i & 0 & 1-i \\ 0 & -i & -1+i & 0 \\ 0 & -1-i & i & 0 \\ 1+i & 0 & -1-i & -i \end{pmatrix},$$

$$T_2' = \begin{pmatrix} 1 & 0 & 0 & -1-i \\ 0 & -1 & 1+i & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad T_3' = \begin{pmatrix} i & 0 & 0 & 1-i \\ 1-i & -i & -1+i & 0 \\ 0 & -1-i & i & 1-i \\ 1+i & 0 & 0 & -i \end{pmatrix},$$

$$T_4' = \begin{pmatrix} 1 & -1+i & 0 & -1-i \\ -1-i & -1 & 1+i & 0 \\ 0 & -1+i & 1 & -1-i \\ 1-i & 0 & -1+i & -1 \end{pmatrix}.$$

LEMMA 6.3

The group G' generated by matrices T_i' satisfies the assumptions of Proposition 6.2.

Proof

Let us define

$$W = \begin{pmatrix} 1 & 1/2-i/2 & -1 & 0 \\ 1 & -1/2+i/2 & 0 & -1-i \\ 0 & -1/2-i/2 & i & 0 \\ 0 & 1/2+i/2 & 0 & 0 \end{pmatrix}.$$

Then for $i = 0, \dots, 4$ it holds that $T_i' = W^{-1} \cdot T_i \cdot W$, where the T_i's are the matrices from the list (2.6). Clearly, the reduction of each T_i' to M^4 (which by abuse of notation we denote by T_i' as well) is the identity on M_0^4, and it remains to check that the isotropy of every element from $M^4 \setminus M_0^4$ is generated by some T_i'. This can be done as follows. One can write $\ker(T_i' - \mathrm{id})$ as $M_0^4 + K_i$ where K_i is a rank 2 free $\mathbb{Z}_2[i]$-module. Next one checks for $i \neq j$ that $K_i + K_j = M^4$ holds, and thus, $K_i \cap K_j = \{0\}$. Finally, because $|(M_0^4 + K_i) \setminus M_0^4| = 48$ it follows that $|M^4| = |M_0^4| + \sum_i |(M_0^4 + K_i) \setminus M_0^4|$, and therefore, every element of $M^4 \setminus M_0^4$ belongs to exactly one $(M_0^4 + K_i) \setminus M_0^4$. We omit calculations. □

COROLLARY 6.4

The quotient $\mathbb{E}^4/G'$ *has a resolution which is a Kummer symplectic* 4*-fold* X *with* $b_2(X) = b_6(X) = 23$ *and* $b_4(X) = 276$.

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